ASIAN J. MATH.
Vol. 2, No. 2, pp. 355–404, June 1998

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POINCARÉ'S THEOREM AND TEICHMÜLLER THEORY FOR OPEN SURFACES*

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Abstract. Let M^2 be an open oriented surface the isolated ends of which are trumpets or half ladders $\sharp_1^\infty T^2, T^2$ the 2-torus. The completed space $\mathcal{M}^r(I, B_k)$ of metrics of bounded geometry splits into components, $\mathcal{M}^r = \sum_i comp(g_i)$. We define for a component $comp(g_0)$ with $K(g_0) \equiv -1, r_{inj}(g_0) > 0, inf\, \sigma_e(\Delta_{g_0}) > 0$ the Teichmüller space $\mathcal{T}^r(comp(g_0)) = comp(g_0)_{-1}/\mathcal{D}_0^{r+1}(g_0)$, where $comp(g_0)_{-1}$ is the submanifold of metrics with $K(g) \equiv -1$ and $\mathcal{D}_0^{r+1}(g_0)$ is the identity component of the diffeomorphism group. Thereafter we show $\mathcal{T}^r \cong (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1} \cong comp(J_0)/\mathcal{D}_0^{r+1}$. Here $comp(1)$ are conformal factors with Sobolev norm $|e^u - 1|_{g_0,r} < \infty$ and $J_0 = J(g_0)$ is the almost complex structure associated to g_0. The first isomorphism is just Poincaré's lemma.

MR classification 58D27, 58D17, 58G03

1. Introduction. The definition and the study of Teichmüller spaces for closed or compact surfaces with boundaries or surfaces with punctures has for a long time been a frequent topic in geometry and analysis. There are many approaches. First we must mention Ahlfors in [1] and Bers in [2] which rely heavily on the theory of quasiconformal maps. Another more geometric fibre bundle approach has been established by Earle and Eells in [10], [11]. Finally, an approach which relies on methods of differential geometry and global analysis has been presented by Fischer and Tromba in [22], [29]. What they are doing is in a certain sense canonical and at the same time very beautiful. Let M^2 be a closed oriented surface of genus $p > 1$, $\mathcal{M}$ its set of Riemannian metrics, $\mathcal{M}^r$ its Sobolev completion, $\mathcal{M}^r_{-1}$ the submanifold of metrics g with scalar curvature $K(g) \equiv -1, \mathcal{P}^r$ the completed space of positive conformal factors, $\mathcal{A}^r$ the completed space of almost complex structures, $\mathcal{D}^{r+1}$ the completed diffeomorphism group, $\mathcal{D}_0^{r+1} \subset \mathcal{D}^{r+1}$ the component of the identity. Then Fischer and Tromba define as Teichmüller space

$$\mathcal{T}^r(M^2) := \mathcal{A}^r/\mathcal{D}_0^{r+1} \tag{1.1}$$

and prove $\mathcal{D}_0^{r+1}$ -equivariant isomorphisms

$$\mathcal{M}^r/\mathcal{P}^r \cong \mathcal{A}^r \tag{1.2}$$

and

$$\mathcal{M}^r_{-1} \cong \mathcal{M}^r/\mathcal{P}^r. \tag{1.3}$$

Hence there are three models for the Teichmüller space:

$$\mathcal{T}^r = \mathcal{A}^r/\mathcal{D}_0^{r+1} \cong (\mathcal{M}^r/\mathcal{P}^r)/\mathcal{D}_0^{r+1} \cong \mathcal{M}^r_{-1}/\mathcal{D}_0^{r+1}.$$

The isomorphism $\mathcal{M}^r_{-1} \cong \mathcal{M}^r/\mathcal{P}^r$ is known as Poincaré's theorem. Thereafter they prove the existence of a slice for the action of $\mathcal{D}_0^{r+1}$ on $\mathcal{M}^r_{-1}$ thus obtaining charts for a manifold structure on $\mathcal{T}^r$. In [29], [30] Tromba proves that $\mathcal{T}^r$ is diffeomorphic to

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* Received December 17, 1997; accepted for publication June 26, 1998.

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whole approach uses standard results of global analysis on compact manifolds, such as the properness of the $\mathcal{D}^{r+1}$ -action on $\mathcal{M}^r$, the closed image property of elliptic operators, the discreteness of the spectrum, the index theorem, the maximum principle and others.

We study Teichmüller spaces for open oriented surfaces of infinite genus M^2. At the beginning it is totally unclear how to define completed spaces $\mathcal{M}^r$, $\mathcal{M}^r_{-1}$, $\mathcal{T}^r$, $\mathcal{A}^r$, $\mathcal{D}^{r+1}$. A second striking obstruction is the fact that the used results, e.g. the properness of the $\mathcal{D}^{r+1}$ -action and the theorems of elliptic theory are totally wrong.

Nevertheless, the general uniformization theorem tells us that there are many complex = almost complex structures and metrics of curvature -1, i.e. there should be a Teichmüller space which "counts" this structures. The main question is how to count them, how to define a Teichmüller space? In this paper, we present a canonical and natural approach but under certain restrictions. We restrict ourselves to open oriented surfaces of the following kind. Start with a closed oriented surface and form the connected sum with a finite number of half ladders $\sharp_1^\infty T^2$, where T^2 is the 2-torus. Now we allow the repeated addition of a finite number of half ladders in such a manner that there arises a surface with at most countably many ends. Surfaces of the admitted topological type can be built up by Y-pieces which guarantees the existence of a metric g_0 satisfying $K(g_0) \equiv -1$ and $r_{inj}(g_0) > 0$. We exclude metric cusps, but we admit additionally metric trumpets, i.e. topological punctures. To define $\mathcal{M}^r$ we restrict to metrics of bounded geometry, i.e. metrics g satisfying

$$\text{(I)} \qquad r_{inj}(M^n, g) = \inf_{x \in M^n} r_{inj}(x) > 0,$$

$$(B_k) \qquad |\nabla^i R^g| \leq C_i, 0 \leq i \leq k.$$

Denote by $\mathcal{M}(I, B_k)$ the set of all such metrics on M^n. (I) implies completeness. We defined in [12] a uniform structure $\mathfrak{U}^r$ and obtained a completion $\mathcal{M}^r(I, B_k), r \leq k$. $\mathcal{M}^r(I, B_k)$ has a representation as topological sum

$$\mathcal{M}^r(I, B_k) = \sum_{i \in I} comp(g_i)$$

and for $k \geq r > \frac{n}{2} + 1$ each component $comp(g_i)$ is a Hilbert manifold. To each g we adapt a diffeomorphism group $\mathcal{D}^{r+1}, k \geq r + 1 > \frac{n}{2} + 1$. The identity component $\mathcal{D}_0^{r+1}(g)$ is an invariant of $comp(g)$. $\mathcal{D}_0^{r+1}$ acts on $comp(g)$ by $(g, f) \to f^*g$. Similarly we define a completed space $\mathcal{P}^r(g)$ of positive conformal factors.

$$\mathcal{P}^r = \sum_i comp(e^{u_i})$$

and $comp(1) \subset \mathcal{P}^r(g)$ is an invariant of $comp(g)$. $comp(1)$ acts on $comp(g)$. If M^n is compact then $\mathcal{M}^r = \mathcal{M}^r(I, B_\infty), \mathcal{M}^r$ and $\mathcal{P}^r$ consist of only one component, $\mathcal{M}^r = comp(g)$ for any $g, \mathcal{P}^r = comp(1)$. Finally we define a complete space $\mathcal{A}^r(g)$ of almost complex structures,

$$\mathcal{A}^r(g) = \sum_i comp(J_i).$$

Return now to M^2 of the above topological type. Denote by $comp(g)_{-1} \subset comp(g)$ the subspace of all metrics $g' \in comp(g)$ such that $K(g') \equiv -1$. Then we would define

$$\mathcal{T}^r(comp(g)) := comp(g)_{-1}/\mathcal{D}_0^{r+1}$$

and expect

$$comp(g)_{-1} \cong comp(g)/comp(1). \tag{1.4}$$

But there are simple examples of components $comp(g)$ with $comp(g)_{-1} = \phi$. Moreover, we don't see any chance to prove (1.4) for arbitrary g. To have $comp(g)_{-1} \neq \phi$, we start with a metric $g_0 \in \mathcal{M}(I, B_\infty)$ with $K(g_0) \equiv -1$. To g_0 we attach an almost complex structure $J_0 = J(g_0) := g_0^{-1}\mu(g_0)$, where $\mu(g_0)$ is the volume form. Then we can summarize our main results in the following

THEOREM. *Suppose* $g_0 \in \mathcal{M}(I, B_\infty), K(g_0) \equiv -1$, $\inf \sigma_e(\Delta g_0) > 0, r > 3$. *Then* $comp(g_0)_{-1} \subset comp(g_0) \subset \mathcal{M}(I, B_\infty)$ *is a submanifold. There is a* $\mathcal{D}_0^{r+1}(g_0)$ *-equivariant isomorphism*

$$comp(g_0)_{-1} \cong comp(g_0)/comp(1) \cong comp(J_0). \tag{1.5}$$

If we define the Teichmüller space $\mathcal{T}^r(comp(g_0))$ *of* $comp(g_0)$ *as*

$$\mathcal{T}^r(comp(g_0)) := comp(J_0)/\mathcal{D}_0^{r+1} \tag{1.6}$$

then

$$\mathcal{T}^r(comp(g_0)) \cong comp(g_0)_{-1}/\mathcal{D}_0^{r+1} \cong (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1}. \tag{1.7}$$

The first isomorphism in (1.5) is Poincaré's theorem for the open case. Its proof occupies the major part of the paper. Moreover, we establish an ILH-version of (1.5)-(1.7). The paper is organized as follows. In section 2 we recall the main facts concerning spaces of Riemannian metrics and Sobolev spaces needed in this paper. In section 3 and 4 we define the space $\mathcal{P}^r$ and $\mathcal{A}^r$ of conformal factors and almost complex structures. Section 5 is devoted to the diffeomorphism group $\mathcal{D}^{r+1}$ and section 6 contains the ILH-version of the considered spaces. In section 7 we prove Poincaré's theorem. The sections 8, 9, 10 are devoted to the proof of (1.5), (1.7). In the concluding section 11 we announce and discuss results concerning the topology of $\mathcal{T}^r(comp(g_0)$ which are the topic of an also long paper in preparation.

We remark, there are other approaches to define a universal Teichmüller space for open surfaces. The advantage of our approach is to couple those metrics resp. complex structures together which belong in a natural sense together, i.e. are elements of the same component in the space of metrics of bounded geometry. For each such component $comp(g_0)_{-1}/\mathcal{D}_0^{r+1}$, there is a good chance to establish a Hilbert manifold structure. The only existing gap is a slice theorem, where parts of a slice theorem are already proven.

But there are uncountably many components containing complete metrics of curvature -1. Fitting them together in a universal Teichmüller space offers absolutely no chance to introduce a manifold structure, modeled over a separable Hilbert space (Sobolev space).

To introduce such a structure and to make Riemannian geometry in the Teichmüller space is one of the advantages of our approach.

The author is deeply indebted to the Max-Planck-Institut für Mathematik for hospitality and good working conditions.

2. Spaces of Riemannian metrics of bounded geometry and Sobolev spaces. Let (M^n, g) be open. Consider the following two conditions (I) and (B_k).

$$r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0, \tag{I}$$

(B_k)

$$|\nabla^i R| \leq C_i, 0 \leq i \leq k,$$

where $r_{inj}(x)$ denotes the injectivity radius at x and R the curvature.

LEMMA 2.1. *If (M^n, g) satisfies (I) then (M^n, g) is complete.*

See [12] for a proof. □

We say (M^n, g) has bounded geometry up to order k if it satisfies (I) and (B_k). Given M^n open and $0 \leq k \leq \infty$. Then there always exists g satisfying (I) and (B_k), i.e. there is no topological obstruction against metrics of bounded geometry of any order.

Set for given M^n

$$\mathcal{M}(I) = \{g | g \quad \text{satisfies} \quad (I)\},$$
$$\mathcal{M}(B_k) = \{g | g \quad \text{satisfies} \quad (B_k)\}$$

and

$$\mathcal{M}(I, B_k) = \mathcal{M}(I) \cap \mathcal{M}(B_k).$$

Denote as above for a tensor t and a metric g by $|t|_{g,x}$ its pointwise and by

$$^b|t|_g := \sup_{x \in M} |t|_{g,x}$$

its supremum norm with respect to g.

LEMMA 2.2. *g and g' are quasi isometric if and only if $^b|g - g'|_g < \infty$ and $^b|g - g'|_{g'} < \infty$.* □

Let

$$^bU(g) = \{g' | ^b|g - g'|_g < \infty \quad \text{and} \quad ^b|g - g'|_{g'} < \infty\} =$$
$$= \text{quasi isometry class of} \quad g.$$

Set for $\delta > 0, p \geq 1, r \in \mathbb{Z}_+$

$$V_\delta = \{(g, g') \in \mathcal{M}(I, B_k)^n | g' \in^b U(g) \quad \text{and}$$

$$|g - g'|_{g,p,r} := (\int (|g - g'|^p_{g,x} + \sum_{i=0}^{r-1} |(\nabla^g)^i (\nabla^g - \nabla g')|^p_{g,x}) dvol_x(g))^{1/p} < \delta\}.$$

THEOREM 2.3. *Assume* $r \leq k, 1 \leq p < \infty$. *Then* $\mathfrak{L} = \{V_\delta\}_{\delta > 0}$ *is a basis for a metrizable uniform structure* $\mathfrak{U}^{p,r}(\mathcal{M}(I, B_k))$ *on* $\mathcal{M}(I, B_k)$.

See [12] for the nontrivial proof. □

Let $\mathcal{M}_r^p(I, B_k) = \mathcal{M}(I, B_k)$ endowed with the uniform topology, $\mathcal{M}^{p,r} = \overline{\mathcal{M}_r^p}$ the completion. If $k \geq r > \frac{n}{p} + 1$ then $\mathcal{M}^{p,r}$ still consists of C^1-metrics, i.e. does not contain semi definite elements. This has been proved by Salomonsen in [26]. We extend the definition above of ${}^bU(g)$ to C^1 metrics.

THEOREM 2.4. *Let* $k \geq r > \frac{n}{p} + 1, g \in \mathcal{M}(I, B_k)$, $U^{p,r}(g) = \{g' \in \mathcal{M}^{p,r}(I, B_k) | g' \in^b U(g)$ *and* $|g - g'|_{g,p,r} < \infty\}$ *and denote by* $comp(g) \subset \mathcal{M}^{p,r}(I, B_k)$ *the component of* g *in* $\mathcal{M}^{p,r}(I, B_k)$. *Then*

$$comp(g) = U^{p,r}(g) \tag{2.1}$$

and $\mathcal{M}^{p,r}(I, B_k)$ *has a representation as topological sum*

$$\mathcal{M}^{p,r}(I, B_k) = \sum_{j \in J} comp(g_j), \tag{2.2}$$

J *an uncountable set.*

The proof is performed in [12]. □

REMARKS. 1. If M^n is compact then the set J consists of one element. 2. If g is non-smooth then there are some small problems to define and to understand $|g - g'|_{g,p,r}$ for $r \geq 2$. In this case one defines $(\nabla^g)^i := (\nabla^{g_0} + (\nabla^g - \nabla^{g_0}))^i$ where $g_0 \in comp(g)$ is smooth and fixed chosen. It is easy to see that $(\nabla^{g_0} + (\nabla^g - \nabla^{g_0}))^i$ makes sense since ∇^{g_0} is a smooth differential operator, $\nabla^g - \nabla^{g_0}$ is a distributional tensor field and $(\nabla^{g_0})^i((\nabla^g - \nabla^{g_0})^j)$ is well defined. We refer to [20] for details. □

Let T_v^u be the bundle of u-fold covariant and v-fold contravariant tensors and define

$$\Omega_r^p(T_v^u, g) = \{t \in C^\infty(T_v^u) | \, |t|_{g,p,r} := = \Big(\int \sum_{i=0}^r |(\nabla^g)^i t|_{g,x}^p dvol_x(g)\Big)^{1/p} < \infty\},$$

$\bar{\Omega}^{p,r}(T_v^u, g) =$ completion of $\Omega_r^p(T_v^u, g)$ with respect to $| \;\; |_{g,p,r}$, $\overset{\circ}{\Omega}{}^{p,r}(T_v^u, g) =$ completion of $C_0^\infty(T_v^u)$ with respect to $| \;\; |_{g,p,r}$ and $\Omega^{p,r}(T_v^u, g) =$ all distributional tensor fields t with $|t|_{g,p,r} < \infty$. Then

$$\overset{\circ}{\Omega}{}^{p,r}(T_v^u, g) \subseteq \bar{\Omega}^{p,r}(T_v^u, g) \subseteq \Omega^{p,r}(T_v^u, g).$$

PROPOSITION 2.5. *Assume* $g \in \mathcal{M}(I, B_k), r \leq k + 2$. *Then*

$$\overset{\circ}{\Omega}{}^{p,r}(T_v^u, g) = \bar{\Omega}^{p,r}(T_v^u, g) = \Omega^{p,r}(T_v^u, g). \tag{2.3}$$

See [13] for a proof. □

Let S^2T^* be the bundle of twofold covariant symmetric tensors. $\Omega^{p,r}(S^2T^*, g)$ is defined as above.

