CUBICAL APPROACH TO DERIVED FUNCTORS

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Abstract

We construct a cubical analog of the Tierney-Vogel theory of simplicial derived functors and prove that these cubical derived functors are naturally isomorphic to their simplicial counterparts. We also show that this result generalizes the well-known fact that the simplicial and cubical singular homologies of a topological space are naturally isomorphic.

1. Introduction

In [**24**] Tierney and Vogel for any functor $F\colon \mathscr{C} \to \mathscr{B}$, where $\mathscr{C}$ is a category with finite limits and a projective class $\mathscr{P}$, and $\mathscr{B}$ is an abelian category, constructed simplicial derived functors and investigated relationships of their theory with other theories of derived functors. Namely, they showed that if $\mathscr{C}$ is abelian and F is additive, then their theory coincides with the classical relative theory of Eilenberg-Moore [**7**], whereas if $\mathscr{C}$ is abelian and F is an arbitrary functor, then it gives a generalization of the theory of Dold-Puppe [**6**]. Besides, they proved that their derived functors are naturally isomorphic to the cotriple derived functors of Barr-Beck [**2**, **3**] if there is a cotriple in $\mathscr{C}$ that realizes the given projective class $\mathscr{P}$.

Almost all homology theories from classical homological algebra like, for example, Hochschild or André-Quillen (co)homologies as well as singular homology theory of topological spaces, can be naturally viewed as special cases of Tierney-Vogel simplicial derived functors ([**24**, (3.1)] and [**3**, (1.3), (1.4), (10.2)]).

It is well known that classical singular homology of topological spaces (usually defined using singular simplices) can be also obtained by means of singular cubes (see e.g., [**17**], [**21**], where the cubical approach is used). A question arises whether other homologies that are special cases of Tierney-Vogel theory admit a cubical description. The aim of this paper is to give a positive answer to this question. More precisely, using projective pseudocubical resolutions (Definition 6.10), we construct a cubical analog of the derived functor theory of Tierney-Vogel and prove the following

Theorem 1.1. *Suppose $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$ in the sense of [**24**, §2], $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \to \mathscr{B}$ a functor. Let $\mathbf{L}_n^\Delta F\colon \mathscr{C} \to \mathscr{B}$, $n \geqslant 0$, be the Tierney-Vogel simplicial derived functors of F, and*

$\mathbf{L}_n^{\square} F \colon \mathscr{C} \to \mathscr{B}$, $n \geqslant 0$, the cubical derived functors of F (Section 6). Then there is an isomorphism

$$\mathbf{L}_n^{\Delta} F(C) \cong \mathbf{L}_n^{\square} F(C), \quad C \in \mathscr{C}, \quad n \geqslant 0,$$

which is natural in F and in C.

Before saying more on the content of the paper itself, let us say a few words about cubical objects and techniques. Simplicial methods have been developed for a long time, and are well-known and successfully used in mathematics. Cubical techniques have seen less success, but developed initially on the one hand by the systematic use of singular cubes in the singular homology theory of topological spaces (see, for example, [**17**], [**21**]), and, on the other hand, by the papers of Kan [**13, 14**] which have related cubical sets to homotopy theory. There are several reasons why people mostly prefer simplicial objects to the cubical ones. For example, a cubical group, in fact even a cubical abelian group can fail to satisfy the Kan condition [**22, 23**]. (This defect is partially eliminated in [**25**] where it is shown that cubical groups which possess connections of [**4**] are Kan.) Another example is the classical bar construction which is naturally simplicial and not cubical. However, cubical objects have also a number of advantages compared to simplicial ones. A prominent example for this is a cubical homotopy which is given by a single morphism in each dimension, whereas a simplicial homotopy requires "many" morphisms.

Although cubical objects are not so heavily used in mathematics as the simplicial ones, the research (see, for example, [**1, 4, 5, 10, 11, 12, 16**]) shows that they are nevertheless interesting and important. In the present paper we show that the cubical theory is good enough to capture almost all (co)homology theories from classical homological algebra.

Now let us list contents of the paper by sections.

In Section 2 we remind the reader of some definitions and constructions needed in the sequel. In particular, we recall the definition of a (pre)cubical object and the construction of the singular cubical set of a topological space.

Section 3 is devoted to a review of the relative Eilenberg-Moore derived functor theory of additive functors from [**7**].

In Section 4 we recall the theory of Tierney-Vogel and prove that the simplicial derived functors of a functor $F \colon \mathscr{C} \to \mathscr{B}$ (where $\mathscr{C}$ is a category with finite limits and a fixed projective class, and $\mathscr{B}$ an abelian category) are just the Eilenberg-Moore derived functors of the unique additive extension $F_{ad} \colon \mathbb{Z}\mathscr{C} \to \mathscr{B}$ of the functor F, where $\mathbb{Z}\mathscr{C}$ is the free preadditive category generated by $\mathscr{C}$. (Note that this statement is essentially due to Barr and Beck [**3**, §5] (see Remark 4.6).)

Next, in Section 5 we define precubical resolutions and prove some of their properties. In particular, we prove Proposition 5.3 which says that our precubical resolutions possess pseudodegeneracies. This is crucial for constructing cubical derived functors as opposed to the simplicial case (see Remark 5.4).

Section 6 is devoted to the (pseudo)cubical normalization functor. This functor is used to define the cubical derived functors at the end of the section.

The main result of the paper (Theorem 1.1) is proved in Section 7. We show that for any functor $F \colon \mathscr{C} \to \mathscr{B}$, where $\mathscr{C}$ is a category with finite limits and a projective class, and $\mathscr{B}$ an abelian category, the cubical derived functors of F are naturally isomorphic

to the Eilenberg-Moore derived functors of the additive extension $F_{ad}\colon \mathbb{Z}\mathscr{C} \to \mathscr{B}$. This together with the aforementioned statement from Section 4 (Proposition 4.5) implies Theorem 1.1.

Finally, the purpose of Section 8 is to show that Theorem 1.1 generalizes the well-known fact that the simplicial and cubical singular homologies of a topological space are naturally isomorphic. The latter is classically proved via the method of acyclic models. We essentially also use an acyclic models argument. However, our approach provides a new conceptual explanation of the coincidence of the simplicial and cubical singular homologies of a topological space.

2.   Preliminaries

Let us recall some definitions and constructions needed in the sequel.

Definition 2.1. A *preadditive category* is a category $\mathscr{A}$ together with the following data:

(i) For any objects X, Y in $\mathscr{A}$, the set of morphisms $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is an abelian group.

(ii) For any morphisms $f, g\colon X \longrightarrow Y$, $h\colon W \longrightarrow X$ and $u\colon Y \longrightarrow Z$ in $\mathscr{A}$, the following hold

$$(f + g)h = fh + gh, \quad u(f + g) = uf + ug.$$

In other words, a preadditive category is just a ring with several objects in the sense of [**18**].

Definition 2.2. Let $\mathscr{A}$ be a preadditive category. A *(non-negatively graded) chain complex* C in $\mathscr{A}$ is a sequence of objects and morphisms

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

in $\mathscr{A}$ such that $d_n d_{n+1} = 0$, $n \geqslant 1$.

Definition 2.3. Let $\mathscr{A}$ be a preadditive category. An *augmented chain complex* over an object $A \in \mathscr{A}$ (or just a complex over A) is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} A$$

in $\mathscr{A}$ such that $d_n d_{n+1} = 0$, $n \geqslant 0$. In other words, an augmented chain complex in $\mathscr{A}$ is a chain complex C in $\mathscr{A}$ together with a morphism $C_0 \xrightarrow{d_0} A$ satisfying $d_0 d_1 = 0$.

Let C and C' be chain complexes in $\mathscr{A}$. A morphism of chain complexes $f\colon C \to C'$ is a sequence of morphisms $\{f_n\colon C_n \to C'_n\}_{n \geqslant 0}$ in $\mathscr{A}$ such that $d_n f_n = f_{n-1} d'_n$, $n \geqslant 1$. A morphism of augmented chain complexes is defined similarly.

Definition 2.4. Let $f, g\colon C \to C'$ be morphisms of chain complexes in a preadditive category $\mathscr{A}$. One says that f and g are *chain homotopic* if there is a sequence $\{h_n\colon C_n \to C'_{n+1}\}_{n \geqslant 0}$ of morphisms in $\mathscr{A}$ such that

$$d'_1 h_0 = f_0 - g_0,$$

$$d'_{n+1} h_n + h_{n-1} d_n = f_n - g_n, \quad n \geqslant 1.$$

Next we recall the well-known definition:

Definition 2.5. A *presimplicial object* S in a category $\mathscr{C}$ is a family of objects $(S_n \in \mathscr{C})_{n \geqslant 0}$ together with face morphisms

$$\partial_i \colon S_n \to S_{n-1} \quad (n \geqslant 1, \ 0 \leqslant i \leqslant n)$$

in $\mathscr{C}$ satisfying

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad i < j.$$

A *simplicial object* is a presimplicial object S together with degeneracy morphisms

$$s_i \colon S_n \to S_{n+1} \quad (0 \leqslant i \leqslant n)$$

in $\mathscr{C}$ satisfying

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & i < j, \\ \mathrm{id} & i = j, \quad j+1, \\ s_j \partial_{i-1} & i > j+1, \end{cases}$$

and

$$s_i s_j = s_{j+1} s_i \qquad i \leqslant j.$$

A morphism $f \colon S \to S'$ between presimplicial objects in a category $\mathscr{C}$ is a family of morphisms $(f_n \colon S_n \to S'_n)_{n \geqslant 0}$ in $\mathscr{C}$ which commute with the face operators. If S and S' are simplicial objects, then the morphisms f_n must commute with the face and degeneracy operators.

Let $\mathscr{C}$ be a category and $C \in \mathscr{C}$ an object. An *augmented (pre)simplicial object* over C is a (pre)simplicial object S in $\mathscr{C}$ together with a morphism $\partial \colon S_0 \to C$ satisfying $\partial \partial_0 = \partial \partial_1$.