THEOREM 2.6. *Assume* $k \geq r > \frac{n}{p} + 1$, $g \in \mathcal{M}(I, B_k)$. *Then* $comp(g) \subset \mathcal{M}^{p,r}(I, B_k)$ *is a Banach manifold and for* $p = 2$ *a Hilbert manifold.*

Proof. $\phi : comp(g) \to \Omega^{p,r}(S^2T^*, g), \phi(g') = g - g'$, is a homeomorphism onto an open subset of $\Omega^{p,r}(S^2T^*, g)$. See [12] for details. □

Define

$$^{b,m}|t|_g = \sum_{i=0}^{m} \sup_{x \in M} |\nabla^i t|_{g,x},$$

$$^{b}_{m}\Omega(T^u_v, g) := \{t \in C^\infty(T^u_v) |\, ^{b,m}|t|_g < \infty\},$$

$^{b,m}\Omega(T^u_v, g)$ = completion of $^{b}_{m}\Omega(T^u_v, g)$ with respect to $^{b,m}|\ \ |_g$ and $^{b,m}\mathring{\Omega}(T^u_v, g)$ = completion of $C^\infty_0(T^u_v)$ with respect to $^{b,m}|\ \ |_g$. Then $^{b,m}\Omega(T^u_v, g) = \{t | t\, C^m$ -tensor field and $^{b,m}|t| < \infty\}$.

THEOREM 2.7. *Assume* (M^n, g) *is open and satisfies* $(I), (B_0)$. *If* $r > \frac{n}{p} + m$, *then there are continuous embeddings*

$$\mathring{\Omega}^{p,r}(T^u_v, g) \hookrightarrow^{b,m} \mathring{\Omega}(T^u_v, g), \tag{2.4}$$

$$\bar{\Omega}^{p,r}(T^u_v, g) \hookrightarrow^{b,m} \Omega(T^u_v, g). \tag{2.5}$$

If, additionally, (M^n, g) *satisfies* $(B_k(M)), k \geq 1, k \geq r, r - \frac{n}{p} \geq s - \frac{n}{q}$, $r \geq s$, $q \geq p$, *then*

$$\Omega^{p,r}(T^u_v) \hookrightarrow \Omega^{q,s}(T^u_v) \tag{2.6}$$

continuously.

We refer to [13], [15] for the proof. □

THEOREM 2.8. *Assume* (M^n, g) *with* (I) *and* (B_k), $0 \leq r \leq r_1, r_2 \leq k$. *If* $r = 0$ *assume*

$$\left\{\begin{array}{l} -\frac{n}{p} < r_1 - \frac{n}{p_1} \\ -\frac{n}{p} < r_2 - \frac{n}{p_2} \\ -\frac{n}{p} \leq r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \\ \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \end{array}\right\} \text{ or } \left\{\begin{array}{c} r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} \\ 0 < r_2 - \frac{n}{p_2} \\ \frac{1}{p} \leq \frac{1}{p_1} \end{array}\right\} \text{ or } \left\{\begin{array}{c} 0 < r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} \leq r_2 - \frac{n}{p_2} \\ \frac{1}{p} \leq \frac{1}{p_2} \end{array}\right\}. \tag{2.7}$$

If $r > 0$ *assume* $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ *and*

$$\left\{\begin{array}{l} r - \frac{n}{p} < r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} < r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \end{array}\right\} \text{ or } \left\{\begin{array}{l} r - \frac{n}{p} \leq r_1 - \frac{n}{p_1} \\ r - \frac{n}{p} \leq r_2 - \frac{n}{p_2} \\ r - \frac{n}{p} < r_1 - \frac{n}{p_1} + r_2 - \frac{n}{p_2} \end{array}\right\}. \tag{2.8}$$

Then the tensor product of tensor fields defines a continuous bilinear map

$$\Omega^{p_1,r_1}(T^u_v) \times \Omega^{p_2,r_2}(T^{u'}_{v'}) \to \Omega^{p,r}(T^u_v \otimes T^{u'}_{v'}). \tag{2.9}$$

Idea of proof. We indicate the proof for $T^u_v = M \times \mathbb{R}$. Assume $u \in \Omega^{p_1,r_1}(M)$, $v \in \Omega^{p_2,r_2}(M)$. We are done if we can show

$$|\nabla^i(u \cdot v)|_{p,0} \le C \cdot |u|_{p_1,r_1} \cdot |v|_{p_2,r_2}, \quad 0 \le i \le r,$$

or

$$|\nabla^j u \otimes \nabla^{i-j} v)|_{p,0} \le C \cdot |u|_{p_1,r_1} \cdot |v|_{p_2,r_2}, \quad 0 \le j \le i \le r.$$

(2.8) follows from Hölder's inequality if e.g. in the case $r > 0$ $\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2}$ and

$$\left\{\begin{array}{l} r - \dfrac{n}{p} < r_1 - \dfrac{n}{p_1} \\ r - \dfrac{n}{p} < r_2 - \dfrac{n}{p_2} \\ r - \dfrac{n}{p} \le r_1 - \dfrac{n}{p_1} + r_2 - \dfrac{n}{p_2} \end{array}\right\} \text{ or } \left\{\begin{array}{l} r - \dfrac{n}{p} \le r_1 - \dfrac{n}{p_1} \\ r - \dfrac{n}{p} \le r_2 - \dfrac{n}{p_2} \\ r - \dfrac{n}{p} < r_1 - \dfrac{n}{p_1} + r_2 - \dfrac{n}{p_2} \end{array}\right\}.$$

□

Quite analogously to T^u_v one defines for Riemannian vector bundles $(E, h, \nabla^h) \to M$ Sobolev spaces $\bar{\Omega}^{p,r}(E, \nabla)$ and ${}^{b,m}\Omega(E, \nabla)$. $(B_k(E))$ means $|(\nabla^h)^i R^h| \le C_i, 0 \le i \le k$. Then 2.8 generalizes to

THEOREM 2.9. *Assume (M^n, g) with $(I), (B_k)$, $(E_i, h_i, \nabla_i) \to M$ with $(B_k), i = 1, 2$. Assume $0 \le r \le r_1, r_2 \le k$ and p, p_1, p_2 as in 2.8. Then there exists a continuous embedding $\Omega^{p_1,r_1}(E_1, \nabla_1) \otimes \Omega^{p_2,r_2}(E_2, \nabla_2) \hookrightarrow \Omega^{p,r}(E_1 \otimes E_2, \nabla_1 \otimes \nabla_2)$. The assertion generalizes to a finite number of bundles.*

We refer to [15] for the proof. □

REMARKS.

1. A special case for E is T^u_v. Here $(B_k(M))$ automatically implies $(B_k(E))$.
2. For $p_1 = p_2 = q = 2, r \ge \bar{r}, r > \frac{n}{2}$, 2.8 implies a bilinear continuous map

$$\bullet : \Omega^{2,r}(M) \times \Omega^{2,\bar{r}}(M) \to \Omega^{2,\bar{r}}(M) \tag{2.10}$$

In particular $\Omega^{2,r}(M)$ becomes a ring for $r > \frac{n}{2}$.

3. (2.4) - (2.6) hold for $\Omega^{p,r}(E), {}^{b,m}\Omega(E)$ correspondingly.
4. $\mathcal{M}^{p,r-1}(I, B_k)$ is still well defined since $k \ge r > \frac{n}{p} + 1$ implies $r - 1 > \frac{n}{p}$. □

A question, which is in the main section 7 of extraordinary meaning, is the invariance of Sobolev spaces under certain changes of the metric and their definition by other differential operators.

THEOREM 2.10. *Assume $k \ge r > \frac{n}{p} + 1, g_0 \in \mathcal{M}(I, B_k)$. Then $\Omega^{p,r}(T^u_v, g_0)$ is an invariant of comp$(g_0) \subset M^{p,r-1}(I, B_k)$, i.e.*

$$\Omega^{p,r}(T^u_v, \nabla^{g_0}, g_0) \cong \Omega^{p,r}(T^u_v, \nabla^g, g) \tag{2.11}$$

as equivalent Banach spaces.

Proof. We have for the pointwise norm $|\ \ |_{g_0} \sim |\ \ |_g$ since g_0 and g are continuous and quasi isometric. Writing

$$\nabla^g = \nabla^{g_0} + (\nabla^g - \nabla^{g_0}) \tag{2.12}$$

and assuming the definition above of ${}^bU(g)$ to C^1 metrics, we obtain for a tensor field τ a pointwise estimate

$$|(\nabla^g)^i \tau| \le P(|\nabla^{g_0})^{j_1}(\nabla^g - \nabla^{g_0})|, |(\nabla^{g_0})^{j_k}\tau|), \tag{2.13}$$

where P is a polynomial in the indicated variables, $j_1 \leq r-1, j_1+j_2 \leq i$, and each monomial satisfies the condition of the module structure theorem and has at least one $|(\nabla^{g_0})^{j_k}\tau|$ as factor. Hence we obtain after $p-th$ power and integration

$$|\tau|_{g,p,r} \leq C_1|\tau|_{g_0,p,r} \tag{2.14}$$

and, for symmetry reasons

$$|\tau|_{g_0,p,r} \leq C_2|\tau|_{g,p,r}, \tag{2.15}$$

$C_i = C_i(g, g_0)$. See [14] for details. □

We remark that in (2.12) - (2.15) we do not really need g smooth. This follows from the remarks after 2.4. In section 7 we consider a slightly more general situation, $g \in comp(g_0)$, $g_t = g_0 + t(g-g_0) = g_0 + th \in comp(g_0)$. Then the constants C_1, C_2 in (2.14), (2.15) will depend on $t, C_i = C_i(g_0, g_t)$. We need in section 7 the existence of constants C_i independent of t which we will now prove. Now and in the sequel we often denote constants in different contexts by the same letter where we are convinced that no confusion will arise.

First, there exist by assumption constants C_1, C_2, $0 < C_1 \leq 1 \leq C_2$,

$$C_1 g_0 \leq g \leq C_2 g_0 \tag{2.16}$$

which implies

$$C_1 g_0 \leq (1-t)g_0 + tg \equiv g_t \leq C_2 g_0, \tag{2.17}$$

and

$$C_1' det\, g_0 \leq det\, g_t \leq C_2' det\, g_0 \tag{2.18}$$

$$C_1'' g_0^{-1} \leq g_t^{-1} \leq C_2'' g_0^{-1}. \tag{2.19}$$

LEMMA 2.11. *If (M^n, g) satisfies (I) and (B_k) and $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}_\alpha$ is a uniformly locally finite cover by normal charts, then there exist constants $C_\beta, C_\beta', C_\gamma'$, β, γ multiindexes, such that*

$$|D^\beta g_{ij}| \leq C_\beta, |D^\beta g^{ij}| \leq C_\beta', |\beta| \leq k, |D^\gamma \Gamma_{ij}^m| \leq C_\gamma', |\gamma| \leq k-1, \tag{2.20}$$

all constants independent of α.

See [17] for a proof. □

COROLLARY 2.12 *Let $g_0 \in \mathcal{M}(I, B_k), g \in comp(g_0) \subset \mathcal{M}^r(I, B_k), k \geq r > \frac{n}{p} + 1, \mathfrak{U} = \{U_\alpha, \phi_\alpha)\}_\alpha$ an atlas of normal charts with respect to g_0 as above. Then, with respect to $\mathfrak{U}$,*

$$|D^\beta g_t^{ij}| \leq C, |\beta| \leq 1. \tag{2.21}$$

Proof. This follows from the definition of g_t^{ij}, (2.19), (2.20), $g \in comp(g_0), g_t = g_0 + t(g-g_0)$ and ${}^{b,1}|g-g_0|_{g_0} < \infty$. □

PROPOSITION 2.13. *Assume g_0, g, k, r as above. Then there exists a constant $C = C(g_0, g)$ independent of t such that*

$$|\nabla^{g_0} - \nabla^{g_t}|_{g_0, r-1} \le C. \tag{2.22}$$

Proof. Pointwise

$$|\nabla^{g_0} - \nabla^{g_t}| = |\Gamma^i_{jm}(g_0) - (\Gamma^i_{jm}(g_0) + \frac{t}{2} g_t^{il}(h_{ej;m} + h_{em;j} + h_{jm;e}))|,$$

where $_{;m} = \nabla^{g_0}_m$, $h = g - g_0$. This and (2.20) for g_0, (2.21) imply

$$|\nabla^{g_0} - \nabla^{g_t}| \le C_0 \cdot t \cdot |\nabla h| \le C_0 |\nabla h|. \tag{2.23}$$

Write $[h] = (h_{ej;m} + h_{em;j} + h_{jm;e})$. Then $\nabla^{g_0}(\nabla^{g_0} - \nabla^{g_t}) = t\nabla^{g_0} g_t^{il}[h] = t(\nabla^{g_t} + (\nabla^{g_0} - \nabla^{g_t}))g_t^{il}[h] = t\{g_t^{ie}\nabla^{g_t}[h] + (\nabla^{g_t} - \nabla^{g_0})g_t^{ie}[h]\}$, i.e.

$$\nabla^{g_0}(\nabla^{g_0} - \nabla^{g_t})| \le C \cdot |\nabla^{g_t}[h]| + C_0 \cdot C \cdot |\nabla h|^2. \tag{2.24}$$

But

$$|\nabla^{g_t}[h]| \le |\nabla^{g_0}[h]| + |(\nabla^{g_t} - \nabla^{g_0})[h]| \le C'|\nabla^2 h| + C_0 \cdot C'' \cdot |\nabla h|^2. \tag{2.25}$$

We infer from (2.24), (2.25)

$$|\nabla^{g_0}(\nabla^{g_0} - \nabla^{g_t})| \le C_2(|\nabla h|^2 + |\nabla^2 h|)$$

An easy induction quite similar to [12], [14] yields

$$|(\nabla^{g_0})^i(\nabla^{g_0} - \nabla^{g_t}| \le P_i(|\nabla^{j_1} h|^{j_h}), \tag{2.26}$$

where P_i is a polynomial in the indicated variables and the monomials satisfy the conditions of the module structure theorem, in particular $j_1 + j_2 \le i + 1 \le r$. (2.26) implies after $p-th$ power and integration (2.22). □

Rewriting $\nabla^{g_t}(\nabla^{g_0} - \nabla^{g_t}) = (\nabla^{g_t} - \nabla^{g_0})(\nabla^{g_0} - \nabla^{g_t}) + \nabla^{g_0}(\nabla^{g_0} - \nabla^{g_t})$ and so on (cf. [12]) and using (2.22) and its proof, we conclude

$$|\nabla^{g_0} - \nabla^{g_t}|_{g_t, r-1} \le C', \tag{2.27}$$

C' independent of t.

COROLLARY 2.14 *Assume g_0, g, k, r as above. Then*

$$\Omega^{p,v}(T^u_v, g_0) \cong \Omega^{p,r}(T^u_v, g_t), \tag{2.28}$$

$$|\quad|_{g_0,p,r} \le C_1 \cdot |\quad|_{g_t,p,r} \tag{2.29}$$

$$|\quad|_{g_t,p,r} \le C_2 \cdot |\quad|_{g_0,p,r} \tag{2.30}$$

which constants $C_i = C_i(g_0, g)$ independent of t. This follows from (2.13) for the pair g_0, g_t and (2.26), (2.27). □

Until now we considered Sobolev spaces based on the covariant derivative ∇^{g_0}, $\Omega^{p,r}(T^u_v, g_0) = \Omega^{p,r}(T^u_v, \nabla^{g_0}, g_0)$. For r even there is another definition of $\Omega^{p,r}$ based on $1, \Delta, \Delta^2, \ldots, \Delta^{r/2}$, $\Delta = \Delta_{g_0} = (\nabla^{g_0})^* \nabla^{g_0}$,

$$|\tau|_{g_0,p,r} = (\int \sum_{i=0}^{r/2} |\Delta^i \tau|^p_{g_0,x} dvol_x(g_0))^{1/p}.$$

THEOREM 2.15. *Assume* $(I), (B_k)$ *for* $(M, g_0), k \geq r, r$ *even. Then*

$$\Omega^{2,r}(M, \nabla^{g_0}, g_0) \cong \Omega^{2,r}(M, \Delta_{g_0}, g_0) \tag{2.31}$$

as equivalent Hilbert spaces.

We refer to [5] for a proof. The main part is that the local Garding's inequality associated with $\mathfrak{U} = \{U_\alpha\}_\alpha$ has constants independent of α. The proof given in [9], [13] contains a mistake. □

There are several techniques to define $\Omega^{2,r}(M, \Delta_{g_0}, g_0)$ for odd r too, e.g. interpolation techniques. (2.31) and its proof, (2.26) - (2.30) imply

THEOREM 2.16. *Assume* (M^n, g_0) *with* (I) *and* (B_k), $k \geq r > \frac{n}{2} + 1$, $g \in$ *comp*$(g_0) \subset \mathcal{M}^r(I, B_k), r$ *even. Then*

$$\begin{aligned} &\Omega^{2,r}(T^u_v, \Delta_{g_0}, g_0) \cong \Omega^{2,r}(T^u_v, \nabla^{g_0}, g_0) \cong \\ &\cong \Omega^{2,r}(T^u_v, \nabla^{g_t}, g_t) \cong \Omega^{2,r}(T^u_v, \Delta_{g_t}, g_t) \end{aligned} \tag{2.32}$$

as equivalent Hilbert spaces with constants independent of t. □

Assume $g_0 \in \mathcal{M}(I, B_k)$ and let $\mathfrak{U} = \{U_\alpha, \phi_\alpha)\}_\alpha$ be a uniformly locally finite atlas of normal charts with respect to g_0 and with radius of $U_\alpha = c < r_{inj}(g_0)$, $\{\psi_\alpha\}_\alpha$ an associated partition of unity with $|\nabla^i \psi_\alpha| \leq C_i, 0 \leq i \leq k+2$. Then, using local Euclidean derivatives, we can define for $r \leq k$ Sobolev spaces $\Omega^r(T^u_v, \mathfrak{U}, \{\psi_\alpha\}_\alpha, g_0)$.

THEOREM 2.17.

$$\Omega^r(T^u_v, \mathfrak{U}, \{\psi_\alpha\}_\alpha, g_0) \cong \Omega^r(T^u_v, \nabla^{g_0}, g_0) \tag{2.33}$$

as equivalent Hilbert spaces.

The proof follows from 2.11. □

3. The space of bounded conformal factors. We now define the space of bounded conformal factors adapted to a Riemannian metric g. Later we assume additionally $g \in \mathcal{M}(I, B_k)$. Let

$$\mathcal{P}_m(g) = \{\varphi \in C^\infty(M) | \inf_{x \in M} \varphi(x) > 0, \sup_{x \in M} \varphi(x) < \infty, |\nabla^i \varphi|_{g,x} \leq C_i, 0 \leq i \leq m\}$$

and set for $r \leq m, r > \frac{n}{p} + 1$

$$\begin{aligned} V_\delta = \{(\varphi, \varphi') \in \mathcal{P}_m(g)^2 | \, |\varphi - \varphi'|_{g,p,r} := \\ = (\int \sum_{i=0}^{r} |(\nabla^g)^i (\varphi - \varphi')|^p_{g,x} dvol_x(g))^{1/p} < \delta\}. \end{aligned}$$

PROPOSITION 3.1. *$\mathfrak{L} = \{V_\delta\}_{\delta>0}$ is a basis for a metrizable uniform structure.*
We omit the very simple proof. □

Let $\bar{\mathcal{P}}^p_{m,r}(g)$ be the completion,

$$C^1\mathcal{P} = \{\varphi \in C^1(M) | \inf_{x\in M} \varphi(x) > 0, \sup_{x\in M} \varphi(x) < \infty\}$$

and set

$$\mathcal{P}^{p,r}_m(g) = \bar{\mathcal{P}}^p_{m,r} \cap C^1\mathcal{P}.$$

$\mathcal{P}^{p,r}_m$ is locally contractible, hence locally arcwise connected and hence components coincide with arc components. Let

$$U^{p,r}_m(\varphi) = \{\varphi' \in \mathcal{P}^{p,r}_m(g) | \, |\varphi - \varphi'|_{g,p,r} < \infty\}$$

and denote by *comp*(φ) the component of φ in $\mathcal{P}^{p,r}_m(g)$.

THEOREM 3.2. *For $\varphi \in \mathcal{P}^{p,r}_m(g)$,*

$$comp(\varphi) = U^{p,r}_m(\varphi) \tag{3.1}$$

and $\mathcal{P}^{p,r}_m(g)$ has a representation as topological sum

$$\mathcal{P}^{p,r}_m(g) = \sum_{i\in I} comp(\varphi_i). \tag{3.2}$$

The proof of (3.1), (3.2) is quite similar to that of (2.1) and (2.2) which is performed in [12]. □

The function identically to 1 is an element of all $\mathcal{P}_m(g), 0 \le m < \infty$. Write $comp^{p,r}_m(1,g)$ for the component of 1 in $\mathcal{P}^{p,r}_m(g)$. Assume $k \ge r > \frac{n}{p} + 1$.

PROPOSITION 3.3. *$comp^{p,r}_m(1,g)$ is an invariant of $comp(g) \subset \mathcal{M}^{p,r}(I, B_k)$, i.e.*

$$comp^{p,r}_m(1,g) = comp^{p,r}_m(1,g') \tag{3.3}$$

for $g' \in comp(g)$.

Proof. We assume without loss of generality g and g' smooth. If not, then we apply the remark 2 after 2.4 and proceed as usual. The proof of 3.3 is quite analogous to that of 2.10. We present it here for completeness. Set $\nabla = \nabla^g, \nabla' = \nabla^{g'}$ and let $\varphi \in comp^{p,r}_m(1,g)$. Then $\varphi \in C^1$ (since $k \ge r > \frac{n}{p} + 1$) and

$$|\varphi - 1|_{g,p,r} = (\int \sum_{i=0}^{r} |\nabla^i(\varphi - 1)|^p_{g,x} dvol_x(g)|^{1/p} < \infty. \tag{3.4}$$

We have to show

$$|\varphi - 1|_{g',p,r} < \infty. \tag{3.5}$$

The pointwise norms $|\nabla'^i(\varphi - 1)|_{g,x}$ and $|\nabla^i(\varphi - 1)|_{g',x}$ are equivalent since g and g' are quasi isometric and we simply write $| \quad |_x \equiv | \quad |$. Then

$$|\nabla'(\varphi - 1)| \le |\nabla' - \nabla| \, |\varphi - 1| + |\nabla(\varphi - 1)|, \tag{3.6}$$

$$\begin{aligned}|\nabla'^2(\varphi-1)| &\le |(\nabla'-\nabla)(\nabla'-\nabla)\varphi| + |(\nabla'-\nabla)\nabla\varphi| + \\ &+ |\nabla(\nabla'-\nabla)\varphi| + |\nabla^2\varphi| \le \\ &\le C(|\nabla'-\nabla|^2|\varphi| + |\nabla'-\nabla||\nabla\varphi| + |\nabla(\nabla'-\nabla)||\varphi| + |\nabla^2\varphi|).\end{aligned} \tag{3.7}$$

A more general formula for $|\nabla'^i(\varphi-1)|$ estimating this by products of the kind

$$|\nabla^{n_1}(\nabla'-\nabla)|\dots|\nabla^{n_s-1}(\nabla'-\nabla)||\nabla^{n_s}(\varphi-1)| \tag{3.8}$$

has been established in [12]. Using (2.1) and the module structure theorem for Sobolev spaces, we obtain

$$(\int(\nabla^{n_1}(\nabla'-\nabla)|\dots|\nabla^{n_{s-1}}(\nabla'-\nabla)||\nabla^{n_s}(\varphi-1)|)^p dvol)^{1/p} < \infty \tag{3.9}$$

and (3.9) can be estimated by the Sobolev norms of $\nabla'-\nabla$ and $\varphi-1$. Hence $\varphi \in com^{p,r}_m(1,g'), comp^{p,r}_m(1,g) \subseteq comp^{p,r}_m(1,g)$. In the same manner we establish the other inclusion. □

REMARKS. Proposition 3.3 does not hold for an arbitrary component $comp^{p,r}_m(\psi,g)$, $\psi \in \mathcal{P}_m(g)$, since $\psi \in \mathcal{P}_m(g)$ does for $j > 2, j \le r \le m$ not imply $\psi \in \mathcal{P}_m(g')$. The latter follows from the fact that we have

$$\int |\nabla^j(\nabla'-\nabla)|^p dvol < \infty$$

but not necessarily

$$\sup_{x\in M}|\nabla^j(\nabla'-\nabla)|_x < \infty.$$

□

In the sequel we restrict ourselves to the case $p = 2$ and write $\Omega^{2,r} \equiv \Omega^r$, $\mathcal{M}^{2,r}(I,B_k) \equiv \mathcal{M}^r(I,B_k)$, $\mathcal{P}^{2,r}_m(g) \equiv \mathcal{P}^r_m(g)$, $|\ \ |_{g,2,r} = |\ \ |_{g,r}$. Next we indicate the structure of $\mathcal{P}^r_m(g)$.

THEOREM 3.4. *Under multiplication $\mathcal{P}^r_m(g)$ is a Hilbert-Lie group.*

Sketch of proof. It follows immediately from the definition, the product and quotient rule and the module structure theorem that $\mathcal{P}^r_m(g)$ is a group. $\mathfrak{L} = \{U_\delta\}_\delta > 0$,

$$U_\delta = \{\varphi \in \mathcal{P}^r_m(g) | \, |\varphi - 1|_{g,r} < \delta\},$$

is a filter basis centred at $1 \in \mathcal{P}^r_m(g)$ that satisfies all axioms for the neighbourhood fiber of 1 of a topological group. Hence $\mathcal{P}^r_m(g)$ is a topological group (cf. [3]). Finally, U_δ is homeomorphic to an open ball in $\Omega^{2,r}(M)$ for $\delta > 0$ sufficiently small and has the structure of a local real Lie group. Hence $\mathcal{P}^r_m(g)$ is a Hilbert-Lie group. □

Assume as always $k \ge r > \frac{n}{2}+1, g \in \mathcal{M}(I,B_k)$ and consider $comp^r_{k+2}(1) \subset \mathcal{P}^r_{k+2}(g), comp(g) \subset \mathcal{M}^r(I,B_k)$.

PROPOSITION 3.5. *a. There is a well defined action*

$$comp^r_{k+2}(1) \times comp(g) \to comp(g)$$
$$(\varphi', g') \to \varphi' \cdot g'.$$

b. The action is smooth, free and proper.

Proof. Let $\varphi' \in comp^r_{k+2}(1) \subset \mathcal{P}^r_{k+2}(g), g' \in comp(g)$. We have to show $\varphi' \cdot g' \in comp(g)$. There exist sequences $\varphi_\nu \underset{|\ |_{g,r}}{\longrightarrow} \varphi', g_\nu \underset{|\ |_{g,r}}{\longrightarrow} g'$, $\varphi_\nu \in comp^r_{k+2}(1) \subset \mathcal{P}_{k+2}(g), g_\nu \in comp(g) \cap \mathcal{M}(I, B_k)$. Then, according to [8], p. 47, Theorem 4.7 and the fact, that g_ν satisfies (I) and $\varphi_\nu \in \mathcal{P}_{k+2}(g)$, we conclude $\varphi_\nu \cdot g_\nu$ satisfies (I). From [23], p. 90 follows that $R^{g_\nu} - R^{\varphi_\nu \cdot g_\nu}$ = sum of terms each of them has bounded derivatives up to order k. Using $\nabla^{g_\nu} - \nabla^{\varphi_\nu \cdot g_\nu}$ = sum of terms each of them has bounded derivatives up to order $k+1$, we see finally that $\varphi_\nu \cdot g_\nu$ satisfies (B_k), i.e. $g_\nu \in \mathcal{M}(I, B_k), \varphi_\nu \in comp(1) \subset \mathcal{P}_{k+2}(g)$ imply $\varphi_\nu \cdot g_\nu \in \mathcal{M}(I, B_k)$. Moreover,

$$\varphi_\nu \cdot g_\nu - g = \varphi_\nu(g_\nu - g) + (\varphi_\nu - 1)g, \quad \varphi_\nu = \varphi_\nu - 1 + 1$$

immediately implies $|\varphi_\nu g_\nu - g|_{g,r} < \infty, \varphi_\nu \cdot g_\nu \in comp(g)$. We conclude from

$$\begin{aligned}
&\varphi_\nu \cdot g_\nu - \varphi' \cdot g' = (\varphi_\nu - \varphi')g_\nu + \varphi'(g_\nu - g'), \\
&g_\nu = (g_\nu - g') + (g' - g) + g, \\
&\varphi'(g_\nu - g') = (\varphi' - 1)(g_\nu - g') + (g_\nu - g')
\end{aligned}$$

and the module structure theorem

$$|\varphi' g' - g|_{g,r} < \infty, \varphi' \cdot g' \in comp(g).$$

b. The smoothness of the action follows from the fact that locally $comp(1)$ and $comp(g)$ can be treated as linear spaces. $\varphi' \cdot g' = g'$ implies $\varphi' \equiv 1$. If

$$\varphi_\nu \cdot g' \to h \tag{3.10}$$

in $comp(g)$, i.e. with respect to $|\ \ |_{g,r}$, then we have also C^1-convergence according to the Sobolev embedding theorem, explicitly

$$\varphi_\nu(x) \to \frac{h_x(v_x, v_x)}{g'_x(v_x, v_x)} \equiv \varphi(x) \tag{3.11}$$

pointwise. It is now very easy to infer from (3.10), (3.11) that $\varphi_\nu \to \varphi$ w.r.t. $|\ \ |_{g,r}$. □

COROLLARY 3.6. *a. The orbits $comp^r_{k+2}(1) \cdot g' \subset comp(g)$ are smooth submanifolds of $comp(g)$.*

b. The quotient space $comp(g)/comp^r_{k+2}(1)$ is a smooth manifold.

c. The projection $\pi : comp(g) \to comp(g)/comp^r_{k+2}(1)$ is a smooth submersion and has the structure of a principal fibre bundle. □

$comp(g)$ has as tangent space at $g' \in comp(g)$ $T_{g'}comp(g) = \Omega^r(S^2T^*, g') \cong \Omega^r(S^2T^*, g)$, where S^2T^* are the symmetric 2-fold covariant tensors. There is an L_2-orthogonal splitting

$$T_{g'}comp(g) = \Omega^{r,c}(S^2T^*, g') \oplus \Omega^{r,T}(S^2T^*, g'), \tag{3.12}$$

where

$$\Omega^{r,c}(S^2T^*, g') = \{h \in \Omega^r(S^2T^*, g') | h(x) = p(x) \cdot g'(x), p \in \Omega^r(M, g')|$$

and

$$\Omega^{r,T}(S^2T^*, g') = \{h \in \Omega^r(S^2T^*, g') | tr_{g'} h = 0\}.$$

The decomposition (3.12) is given by

$$h = \frac{1}{n}(tr_{g'} h) \cdot g' + (h - \frac{1}{n}(tr_{g'} h) g').$$

See [29] p. 19 for further details.