Any presimplicial object in a preadditive category gives rise to a chain complex in the following way:

Definition 2.6. Let S be a presimplicial object in a preadditive category $\mathscr{A}$. The *unnormalized chain complex* $U(S)$ associated to S is defined by

$$U(S)_n = S_n, \quad n \geqslant 0,$$

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_i \colon U(S)_n \longrightarrow U(S)_{n-1}, \quad n > 0.$$

The presimplicial identities imply that $d^2 = 0$.

Further we recall some basic definitions from the cubical theory.

Definition 2.7. A *precubical object* X in a category $\mathscr{C}$ is a family of objects $(X_n \in$

$\mathscr{C})_{n \geqslant 0}$ together with morphisms

$$\partial_i^0, \partial_i^1 \colon X_n \to X_{n-1} \quad (n \geqslant 1, \quad 1 \leqslant i \leqslant n)$$

in $\mathscr{C}$ satisfying

$$\partial_i^\alpha \partial_j^\varepsilon = \partial_{j-1}^\varepsilon \partial_i^\alpha \quad i < j, \quad \alpha, \varepsilon \in \{0, 1\}.$$

The morphisms ∂_i^0 and ∂_i^1 are called *face operators*.

Definition 2.8 ([**13**])**.** A *cubical object* X in a category $\mathscr{C}$ is a family of objects $(X_n \in \mathscr{C})_{n \geqslant 0}$ together with morphisms

$$\partial_i^0, \partial_i^1 \colon X_n \to X_{n-1} \quad (n \geqslant 1, \quad 1 \leqslant i \leqslant n)$$

and

$$s_i \colon X_{n-1} \to X_n \quad (n \geqslant 1, \quad 1 \leqslant i \leqslant n)$$

in $\mathscr{C}$ which satisfy

$$\partial_i^\alpha \partial_j^\varepsilon = \partial_{j-1}^\varepsilon \partial_i^\alpha \quad i < j, \quad \alpha, \varepsilon \in \{0, 1\}.$$

$$s_i s_j = s_{j+1} s_i \quad i \leqslant j$$

and

$$\partial_i^\alpha s_j = \begin{cases} s_{j-1} \partial_i^\alpha & i < j, \\ \mathrm{id} & i = j, \\ s_j \partial_{i-1}^\alpha & i > j, \end{cases}$$

where $\alpha \in \{0, 1\}$. The morphisms s_i are called *degeneracy operators*.

Let $\mathscr{C}$ be a category and C an object in $\mathscr{C}$. An *augmented (pre)cubical object* over C is a (pre)cubical object X in $\mathscr{C}$ together with a morphism $\partial \colon X_0 \to C$ satisfying $\partial \partial_1^0 = \partial \partial_1^1$. We will denote such an object by $X \xrightarrow{\ \partial\ } C$.

A morphism $f \colon X \to X'$ between precubical objects in a category $\mathscr{C}$ is a family of morphisms $(f_n \colon X_n \to X'_n)_{n \geqslant 0}$ in $\mathscr{C}$ which commute with the face operators. If X and X' are cubical objects, then the morphisms f_n must commute with the face and degeneracy operators. A morphism between augmented (pre)cubical objects $X \xrightarrow{\ \partial\ } C$ and $X' \xrightarrow{\ \partial'\ } C'$ in $\mathscr{C}$ is a morphism

$$\overline{f} = (f_n \colon X_n \to X'_n)_{n \geqslant 0}$$

between the (pre)cubical objects X and X' together with a morphism $f \colon C \to C'$ satisfying $f \partial = \partial' f_0$.

Definition 2.9. Let X be a precubical object in a preadditive category $\mathscr{A}$. The unnormalized chain complex $C(X)$ associated to X is defined by

$$C(X)_n = X_n, \quad n \geqslant 0,$$

$$d_n = \sum_{i=1}^{n} (-1)^i (\partial_i^1 - \partial_i^0) \colon C(X)_n \longrightarrow C(X)_{n-1}, \quad n > 0.$$

The precubical identities show that $d^2 = 0$.

Example 2.10. Let I^n denote the standard n-cube, $n \geqslant 0$. There are standard maps

$$\delta_\varepsilon^i \colon I^{n-1} \to I^n, \quad n \geqslant 1, \quad \varepsilon \in \{0,1\} \quad 1 \leqslant i \leqslant n,$$

and

$$\eta^i \colon I^n \to I^{n-1}, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant n,$$

defined by

$$\delta_\varepsilon^i(t_1, \cdots, t_{n-1}) = (t_1, \cdots, t_{i-1}, \varepsilon, t_i, \cdots, t_{n-1}),$$

and

$$\eta^i(t_1, \cdots t_n) = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n),$$

respectively. The spaces $\{I^n\}_{n \geqslant 0}$ together with these maps form the *standard cocubical object* in the category **Top** of topological spaces. This cocubical object gives rise to a classical example of a cubical object, the singular cubical set $Q(Y)$ of a topological space Y. The precise construction is:

$$Q_n(Y) = \mathrm{Hom}_{\mathbf{Top}}(I^n, Y),$$

$$\partial_i^\varepsilon(f \colon I^n \to Y) = f\delta_\varepsilon^i, \quad \varepsilon \in \{0,1\}, \quad n \geqslant 1 \quad 1 \leqslant i \leqslant n,$$

$$s_i(f \colon I^{n-1} \to Y) = f\eta^i, \quad n \geqslant 1 \quad 1 \leqslant i \leqslant n.$$

Note that $Q(Y)$ is used to define the singular cubical homology of the space Y (see [**17**, II.§2] or Example 6.9 below).

3. Partially defined Eilenberg-Moore derived functors

The following definitions are well-known.

Definition 3.1 ([**7**, I.3]). Let $\mathscr{A}$ be a preadditive category and $\mathscr{P}$ a class of objects in $\mathscr{A}$ (which need not be a "projective class" in any sense). A complex

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} A$$

over $A \in \mathscr{A}$ is said to be $\mathscr{P}$-*acyclic* if for any $Q \in \mathscr{P}$ the sequence of abelian groups

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{A}}(Q, C_1) \xrightarrow{\partial_{1*}} \mathrm{Hom}_{\mathscr{A}}(Q, C_0) \xrightarrow{\partial_{0*}} \mathrm{Hom}_{\mathscr{A}}(Q, A) \longrightarrow 0$$

is exact.

Definition 3.2 ([**7**, I.3]). Let $\mathscr{A}$ be a preadditive category and $\mathscr{P}$ a class of objects in $\mathscr{A}$. A $\mathscr{P}$-*resolution* of an object $A \in \mathscr{A}$ is a $\mathscr{P}$-acyclic complex

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A$$

over A with $P_n \in \mathscr{P}$, $n \geqslant 0$.

Note that an object $A \in \mathscr{A}$ need not necessarily possess a $\mathscr{P}$-resolution.

There is a comparison theorem for $\mathscr{P}$-resolutions which can be proved using the standard homological algebra arguments (see e.g., [**26**, 2.2.7]). More precisely, the following is valid.

Proposition 3.3 (Comparison theorem)**.** *Let* $P_* \longrightarrow A$ *be a complex over* $A \in \mathscr{A}$ *consisting of objects of* $\mathscr{P}$, *and let* $Q_* \longrightarrow B$ *be a* $\mathscr{P}$-*acyclic complex. Then any morphism* $f \colon A \longrightarrow B$ *can be extended to a morphism of augmented chain complexes*

$$
\begin{array}{ccc}
P_* & \longrightarrow & A \\
\overline{f} \downarrow & & \downarrow f \\
Q_* & \longrightarrow & B.
\end{array}
$$

Moreover, any two such extensions are chain homotopic (see Definition 2.4).

Suppose $\mathscr{A}$ is a preadditive category, $\mathscr{P}$ a class of objects in $\mathscr{A}$, $\mathscr{B}$ an abelian category, $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ an additive functor, and $\mathscr{A}'$ the full subcategory of those objects in $\mathscr{A}$ which possess $\mathscr{P}$-resolutions. Recall that Proposition 3.3 allows one to construct the left derived functors $\mathbf{L}_n^{\mathscr{P}} F \colon \mathscr{A}' \longrightarrow \mathscr{B}$, $n \geqslant 0$, of F with respect to the class $\mathscr{P}$ as follows. If $A \in \mathscr{A}'$, choose (once and for all) a $\mathscr{P}$-resolution $P_* \longrightarrow A$ and define

$$
\mathbf{L}_n^{\mathscr{P}} F(A) = H_n(F(P_*)), \quad n \geqslant 0.
$$

Remark 3.4. If $\mathscr{P}$ is a projective class in the sense of [**7**], then $\mathbf{L}_n^{\mathscr{P}} F$, $n \geqslant 0$, are exactly the derived functors introduced in [**7**, I.3]. Note that in this case $\mathscr{A}' = \mathscr{A}$, i.e., the functors $\mathbf{L}_n^{\mathscr{P}} F$ are defined everywhere.

Next we recall

Definition 3.5 ([**7**, I.2])**.** Let $\mathscr{A}$ be a preadditive category and $\mathscr{P}$ a class of objects of $\mathscr{A}$. A sequence

$$
X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z
$$

in $\mathscr{A}$ is said to be $\mathscr{P}$-*exact* if $gf = 0$ and the sequence of abelian groups

$$
\mathrm{Hom}_{\mathscr{A}}(P, X) \xrightarrow{\ f_*\ } \mathrm{Hom}_{\mathscr{A}}(P, Y) \xrightarrow{\ g_*\ } \mathrm{Hom}_{\mathscr{A}}(P, Z)
$$

is exact for any $P \in \mathscr{P}$.

Definition 3.6 ([**7**, I.2])**.** A *closure* of a class $\mathscr{P}$, denoted by $\overline{\mathscr{P}}$, is the class of all those objects $Q \in \mathscr{A}$ for which

$$
\mathrm{Hom}_{\mathscr{A}}(Q, X) \xrightarrow{\ f_*\ } \mathrm{Hom}_{\mathscr{A}}(Q, Y) \xrightarrow{\ g_*\ } \mathrm{Hom}_{\mathscr{A}}(Q, Z)
$$

is exact whenever $X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z$ is $\mathscr{P}$-exact.