COROLLARY 3.7. *For* $[g'] = comp^r_{k+2}(1) \cdot g'$

$$T_{g''}(comp^r_{k+2}(1) \cdot g') = \Omega^{r,c}(S^2T^*, g'') \tag{3.13}$$

and

$$T_{[g']} comp(g)/comp^r_{k+2}(1) = \Omega^{r,T}(S^2T^*, g'). \tag{3.14}$$

□

4. The space of almost complex structures. Consider M^{2m} open, oriented, with some fixed Riemannian metric g. Denote by $\Omega(Aut\, TM) \equiv C^\infty(Aut\, TM) \subset \Omega(T^1_1(M)) \equiv C^\infty(T^1_1)$ the set of all smooth automorphisms of TM covering id_M.

$$\mathcal{A} = \{J \in \Omega(Aut\, TM) | J^2 = -id_{TM}, J \quad \text{compatible with the fixed orientation}\}$$

is the subset of almost complex structures. Here J is compatible with the fixed orientation if each basis of the kind $X_1, \ldots, X_m, JX_1, \ldots, JX_m$ gives the fixed orientation. g induces a metric connection ∇^g on T^1_1. Assume g with (I) and $(B_k), k \geq r > \frac{n}{2}+1, \delta > 0$ and set

$$V_\delta = \{(J, J') \in \mathcal{A}^2 | |J - J'|_{g,r} < \delta\}.$$

LEMMA 4.1. *$\mathfrak{L} = \{V_\delta\}_{\delta > 0}$ is a basis for a metrizable uniform structure.* □

Denote by $\mathcal{A}^r = \mathcal{A}^r(g)$ the completion.

PROPOSITION 4.2. *$\mathcal{A}^r(g)$ has a representation as a topological sum*

$$\mathcal{A}^r = \sum_{i \in I} comp(J_i) \tag{4.1}$$

where the component $comp(J)$ is given by

$$comp(J) = \{J' \in \mathcal{A}^r | |J - J'|_{g,r} < \infty\}. \tag{4.2}$$

□

PROPOSITION 4.3. *Each component has the structure of a Hilbert manifold of class $k - r$.*

Proof. $\mathcal{A}^r$ can be considered as the space of sections of a bundle $B \to M$ with fibre $GL^+(2m, R)/GL(m, \mathbb{C})$, where B can be endowed with a metric of bounded geometry of order $k - 1$ associated to the Sasaki metric on TM. Then the result follows from [14]. □

REMARKS. For $dim\, M = 2$, we give below another equivalent description. □

PROPOSITION 4.4. *$\mathcal{A}^r(g)$ is an invariant of $comp(g) \subset \mathcal{M}^r(I, B_k)$, i.e. for $g' \in comp(g)$,*

$$\mathcal{A}^r(g) = \mathcal{A}^r(g'). \tag{4.3}$$

□

5. Diffeomorphism groups on open manifolds. Let $(M^n, g), (N^{n'}, h)$ be open, satisfying (I) and (B_k) and let $f \in C^\infty(M, N)$. Then the differential $df = f_* = Tf$ is a section of $T^*M \otimes f^*TN$. f^*TN is endowed with the induced connection $f^*\nabla^h$. The connections ∇^g and $f^*\nabla^h$ induce connections ∇ in all tensor bundles $T^q_s(M) \otimes f^*T^u_v(N)$. Therefore, $\nabla^m df$ is well defined. Assume $m \leq k$. We denote by $C^{\infty,m}(M, N)$ the set of all $f \in C^\infty(M, N)$ satisfying

$${}^{b,m}|df| = \sum_{i=0}^{m-1} \sup_{x \in M} |\nabla^i df|_x < \infty.$$

Let $Y \in \Omega(f^*TN) \equiv C^\infty(f^*TN)$. Then Y_x can be written as $(Y_{f(x)}, x)$ and we define a map $g_Y : M \to N$ by

$$g_Y(x) := (exp\,Y)(x) := exp\,Y_x := exp_{f(x)} Y_{f(x)}.$$

Then the map g_Y defines an element of $C^\infty(M, N)$. More generally we have:

PROPOSITION 5.1. *Assume $m \leq k$ and ${}^{b,m}|Y| = \sum_{i=0}^m \sup_{x \in M} |\nabla^i Y|_x < \delta_N < r_{inj}(N)$, $f \in C^{\infty,m}(M, N)$. Then*

$$g_Y \equiv exp\,Y \in C^{\infty,m}(M, N).$$

We refer to [14] for a proof. The main point is, that one shows

$$|\nabla^\mu d\,exp\,Y| \leq P_\mu(|\nabla^i df|, |\nabla^j Y|),\ i \leq \mu, j \leq \mu + 1, \tag{5.1}$$

where the P_μ are certain universal polynomials in the indicated variables without constant term and each term has at least one $|\nabla^j Y|, 0 \leq i \leq \mu + 1$, as a factor. □

Now consider manifolds of maps in the L_p-category. According to the Sobolev embedding theorem, for $r > \frac{n}{p} + s, Y \in \Omega^{p,r}(f^*TN)$ arbitrary, there exists a constant D such that

$${}^{b,s}|Y| \leq D \cdot |Y|_{p,r}, \tag{5.2}$$

where $|Y|_{p,r} = (\int \sum_{i=0}^r |\nabla^i Y|^p dvol)^{1/p}$. Set for $\delta > 0$, $\delta \cdot D \leq \delta_N < r_{inj}(N)/2$, $1 \leq p < \infty$, $k \geq m \geq r > \frac{n}{p} + 1$

$$V_\delta = \{(f, g) \in C^{\infty,m}(M, N)^2 |\ \text{there exists a}\ \ Y \in \Omega^p_r(f^*TN)\ \ \text{such that}\ \ g = g_Y = exp\,Y\ \ \text{and}\ \ |Y|_{p,r} < \delta\}.$$

THEOREM 5.2. *$\mathfrak{L} = \{V_\delta\}_{0<\delta<r_{inj}(N)/2D}$ is a basis for a metrizable uniform structure $\mathfrak{U}^{p,r}(C^{\infty,m}(M,N))$.*

The proof essentially uses several iterated estimates of type (5.1) and others, where the arising polynomials P_μ, Q_μ are p-integrable. It is rather complicated, occupies 40 pages and is performed in [14]. □

Let ${}^m\Omega^{p,r}(M,N)$ be the completion of $C^{\infty,m}(M,N)$ with respect to this uniform structure. From now on we assume $r = m$ and denote $\Omega^{p,r}(M,N) \equiv^r \Omega^{p,r}(M,N)$.

THEOREM 5.3. *Let $(M^n,g),(N^{n'},h)$ be open, satisfying (I) and (B_k), $r \le k$, $1 \le p < \infty$, $r > \frac{n}{p}+1$. Then each component of $\Omega^{p,r}(M,N)$ is a C^{k+1-r} -Banach manifold, and for $p = 2$ it is a Hilbert manifold.*

We refer to [14] for the proof. □

Let (M^n,g) be open, satisfying (I) and $(B_k), k,p,r$ as above. Set

$$\mathcal{D}^{p,r}(g) = \{f \in \Omega^{p,r}(M,M) | f \quad \text{is injective, surjective,} \\ \text{preserves orientation and} \quad |\lambda|_{min}(df) > 0\}.$$

Here $|\lambda|_{min}(df)$ is defined as follows. Fix at any point $x \in M$ the class of orthonormal bases. Let $e_1,\ldots,e_n \in T_xM$ and $f_1,\ldots,f_n \in T_{f(x)}M$ be such bases. $df|_x$ now can be described as a nonsingular matrix. $\lambda_i(df)$ are defined as the diagonal elements of the Jordan normal forn. $|\lambda|_{min}(df)_x$ is the smallest absolute value of the λ_i's. It is invariant defined, i.e. it does not depend on the choice of the orthonormal bases since the Jordan normal form of a matrix is a similarity invariant.

THEOREM 5.4. *$\mathcal{D}^{p,r}$ is open in $\Omega^{p,r}(M,M)$; in particular, each component is a C^{k+1-r} -Banach manifold, and for $p = 2$ it is a Hilbert manifold.* □

THEOREM 5.5. *Assume $(M^n,g),k,p,r$ as above.*

a. Assume $f,g \in \mathcal{D}^{p,r}, g \in comp(id_M) \subset \mathcal{D}^{p,r}$. Then $g\circ f \in \mathcal{D}^{p,r}$ and $g \circ f \in comp(f)$.
b. Assume $f \in comp(id_M) \subset \mathcal{D}^{p,r}$. Then $f^{-1} \in comp(id_M) \subset \mathcal{D}^{p,r}$.
c. $comp(id_M)$ is a metrizable topological group.

We refer to [14] for the proof. □

Denote $\mathcal{D}_0^{p,r} \equiv comp(id_M)$.

THEOREM 5.6. *(α -lemma). Assume $r \le k, r > \frac{n}{p}+1, f \in D^{p,r}$. Then the right multiplication $\alpha_f : \mathcal{D}_0^{p,r} \to \mathcal{D}^{p,r}$, $\alpha_f(g) = g \circ f$, is of class C^{k+1-r}.*

THEOREM 5.7. *(ω-lemma). Let $k+1-(r+s) > s, f \in \mathcal{D}_0^{p,r+s}, r > \frac{n}{p}+1$. Then the left multiplication $\omega_f : \mathcal{D}^{p,r} \to \mathcal{D}^{p,r}, \omega_p(g) = f \circ g$, is of class C^s.*

The proofs are performed in [19]. □

We defined for $C^{\infty,m}$ a uniform structure $\mathfrak{U}^{p,r}$. Consider now

$$C^{\infty,\infty}(M,N) = \cap_m C^{\infty,m}(M,N).$$

Then we have an inclusion

$$i : C^{\infty,\infty}(M,N) \to C^{\infty,m}(M,N)$$

and

$$i \times i : C^{\infty,\infty}(M,N)^2 \to C^{\infty,m}(M,N)^2,$$

hence a well defined uniform structure $\mathfrak{U}^{\infty,p,r} = (i \times i)^{-1}\mathfrak{U}^{p,r}$ (cf. [28], p. 108-109). After completion we obtain once again the manifolds of mappings $\Omega^{\infty,p,r}(M,N)$, where $f \in \Omega^{\infty,p,r}(M,N)$ if and only if for every $\varepsilon > 0$ there exists an $\tilde{f} \in C^{\infty,\infty}(M,N)$ and a $Y \in \Omega^{p,r}(\tilde{f}^*TN)$ such that $f = exp\,Y$ and $|Y|_{p,r} \leq \varepsilon$. Moreover, each connected component of $\Omega^{\infty,p,r}(M,N)$ is a Banach manifold and $T_f\Omega^{\infty,p,r}(M,N) = \Omega^{p,r}(f^*TN)$. In the notation above $\Omega^{\infty,p,r} \equiv {}^{\infty}\Omega^{p,r}$. As above we set

$$\mathcal{D}^{\infty,p,r}(M,g) = \{f \in \Omega^{\infty,p,r}(M,M) | f \quad \text{is injective, surjective,}$$
$$\text{preserves orientation and} \quad |\lambda|_{min}(df) > 0\}.$$

We assume $p = 2$ and write

$$\Omega^{\infty,r}(M,N) \equiv \Omega^{\infty,p,r}(M,N)$$

and

$$\mathcal{D}^{\infty,r}(M,g) \equiv \mathcal{D}^{\infty,2,r}(M,g).$$

The only difference between our former construction and the new one is the fact that the spaces $\Omega^{\infty,r}$ are based on maps which are bounded up to arbitrary high order. For compact manifolds we have

$$C^{\infty}(M,N) = C^{\infty,r}(M,N) = C^{\infty,\infty}(M,N), \Omega^{\infty,r}(M,N) = \Omega^{r}(M,N)$$

and $\mathcal{D}^{\infty,r}(M,g) = \mathcal{D}^r(M,g)$ for all r. For open manifolds we have strong inclusions $C^{\infty,\infty} \subset C^{\infty,r}$ and $\mathcal{D}^{\infty,r} \subset \mathcal{D}^r$. It is very easy to construct a diffeomorphism $f \in C^{\infty,1}(\mathbb{R},\mathbb{R})$ such that $f \notin C^{\infty,2}(\mathbb{R},\mathbb{R})$. This supports the conjecture that the inclusion $\mathcal{D}^{r+s} \hookrightarrow \mathcal{D}^r, s \geq 1$, is not dense. We settle this question in a forthcoming paper. The space $\mathcal{D}^{\infty,r+s}$ is densely and continuously embedded into $\mathcal{D}^{\infty,r}$. This follows easily from the corresponding properties for Sobolev spaces. The components of the identity have special nice properties:

PROPOSITION 5.8. *Assume the conditions for defining $\mathcal{D}^r$. Then*

$$\mathcal{D}_0^{\infty,r} = \mathcal{D}_0^r. \tag{5.3}$$

Proof. Let $f \in \mathcal{D}_0^r$. Given any $\delta < r_{inj}/D$, there exist vector fields $X_1, \dots, X_m \in \Omega^r(TM), |X_\mu|_r < \delta, \mu = 1, \dots, m$, $f = exp\,X_m \circ \dots \circ exp\,X_1$,^{b,1} $|X| \leq D|X|_r$. We are done if we can show that for $X \in \Omega^r(TM), |X|_r < \delta$ and given $\varepsilon > 0$ there exists a diffeomorphism $f_X \in C^{\infty,\infty}$ and $Y \in \Omega^r(f_X^*TM) = \Omega^r(TM)$ with $|Y|_r < \varepsilon$ such that $exp\,X = exp\,Y \equiv exp_{f_X} Y \circ f_X$. But this is very easy. For ε_1 arbitrary small, there exists a smooth vector field $Y_1 \in C_0^{\infty}(TM)$ with compact support such that $|X - Y_1|_r < \varepsilon_1$. Choosing ε_1 sufficiently small, there exists a unique vector field $Y \in \Omega^r((exp\,Y_1)^*TM)$ such that $exp\,Y \equiv exp_{exp\,Y_1} Y \circ exp\,Y_1 = exp\,X$ and $|Y|_r \leq Q_r(\varepsilon_1)$, where Q_r is a polynomial without constant term. This follows from the geodesic triangle argument of [14]. Hence, for ε_1 sufficiently small we have $|Y|_v < \varepsilon$. We set $f_X = exp\,Y_1$. For $f = exp\,X_m \circ \dots \circ exp\,X_1$ we apply the techniques of the proof for $\mathcal{D}_0^r$ being a group of [14] and obtain for any given small $\varepsilon > 0$ a representation $f = exp_{\tilde{f}} Y \circ \tilde{f}$ with $f \in C^{\infty,\infty}$, $Y \in \Omega^r(\tilde{f}^*TM), |Y|_r < \varepsilon$ and $\tilde{f}$ is built up from the $f_{X_\mu} \in C^{\infty,\infty}$. □

REMARKS. 1. A detailed proof of proposition 5.8 would occupy dozens of pages but the arguments needed are all contained in [14]. 2. The essential reason for the special good property of $\mathcal{D}_0^r$ is that $id \in C^{\infty,\infty}(M,M)$. For diffeomorphisms in other components of $\mathcal{D}^r$ this is in general wrong.

PROPOSITION 5.9. *For $g \in \mathcal{M}(I,B_k), \mathcal{D}_0^r(M,g)$ is an invariant of $comp(g)$, i.e. if $g' \in comp(g)$ then*

$$\mathcal{D}_0^r(M,g) = \mathcal{D}_0^r(M,g'). \tag{5.4}$$

Proof. We restrict to the case $g' \in comp(g) \cap \mathcal{M}(I,B_k)$. The more general case induces rather delicate approximation procedures but is also true. Already the definitions are much more involved. The assertion follows immediately from

$$^{b,m}|d\, id_M|_g \sim^{b,m} |d\, id_M|_{g'}, \tag{5.5}$$

$$\Omega^r(T^*M,g) \sim \Omega^r(T^*M,g'), \tag{5.6}$$

$$\Omega^r(T^*M,g) \sim \Omega^r((exp\, X)^*M,(exp\, X)^*g). \tag{5.7}$$

(5.5) holds since g and g' are quasi isometric. (5.6) is theorem 2.10 and (5.7) is the last equation on p. 292 of [14]. □

Assume now $k \geq r, r > \frac{n}{2}+1, g \in \mathcal{M}(I,B_{k+1})$.