Clearly, $\mathscr{P} \subseteq \overline{\mathscr{P}}$ and $\mathscr{P}$-exactness is equivalent to $\overline{\mathscr{P}}$-exactness. In particular, $\overline{\overline{\mathscr{P}}} = \overline{\mathscr{P}}$.

Note that if a preadditive category $\mathscr{A}$ has a terminal object, then any $\mathscr{P}$-resolution is a $\overline{\mathscr{P}}$-resolution as well. This together with 3.3 implies the following

Proposition 3.7. *Let* $\mathscr{A}$ *be a preadditive category with a terminal object,* $\mathscr{P}$ *a class of objects in* $\mathscr{A}$, $\mathscr{B}$ *an abelian category,* $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ *an additive functor, and* A *an object in* $\mathscr{A}$ *which possesses a* $\mathscr{P}$-*resolution. Then there is a natural isomorphism*

$$
\mathbf{L}_n^{\mathscr{P}} F(A) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} F(A), \quad n \geqslant 0.
$$

4. Simplicial derived functors and Eilenberg-Moore derived functors

In this section we review the construction of simplicial derived functors from [**24**, §2] and show that they can be obtained as derived functors of an additive functor.

Definition 4.1. Let $\mathscr{P}$ be a class of objects in a category $\mathscr{C}$.

(i) A morphism $f\colon C \to C'$ in $\mathscr{C}$ is said to be $\mathscr{P}$-*epimorphic* if

$$\operatorname{Hom}_{\mathscr{C}}(P, f)\colon \operatorname{Hom}_{\mathscr{C}}(P, C) \to \operatorname{Hom}_{\mathscr{C}}(P, C')$$

is surjective for all $P \in \mathscr{P}$.

(ii) The class $\mathscr{P}$ is called a *projective class* if for each $C \in \mathscr{C}$ there exists a $\mathscr{P}$-epimorphism $e\colon P \to C$ with $P \in \mathscr{P}$.

Definition 4.2 ([**24**, (2.1)]). Let $\mathscr{C}$ be a category and

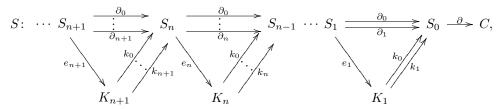$$f_0, f_1, \cdots, f_n\colon C \to D$$

a sequence of morphisms in $\mathscr{C}$, $n \geqslant 0$. A *simplicial kernel* of the sequence $(f_0, f_1, \cdots, f_n)$ is a sequence

$$k_0, k_1, \cdots, k_{n+1}\colon K \to C$$

such that

(i) $f_i k_j = f_{j-1} k_i$ for $0 \leqslant i < j \leqslant n+1$,

(ii) if $h_0, h_1, \cdots, h_{n+1}\colon L \to C$ is any other sequence satisfying identities $f_i h_j = f_{j-1} h_i$ for $0 \leqslant i < j \leqslant n+1$, then there exists a unique morphism $h\colon L \to K$ in $\mathscr{C}$ with $k_i h = h_i$, $0 \leqslant i \leqslant n+1$.

Let $\mathscr{P}$ be a projective class in $\mathscr{C}$, $S \xrightarrow{\partial} C$ an augmented presimplicial object over $C \in \mathscr{C}$, and suppose that $\mathscr{C}$ has finite limits. There is a factorization ([**24**, (2.3)])

where $K_1 \xrightarrow[k_1]{k_0} S_0$ is a kernel pair of $S_0 \xrightarrow{\partial} C$, and

$$k_0, k_1, \cdots, k_n\colon K_n \to S_{n-1}$$

a simplicial kernel of $(\partial_0, \cdots, \partial_{n-1})$ for $n \geqslant 2$.

One says that $S \xrightarrow{\partial} C$ is a $\mathscr{P}$-projective presimplicial resolution if $S_n \in \mathscr{P}$, $n \geqslant 0$, and ∂ and e_n, $n \geqslant 1$, are $\mathscr{P}$-epimorphic.

Note that if $\mathscr{C}$ is a category with finite limits and a projective class $\mathscr{P}$, then any $C \in \mathscr{C}$ has a $\mathscr{P}$-projective resolution.

Now we are ready to recall the construction of simplicial derived functors. Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. The simplicial derived functors $\mathbf{L}_n^\Delta F$ of F with respect to the class $\mathscr{P}$ are defined as follows [**24**, §2]. For any object $C \in \mathscr{C}$, choose (once and for all) a $\mathscr{P}$-projective presimplicial resolution

$$S \xrightarrow{\ \partial\ } C$$

of C and define

$$\mathbf{L}_n^\Delta F(C) = H_n(U(F(S))), \quad n \geqslant 0.$$

By the comparison theorem for projective presimplicial resolutions [**24**, (2.4) Theorem], the objects $\mathbf{L}_n^\Delta F(C)$ are well-defined and functorial in F and C.

We will now show that the derived functors $\mathbf{L}_n^\Delta F$ can be obtained as the derived functors of some additive functor. First recall

Lemma 4.3. *Let* $S \xrightarrow{\ \partial_0\ } S_{-1}$ *be an augmented presimplicial set. Suppose that the map* $\partial_0\colon S_0 \longrightarrow S_{-1}$ *is surjective and the following extension condition holds: For any* $n \geqslant 0$ *and any collection of* $n+2$ *elements* $x_i \in S_n$, $0 \leqslant i \leqslant n+1$, *satisfying*

$$\partial_i x_j = \partial_{j-1} x_i, \quad 0 \leqslant i < j \leqslant n+1,$$

there exists $x \in S_{n+1}$ *such that* $\partial_i x = x_i$, $0 \leqslant i \leqslant n+1$. *Then the augmented chain complex*

$$U(\mathbb{Z}[S]) \longrightarrow \mathbb{Z}[S_{-1}]$$

is chain contractible ($\mathbb{Z}[X]$ *denotes the free abelian group generated by* X*). In particular, it has trivial homology in each dimension.*

The proof is standard (one constructs inductively a presimplicial contraction).

Example 4.4. Let $S \xrightarrow{\ \partial\ } C$ be a $\mathscr{P}$-projective presimplicial resolution of C and suppose $P \in \mathscr{P}$. Then the augmented presimplicial set

$$\operatorname{Hom}_{\mathscr{C}}(P, S) \xrightarrow{\ \partial_*\ } \operatorname{Hom}_{\mathscr{C}}(P, C)$$

satisfies the conditions of 4.3. Consequently, the homologies of the augmented chain complex

$$U(\mathbb{Z}[\operatorname{Hom}_{\mathscr{C}}(P, S)]) \longrightarrow \mathbb{Z}[\operatorname{Hom}_{\mathscr{C}}(P, C)]$$

vanish.

Now suppose again that $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Let $\mathbb{Z}\mathscr{C}$ denote the free preadditive category generated by $\mathscr{C}$ [**18**, §1], i.e., the objects of $\mathbb{Z}\mathscr{C}$ are those of $\mathscr{C}$, and for any objects C and D in $\mathscr{C}$, $\operatorname{Hom}_{\mathbb{Z}\mathscr{C}}(C, D)$ is the free abelian group generated by $\operatorname{Hom}_{\mathscr{C}}(C, D)$. The composition of morphisms in $\mathbb{Z}\mathscr{C}$ is induced by that in $\mathscr{C}$. Clearly, $\mathscr{C}$ is a subcategory of $\mathbb{Z}\mathscr{C}$. Further, since the category $\mathscr{B}$ is abelian (and

therefore additive), the functor $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ can be uniquely extended to an additive functor

$$F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}.$$

The following proposition relates the simplicial derived functors of F to the Eilenberg-Moore derived functors of F_{ad}.

Proposition 4.5. *Suppose $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Then:*

(i) For any $\mathscr{P}$-projective presimplicial resolution $S \longrightarrow C$, the augmented chain complex

$$U(S) \longrightarrow C$$

in $\mathbb{Z}\mathscr{C}$ is a $\mathscr{P}$-resolution of C in the sense of Definition 3.2.

(ii) For any $C \in \mathscr{C}$, there is a natural isomorphism

$$\mathbf{L}_n^\Delta F(C) \cong \mathbf{L}_n^{\mathscr{P}} F_{ad}(C), \quad n \geqslant 0.$$

Proof. The first claim immediately follows from 4.4 and the definition of $\mathbb{Z}\mathscr{C}$. The second claim is a consequence of the first one and the definition of F_{ad}. Indeed, if $S \longrightarrow C$ is a $\mathscr{P}$-projective presimplicial resolution of C, then we have

$$\mathbf{L}_n^\Delta F(C) = H_n(U(F(S))) = H_n(F_{ad}(U(S))) = \mathbf{L}_n^{\mathscr{P}} F_{ad}(C). \qquad \square$$

Remark 4.6. Proposition 4.5 is essentially due to Barr and Beck [**3**, §5]. More precisely, in the case when the projective class $\mathscr{P}$ comes from a cotriple (see [**24**, §3]) the above statement is proved in [**3**, §5]. (The cotriple derived functor theory of Barr-Beck is a special case of the Tierney-Vogel theory [**24**, §3].) Thus 4.5 is a simple generalization of the result of Barr and Beck.

5. Cubical resolutions

This section is devoted to the construction and properties of cubical resolutions which we use to define cubical derived functors in Section 6.