PROPOSITION 5.10. *$\mathcal{D}_0^{r+1}(g)$ acts on $comp(g) \subset \mathcal{M}^r(I,B_k)$.*

Proof. We have to show $g' \in comp(g), f \in \mathcal{D}_0^{r+1}(g)$ imply $f^*g' \in comp(g)$. There exists a sequence $(g_\nu)_\nu, g_\nu \in comp(g) \cap \mathcal{M}(I,B_k), g_\nu \underset{|\ |_{g,r}}{\longrightarrow} g'$. We start with $f = exp\, X, X \in \Omega^r(TM)$. X can be approximated by $(X_\mu)_\mu, X_\mu \in C_0^\infty(TM), X_\mu \underset{|\ |_{g,r}}{\longrightarrow} X$. Set $f_\mu = exp\, X_\mu$. Consider the diagonal sequence $(f_\nu^* g_\nu)_\nu$. Then $f_\nu^* g_\nu \in comp(g) \cap \mathcal{M}(I,B_k)$ which follows from $g_\nu \in comp(g) \cap \mathcal{M}(I,B_k)$ and $X_\nu \in C_0^\infty(TM)$. We are done if we can show

$$f_\nu^* g_\nu \underset{|\ |_{g,r}}{\longrightarrow} f^*g' \tag{5.8}$$

and

$$|f^*g' - g'|_{g,r} < \infty. \tag{5.9}$$

Write

$$f_\nu^* g_\nu - f^*g' = (f_\nu^* - f^*)g_\nu + f^*(g_\nu - g'), \tag{5.10}$$

$$g_\nu = (g_\nu - g) + g, \tag{5.11}$$

$$f^* = (f^* - id^*) + id^*. \tag{5.12}$$

Inserting (5.11), (5.12) into (5.10), using the (rather delicate) proof of theorem 3.1 of [21], the r-boundedness of g and id^* and the module structure theorem, we obtain $|f_\nu^* g_\nu - f^*g'|_{g,r} \underset{\nu\to\infty}{\longrightarrow} 0$, i.e. (5.8). Write

$$f^*g' - g' = (f^* - f_\nu)g' + f_\nu^*(g' - g_\nu) + (f_\nu^* - 1)g_\nu + (g_\nu - g'), \tag{5.13}$$

$$g' = (g' - g) + g, \tag{5.14}$$

$$f_\nu^* = (f_\nu^* - id^*) + id^*, \tag{5.15}$$

$$g_\nu = (g_\nu - g) + g. \tag{5.16}$$

Inserting (5.14) - (5.16) into (5.13), we obtain by the same arguments

$$|f^* g' - g'|_{g,r} < \infty.$$

Assume now $f = exp\, X_2 \circ exp\, X_1$. Replacing g' of the first case by $(expX_2)^* g'$ and applying the same procedure, we obtain again $f^* g' \in comp(g)$. For $f = exp\, X_n \circ \ldots \circ exp\, X_1$ we perform induction. □

6. The ILH-version of the considered spaces. For metrics g satisfying the conditions (I) and

$$(B_\infty) \qquad |\nabla^i R| \leq C_i, i = 0, 1, n, \ldots$$

we have additional structures. Then $\mathcal{D}_0^r(g) \equiv \mathcal{D}_0^{\infty,r}(g)$ is defined for all $r > \frac{n}{2} + 1$. As we shall see now, we can form $\mathcal{D}_0^\infty(g) = \varprojlim_r \mathcal{D}_0^r(g)$ which is an ILH-group. To make this clear, we recall some definitions which are a little bit different from them originally given a long time ago by Omori. We adapt to [27].

A collection of groups $\{G^\infty, G^r | r \geq r_0\}$ is called an ILH-Lie group if it satisfies the following connections.

1. Each G^r is a Hilbert manifold of class $C^{k(r)}$ modelled by a Hilbert space E^r and $k(r) \to \infty$ as $r \to \infty$.
2. For each $r \geq r_0$ there are linear continuous, dense inclusions $E^{r+1} \hookrightarrow E^r$ and dense inclusions of class $C^{k(r)}$ $G^{r+1} \hookrightarrow G^r$.
3. Each G^r is a topological group and $G^\infty = \varprojlim_r G^r$ is a topological group with the inverse limit topology.
4. If (U^r, φ^r, E^r) is a chart of G^r, then $(U^r \cap G^t, \varphi^r|_{U^r \cap G^t}, E^t)$ is a chart for G^t, for all $t \geq r$.
5. The multiplication $\mu : G^\infty \times G^\infty \to G^\infty$ extends to a C^s -map $\mu : G^{r+s} \times G^r \to G^r$ for all r with $s \leq k(r)$.
6. Inversion $\nu : G^\infty \to G^\infty$ extends to a C^s -map $\nu : G^{r+s} \to G^r$ for all r with $s \leq k(r)$.
7. Right multiplication R_g by $g \in G^r$ extends to a $C^{k(r)}$ -map $R_g : G^r \to G^r$.

THEOREM 6.1. *Assume (M^n, g) oriented, open with (I) and (B_∞). Set $\mathcal{D}_0^\infty(g) := \varprojlim_r \mathcal{D}_0^r(g)$ with the inverse limit topology. Then $\{\mathcal{D}_0^\infty(g), \mathcal{D}_0^r(g) | r > \frac{n}{2} + 1\}$ is an ILH-Lie group.*

Proof. In this case $k(r) = k - r + 1 = \infty - r + 1 = \infty$. 1. $\mathcal{D}_0^r$ is a Hilbert manifold of class C^∞ modelled on $E^r = \Omega^r(TM, g) = T_e \mathcal{D}_0^r$, $r > \frac{n}{2} + 1$. 2. The inclusions $\Omega^{r+1}(TM) \hookrightarrow \Omega^r(TM)$ are dense and continuous. Using charts,

$$B_\delta(0) \subset T_f\mathcal{D}_0^{r+1} \overset{exp_f^{r+1}}{\longrightarrow} U_\delta^{r+1} \subset \mathcal{D}_0^{r+1} \overset{i}{\longrightarrow} U^r \to$$
$$\overset{(exp_f^r)^{-1}}{\longrightarrow} B_\delta(0) \subset T_f\,\mathcal{D}_0^r \tag{6.1}$$

and $k = \infty$, we obtain that i is dense and C^∞ since $(exp_f^r)^{-1} \circ i \circ exp_f^{r+1}$ is of class C^∞. 3. Each $\mathcal{D}_0^r$ is a topological group and $\mathcal{D}_0^\infty = \varprojlim \mathcal{D}_0^r$ by definition. 4. follows from (6.1) replacing $r+1$ by t. 5. follows from 5.6 using $k = \infty$. 6. can be proved quite similar (cf. [14], (6.8) - (6.11) and the proof of 6.5). 7. follows from 5.6. □

PROPOSITION 6.2. *$f \in \mathcal{D}_0^\infty(g)$ if and only if f is a C^∞ -diffeomorphism satisfying ${}^{b,m}|df| < \infty$ for all m, $|\lambda|_{min}(df) > 0$ and which is homotopic in this set (with respect to the inverse limit topology) to the identity.* □

Omitting all group properties in the above definition, we obtain an ILH-manifold. Similarly one defines ILB-Lie groups (cf.[27]). Set $\mathcal{D}_0^{p,\infty} = \varprojlim \mathcal{D}_0^{p,r}$.

THEOREM 6.3. *$\{D_0^{p,\infty}, D_0^{p,r}(r > \frac{n}{2}+1\}$ is an ILB-Lie group.* □

Furthermore, quite natural one defines C^k-ILH maps between ILH-manifolds and ILH-principal fibre bundles $P \overset{\pi}{\longrightarrow} P/G$ of class C^k. Consider $g \in \mathcal{M}(I, B_\infty)$, $comp^r(g) \subset \mathcal{M}^r(I, B_\infty)$, $comp^\infty(g) := \varprojlim_r comp^r(g)$, $\mathcal{P}^r_\infty(g)$, $\mathcal{P}^\infty_\infty(g) = \varprojlim_r \mathcal{P}^r_\infty(g)$, $comp^\infty_\infty(1) \subset \mathcal{P}^\infty_\infty(g)$.

THEOREM 6.4. *$\{comp^\infty(g), comp^r(g)|r \geq \frac{n}{2}+n\}$, $\{comp^\infty_\infty(1), comp^r_\infty(1)|r > \frac{n}{2}+1\}$ are ILH-manifolds and $comp^\infty(g) \to comp^\infty(g)/comp^\infty_\infty(1)$ is an ILH-bundle.*
□

7. The space of hyperbolic metrics for $n = 2$. We will show that for certain classes of open surfaces, a suitable metric g_0 and the space $comp(g_0)_{-1} \subset comp(g_0)$ of constant scalar curvature -1 holds

$$comp(g_0)_{-1} \cong comp(g_0)/comp(1) \tag{7.1}$$

where these spaces are manifolds and $\mathcal{D}_0^r(g_0)$ -equivariant diffeomorphic to a certain component in the space of almost complex structures. $comp_{-1}(g_0)/\mathcal{D}_0^r(g_0)$ will be one of our models for the Teichmüller space.

We consider open surfaces M^2. Each such surface has ends. We admit punctures as ends. If each end is isolated then M^2 has a finite number of ends, each of them is given by an infinite half ladder $= \overset{\infty}{\underset{n=1}{\sharp}} T^2$, where T^2 is the 2-Torus or it is given by a puncture. If M^2 has an infinite number of ends then there exists at least one non-isolated end, i.e. an end that has no neighbourhood which is not a neighbourhood of another end. This occurs e.g. if we have repeated branchings of half ladders. In any case, such a surface can be built up by Y-pieces or so called trumpets which we explain now. We follow the representation given in [6].

LEMMA 7.1. *Let a, b, c be arbitrary positive real numbers. There exists a right angled geodesic hexagon in the hyperbolic plane with pairwise non-adjacent sides of length a, b, c.* □

Next we paste two copies of such a hexagon together along the remaining three sides to obtain a hyperbolic surface Y with three closed boundary geodesics of length $2a, 2b, 2c$. They determine Y up to isometry (Theorem 3.17 of [6]).

Two different Y-pieces can be glued along their boundary geodesics if they have the same length. The same holds for two "legs" of same boundary length of one Y-

piece. It is a deep result of hyperbolic geometry that one obtains as a result smooth hyperbolic surfaces. Moreover, we can perform gluing with an additional twisting (cf. [6]). But here we consider gluings without twisting, at least for our starting metric g_0. As a well known matter of fact, any topologically given open surface of the above kind can be built up by Y-pieces and trumpets and we obtain in this way a hyperbolically metrized surface (M^2, g_0). The lengths of all closed boundary geodesics can be chosen in such a manner (and $\geq a > 0$) that $r_{inj}(M^2, g_0) > 0$, (cf. [6]) i.e. $g_0 \in \mathcal{M}(I, B_\infty)$.

Given an open surface M^2 of the above type, i.e. M^2 is the connected sum of a closed surface with an infinite number of half ladders with possibly infinitely many punctures, fix in this case a hyperbolic metric $g_0 \in \mathcal{M}(I, B_\infty)$. Later we must impose that this lengths must grow suitably. Consider $\mathcal{P}_\infty(g_0) = \bigcap_m \mathcal{P}_m(g_0)$, $\mathcal{P}^r_\infty(g_0)$ defined by the induced uniform structure. It is a very simple fact that $comp^r_k(1, g_0) \subset \mathcal{P}^r_k(g_0)$ and $comp^r_\infty(1, g_0) \subset \mathcal{P}^r_\infty(g_0)$ coincide, $k \geq 1$. We fix $r > 3$ and write $comp(1) = comp^r(1, g_0)$. Consider $comp(g_0) \subset \mathcal{M}^r(I, B_\infty)$. As we already know, $comp(1)$ acts on $comp(g_0)$ and $comp(g_0)/comp(1)$ is a Hilbert manifold. Let $comp(g_0)_{-1} \subset comp(g_0)$ be the subspace of all metrics $g \in comp(g_0)$ such that the scalar curvature $K(g)$ equals -1. Since we assume $r > 3 = \frac{2}{2} + 2$, g is at least of class C^2 and $K(g)$ is well defined. Usually $K(g)$ denotes the sectional curvature but we use it for scalar curvature which is twice the sectional curvature. We could also work with sectional curvature but then in the differential equation below appears a factor 2 which we should take into account in all calculations. Only for this reason we decided to work with scalar curvature. Both approaches are trivially equivalent.

We wish to show that $comp(g_0)_{-1} \subset comp(g_0) \subset \mathcal{M}^r(I, B_\infty)$ is a smooth submanifold of $comp(g_0)$ which is diffeomorphic to $comp(g_0)/comp(1)$. This is a rather deep fact which requires a series of preliminaries and is valid only under an additional spectral assumption. Let $g \in comp(g_0)$. Then, according to (2.32), Δ_g maps $\Omega^r = \Omega^r(M, \nabla^{g_0}, g_0)$ into $\Omega^{r-2} \subset L_2(M, g_0)$.

LEMMA 7.2. *$\Delta_g + 1$ is surjective.*

Proof. Consider $\Delta_g + 1$ with domain $\Omega^r \subset \Omega^{r-2}$. Then the closure of $(\Omega^r, |\ \ |_{r-2})$ with respect to $|\cdot|_{r-2} + |(\Delta_g + 1)\cdot|_{r-2}$ is just Ω^r, i.e. $\Delta_g + 1$ is a closed operator in the Hilbert space Ω^{r-2}. Moreover, $|(\Delta_g + 1)\varphi|_{r-2} \geq c \cdot |\varphi|_{r-2}, c = 1, \varphi \in \Omega^r$. Hence $(\Delta_g + 1)\varphi_i \to \psi$ gives φ_i Cauchy and $\varphi_i \to \varphi$ in Ω^{r-2}. $\Delta_g + 1$ is closed, hence $(\Delta_g + 1)\varphi = \psi, im(\Delta_g + 1)$ closed. Finally, the orthogonal complement of $im(\Delta_g + 1)$ in Ω^{r-2} is $\{0\}$ since the adjoint (in Ω^{r-2}) operator to $\Delta_g + 1$ has no kernel. □

Let $h \in T_g\, comp(g_0) = \Omega^r(S^2T^*, g)$. For h the divergence $\delta_g h$ is defined by $(\delta_g h)_j = \nabla^k h_{jk} = g^{ik}\nabla^g_i h_{jk}$. For $\omega = \omega_i dx^i$ a 1-form and $X_\omega = \omega^i \frac{\partial}{\partial x^i}$ the corresponding vector field the divergence δ_w is defined by $\delta_g \omega := \delta_g X_\omega = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\omega^i \sqrt{g})$. Hence for $h \in \Omega^r(S^2T^*, g)$ the expression $\delta_g\delta_g h$ is well defined. As we already mentioned, for $r > 3 = \frac{2}{2} + 2, g \in comp(g_0)$ is at least of class C^2 and the scalar curvature $K(g)$ is well defined.

LEMMA 7.3. $K(g) - (-1) = K(g) - K(g_0) \in \Omega^{r-2}$.

This follows immediately from the topology in $comp(g_0)$ and the definition of $K(g)$. □

Consider the C^∞ -map

$$\psi : comp(g_0) \to \Omega^{r-2}(M, g_0)$$
$$g \to K(g) - (-1).$$

Then $comp(g_0)_{-1} = \psi^{-1}(0)$.

THEOREM 7.4. *$comp(g_0)_{-1} \subset comp(g_0)$ is a smooth submanifold.*

Proof. It suffices to show, 0 is a regular value for ψ, i.e. if $K(g) = -1$ for some g then $D\psi|_g : T_g\, comp(g_0) \to \Omega^{r-2}(M, g_0)$ is surjective. Hence we have to calculate $D\psi|_g(h), h \in T_g\, comp(g_0) = \Omega^r(S^2T^*, g)$. This has been done in [29],

$$D\psi|_g(h) = \Delta_g(tr_g h) + \delta_g \delta_g h + \frac{1}{2} tr_g h. \tag{7.2}$$

$D\psi|_g$ is already surjective if the restriction to h of the kind $h = \lambda \cdot g, \lambda \in \Omega^r(M)$, is surjective. Then (7.2) becomes

$$D\psi|_g(\lambda \cdot g) = \Delta_g \lambda + \lambda = (\Delta_g + 1)\lambda,$$

but $\Delta_g + 1$ is surjective according to 7.2. □

Next we prepare Poincaré's theorem which roughly spoken asserts $comp(g_0)_{-1} \cong comp(g_0)/comp(1)$. Denote by $\sigma_e(\Delta)$ the essential spectrum of Δ. Here we omit the bar in the unique self adjoint extension $\bar{\Delta}$ which equals to the closure.