Definition 5.1. Let $\mathscr{C}$ be a category and

$$f_1^0, \ldots, f_n^0, f_1^1, \ldots, f_n^1\colon C \to D$$

a sequence of morphisms in $\mathscr{C}$, $n \geqslant 1$. A *cubical kernel* of the sequence $(f_1^0, \ldots, f_n^0, f_1^1, \ldots, f_n^1)$ is a sequence

$$k_1^0, \ldots, k_n^0, k_{n+1}^0, k_1^1, \ldots, k_n^1, k_{n+1}^1\colon K \to C$$

of morphisms such that

(i) $f_i^\varepsilon k_j^\alpha = f_{j-1}^\alpha k_i^\varepsilon$ for $1 \leqslant i < j \leqslant n+1$, $\varepsilon, \alpha \in \{0,1\}$;

(ii) if $h_1^0, \ldots, h_n^0, h_{n+1}^0, h_1^1, \ldots, h_n^1, h_{n+1}^1\colon L \to C$ is any other sequence satisfying identities $f_i^\varepsilon h_j^\alpha = f_{j-1}^\alpha h_i^\varepsilon$ for $1 \leqslant i < j \leqslant n+1$, $\varepsilon, \alpha \in \{0,1\}$, then there exists a unique morphism $h\colon L \to K$ in $\mathscr{C}$ with $k_i^\varepsilon h = h_i^\varepsilon$, $1 \leqslant i \leqslant n+1$, $\varepsilon \in \{0,1\}$.

It immediately follows from the definition that the cubical kernel is just a special kind of finite limit.

Now suppose $\mathscr{C}$ is a category with finite limits, let $\mathscr{P}$ be a projective class in $\mathscr{C}$ (see Definition 4.1), and $X \xrightarrow{\ \partial\ } C$ be an augmented precubical object over $C \in \mathscr{C}$. Then we have a factorization

$$X\colon \ \cdots\ X_{n+1} \ \genfrac{}{}{0pt}{}{\partial_1^0}{\vdots\ \partial_{n+1}^1} \ X_n \ \genfrac{}{}{0pt}{}{\partial_1^0}{\vdots\ \partial_n^1} \ X_{n-1} \ \cdots X_1 \ \genfrac{}{}{0pt}{}{\partial_1^0}{\partial_1^1} \ X_0 \ \xrightarrow{\ \partial\ } \ C,$$

where $K_1 \underset{k_1^1}{\overset{k_1^0}{\rightrightarrows}} X_0$ is a kernel pair of $X_0 \xrightarrow{\ \partial\ } C$, and
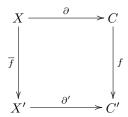
$$k_1^0, \ldots, k_n^0, k_1^1, \ldots, k_n^1 \colon K_n \to X_{n-1}$$

is a cubical kernel of $(\partial_1^0, \ldots, \partial_{n-1}^0, \partial_1^1, \ldots, \partial_{n-1}^1)$ for $n \geqslant 2$. We say that

1) $X \xrightarrow{\ \partial\ } C$ is $\mathscr{P}$-*projective* if each $X_n \in \mathscr{P}$;

2) $X \xrightarrow{\ \partial\ } C$ is $\mathscr{P}$-*exact* if ∂ and e_n $(n \geqslant 1)$ are $\mathscr{P}$-epimorphic;

3) $X \xrightarrow{\ \partial\ } C$ is a $\mathscr{P}$-*projective* resolution of C if it is $\mathscr{P}$-projective and $\mathscr{P}$-exact.

Obviously, if $\mathscr{C}$ is a category with finite limits and a projective class $\mathscr{P}$, then each $C \in \mathscr{C}$ has a $\mathscr{P}$-projective resolution. Moreover, the following comparison theorem shows that such a resolution is unique up to precubical homotopy equivalence.

Theorem 5.2. *Let $X \xrightarrow{\ \partial\ } C$ be $\mathscr{P}$-projective and $X' \xrightarrow{\ \partial'\ } C'$ be $\mathscr{P}$-exact. Then any morphism $f\colon C \to C'$ in $\mathscr{C}$ can be extended to a precubical morphism*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \partial\ } & C \\
\Big\downarrow{\overline{f}} & & \Big\downarrow{f} \\
X' & \xrightarrow{\ \partial'\ } & C'
\end{array}
$$

over f (i.e., $\overline{f}$ and f form a morphism of augmented precubical objects). Furthermore, any two such extensions are precubically homotopic. That is, if $\overline{f}, \overline{g}\colon X \to X'$ are two extensions of f, then there exist morphisms $h_n\colon X_n \to X'_{n+1}$, $n \geqslant 0$, in $\mathscr{C}$ such that

$$\partial_1^0 h_n = f_n, \quad \partial_1^1 h_n = g_n, \quad n \geqslant 0,$$
$$\partial_i^\varepsilon h_n = h_{n-1}\partial_{i-1}^\varepsilon, \quad n \geqslant 1, \quad 1 < i \leqslant n+1, \quad \varepsilon \in \{0,1\}.$$
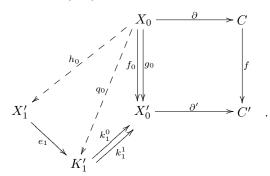
Proof. We construct the extension $\overline{f} = (f_n\colon X_n \to X'_n)$ and show its uniqueness up to precubical homotopy by induction on n. Since X_0 is $\mathscr{P}$-projective and $\partial'\colon X'_0 \to$

C' is $\mathscr{P}$-epimorphic, there is $f_0\colon X_0 \to X_0'$ with $\partial' f_0 = f\partial$. Next, one has $\partial' f_0 \partial_1^0 = f\partial\partial_1^0 = f\partial\partial_1^1 = \partial' f_0 \partial_1^1$. Therefore $f_0\partial_1^0 = k_1^0\varphi_1$ and $f_0\partial_1^1 = k_1^1\varphi_1$ for a uniquely defined $\varphi_1\colon X_1 \to K_1'$. As X_1 is $\mathscr{P}$-projective and $e_1\colon X_1' \to K_1'$ is $\mathscr{P}$-epimorphic, there exists $f_1\colon X_1 \to X_1'$ with $e_1 f_1 = \varphi_1$, and we have $\partial_1^0 f_1 = k_1^0 e_1 f_1 = k_1^0\varphi_1 = f_0\partial_1^0$ and $\partial_1^1 f_1 = k_1^1 e_1 f_1 = k_1^1\varphi_1 = f_0\partial_1^1$. Thus f_0 and f_1 are constructed. Inductively, suppose given morphisms $f_r\colon X_r \to X_r'$ for $r \leqslant n$ so that $\partial_i^\omega f_r = f_{r-1}\partial_i^\omega, 1 \leqslant i \leqslant r, \omega \in \{0,1\}$. Then

$$\partial_i^\omega f_n \partial_j^\alpha = f_{n-1}\partial_i^\omega\partial_j^\alpha = f_{n-1}\partial_{j-1}^\alpha\partial_i^\omega = \partial_{j-1}^\alpha f_n\partial_i^\omega, 1 \leqslant i < j \leqslant n+1, \omega, \alpha \in \{0,1\}.$$

Hence there is a unique $\varphi_{n+1}\colon X_{n+1} \to K_{n+1}'$ such that $k_i^\omega\varphi_{n+1} = f_n\partial_i^\omega, 1 \leqslant i \leqslant n+1, \omega \in \{0,1\}$. Since X_{n+1} is $\mathscr{P}$-projective and $e_{n+1}\colon X_{n+1}' \to K_{n+1}'$ is $\mathscr{P}$-epimorphic, there exists $f_{n+1}\colon X_{n+1} \to X_{n+1}'$ with $e_{n+1}f_{n+1} = \varphi_{n+1}$. Then we have $\partial_i^\omega f_{n+1} = k_i^\omega e_{n+1} f_{n+1} = k_i^\omega\varphi_{n+1} = f_n\partial_i^\omega, 1 \leqslant i \leqslant n+1, \omega \in \{0,1\}$. This completes the inductive step and proves the existence of $\overline{f}$.

Now suppose $\overline{g} = (g_n\colon X_n \to X_n')$ is another extension of $f\colon C \to C'$. We want to construct $h = (h_n\colon X_n \to X_{n+1}')$ with $\partial_1^0 h_n = f_n, \partial_1^1 h_n = g_n, n \geqslant 0$, and $\partial_i^\varepsilon h_n = h_{n-1}\partial_{i-1}^\varepsilon, 1 < i \leqslant n+1, \varepsilon \in \{0,1\}$. For $n = 0$ consider diagram

As $\partial' f_0 = \partial' g_0$, there exists $q_0\colon X_0 \to K_1'$ such that $k_1^0 q_0 = f_0$ and $k_1^1 q_0 = g_0$. Next, since X_0 is $\mathscr{P}$-projective and e_1 is $\mathscr{P}$-epimorphic, there is $h_0\colon X_0 \to X_1'$ with $e_1 h_0 = q_0$. This and the two previous equalities give $\partial_1^0 h_0 = k_1^0 e_1 h_0 = k_1^0 q_0 = f_0$ and $\partial_1^1 h_0 = k_1^1 e_1 h_0 = k_1^1 q_0 = g_0$. Thus h_0 is constructed. Inductively, suppose given $h_0, h_1, \ldots, h_{n-1}$ with the required properties. (See Figure 1.) Define $\psi_i^\varepsilon\colon X_n \to X_n', 1 \leqslant i \leqslant n+1, \varepsilon \in \{0,1\}$, as follows:

$$\psi_1^0 = f_n, \quad \psi_1^1 = g_n, \quad \psi_i^\varepsilon = h_{n-1}\partial_{i-1}^\varepsilon, \quad 1 < i \leqslant n+1, \quad \varepsilon \in \{0,1\}.$$

By the induction assumption,

$$\partial_1^0 \psi_j^\varepsilon = \partial_1^0 h_{n-1}\partial_{j-1}^\varepsilon = f_{n-1}\partial_{j-1}^\varepsilon = \partial_{j-1}^\varepsilon f_n = \partial_{j-1}^\varepsilon \psi_1^0,$$
$$1 < j \leqslant n+1, \quad \varepsilon \in \{0,1\}, \quad n \geqslant 1,$$
$$\partial_1^1 \psi_j^\varepsilon = \partial_1^1 h_{n-1}\partial_{j-1}^\varepsilon = g_{n-1}\partial_{j-1}^\varepsilon = \partial_{j-1}^\varepsilon g_n = \partial_{j-1}^\varepsilon \psi_1^1,$$
$$1 < j \leqslant n+1, \quad \varepsilon \in \{0,1\}, \quad n \geqslant 1,$$
$$\partial_i^\alpha \psi_j^\varepsilon = \partial_i^\alpha h_{n-1}\partial_{j-1}^\varepsilon = h_{n-2}\partial_{i-1}^\alpha\partial_{j-1}^\varepsilon = h_{n-2}\partial_{j-2}^\varepsilon\partial_{i-1}^\alpha = \partial_{j-1}^\varepsilon h_{n-1}\partial_{i-1}^\alpha$$
$$= \partial_{j-1}^\varepsilon \psi_i^\alpha, \quad 1 < i < j \leqslant n+1, \quad \alpha, \varepsilon \in \{0,1\}, \quad n > 1.$$

That is $\partial_i^\alpha \psi_j^\varepsilon = \partial_{j-1}^\varepsilon \psi_i^\alpha$ for $1 \leqslant i < j \leqslant n+1, \alpha, \varepsilon \in \{0,1\}$. Therefore $k_i^\varepsilon q_n = \psi_i^\varepsilon, 1 \leqslant$
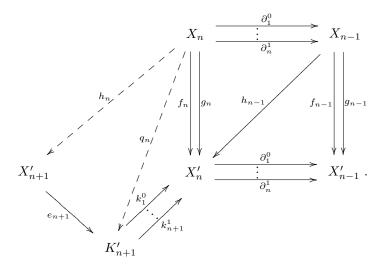
Figure 1: The inductive step in the construction of precubical homotopy.