PROPOSITION 7.5. *$\sigma_e(\Delta_{g_0})$ is an invariant of $comp(g_0)$, i.e. for $g \in comp(g_0)$,*

$$\sigma_e(\Delta_g) = \sigma_e(\Delta_{g_0}).$$

Proof. Let $\lambda \in \sigma_e(\Delta_{g_0})$ and $(\varphi_\nu)_\nu$ be a Weyl sequence for λ, i.e. $\varphi_\nu \in D_{\bar{\Delta}_{g_0}}$, bounded, not precompact and $\lim_{\nu \to \infty} (\Delta_{g_0} - \lambda)\varphi_\nu = O$. Then, according to (2.32), $(\varphi_\nu)_\nu \subset D_{\bar{\Delta}_g}$ is bounded and not precompact with respect to $L_2(M, g)$. Writing $\Delta_g - \lambda = \Delta_{g_0} - \lambda + \Delta_g - \Delta_{g_0}$, it is possible to show $\lim_{\nu \to \infty} (\Delta_g - \Delta_{g_0})\varphi_\nu = 0$, i.e. $\sigma_e(\Delta_{g_0}) \subseteq \sigma_e(\Delta_g)$. By symmetry we conclude $\sigma_e(\Delta_{g_0}) = \sigma_e(\Delta_g)$. We refer to [7], [18] for details. □

LEMMA 7.6. *Assume $inf\, \sigma_e(\Delta_{g_0} > 0$. Then $inf\, \sigma(\Delta_g) > 0$ for all $g \in comp(g_0)$, where σ denotes the spectrum.*

Proof. According to 7.5 inf $\sigma_e(\Delta_{g_0}) = inf \sigma_e(\Delta_g)$. From $g \in \mathcal{M}(I, B_\infty), g \in comp(g_0) \subset \mathcal{M}^r(I, B_\infty), r > 3$ follows that g satisfies (I) and (B_0) which implies $vol(M^2, g) = \infty$. Hence $\lambda = 0$ cannot be an eigenvalue. All other spectral values between 0 and $inf \sigma_e(\Delta_g)$ belong to the purely discrete point spectrum $\sigma_{pd}(\Delta_g)$, i.e. $inf \sigma(\Delta_g) > 0$. □

Now we state the first main theorem of this section.

THEOREM 7.7. *Assume (M^2, g_0) with g_0 smooth, $K(g_0) \equiv -1$, $r_{inj}(M^2, g_0) > 0$, $inf \sigma_e(\Delta_{g_0}) > 0$. Let $g \in comp(g_0) \subset \mathcal{M}^r(I, B_\infty), r > 3$. Then there exists a unique $\rho \in comp(1) \subset \mathcal{P}^r_\infty(g_0)$ such that $K(\rho \cdot g) \equiv -1$.*

Proof. Let $\rho = e^u$. For the existence we have to solve the PDE

$$\Delta_g u + K(g) + e^u = 0. \tag{7.3}$$

We seek for a solution $u \in \Omega^r(M, g_0)$. $u \in \Omega^r(M, g_0), r > 3$ imply $e^u - 1 \in \Omega^r$ as we will see below. (7.3) has a solution according to the general uniformization theorem. But this theorem does not provide $u \in \Omega^r$. Therefore we have to sharpen our considerations. The existence will be established by the implicit function theorem and a version of the continuity method. Consider $g_t = (1-t)g_0 + tg = g_0 + t(g - g_0) = g_0 + th \in comp(g_0)$ and the map

$$\begin{aligned} F : [0,1] \times \Omega^r &\to \Omega^{r-2} \\ (t,u) \to F(t,u) &= \Delta_{g_t} u + K(g_t) + e^u = \\ &= \Delta_{g_t} u + (K(g_t) - (-1)) + e^u - 1. \end{aligned} \tag{7.4}$$

We want to show that there exists a unique $u_1 \in \Omega^r(M, g_0)$ such that $F(1, u_1) = 0$. For this we consider the set

$$\mathcal{S} = \{t \in [0,1] | \quad \text{There exists} \quad u_t \in \Omega^r \quad \text{such that} \quad F(t, u_t) = 0\}$$

and we want to show $\mathcal{S} = [0,1]$. We start with $\mathcal{S} \neq \phi$. For $t = 0, g_t = g_0, K(g_0) = -1$ and $u_0 \equiv 0$ satisfies (7.3). Moreover,

$$F_u(0,0) = D_2 F|_{(0,0)} = \Delta_{g_0} + 1 \tag{7.5}$$

is bijective between Ω^r and Ω^{r-2}, as we have already seen. Hence there exist $\delta > 0, \varepsilon > 0$ such that for $t \in]0, \delta[$ there exists a unique $u_t \in U_\varepsilon(0) \subset \Omega^r$ with

$$F(t, u_t) = 0. \tag{7.6}$$

By the same consideration we can show that $\mathcal{S}$ is open in $[0,1]$. To show $\mathcal{S} = [0,1]$ we should show $\mathcal{S}$ is closed. This would be done if we could prove the following. Assume $t_1 < t_2 < \dots, t_\nu \in \mathcal{S}, t_\nu \to t_0$, then $t_0 \in \mathcal{S}$. The canonical procedure to prove this would be to prove

$$(u_{t_\nu})_\nu \quad \text{is a Cauchy sequence in} \quad \Omega^r, u_{t_\nu} \to u_{t_0}, \tag{7.7}$$

$$\Delta_{g_{t_0}} u_{t_0} + K(g_{t_0}) \in e^{u_{t_0}} = 0. \tag{7.8}$$

We prefer a slightly other version of this establishing the following

PROPOSITION 7.8. *There exists a* $\delta > 0, \delta$ *independent of* t_0*, such that* $t_0 \in \mathcal{S}$ *implies* $]t - \delta_0, t_0 + \delta[\cap [0,1] \subset \mathcal{S}$.

We will see later that the proof of 7.8 is equivalent to that of (7.7) and (7.8). The proof of 7.8 is based on careful estimates in the implicit function theorem to which we turn now our attention. Roughly speaking, the proof goes as follows.

Let $t_0 \in \mathcal{S}, u_{t_0} \in \Omega^r$,

$$F(t_0, u_{t_0}) = \Delta_{g_{t_0}} u_{t_0} + K(g_{t_0}) + e^{u_{t_0}} = 0.$$

Set $g(t,u) := F_u(t_0, u_{t_0})u - F(t,u)$. Then $F(t,u) = 0$ is equivalent to

$$u = F_u(t_0, u_{t_0})^{-1} g(t,u). \tag{7.9}$$

If we define $T_t u := F_u(t_0, u_{t_0})^{-1} g(t,u)$, then we are done if we can find for any $t_0 \in \mathcal{S}$ a complete metric subspace $M_{t_0,\delta_1} \subset \Omega^r(M, g_0)$ such that

$$T_t : M_{t_0,\delta_1} \to M_{t_0,\delta_1} \tag{7.10}$$

and

$$T_t \quad \text{is contracting} \tag{7.11}$$

for all $t \in]t_0 - \delta, t_0 + \delta[\cap [0,1], \delta$ independent of t_0. Indeed, in this case T_t would have a unique fixed point u_t solving (7.6).

We now prepare the construction of M_{t_0,δ_1} and the proof of (7.10), (7.11) by a series of estimates. First we apply the mean value theorem. From $g_u(t,v) = F_u(t_0, u_{t_0}) - F_u(t,v)$ follows

$$\begin{aligned} |g(t,u) - g(t,v)|_{r-2} \leq \sup_{0<\vartheta<1} |g_u(t, v + \vartheta(u-v))|_{r-2} \cdot |u - v|_r, \\ |T_t u - T_t v|_r \leq |(\Delta_{g_{t_0}} + e^{u_{t_0}})^{-1}|_{r-2,r} \cdot \\ \cdot \sup_{0<\vartheta<1} |(\Delta_{g_{t_0}} - \Delta_{g_t}) + ((e^{u_{t_0}} - e^{v+\vartheta(u-v)}) \cdot)|_{r,r-2} \cdot |u - v|_r, \end{aligned} \tag{7.12}$$

where $| \quad |_{i,j}$ denotes the operator norm $\Omega^i(M, g_0) \to \Omega^j(M, g_0)$. We estimate

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}} \cdot))^{-1}|_{r-2,r} \cdot |\Delta_{g_{t_0}} - \Delta_{g_t}|_{r,r-2} \tag{7.13}$$

and

$$\begin{aligned} |(\Delta_{g_{t_0}} + (e^{u_{t_0}} \cdot))^{-1} (e^{u_{t_0}} \cdot)|_{r-2,r} \cdot \\ \cdot |(1 - e^{v - u_{t_0} + \vartheta(u - u_{t_0} - (v - u_{t_0}))}) \cdot |_{r,r-2} \end{aligned} \tag{7.14}$$

and start with (7.13). In the sequel, the same letters for constants in different inequalities can denote different constants. The key role in all following considerations plays the Lipschitz continuity of $|\Delta_{g_t}|_{i,j}$.

LEMMA 7.9. *Assume g_0, g, t, t_0, r as above. Then there exists a constant $C = C(g_0, r, |g - g_0|_{g_0,r}) > 0$ such that*

$$|\Delta_{g_{t_0}} - \Delta_{g_t}|_{r,r-2} \leq C \cdot |t_0 - t|. \tag{7.15}$$

Proof. Set $\Delta(\tau) := \Delta_{g_\tau} = \Delta_{g_0 + \tau(g - g_0)} = \Delta_{g_0 + \tau \cdot h}$. Then $|\Delta_{g_{t_0}} - \Delta_{g_t}|_{i,j} \leq |\Delta'(t + \vartheta(t_0 - t))|_{i,j} \cdot |(t_0 - t)|$. We calculate and estimate $\Delta'(\tau)$. Locally,

$$\begin{aligned}
\Delta'(\tau) &= -[(\frac{1}{\sqrt{g_\tau}})'\partial_i\sqrt{g_\tau}g_\tau^{ij}\partial_j + \frac{1}{\sqrt{g_\tau}}\partial_i(\sqrt{g_\tau})'g_\tau^{ij}\partial_j + \\
&+ \frac{1}{\sqrt{g_\tau}}\partial_i\sqrt{g_\tau}(g_\tau^{ij})'\partial_j], \\
(\sqrt{g_\tau})' &= \frac{1}{2}\sqrt{g_\tau}tr_{g_\tau}h, \\
(\frac{1}{\sqrt{g_\tau}})' &= -\frac{1}{2}\frac{1}{\sqrt{g_\tau}}tr_{g_\tau}h \\
(g_\tau^{ij})' &= -g_\tau^{ik}g_\tau^{je}h_{ke} \equiv -h^{ij(\tau)}, \\
\Delta'(\tau)w &= (-\frac{1}{2}tr_{g_\tau}h\cdot\Delta(\tau) - \frac{1}{2}\frac{1}{\sqrt{g_\tau}}\partial_i\sqrt{g_\tau}tr_{g_\tau}h\, g_\tau^{ij}\partial_j + \\
&+ \frac{1}{\sqrt{g_\tau}}\partial_i\sqrt{g_\tau}h^{ij(\tau)}\partial_j)w = \\
&= -\frac{1}{2}(\nabla^{g_\tau}tr_{g_\tau}h, \nabla^{g_\tau}w)_{g_\tau} + (\frac{1}{\sqrt{g_\tau}}\partial_i\sqrt{g_\tau}h^{ij(\tau)}\partial_j)w. \qquad (7.16)
\end{aligned}$$

We estimate the first term on the right hand side of (7.16), using

$$\nabla_k^{g_\tau}tr_{g_\tau}h = \nabla_k^{g_\tau}\, g_\tau^{ij}h_{ij} = g_\tau^{ij}\,\nabla_k^{g_\tau}\, h_{ij} = g_\tau^{ij}h_{ij;k},$$

or more general,

$$\begin{aligned}
\nabla^{g_\tau}tr_{g_\tau}h &= tr_{g_\tau}(\nabla^{g_\tau}h), \\
(\nabla^{g_\tau})^i tr_{g_\tau}h &= tr_{g_\tau}(\nabla^{g_\tau})^i,
\end{aligned}$$

where here tr_{g_τ} refers to the trace with respect to the first two indices. Moreover

$$|(\nabla^{g_\tau})^i tr_{g_\tau}h|_{g_\tau} = |tr_{g_\tau}(\nabla^{g_\tau})^i h|_{g_\tau} \le C_1|(\nabla^{g_\tau})^i h|_{g_\tau},$$

and, according to 2.14,

$$\begin{aligned}
(\int |(\nabla^{g_\tau})^i h|^2_{g_\tau,x}dvol_x(g_\tau))^{1/2} &\le C_{2,i}|h|_{g_0,r}, i \le r, \\
(\int |(\nabla^{g_\tau})^i tr_{g_\tau}h|^2_{g_\tau,x}dvol_x(g_\tau))^{1/2} &\le C_{e,i}|h|_{g_0,r}, i \le r. \qquad (7.17)
\end{aligned}$$

We infer from (7.17), 2.9, 2.14

$$
\begin{aligned}
&|-\frac{1}{2}(\nabla^{g_\tau}h,\nabla^{g_\tau}w)|_{g_0,r-2}\le C_1|(\nabla^{g_\tau}tr_{g_\tau}h,\nabla^{g_\tau}w)|_{g_\tau,r-2}=\\
&=C_1(\int\sum_{i=0}^{r-2}|(\nabla^{g_\tau})^i(\nabla^{g_\tau}tr_{g_\tau}h,\nabla^{g_\tau}w)_{g_\tau}|^2_{g_\tau,x}dvol_x(g_\tau))^{1/2}=\\
&=C_1(\int\sum_{i=0}^{r-1}\sum_{j+k=i}|(tr_{g_\tau}(\nabla^{g_\tau})^{j+1}h,(\nabla^{g_\tau})^{k+1}w)_{g_\tau}|^2_{g_\tau,x}dvol_x(g_\tau))^{1/2}\le\\
&\le C_2(\int\sum_{j+k=i}|(\nabla^{g_\tau})^{j+1}h|^2_{g_\tau,x}\cdot|(\nabla^{g_\tau})^{k+1}w|^2_{g_\tau,x}dvol_x(g_\tau))^{1/2}\le\\
&\le C_3|h|_{g_\tau,r-1}\cdot|w|_{g_\tau,r-1}\le C_4(g_0,h,r)\cdot|w|_{g_0,r-1}.
\end{aligned}
\tag{7.18}
$$

Hence there remains to estimate

$$|\frac{1}{\sqrt{g_\tau}}\partial_i\sqrt{g_\tau}h^{ij(\tau)}\partial_j w|_{g_0,r-2}. \tag{7.19}$$

$$\frac{1}{\sqrt{g_\tau}}\partial_i\sqrt{g_\tau}h^{ij(\tau)}\partial_j w=$$

$$=\frac{1}{\sqrt{g_\tau}}(\partial_i\sqrt{g_\tau})h^{ij(\tau)}\partial_j w+ \tag{7.20}$$

$$+\partial_i h^{ij(\tau)}\partial_j w+ \tag{7.21}$$

$$+h^{ij(\tau)}\partial_i\partial_j w. \tag{7.22}$$

One way to estimate (7.20) - (7.22) in the $|\ \ |_{g_0,r-2}$ -norm is to introduce a cover $\mathfrak{U}=\{(U_\alpha,\phi_\alpha)\}_\alpha,\{\psi_\alpha\}_\alpha$ and to apply (2.33). We present a more covariant procedure of estimation. For abbreviation, $\nabla=\nabla(\tau)=\nabla^{g_\tau},h^{ij}=h^{ij(\tau)}=g_\tau^{ki}g_\tau^{lj}h_{kl}$, $h_{kl}=(g-g_0)_{kl},\Gamma^k_{ij}=\Gamma^k_{ij}(\tau)$.