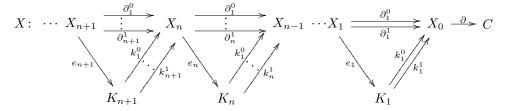$i \leqslant n+1, \varepsilon \in \{0,1\}$, for a uniquely defined $q_n \colon X_n \to K'_{n+1}$. Since X_n is $\mathscr{P}$-projective and e_{n+1} is $\mathscr{P}$-epimorphic, there is $h_n \colon X_n \to X'_{n+1}$ with $e_{n+1}h_n = q_n$. Now we have

$$\partial_1^0 h_n = k_1^0 e_{n+1} h_n = k_1^0 q_n = \psi_1^0 = f_n,$$
$$\partial_1^1 h_n = k_1^1 e_{n+1} h_n = k_1^1 q_n = \psi_1^1 = g_n,$$
$$\partial_i^\varepsilon h_n = k_i^\varepsilon e_{n+1} h_n = k_i^\varepsilon q_n = \psi_i^\varepsilon = h_{n-1}\partial_{i-1}^\varepsilon,$$
$$1 < i \leqslant n+1, \quad \varepsilon \in \{0,1\}.$$

This finishes the inductive step and completes the proof of the theorem. □

The following proposition is crucial for constructing our cubical derived functors.

Proposition 5.3. *Suppose that*

is a $\mathscr{P}$-projective resolution of $C \in \mathscr{C}$. Then X has pseudodegeneracy operators, i.e., there exist $s_i \colon X_n \to X_{n+1}$, $n \geqslant 0$, $1 \leqslant i \leqslant n+1$, satisfying

$$\partial_i^\alpha s_j = \begin{cases} s_{j-1}\partial_i^\alpha & i < j, \\ \mathrm{id} & i = j, \\ s_j \partial_{i-1}^\alpha & i > j, \end{cases}$$

where $\alpha \in \{0,1\}$.

Proof. As $K_1 \underset{k_1^1}{\overset{k_1^0}{\rightrightarrows}} X_0$ is a kernel pair of $X_0 \xrightarrow{\partial} A$, there is $\varphi_1 \colon X_0 \to K_1$ such that

$k_1^0 \varphi_1 = \mathrm{id}$ and $k_1^1 \varphi_1 = \mathrm{id}$. On the other hand, since X_0 is $\mathscr{P}$-projective and e_1 is $\mathscr{P}$-epimorphic, $e_1 s_1 = \varphi_1$ for some $s_1 \colon X_0 \to X_1$ and we have $\partial_1^0 s_1 = k_1^0 e_1 s_1 = k_1^0 \varphi_1 = \mathrm{id}$ and $\partial_1^1 s_1 = k_1^1 e_1 s_1 = k_1^1 \varphi_1 = \mathrm{id}$. Thus $s_1 \colon X_0 \to X_1$ is constructed. Inductively, suppose given $s_1 \colon X_0 \to X_1$, $s_1, s_2 \colon X_1 \to X_2, \ldots, s_1, \ldots, s_n \colon X_{n-1} \to X_n$ with the required properties. Fix j, $1 \leqslant j \leqslant n+1$, and define $\psi_{ij}^\varepsilon \colon X_n \to X_n$, $1 \leqslant i \leqslant n+1$, $\varepsilon \in \{0,1\}$, by

$$\psi_{ij}^\varepsilon = \begin{cases} \mathrm{id} & i = j, \\ s_{j-1} \partial_i^\varepsilon & i < j, \\ s_j \partial_{i-1}^\varepsilon & i > j. \end{cases}$$

Using the induction assumption, one checks that $\partial_i^\varepsilon \psi_{mj}^\alpha = \partial_{m-1}^\alpha \psi_{ij}^\varepsilon$ for $1 \leqslant i < m \leqslant n+1$, $\varepsilon, \alpha \in \{0,1\}$. Then there exists $\varphi_j \colon X_n \to K_{n+1}$ such that $k_i^\varepsilon \varphi_j = \psi_{ij}^\varepsilon$, $1 \leqslant i \leqslant n+1$, $\varepsilon \in \{0,1\}$. Since X_n is $\mathscr{P}$-projective and e_{n+1} is $\mathscr{P}$-epimorphic, there is $s_j \colon X_n \to X_{n+1}$ with $e_{n+1} s_j = \varphi_j$. According to this, for $i = 1, \ldots, n+1$ and $\varepsilon \in \{0,1\}$, one has

$$\partial_i^\varepsilon s_j = k_i^\varepsilon e_{n+1} s_j = k_i^\varepsilon \varphi_j = \psi_{ij}^\varepsilon = \begin{cases} \mathrm{id} & i = j, \\ s_{j-1} \partial_i^\varepsilon & i < j, \\ s_j \partial_{i-1}^\varepsilon & i > j. \end{cases}$$

Thus we have constructed $s_1, \ldots, s_{n+1} \colon X_n \to X_{n+1}$ with the desired properties. $\square$

Remark 5.4. Suppose $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F \colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. As we will see in Section 6, Proposition 5.3 is essential for constructing the cubical derived functors of $F \colon \mathscr{C} \longrightarrow \mathscr{B}$. This contrasts with the fact that the construction of the derived functors by Tierney and Vogel does not use the existence of pseudodegeneracies in $\mathscr{P}$-projective presimplicial resolutions. On the other hand, as shown in [**22**], the unnormalized chain complex functor C (see Definition 2.9) sends precubically homotopic morphisms of precubical objects to chain homotopic morphisms. This together with Theorem 5.2 allows us to conclude that one does not need Proposition 5.3 to prove that the functors

$$\bar{L}_n^\square F(C) = H_n(C(F(X))), \ n \geqslant 0,$$

where X is a $\mathscr{P}$-projective precubical resolution of $C \in \mathscr{C}$, are well defined. But in this way one obtains "wrong" derived functors by the following reason. One can easily see that $\bar{L}_n^\square F(Q) \cong F(Q)$ for any $Q \in \mathscr{P}$ and any $n \geqslant 0$. Thus in general the higher ($n > 0$) derived functors $\bar{L}_n^\square F$ do not vanish on $\mathscr{P}$-projectives, i.e., the crucial property of derived functors is not satisfied. In order to obtain reasonable derived functors one has to use a normalization procedure (just as in the case of cubical singular homology [**17**, II.§2] which we discuss in detail in the next section).

6. Pseudocubical objects in idempotent complete preadditive categories and cubical derived functors

We start with the following essential definitions.

Definition 6.1. A *pseudocubical object* X in a category $\mathscr{C}$ is a family of objects $(X_n \in \mathscr{C})_{n \geqslant 0}$ together with face operators

$$\partial_i^0, \partial_i^1 \colon X_n \longrightarrow X_{n-1}, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant n,$$

and pseudodegeneracy operators

$$s_i \colon X_{n-1} \longrightarrow X_n, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant n,$$

satisfying the pseudocubical identities

$$\partial_i^\alpha \partial_j^\varepsilon = \partial_{j-1}^\varepsilon \partial_i^\alpha \qquad i < j, \quad \alpha, \varepsilon \in \{0, 1\},$$

and

$$\partial_i^\alpha s_j = \begin{cases} s_{j-1} \partial_i^\alpha & i < j, \\ \text{id} & i = j, \\ s_j \partial_{i-1}^\alpha & i > j, \end{cases}$$

for $\alpha \in \{0, 1\}$.

Note that, in general, the pseudodegeneracies need not satisfy the identities

$$s_i s_j = s_{j+1} s_i, \quad i \leqslant j$$

(see Definition 2.8).

Definition 6.2. Let $\mathscr{C}$ be a category and $C \in \mathscr{C}$ an object. An *augmented pseudocubical object* over C is a pseudocubical object X in $\mathscr{C}$ together with a morphism $\partial \colon X_0 \to C$ satisfying $\partial \partial_1^0 = \partial \partial_1^1$.

Important examples of augmented pseudocubical objects are provided by Proposition 5.3.

Now we recall the definition and some properties of the normalized chain complex of a pseudocubical object in the general setting of idempotent complete preadditive categories. (Note that the normalized chain complex of a cubical object in an abelian category was originally introduced by Świątek in [**22**].)

Definition 6.3 (see e.g., [**8**], [**15**]). A preadditive category $\mathscr{A}$ is *idempotent complete* if any idempotent $p \colon E \longrightarrow E$ in $\mathscr{A}$ (i.e., $p^2 = p$) has a kernel. That is, there is a morphism

$$i \colon \mathrm{Ker}(p) \longrightarrow E$$

with $pi = 0$, and for any morphism $f \colon F \to E$, satisfying $pf = 0$, there is a unique morphism $F \xrightarrow{\ g\ } \mathrm{Ker}(p)$ such that $ig = f$.