$$
\begin{aligned}
&\frac{1}{\sqrt{g_\tau}}(\partial_i\sqrt{g_\tau})h^{ij}\partial_j w=\Gamma^k_{ik}h^{ij}\partial_j w=\Gamma^k_{ik}h^i_l g_\tau^{lj}\partial_j w=\\
&=\Gamma^k_{ik}h^i_l(\nabla w)^l,
\end{aligned}
\tag{7.23}
$$

$$
\begin{aligned}
&(\partial_i h^{ij})\partial_j w=\partial_i(h^i_l g_\tau^{lj})\partial_j w=(\partial_i h^i_l)(\nabla w)^l+\\
&+h^i_l(\partial_i g_\tau^{lj})\partial_j w=\nabla_i h^i_l(\nabla w)^l-(\Gamma^i_{is}h^s_l-\Gamma^s_{il}h^i_s)(\nabla w)^l-\\
&-(h^i_l\Gamma^l_{is}g_\tau^{sj}+h^i_l\Gamma^j_{is}g^{ls})\partial_j w=\\
&=(\delta_{g_T}h,\nabla w)_{g_T}-(\Gamma^i_{is}h^s_l-\Gamma^s_{il}h^i_s)(\nabla w)^l-\\
&-(h^i_l\Gamma^l_{is}(\nabla w)^s+h^{ij}(\partial_i\partial_j w-\nabla_i\nabla_j w)),
\end{aligned}
\tag{7.24}
$$

where we used for the components of a covariant derivative

$$\nabla_i \partial_s w = \partial_i \partial_s w - \Gamma^j_{is} \partial_j w,$$
$$\Gamma^j_{is} \partial_j w = \partial_i \partial_s w - \nabla_i \nabla_s w.$$

Adding (7.23), (7.24), (7.22), yields

$$\begin{aligned} &\frac{1}{\sqrt{g_\tau}} \partial_i \cdot \sqrt{g_\tau} h^{ij} \partial_j w = \\ &= \Gamma^k_{ik} h^i_l (\nabla w)^l + (\delta_{g_\tau} h, \nabla w)_{g_\tau} - \Gamma^i_{is} h^s_l (\nabla w)^l + \\ &+ \Gamma^s_{il} h^i_s (\nabla w)^l - h^i_l \Gamma^l_{is} (\nabla w)^s - h^{ij} \partial_i \partial_j w + \\ &+ h^{ij} \nabla_i \cdot \nabla_j w + h^{ij} \partial_i \partial_j w = \\ &= (\delta_{g_\tau} h, \nabla w)_{g_\tau} + h^{ij} \nabla_i \nabla_j w. \end{aligned} \tag{7.25}$$

We write

$$h^{ij} \nabla_i \nabla_j w = (h_{ij}, \nabla_i \nabla_j w)_{g_\tau} \tag{7.26}$$

Using

$$\nabla^2_{V,W} = \nabla_V \nabla_W - \nabla_{\nabla_V, W} \tag{7.27}$$

we can rewrite (7.26) as

$$h^{ij} \nabla_i \nabla_j w = (h, \nabla^2 w)_{g_\tau} + (h_{ij}, \nabla_{\nabla_i \partial_j} w)_{g_\tau}. \tag{7.28}$$

(7.27) and hence (7.28) has a generalization to higher covariant derivatives (cf. [14]). From this, $g_\tau \in comp(g_0)$, pointwise estimates for $\nabla_{\nabla_i \partial_j}$ and other mixed derivatives with respect to g_0, corresponding Sobolev estimates with respect to g_τ ($\nabla^{g_\tau} = \nabla^{g_0} + \nabla^{g_\tau} - \nabla^{g_0}$ etc.), the module structure theorem and 2.16, 2.17 we obtain finally

$$\begin{aligned} &|\frac{1}{\sqrt{g_\tau}} \partial_i \cdot \sqrt{g_\tau} h^{ij(\tau)} \partial_j w|_{g_0, r-2} \le \\ &\le C_1 |\frac{1}{\sqrt{g_\tau}} \partial_i \sqrt{g_\tau} h^{ij(\tau)} \partial_j w|_{g_\tau, r-2} = \\ &= C_1 (\int \sum_{s=0}^{r-2} |(\nabla)^s ((\delta_{g_\tau} h, \nabla w)_{g_\tau} + (h_{ij}, \nabla_i \nabla_j w)_{g_\tau})|^2 dvol(g_\tau))^{1/2} \le \\ &\le C_2 |h|_{g_\tau, r-1} \cdot |w|_{g_\tau, r} \le \\ &\le C_3 |h|_{g_0, r-1} \cdot |w|_{g_0, r} \end{aligned} \tag{7.29}$$

Here we again used $|(\nabla)^s \delta_{g_\tau} h| \le C \cdot |\nabla^{s+1} h|$. (7.18) and (7.29) imply

$$|(\Delta_{g_{t_0}} - \Delta_{g_t}) w|_{r-2} \le |t_0 - t| \cdot C(g_0, h, r) \cdot |w|_{g_0, r},$$

i.e. $|\Delta_{g_{t_0}} - \Delta_{g_t}|_{r, r-2} \le C \cdot |t_0 - t|$, where C depends on g_0, h, r but is independent of t. This finishes the proof of 7.9. □

Now we continue to estimate (7.13) and have to estimate

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}}\cdot))^{-1}|_{r-2,r}$$

First we recall that Δ_{g_t} is self adjoint on $\Omega^2(M, \Delta_{g_t}, g_t) = \Omega^2(M, \Delta_{g_0}, g_0) \subset L_2(M) = \Omega^0(M)$. For $u \in \Omega^r, r > 3$, the operator $v \to e^u \cdot v$ is symmetric and bounded on L_2. Hence $\Delta_{g_t} + e^u$ is self adjoint.

LEMMA 7.10. *There exists a constant $c > 0$ such that $inf\, \sigma(\Delta_{g_t}) \geq c, 0 \leq t \leq 1$.*

Proof. Assume the converse. Then there exists a convergent sequence $t_i \to t^*$ in $[0,1]$ such that $\lambda_{min}(\Delta_{g_{t_i}}) \to 0$. Here $\lambda_{min}(\Delta_{g_{t_i}})$ is the minimal spectral value of $\Delta_{g_{t_i}}$. It is > 0 and either equal to $inf\, \sigma_e(\Delta_{g_t})$ or an isolated eigenvalue of finite multiplicity. According to 7.9, $\Delta_{g_{t_i}} \to \Delta_{g_{t^*}}$ in the generalized sense of [24], IV, § 2.6. Then, according to [24], V, § 4, remark 4.9, $\lambda_{min}(\Delta_{g_{t_i}}) \to \lambda_{min}(\Delta_{g_{t^*}})$, i.e. necessary $\lambda_{min}(\Delta_{g_{t^*}}) = 0$, a contradiction. □

COROLLARY 7.11. *For arbitrary $t \in [0,1], u \in \Omega^r$*

$$inf\, \sigma(\Delta_{g_t} + e^u) \geq c,$$
$$\Delta_{g_t} + e^u = \int_c^\infty \lambda dE_\lambda(t,u),$$
$$(\Delta_{g_t} + e^u)^{-1} = \int_c^\infty \lambda^{-1} dE_\lambda(t,u),$$

$(\Delta_{g_t} + e^u)^{-1}$ is a bounded operator on L_2 and, according to [24], p.357, (5.17), the operator norm of $(\Delta_{g_t} + e^u)^{-1}$ is $\leq \frac{1}{c}$. □

We want to prove more and to estimate

$$((\Delta_{g_t} + e^u)^{-1}|_{r-2,r}. \tag{7.30}$$

First we have to assure that (7.30) makes sense.

LEMMA 7.12. *For $u \in \Omega^r, r > 3$, the map $v \to e^u \cdot v$ is a bounded map $\Omega^i \to \Omega^i, i \leq r$, with*

$$|e^u|_{i,i} \leq C(u,i) \leq C(i) \cdot sup\, e^u \cdot |u|_r. \tag{7.31}$$

Proof. This follows immediately from 2.7, 2.8. □

COROLLARY 7.13. *The Sobolev spaces based on the operators Δ_{g_t} and $\Delta_{g_t} + e^u$ are equivalent for $i \leq r$,*

$$\Omega^i(M^2), \Delta_{g_s}, g_s) \cong \Omega^i(M^2, \Delta_{g,t} + e^u), i \leq r. \tag{7.32}$$

□

REMARKS. The heart of the estimate for (7.30) consists in proving that the constants arising in (7.31), (7.32) can be chosen independently of t and u if u solves

$$F(t,u) \equiv \Delta_{g_t} u + K(g_t) + e^u = 0.$$

□

Consider $\Omega^r \subset \Omega^2 \subset \Omega^0 = L_2, \Omega^{r-2} \subset L_2$ and assume r even.

LEMMA 7.14. *$\Delta_{g_t} + e^u : \Omega^2 \to \Omega^0 = L_2$ induces a bijective morphism between $\Omega^r \subset \Omega^2$ and $\Omega^{r-2} \subset \Omega^0$.*

Proof. Surely, $\Delta_{g_t} + e^u$ maps $\Omega^r \subset \Omega^2$ into $\Omega^{r-2} \subset \Omega^0 = L_2$. This map is injective according to 7.10. It is surjective: Let $v \in \Omega^{r-2} \subset \Omega^0$. Then $(\Delta_{g_t} + e^u)^{-1}v \in \Omega^2$, $(\Delta_{g_t} + e^u)^i((\Delta_{g_t} + e^u)^{-1}v) = (\Delta + e^u)^{i-1}v$ is square integrable $i \leq \frac{r}{2}$. The assertion now follows from 7.13. □

Now we state our main

PROPOSITION 7.15. *Assume $r > 3$ even. Then there exists a constant $C = C(g_0, g) > 0$, independent of t, such that*

$$|(\Delta_{g_t} + e^{u_t})^{-1}|_{r-2,r} \leq C \tag{7.33}$$

for any solution $u_t \in \Omega^r = \Omega^r(M, g_0)$ of $\Delta_{g_t} u_t + K(g_t) + e^{u_t} = 0$.

Proof. We would be done if we could show

$$|(\Delta_{g_t} + e^{u_t})^{-1}v|_0 \leq C_0 |v|_0 \tag{7.34}$$

$$|\Delta^i_{g_t}(\Delta_{g_t} + e^{u_t})^{-1}v|_0 \leq C_i |v|_{2i-2} \leq C_i |v|_{r-2}, 1 \leq i \leq \frac{r}{2}, \tag{7.35}$$

$C_i = C_i(g_0, g), |\ \ |_j = |\ \ |_{g_0,j}$. We perform induction. (7.34) follows from (7.11). Consider $i = 1$ in (7.35) and denote $\Delta_{g_t} + e^{u_t} \equiv \Delta + e^u$. Then

$$\Delta(\Delta + e^u)^{-1}v = v - (e^u) \circ (\Delta + e^u)^{-1}v. \tag{7.36}$$

LEMMA 7.16. *There exists a constant $D > 0$ independent of t such that*

$$sup\, e^{u_t} \leq D \tag{7.37}$$

for any solution of $\Delta_{g_t} u_t + K(g_t) + e^{u_t} = 0$.

Proof. Let (M^2, g) be a Riemannian 2-manifold, oriented. Then g defines an integrable almost complex structure J_g such that (M^2, g, J_g) is Kählerian. Moreover, $J_g = J_e u \cdot g$. Consider now our case $id : (M, g_t, J_{g_t}) \to (M, e^{u_t} \cdot g_t, J_{g_t})$. id is a nonconstant holomorphic map. We repeat Yau's

GENERAL SCHWARZ LEMMA. *Let (M, g) and (N, h) complete Riemannian surfaces with sectional curvatures K_M and K_N and $f : M \to N$ a nonconstant holomorphic map. Assume $K_M \geq K_1$ and $K_N \leq K_2 < 0$. Then $K_1 < 0$ and*

$$f^*h \leq \frac{K_1}{K_2} \cdot g \tag{7.38}$$

See [32] for a proof. □

(7.38) implies in our case with $id : (M, g_t) \to (M, e^{u_t} \cdot g_t)$

$$e^u \leq -\inf_{x \in M} K(g_t)(x)/2, \tag{7.39}$$

where in (7.39) K denotes the scalar curvature = 2· sectional curvature. $g_t \in comp(g_0), K(g_0) \equiv -1$ and $r > 3$ imply $\inf_{x \in M} K(g_t)(x) \leq -1$ but we must prove that $\inf_{x \in M} K(g_t)(x)$ really exists. This is the content of

LEMMA 7.17. *There exists a constant $D_1 > 0$ independent of t such that*

$$|K(g_t)(x)| \leq D_1 \quad \text{for all} \quad t \in [0, 1], x \in M. \tag{7.40}$$

Proof. (7.40) would follow if we could prove

$$^{b}|-1-K(g_t)| \equiv^{b,0} |-1-K(g_t)| \leq D_2. \tag{7.41}$$

but this follows immediately from the facts $g, g_t = g_0 + t(g - g_0) \in comp(g) \subset \mathcal{M}^r(I, B_\infty)$, $r > 3 = \frac{2}{2} + 2$, ${}^{b,2}|g_t - g_0|_{g_0} = t \cdot{}^{b,2} |g - g_0| \leq D_3 \cdot t \cdot |g - g_0|_{g_0,r}$, (2.34) and scalar curvature has an expression by derivatives of order ≤ 2 of the metric. This proves (7.40) and hence (7.37). □

Now, according to (7.36), 2.14,

$$\begin{aligned}|\Delta(\Delta + e^u)^{-1}v|_0 &\leq |v|_0 + D|(\Delta + e^u)^{-1}v|_0 \leq \\ &\leq |v|_0 + D \cdot C_0|v|_0 = C_1|v|_0\end{aligned}$$

which finishes the proof of (7.35) for $i = 1$. Assume now

$$|\Delta^j(\Delta + e^u)^{-1}v|_0 \leq C_j \cdot |v|_{j-2}, j \leq i-1, i \leq \frac{r}{2}. \tag{7.42}$$

Then

$$\begin{aligned}\Delta^i(\Delta + e^u)^{-1}v &= \Delta^{i-1}(\Delta(\Delta + e^u)^{-1}v) = \\ &= \Delta^{i-1}v - \Delta^{i-1}((e^u\cdot)(\Delta + e^u)^{-1}v).\end{aligned} \tag{7.43}$$

Clearly,

$$|\Delta^{i-1}v|_0 \leq |v|_{g_t,2i-2} \leq C \cdot |v|_{g_0,2i-2}, \tag{7.44}$$

hence we have to estimate

$$\Delta^{i-1}((e^u)(\Delta + e^u)^{-1}v). \tag{7.45}$$

As follows from

$$\Delta(v \cdot w) = v \cdot \Delta w + w\Delta v - 2(\nabla u, \nabla w), \tag{7.46}$$

$$\Delta e^u = e^u(\Delta u - |\nabla u|^2) \tag{7.47}$$

and the induction assumption applied to $\Delta^j(\Delta + e^u)^{-1}v$, we have a desired estimate for (7.45) if we have an estimate for $|u|_0, |\Delta u|_0, \ldots, |\Delta^{i-1}u|_0$, independent of $t, u = u_t$ solution of $\Delta_{g_t}u_t + K(g_t) + e^{u_t} = 0$. The proof of 7.15 would be finished if we could prove

PROPOSITION 7.18. *Assume $r > 3$ even. Then there exist constants $D_i = D_i(g, g_0)$ independent of t, such that*

$$|\Delta^i_{g_0}u|_0 \leq D_i, i \leq \frac{r}{2}, \tag{7.48}$$

for $u = u_t$ a solution of $\Delta_{g_t}u_t + K(g_t) + e^{u_t} = 0$.