The following two propositions are well known (see e.g., [**15**]).

Proposition 6.4. *Suppose $\mathscr{A}$ is an idempotent complete preadditive category and $p \colon E \to E$ an idempotent in $\mathscr{A}$. Then there is a diagram*

$$\mathrm{Ker}(p) \underset{i^1}{\overset{\pi^1}{\rightleftarrows}} E \underset{i^2}{\overset{\pi^2}{\rightleftarrows}} \mathrm{Ker}(1-p)$$

such that

$$\begin{aligned} \pi^1 i^1 &= 1, \quad \pi^2 i^2 = 1, \\ \pi^1 i^2 &= 0, \quad \pi^2 i^1 = 0, \\ i^1 \pi^1 &= 1 - p, \quad i^2 \pi^2 = p. \end{aligned}$$

In particular, the coproduct $\mathrm{Ker}(p) \oplus \mathrm{Ker}(1-p)$ exists in $\mathscr{A}$ and is isomorphic to E.

Proposition 6.5. *Let $\mathscr{A}$ be a preadditive category. Then there exists an idempotent complete preadditive category $\widetilde{\mathscr{A}}$ (called the idempotent completion of $\mathscr{A}$) and a full additive embedding*

$$\varphi \colon \mathscr{A} \to \widetilde{\mathscr{A}}$$

satisfying the following universal property: For any idempotent complete preadditive category $\mathscr{D}$ and an additive functor $\mathscr{A} \overset{\psi}{\longrightarrow} \mathscr{D}$, there is an additive functor $\psi' \colon \widetilde{\mathscr{A}} \to \mathscr{D}$ which makes the diagram

$$\begin{array}{ccc} \mathscr{A} & \overset{\varphi}{\longrightarrow} & \widetilde{\mathscr{A}} \\ & {\scriptstyle \psi} \searrow & \downarrow {\scriptstyle \psi'} \\ & & \mathscr{D} \end{array}$$

commute up to a natural equivalence, and which is unique up to a natural isomorphism.

Let X be a pseudocubical object in an idempotent complete preadditive category $\mathscr{D}$. We want to construct the normalized chain complex of X as a kernel of a certain idempotent endomorphism of the unnormalized chain complex $C(X)$ (see 2.9). Define

$$\sigma_n^X = (1 - s_1 \partial_1^1)(1 - s_2 \partial_2^1) \cdots (1 - s_n \partial_n^1) \colon X_n \to X_n, \quad n \geqslant 0, \quad (\sigma_0 = 1).$$

By the pseudocubical identities $(1 \leqslant i \leqslant n)$,

$$\begin{aligned} \partial_i^1 \sigma_n^X &= \partial_i^1 (1 - s_1 \partial_1^1)(1 - s_2 \partial_2^1) \cdots (1 - s_n \partial_n^1) \\ &= (1 - s_1 \partial_1^1) \partial_i^1 (1 - s_2 \partial_2^1) \cdots (1 - s_n \partial_n^1) = \cdots \\ &= (1 - s_1 \partial_1^1) \cdots (1 - s_{i-1} \partial_{i-1}^1) \partial_i^1 (1 - s_i \partial_i^1) \cdots (1 - s_n \partial_n^1) \\ &= (1 - s_1 \partial_1^1) \cdots (1 - s_{i-1} \partial_{i-1}^1)(\partial_i^1 - \partial_i^1)(1 - s_{i+1} \partial_{i+1}^1) \cdots (1 - s_n \partial_n^1) = 0. \end{aligned}$$

This implies

$$(1 - s_i \partial_i^1) \sigma_n^X = \sigma_n^X$$

for $1 \leqslant i \leqslant n$, and thus

$$\sigma_n^X \sigma_n^X = (1 - s_1 \partial_1^1)(1 - s_2 \partial_2^1) \cdots (1 - s_n \partial_n^1) \sigma_n^X =$$
$$(1 - s_1 \partial_1^1)(1 - s_2 \partial_2^1) \cdots (1 - s_{n-1} \partial_{n-1}^1) \sigma_n^X = \cdots = (1 - s_1 \partial_1^1) \sigma_n^X = \sigma_n^X.$$

Further, another routine calculation using pseudocubical identities shows that the morphisms σ_n^X, $n \geqslant 0$, are compatible with the differentials of $C(X)$. Hence one gets an idempotent endomorphism of $C(X)$ which we denote by

$$\sigma^X \colon C(X) \to C(X).$$

Since the category $\mathscr{D}$ is idempotent complete, the chain map σ^X has a kernel $\operatorname{Ker} \sigma^X$ in the category of non-negative chain complexes in $\mathscr{D}$. Furthermore, by 6.4, there is a diagram in the category of chain complexes

$$\operatorname{Ker}(\sigma^X) \underset{i^1}{\overset{\pi^1}{\rightleftarrows}} C(X) \underset{i^2}{\overset{\pi^2}{\rightleftarrows}} \operatorname{Ker}(1 - \sigma^X)$$

such that

$$\pi^1 i^1 = 1, \quad \pi^2 i^2 = 1,$$
$$\pi^1 i^2 = 0, \quad \pi^2 i^1 = 0,$$
$$i^1 \pi^1 = 1 - \sigma^X, \quad i^2 \pi^2 = \sigma^X.$$

Definition 6.6. Let X be a pseudocubical object in an idempotent complete preadditive category $\mathscr{D}$. The chain complex $\operatorname{Ker}(1 - \sigma^X)$, denoted by $N(X)$, is called the *normalized chain complex* of X.

Remark 6.7. If $\mathscr{D}$ is an abelian category, then $N(X)$ admits the following description:

$$N(X)_0 = X_0, \quad N(X)_n = \bigcap_{i=1}^{n} \operatorname{Ker}(\partial_i^1), \quad n > 0,$$

$$d_n = \sum_{i=1}^{n} (-1)^{i+1} \partial_i^0 \colon N(X)_n \to N(X)_{n-1}, \quad n > 0.$$

Thus in the abelian case one does not need pseudodegeneracies to define $N(X)$.

The functor N sends precubically homotopic morphisms to chain homotopic morphisms. This is proved in [**22**] for cubical objects in abelian categories. The proof, which we repeat here for completeness, shows that in fact one has:

Proposition 6.8. *Let X be a pseudocubical object and Y a precubical object in an abelian category, and $f, g \colon X \to Y$ precubical morphisms. If f and g are precubically homotopic (see Proposition 5.2), then $N(f)$ and $N(g) \colon N(X) \to N(Y)$ are chain homotopic.*

Proof. Let $h = (h_n \colon X_n \to Y_{n+1})_{n \geqslant 0}$ be a precubical homotopy from f to g. For simplicity, we denote by $i_n \colon N_n(X) \to X_n$ (without the superscript 2), $n \geqslant 0$, the n-th level of the canonical morphism $N(X) \xrightarrow{i^2} X$. One checks that $\partial_i^1 (h_n - g_{n+1}s_1)i_n = 0$, $1 \leqslant i \leqslant n+1$, $n \geqslant 0$. Consequently, we have morphisms

$$t_n = (h_n - g_{n+1}s_1)i_n \colon N_n(X) \to N_{n+1}(Y), \quad n \geqslant 0.$$

Clearly,

$$d_1 t_0 = \partial_1^0 (h_0 - g_1 s_1)i_0 = (f_0 - g_0)i_0 = N_0(f) - N_0(g).$$

Furthermore, for all $n \geqslant 1$, we have

$$
dt_n + t_{n-1}d = \left(\sum_{i=1}^{n+1} (-1)^{i+1} \partial_i^0 \right) (h_n - g_{n+1}s_1)i_n
$$

$$
+ (h_{n-1} - g_n s_1) \left(\sum_{i=1}^{n} (-1)^{i+1} \partial_i^0 \right) i_n
$$

$$
= f_n i_n - g_n i_n + \left(\sum_{i=2}^{n+1} (-1)^{i+1} \partial_i^0 \right) (h_n - g_{n+1}s_1)i_n
$$

$$
+ \left(\sum_{i=1}^{n} (-1)^{i+1} \partial_{i+1}^0 \right) (h_n - g_{n+1}s_1)i_n
$$

$$
= f_n i_n - g_n i_n + \left(\sum_{i=2}^{n+1} (-1)^{i+1} \partial_i^0 \right) (h_n - g_{n+1}s_1)i_n
$$

$$
+ \left(\sum_{i=2}^{n+1} (-1)^{i} \partial_i^0 \right) (h_n - g_{n+1}s_1)i_n
$$

$$
= f_n i_n - g_n i_n = N_n(f) - N_n(g).
$$

Hence $t = (t_n \colon N_n(X) \to N_{n+1}(Y))_{n \geqslant 0}$ is a chain homotopy from $N(f)$ to $N(g)$. $\square$

Example 6.9. Let Y be a topological space and $\mathbb{Z}[Q(Y)]$ denote the free cubical abelian group generated by the singular cubical set $Q(Y)$ (see 2.10). Then the homology $H_n(N(\mathbb{Z}[Q(Y)]))$, $n \geqslant 0$, is the n-th cubical singular homology of the space Y.

Definition 6.10. Suppose $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$ and $X \xrightarrow{\partial} C$ an augmented pseudocubical object in $\mathscr{C}$. One says that $X \xrightarrow{\partial} C$ is a *$\mathscr{P}$-projective pseudocubical resolution* of C if $X \xrightarrow{\partial} C$ is a precubical resolution of C after forgetting the pseudodegeneracies.

Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F \colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Then the cubical derived functors $\mathbf{L}_n^{\square} F$ of F with respect to the class $\mathscr{P}$ are defined as follows. For any object $C \in \mathscr{C}$, choose (once and for all) a $\mathscr{P}$-projective pseudocubical resolution

$$
X \xrightarrow{\partial} C
$$

of C (Existence of such a resolution follows from 5.3.) and define

$$
\mathbf{L}_n^{\square} F(C) = H_n(N(F(X))), \quad n \geqslant 0.
$$

Theorem 5.2 and the homotopy invariance of the functor N (Proposition 6.8) imply that the objects $\mathbf{L}_n^{\square} F(C)$ are well-defined and functorial in F and C. In particular, we see that Proposition 5.3 is essential for constructing the cubical derived functors since pseudodegeneracies are needed in the proof of 6.8.

The following lemma is the main technical tool for proving a cubical analog of Proposition 4.5.

Lemma 6.11. *Let $F\colon \mathscr{D} \to \mathscr{D}'$ be an additive functor between idempotent complete preadditive categories. Then for any pseudocubical object X in $\mathscr{D}$, there is a natural isomorphism*

$$F(N(X)) \cong N(F(X))$$

of chain complexes in $\mathscr{D}'$.

Proof. Applying the additive functor F to the diagram

$$\mathrm{Ker}(\sigma^X) \underset{i^1}{\overset{\pi^1}{\rightleftarrows}} C(X) \underset{i^2}{\overset{\pi^2}{\rightleftarrows}} \mathrm{Ker}(1 - \sigma^X) = N(X),$$

we get a diagram

$$F(\mathrm{Ker}(\sigma^X)) \underset{F(i^1)}{\overset{F(\pi^1)}{\rightleftarrows}} C(F(X)) \underset{F(i^2)}{\overset{F(\pi^2)}{\rightleftarrows}} F(N(X))$$

(in $\mathscr{D}'$) whose morphisms satisfy the following identities:

$$
\begin{aligned}
F(\pi^1)F(i^1) = 1, \quad & F(\pi^2)F(i^2) = 1, \\
F(\pi^1)F(i^2) = 0, \quad & F(\pi^2)F(i^1) = 0, \\
F(i^1)F(\pi^1) &= 1 - F(\sigma^X), \\
F(i^2)F(\pi^2) &= F(\sigma^X).
\end{aligned}
$$

Besides, it follows from the additivity of F that $F(\sigma^X) = \sigma^{F(X)}$, and hence we obtain

$$F(i^2)F(\pi^2) = \sigma^{F(X)}.$$

This finally implies that

$$F(N(X)) \cong \mathrm{Ker}(1 - \sigma^{F(X)}) = N(F(X)). \quad \square$$

7. Cubical derived functors and Eilenberg-Moore derived functors

Let $\mathscr{C}$ be a category with finite limits, $\mathscr{P}$ a projective class, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \to \mathscr{B}$ a functor. In this section we prove that for any object $C \in \mathscr{C}$, there is a natural isomorphism

$$\mathbf{L}_n^{\square} F(C) \cong \mathbf{L}_n^{\mathscr{P}} F_{ad}(C), \quad n \geqslant 0,$$

where $F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}$ is the additivization of F (see Section 4). This together with 4.5 obviously implies Theorem 1.1.

The proof of this isomorphism is similar to that of 4.5. However, the arguments become a little bit complicated in the cubical setting as we have to consider normalized chain complexes in order to get the "right" homology.

Proposition 7.1. *Suppose $\mathscr{A}$ is a preadditive category, $\mathscr{P}$ a class of objects in $\mathscr{A}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ an additive functor. Suppose further that $\overline{\mathscr{P}}$ is the closure of the class $\mathscr{P}$ in the idempotent completion $\widetilde{\mathscr{A}}$ of $\mathscr{A}$ (see 6.5), and*

$\widetilde{F}\colon \widetilde{\mathscr{A}} \to \mathscr{B}$ the extension of F. Then for any $A \in \mathscr{A}$ which possesses a $\mathscr{P}$-resolution, there is a natural isomorphism

$$\mathbf{L}_n^{\mathscr{P}} F(A) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F}(A), \quad n \geqslant 0.$$

Proof. Since $\widetilde{\mathscr{A}}$ has a zero object, any $\mathscr{P}$-resolution in $\mathscr{A}$ is a $\overline{\mathscr{P}}$-resolution in $\widetilde{\mathscr{A}}$. The rest follows from 3.7. □

Corollary 7.2. *Assume that $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Assume further that $\widetilde{F_{ad}}\colon \widetilde{\mathbb{Z}\mathscr{C}} \to \mathscr{B}$ is the extension of $F_{ad}\colon \mathbb{Z}\mathscr{C} \longrightarrow \mathscr{B}$ to the idempotent completion $\widetilde{\mathbb{Z}\mathscr{C}}$ of $\mathbb{Z}\mathscr{C}$, and $\overline{\mathscr{P}}$ the closure of $\mathscr{P}$ in $\widetilde{\mathbb{Z}\mathscr{C}}$. Then for any object $C \in \mathscr{C}$, there is a natural isomorphism*

$$\mathbf{L}_n^{\mathscr{P}} F_{ad}(C) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F_{ad}}(C), \quad n \geqslant 0.$$

Next we need the following technical lemma:

Lemma 7.3. *Let $X \xrightarrow{\partial} X_{-1}$ be an augmented pseudocubical set. Suppose that $\partial\colon X_0 \longrightarrow X_{-1}$ is surjective and the following conditions hold:*

(i) *For any $x, y \in X_0$, satisfying $\partial x = \partial y$, there exists $z \in X_1$ such that $\partial_1^0 z = x$ and $\partial_1^1 z = y$.*

(ii) *For any $n \geqslant 1$ and any collection of $2n+2$ elements $x_i^{\varepsilon} \in X_n$, $1 \leqslant i \leqslant n+1$, $\varepsilon \in \{0,1\}$, satisfying*

$$\partial_i^{\alpha} x_j^{\varepsilon} = \partial_{j-1}^{\varepsilon} x_i^{\alpha}, \quad 1 \leqslant i < j \leqslant n+1, \quad \alpha, \varepsilon \in \{0,1\},$$

there exists $x \in X_{n+1}$, such that

$$\partial_i^{\varepsilon} x = x_i^{\varepsilon}, \quad 1 \leqslant i \leqslant n+1, \quad \varepsilon \in \{0,1\}.$$

Then the augmented normalized chain complex

$$N(\mathbb{Z}[X]) \to \mathbb{Z}[X_{-1}]$$

is chain contractible and therefore it has trivial homology in each dimension.

We omit the routine details of the proof here. Note only that the main idea is to construct inductively a precubical homotopy equivalence between X and the constant cubical object determined by X_{-1} and then use the homotopy invariance of the functor N (Proposition 6.8).

Example 7.4. Let $X \xrightarrow{\partial} C$ be a $\mathscr{P}$-projective pseudocubical resolution of C and P an object from $\mathscr{P}$. Then the augmented pseudocubical set

$$\mathrm{Hom}_{\mathscr{C}}(P, X) \xrightarrow{\partial_*} \mathrm{Hom}_{\mathscr{C}}(P, C)$$

satisfies the conditions of 7.3. Consequently, the homologies of the augmented chain complex

$$N(\mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(P, X)]) \longrightarrow \mathbb{Z}[\mathrm{Hom}_{\mathscr{C}}(P, C)]$$

vanish.

We are now ready to prove the main result of this section.

Proposition 7.5. *Suppose $\mathscr{C}$ is a category with finite limits, $\mathscr{P}$ a projective class in $\mathscr{C}$, $\mathscr{B}$ an abelian category, and $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ a functor. Then:*

(i) For any $\mathscr{P}$-projective pseudocubical resolution $X \to C$, the augmented chain complex

$$N(X) \to C$$

in the category $\widetilde{\mathbb{Z}\mathscr{C}}$ is a $\overline{\mathscr{P}}$-resolution of C in the sense of 3.2. ($\overline{\mathscr{P}}$ is the closure of $\mathscr{P}$ in $\widetilde{\mathbb{Z}\mathscr{C}}$.)

(ii) For any $C \in \mathscr{C}$, there is a natural isomorphism

$$\mathbf{L}_n^{\square} F(C) \cong \mathbf{L}_n^{\mathscr{P}} F_{ad}(C), \quad n \geqslant 0.$$

Proof. Since $N(X)_n$ is a retract of X_n and $\overline{\mathscr{P}}$ is closed under retracts, $N(X)_n \in \overline{\mathscr{P}}$ for all $n \geqslant 0$. Further, by Lemma 6.11, one has a natural isomorphism of augmented chain complexes

$$
\begin{array}{ccc}
\operatorname{Hom}_{\widetilde{\mathbb{Z}\mathscr{C}}}(Q, N(X)) & \longrightarrow & \operatorname{Hom}_{\widetilde{\mathbb{Z}\mathscr{C}}}(Q, C) \\
\downarrow{\scriptstyle \cong} & & \| \\
N(\mathbb{Z}[\operatorname{Hom}_{\mathscr{C}}(Q, X)]) & \longrightarrow & \mathbb{Z}[\operatorname{Hom}_{\mathscr{C}}(Q, C)]
\end{array}
$$

for any $Q \in \mathscr{P}$. It follows from 7.4 that the lower chain complex is acyclic and thus so is the upper one. Consequently, the augmented chain complex $N(X) \to C$ in $\widetilde{\mathbb{Z}\mathscr{C}}$ is $\mathscr{P}$-acyclic or, equivalently, $\overline{\mathscr{P}}$-acyclic. This completes the proof of (i).

Let us prove (ii). By Corollary 7.2, it suffices to get a natural isomorphism

$$\mathbf{L}_n^{\square} F(C) \cong \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F_{ad}}(C).$$

Choose any $\mathscr{P}$-projective pseudocubical resolution $X \to C$. Lemma 6.11 together with (i) gives

$$\mathbf{L}_n^{\square} F(C) = H_n(N(F(X))) = H_n(N(\widetilde{F_{ad}}(X))) \cong H_n(\widetilde{F_{ad}}(N(X))) = \mathbf{L}_n^{\overline{\mathscr{P}}} \widetilde{F_{ad}}(C). \quad \square$$

Clearly, Theorem 1.1 is an immediate consequence of 4.5 and 7.5.

8. Connection with topology

In this section we show that Theorem 1.1 generalizes the well-known fact that the cubical and simplicial singular homologies of a topological space are naturally isomorphic (see 6.9). (For basic properties of the cubical singular homology see, for example, [**17**].) For this we need spectral sequences of bipseudocubical objects, (pseudo)cubical analogs of spectral sequences of bi(pseudo)simplicial objects (see e.g., [**9**, IV.2]). Note that the identification of E^2-terms in the (pseudo)cubical setting is technically a little bit complicated compared to its simplicial counterpart as one has to take care of the normalizations.

Before starting to discuss spectral sequences we recall the following

Definition 8.1. A *bipseudocubical object* X in a category $\mathscr{C}$ is a family of objects $(X_{nm} \in \mathscr{C})_{n,m \geqslant 0}$ together with horizontal and vertical face operators

$$\partial_i^{0,h}, \partial_i^{1,h} \colon X_{nm} \longrightarrow X_{n-1,m}, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant n, \quad m \geqslant 0,$$

$$\partial_i^{0,v}, \partial_i^{1,v} \colon X_{nm} \longrightarrow X_{n,m-1}, \quad m \geqslant 1, \quad 1 \leqslant i \leqslant n, \quad n \geqslant 0,$$

and horizontal and vertical pseudodegeneracy operators

$$s_i^h \colon X_{n-1,m} \longrightarrow X_{nm}, \quad n \geqslant 1, \quad 1 \leqslant i \leqslant n, \quad m \geqslant 0,$$

$$s_i^v \colon X_{n,m-1} \longrightarrow X_{nm}, \quad m \geqslant 1, \quad 1 \leqslant i \leqslant m, \quad n \geqslant 0,$$

such that horizontal and vertical operators commute and the pseudocubical identities (6.1) hold horizontally and vertically. If, in addition, the pseudodegeneracies are degeneracies (see 2.8) in each direction, then we say that X is a *bicubical* object.

Let X be a bipseudocubical object in an abelian category $\mathscr{A}$. Then there is an associated first quadrant double complex NNX of X defined as follows:

$$NNX_{00} = X_{00}, \quad NNX_{n0} = \bigcap_{i=1}^{n} \mathrm{Ker}(\partial_i^{1,h}), \quad n > 0,$$

$$NNX_{0m} = \bigcap_{i=1}^{m} \mathrm{Ker}(\partial_i^{1,v}), \quad m > 0,$$

$$NNX_{nm} = \bigcap_{i=1}^{n} \mathrm{Ker}(\partial_i^{1,h}) \cap \bigcap_{i=1}^{m} \mathrm{Ker}(\partial_i^{1,v}), \quad n > 0, \quad m > 0,$$

$$d^h = \sum_{i=1}^{n} (-1)^{i+1} \partial_i^{0,h} \colon NNX_{nm} \to NNX_{n-1,m}, \quad n > 0, \quad m \geqslant 0,$$

$$d^v = \sum_{i=1}^{m} (-1)^{n+i+1} \partial_i^{0,v} \colon NNX_{nm} \to NNX_{n,m-1}, \quad n \geqslant 0, \quad m > 0.$$

Proposition 8.2. *Let X be a bipseudocubical object in an abelian category. Then there are two first quadrant spectral sequences*

$$^I E_{pq}^2 = H_p^h(N^h H_q^v(N^v X)) \Longrightarrow H_{p+q}(\mathrm{Tot}\, NNX),$$

and

$$^{II} E_{pq}^2 = H_p^v(N^v H_q^h(N^h X)) \Longrightarrow H_{p+q}(\mathrm{Tot}\, NNX).$$

(Here N^h and N^v stand for the horizontal and vertical normalizations, respectively, and $\mathrm{Tot}\, NNX$ denotes the total complex of NNX.)

Proof. These spectral sequences are the two spectral sequences associated to the double complex NNX (see e.g., [**26**, 5.6]). By Lemma 6.11, normalization and homology in different directions commute. This gives the desired identifications of E^2-terms. $\square$

Example 8.3. Suppose $E \xrightarrow{\ f\ } B$ is a Serre fibration. Let $Q_{pq}(f)$ denote the set of

all commutative diagrams of the form

$$\begin{array}{ccc} I^p \times I^q & \xrightarrow{\sigma} & E \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ I^p & \xrightarrow{\tau} & B. \end{array}$$

The sets $(Q_{pq}(f))_{p,q \geqslant 0}$ together with the operators

$$\partial_i^{\varepsilon,h}(\sigma,\tau) = (\sigma(\delta_\varepsilon^i \times 1), \tau\delta_\varepsilon^i), \quad \partial_j^{\varepsilon,v}(\sigma,\tau) = (\sigma(1 \times \delta_\varepsilon^j), \tau), \quad \varepsilon \in \{0,1\},$$

$$s_i^h(\sigma,\tau) = (\sigma(\eta^i \times 1), \tau\eta^i), \quad s_j^v(\sigma,\tau) = (\sigma(1 \times \eta^j), \tau),$$

(see Example 2.10) form a bicubical set which we denote by $Q(f)$. By Proposition 8.2, the free bicubical abelian group $\mathbb{Z}[Q(f)]$ yields two spectral sequences. The second spectral sequence collapses and the first one is the Serre spectral sequence of

$E \xrightarrow{f} B$. Note that this construction of the Serre spectral sequence is the original one used by Serre in [**21**].

Now, using Theorem 1.1, we compare the cubical and simplicial singular homologies of a topological space. Let Δ^n, $n \geqslant 0$, denote the standard n-simplex. The class $\mathscr{P}_\Delta$ of all possible disjoint unions of standard simplices is a projective class (in the sense of Definition 4.1) in the category **Top** of topological spaces. (Moreover, in fact, it comes from a cotriple [**3**, (10.2)].) Indeed, for any space Y, the map

$$\bigsqcup_{\substack{\Delta^n \to Y, \\ n \geqslant 0}} \Delta^n \to Y,$$

where the disjoint union is taken over all possible continuous maps $\Delta^n \to Y$, $n \geqslant 0$, is a $\mathscr{P}_\Delta$-epimorphism. Consider the functor

$$F \colon \mathbf{Top} \to \mathbf{Ab}, \quad F(Y) = H_0^\Delta(Y,A) = \mathbb{Z}[\pi_0 Y] \otimes A,$$

where **Ab** is the category of abelian groups, $H_*^\Delta(Y,A)$ the simplicial singular homology of Y with coefficients in an abelian group A, and $\pi_0 Y$ the set of path components of Y. It follows from [**3**, (10.2)], and [**24**, (3.1)] that there is a natural isomorphism

$$\mathbf{L}_n^\Delta F(Y) \cong H_n^\Delta(Y,A), \quad n \geqslant 0,$$

where the simplicial derived functors are taken with respect to the projective class $\mathscr{P}_\Delta$ (cf. [**19**], [**20**]). We sketch the proof of this natural isomorphism along the lines of [**3**, (10.2)]. The standard cosimplicial object $\Delta^\bullet$ gives rise to an augmented simplicial functor

$$F_\bullet^\Delta \to F, \quad F_n^\Delta(Y) = \mathbb{Z}[\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)] \otimes A.$$

Further, suppose $S \to Y$ is a $\mathscr{P}_\Delta$-projective presimplicial resolution of Y. Evaluating $F_\bullet^\Delta$ on S yields a bipresimplicial abelian group. It is easily seen that both resulting spectral sequences collapse at E^2 and give the desired isomorphism.

Similarly, one can describe the cubical singular homologies $H_n^\square(Y,A)$, $n \geqslant 0$, as cubical derived functors of the functor $F(Y) = \mathbb{Z}[\pi_0 Y] \otimes A$. To show this we use the

class $\mathscr{P}_\square$ consisting of all possible disjoint unions of standard cubes. The class $\mathscr{P}_\square$ is a projective class in **Top** and there is a natural isomorphism

$$\mathbf{L}_n^\square F(Y) \cong H_n^\square(Y, A), \quad n \geqslant 0,$$

where the cubical derived functors are taken with respect to $\mathscr{P}_\square$. Indeed, the standard cocubical object $I^\bullet$ (see Example 2.10) yields an augmented cubical functor

$$F_\bullet^\square \to F, \quad F_n^\square(Y) = \mathbb{Z}[\mathrm{Hom}_{\mathbf{Top}}(I^n, Y)] \otimes A.$$

Further, suppose $X \to Y$ is a $\mathscr{P}_\square$-projective pseudocubical resolution of Y. After evaluating $F_\bullet^\square$ on X we get a bipseudocubical abelian group $F_\bullet^\square(X)$. By Proposition 8.2, we have two spectral sequences associated to $F_\bullet^\square(X)$. Both spectral sequences have only one possible non-trivial row at the E^2-stage and thus they collapse, giving natural isomorphisms

$$H_n^\square(Y, A) \cong H_n(\mathrm{Tot}\, NNF_\bullet^\square(X)), \quad n \geqslant 0,$$

and

$$\mathbf{L}_n^\square F(Y) \cong H_n(\mathrm{Tot}\, NNF_\bullet^\square(X)), \quad n \geqslant 0.$$

Combining these two isomorphisms we get the desired isomorphism.

Note that the class $\mathscr{P} = \mathscr{P}_\Delta \cup \mathscr{P}_\square$ is also a projective class in **Top**. Obviously, the simplicial derived functors with respect to the class $\mathscr{P}_\Delta$ are naturally isomorphic to the simplicial derived functors with respect to $\mathscr{P}$. Similarly, the cubical derived functors with respect to the class $\mathscr{P}_\square$ are naturally isomorphic to the cubical derived functors with respect to $\mathscr{P}$. Thus, by 1.1, there is a natural isomorphism

$$\mathbf{L}_n^\Delta F(Y) \cong \mathbf{L}_n^\square F(Y), \quad n \geqslant 0,$$

for any topological space Y, i.e.,

$$H_n^\Delta(Y, A) \cong H_n^\square(Y, A), \quad n \geqslant 0.$$

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