Proof. According to 2.16, we are done if we could show $|\Delta^i_{g_t}u|_0 \leq D_i$ and write in the sequel simply $u \equiv u_t, \Delta \equiv \Delta_{g_t}, K \equiv K(g_t)$. Then

$$\Delta u + K + e^u = 0$$

is equivalent to

$$(\Delta + \frac{e^u - 1}{u})u = -(K+1),$$

i.e.

$$u = (\Delta + \frac{e^u - 1}{u})^{-1}(-(K+1)). \tag{7.49}$$

Here $\frac{e^u-1}{u}$ is well defined, ≥ 0 and $(\Delta + \frac{e^u-1}{u})^{-1}$ is a well defined bounded operator according to 7.11. We would be done for $i = 0$ in (7.48) if we could show $|K(g_t) - (-1)|_0 \leq C = C(g, g_0)$ independent of t. We prove more general

LEMMA 7.19. *Let* $t, t_0 \in \{0, 1\}$. *Then*

$$|K(g_{t_0}) - K(g_t)|_{r-2} \leq |t_0 - t| \cdot C, \tag{7.50}$$

$C = C(g_0, g)$ *independent of* t.

Proof. According to the mean value theorem for maps into affine Banach spaces,

$$|K(g_{t_0}) - K(g_t)|_{r-2} \leq |t_0 - t| \cdot \sup_{t_0 < \tau < t} |K'(g_\tau)|_{r-2}. \tag{7.51}$$

$$K'(g_\tau) = \frac{d}{d\sigma} K(g_0 + \tau h + \sigma h)|_{\sigma=0} =$$
$$= \tau(\Delta_{g_\tau} tr_{g_\tau} h + \delta_{g_\tau} \delta_{g_\tau} h - \frac{1}{2} K(g_\tau) tr_{g_\tau} h),$$

hence

$$|K'(g_\tau)|_i \leq \tau \cdot (C_i' |h|_{i+2} + \frac{1}{2} |K(g_\tau) tr_{g_\tau} h|_i). \tag{7.52}$$

We have to estimate $|K(g_\tau) tr_{g_\tau} h|_i$. For $i = 0$, i.e. $|\ \ |_0$, there does not arise any problem since $|K(g_\tau) \leq C_0''$, C_0'' independent of τ and $|tr_{g_\tau} h|_0 \leq C_0''' \cdot |h|_0$. We continue with $i = 2$ to indicate the general rule.

$$K(g_\tau) = 2\, R_{1212}(g_\tau)(det(g_\tau),$$
$$\frac{1}{2} \Delta\, K(g_\tau) = \frac{1}{2} \Delta(K(g_\tau) + 1) = \frac{1}{2} \Delta(K(g_\tau) - K(g_0)) =$$
$$= \frac{1}{2} \Delta[R_{1212}(g_\tau)(det(g_0) - det(g_\tau)) +$$
$$+ (R_{1212}(g_\tau) - R_{1212}(g_0)) det(g_0))/det(g_0) \cdot det(g_\tau)], \tag{7.53}$$

where $\Delta = \Delta_{g_0}$. Choose an atlas $\mathfrak{U} = \{(U_\alpha, \phi_{\alpha)}\}_\alpha$ as in section 2. Then $g_{0,ij}$, g_0^{ij}, $det\, g_0$ and all of its derivatives are bounded,

$$det(g_0) \geq c > 0. \tag{7.54}$$

$r > 3$ and $g_\tau = g_0 + \tau h, |h|_r < \infty$ imply

$$g_{\tau,ij}, g_\tau^{ij}, det(g_\tau) \quad \text{bounded}, \quad det(g_\tau) \geq c' > 0 \tag{7.55}$$

There holds

$$\Gamma^i_{jk}(g_\tau) \equiv \Gamma^i_{jk}(g_0 + \tau h) = \Gamma^i_{jk}(g_0) + \\ + \frac{1}{2} g^{ie}_\tau (\tau h_{ej;k} + \tau h_{ek;j} + \tau h_{jk;e}) \tag{7.56}$$

and

$$R^\alpha_{\beta\gamma\delta}(g_\tau) = (\partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\rho\gamma} \Gamma^\rho_{\beta\delta} - \Gamma^\alpha_{\rho\delta} \Gamma^\rho_{\beta\gamma})(g_\tau), \tag{7.57}$$

where $;j$ denotes $\nabla^{g_0}_j$. Finally we conclude from (7.53)-(7.57), $|h|_r < \infty, \nabla^{g_0} = \nabla^{g_\tau} + \nabla^{g_0} - \nabla^{g_\tau}$, the module structure theorem, 2.16 and 2.17 that

$$|\Delta K(g_\tau)|_0 \le D_2(|h|_4) \tag{7.58}$$

D_2 a polynomial in $|h|_r$. Similar for higher derivatives,

$$|\Delta^j K(g_\tau)|_0 \le D_{2j}(|h|_{2j+2}) \tag{7.59}$$

We omit the very long but rather simple details. This finishes the proof of 7.19. □

Hence

$$|u|_0 = |(\Delta + \frac{e^u - 1}{u})^{-1}(-(K+1))|_0 \le \frac{C}{c} = D_0. \tag{7.60}$$

Next we study Δu to indicate the general rule.

$$\Delta u = \Delta((\Delta + \frac{e^u - 1}{u})^{-1} - (K+1)) = \\ = (\Delta + \frac{e^u - 1}{u})(\Delta + \frac{e^u - 1}{u})^{-1}(-(K+1) - (\frac{e^u - 1}{u})(\Delta + \frac{e^u - 1}{u})^{-1}(-(K+1)) \\ = -(K+1) + (\frac{e^u - 1}{u})(\Delta + \frac{e^u - 1}{u}))^{-1}(-(K+1)) \tag{7.61}$$

$\frac{e^u - 1}{n}$ can even pointwise be estimated by a constant independent of t: Let $|u(x)| \ge 1$. Then, according to (7.37),

$$|\frac{e^{u(x)} - 1}{u(x)}| \le |e^{u(x)} - 1| \le D + 1 = C'.$$

If $|u(x)| < 1$, then $|\frac{e^{u(x)} - 1}{u(x)}| \le \sum_{i=1}^\infty \frac{1}{i!} < e = C''$. Hence $|\Delta u|_0 \le D_2$. Assume now

$$|\Delta^j u|_0 \le D_j, j \le i - 1, i \le \frac{r}{2},$$

and consider $\Delta^i u$. According to (7.61),

$$\Delta^i u = -\Delta^{i-1}(K+1) - \Delta^{i-1} \circ (\frac{e^u - 1}{u} \cdot)((\Delta + \frac{e^u - 1}{u})^{-1}(K+1)) \tag{7.62}$$

for $i \ge 2$. 7.19 yields $|\Delta^{i-1}(K+1)|_0 \le D'$. If we write $\Delta^i u$ to determine a Sobolev norm, this means $\Delta^i_{g_0} u$ since our general reference Sobolev norm is $| \quad |_{g_0, j}, j \le r$. But for the calculations in the sequel we have often to work with $\Delta^i_{g_t}$ since then formulas become easier. But this does not touch the proof of our desired a priori Sobolev estimates according to 2.16.

We have to find an a priori estimate

$$|\Delta^{i-1}(\frac{e^u-1}{u}\cdot((\Delta+\frac{e^u-1}{u})^{-1}(K+1)))|_0 \le D'', \tag{7.63}$$

$D'' = D''(g, g_0)$ independent of t. Consider $\Delta^{i-1}(v \cdot w)$. In our case $v = \frac{e^u-1}{u}$, $w = (\Delta + \frac{e^u-1}{u})^{-1}(K+1)$. We obtain from

$$\Delta(v\cdot w) = v\Delta w + w\Delta v - 2(\nabla v, \nabla w) \tag{7.64}$$

that $\Delta^{i-1}(v \cdot w)$ has a representation

$$\Delta^{i-1}(v\cdot w) = \sum_{j+k=i-1} \Delta^j v \cdot \Delta^k w + \quad \text{sum of mixed terms.} \tag{7.65}$$

It follows from (2.32), (I), (B_∞) for g_0 and the module structure theorem that a priori estimates for all

$$|\Delta^j v \cdot \Delta^k w|_0$$

imply such estimates for all mixed terms too.

REMARKS. We could also work exclusively with covariant derivatives. But then all of our expressions grow rapidly. Therefore we decided to work only with every second derivative, i.e. to work with the Δ's. □

Consider now all products

$$\Delta^j(\frac{e^u-1}{u})\cdot\Delta^k(\Delta+\frac{e^u-1}{u})^{-1}(K+1), j+k=i-1.$$

$\frac{e^u-1}{u} = 1 + \frac{u}{2!} + \frac{u^2}{3!} + \dots$ and $\frac{u}{2!} + \frac{u^2}{3!} + \dots$ converges in Ω^r since all $u^i \in \Omega^r$, $|u^i|_r \le C^{i-1}$. $|u|_r^i$ and $\frac{|u|_r}{2!} + \frac{C|u|_r^2}{3!} + \frac{C^2|u|_r^3}{4!} + \dots$ converges. We have already seen

$$|\frac{e^u-1}{u}| \le C_0'.$$

Using $\Delta u^k = -\nabla^j\nabla_j u^k = -k(k-1)u^{k-2}|\nabla u|^2 + ku^{k-1}\Delta u$, we see that at least formally

$$\begin{aligned}\Delta(\frac{e^u-1}{u}) &= \Delta u(\frac{1}{2!} + \frac{2u}{3!} + \frac{3u^2}{4!} + \dots)\\ &\quad - |\nabla u|^2(\frac{2}{3!} + \frac{2\cdot3\cdot u}{4!} + \frac{3\cdot4\cdot u^2}{5!} + \dots).\end{aligned} \tag{7.63}$$

But the same argument as above and the module structure theorem yields $\Delta(\frac{e^u-1}{u})$ and its series (7.63) is a well defined element of Ω^{r-2}. We want to establish an a priori estimate for $|\Delta(\frac{e^u-1}{u})|_0$. We already proved

$$|\Delta u|_0 \le D_2 \tag{7.64}$$

which implies

$$|\Delta u \cdot \frac{1}{2!}|_0 \leq D_2/2. \tag{7.65}$$

We continue to establish an a priori estimate for

$$|\Delta u(\frac{1}{2!} + \frac{2u}{3!} + \frac{3u^2}{4!} + \dots)|_0 \tag{7.66}$$

The a priori estimate for $|u|_0$ and $|\Delta u|_0$ yield such an estimate for $|u|_2$.

$$|u|_2 \leq D_2'. \tag{7.67}$$

According to remark 2 after 2.8, Ω^2 is an algebra and we have an estimate

$$|u^2|_2 \leq C_2 |u|_2^2, |u^k|_2 \leq C_2^{k-1} |u|_2^k,$$

together with (7.67),

$$|u^k|_2 \leq C_2^{k-1} D_2'^k.$$

Hence $\frac{2u}{3!} + \frac{3u^2}{4!} + \dots$ is a well defined element of Ω^2 (even of Ω^r as we have seen) and there exists an estimate

$$|(\frac{2u}{3!} + \frac{3u^2}{4!} + \dots)|_2 \leq \frac{2D_2'}{3!} + \frac{3C_2 D_2'^2}{4!} + \dots = D_2''. \tag{7.68}$$

Now we apply once again the module structure theorem 2.8. In our case $n = 2$, $p_1 = p_2 = p = 2$, $\frac{n}{p_1} = \frac{n}{p_2} = \frac{n}{p} = 1$, $\Delta u \in \Omega^0 = L_2$, $r_1 = 0 < 1$, $(\frac{2u}{3!} + 3n^2 4! + \dots) \in \Omega^2$, $r_2 = 2$, $r = 0$, then, according to 2.8,

$$|\Delta u \cdot (\frac{2u}{3!} + \frac{3u^2}{4!} + \dots)|_0 \leq C \cdot |\Delta u|_0 \cdot |(\frac{2u}{3!} + \frac{3u^2}{4!} + \dots)|_2 \leq$$
$$\leq C \cdot D_2 \cdot D_2'',$$

together with (7.65),

$$|\Delta u \cdot (1 + \frac{2u}{3!} + \frac{3u^2}{4!} + \dots)|_0 \leq D_2/2 + C \cdot D_2 \cdot D_2'' = D_2'''. \tag{7.69}$$

Quite similar we manage the second term in (7.63) using that $\nabla u \in \Omega^1$, $|\nabla u|^2 \in L_2, ||\nabla u|2|_0 \leq C_1 |\nabla u|_1^2 \leq C_1 |u|_2^2$ and again $(\frac{2 \cdot 3u}{4!} + \frac{3 \cdot 4u^2}{5!} + \dots) \in \Omega^2$. We obtain

$$||\nabla u|^2 (\frac{2}{3!} + \frac{2 \cdot 3 \cdot u}{4!} + \frac{3 \cdot 4 \cdot u^2}{5!} + \dots)|_0 \leq D_2^{(4)}, \tag{7.70}$$

i.e.

$$|\Delta(\frac{e^u - 1}{u})|_0 \leq D_2^{(5)}. \tag{7.71}$$

Now it is every easy to recognize the general rule. One forms $\Delta^j(\frac{e^u-1}{u})$, obtains a finite sum of factors $\times$ series, the factors are in $L_2 = \Omega_0$ and have an a priori L_2-estimate coming from $|\Delta^k u|_0 \leq D_k, k \leq j-1$. The series are in Ω^2 and have an a priori $| \quad |_2$ -estimate which yields together

$$|\Delta^j(\frac{e^u-1}{u})|_0 \leq D'_j, j \leq i-1, \tag{7.72}$$

$D'_j = D'_j(g, g_0)$ independent of t. Finally we want to establish a priori estimates for

$$\Delta^k((\Delta + \frac{e^u-1}{u})^{-1}(K+1)), k \leq i-1. \tag{7.73}$$

But if we replace in (7.42)-(7.47) e^u by $\frac{e^u-1}{u}$, then we see that we get a priori estimates if we have such estimates for

$$|\Delta^l(\frac{e^u-1}{u})|_0, l \leq k, k \leq i-1.$$

But these we have just established. i.e. we obtain

$$|\Delta^k((\Delta + \frac{e^u-1}{u})^{-1}(K+1))|_0 \leq E_k, k \leq i-1.$$

$K+1 \in \Omega^{r-2}, (\Delta + \frac{e^u-1}{u})^{-1}(K+1) \in \Omega^r$, $\Delta^k((\Delta + \frac{e^u-1}{u})^{-1}(K+1)) \in \Omega^2$ since $k \leq i-1 \leq \frac{r}{2}-1$, $\Delta^j(\frac{e^u-1}{u}) \in \Omega^0 = L_2$. Applying once again 2.8, we obtain

$$|\Delta^j(\frac{e^u-1}{u}) \cdot \Delta^k((\Delta + \frac{e^u-1}{u})^{-1}(K+1))|_0 \leq F_{j,k},$$

$F_{j,k} = F_{j,k}(g, g_0)$ independent of t. Quite similar we conclude

$$|\text{mixed terms}|_0 \leq F.$$

Hence

$$|\Delta^{i-1}(\frac{e^u-1}{u} \cdot ((\Delta + \frac{e^u-1}{u})^{-1}(K+1))|_0 \leq F + \sum_{j+k=i-1} F_{j,k},$$

together with (7.50),

$$|\Delta^i u|_0 \leq D_i \tag{7.74}$$

$D_i = D_i(g, g_0)$ independent of $t, i \leq \frac{r}{2}$. This proves 7.18, hence (7.35) and our main proposition 7.15. □

COROLLARY 7.20 *There exists a constant $C = C(g, g_0)$ such that*

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}}\cdot))^{-1}|_{r-2,r} \cdot |\Delta_{g_{t_0}} - \Delta_{g_t}|_{r,r-2} \leq C \cdot |t - t_0|. \tag{7.75}$$

□

The estimate of the first factor of (7.14) is already done,

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}}\cdot))^{-1}(e^{u_{t_0}}\cdot)|_{r-2,r} \leq$$
$$\leq |(\Delta_{g_{t_0}} + (e^{u_{t_0}}\cdot))^{-1}|_{r-2,r} \cdot |(e^{u_{t_0}}\cdot)|_{r-2,r-2}.$$

According to (7.33),

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}}\cdot))^{-1}|_{r-2,2} \leq C_1, \tag{7.76}$$

and, according to (7.31), (7.37) and $|\Delta^j u|_0 \le D_j, 0 \le j \le \frac{r}{2}$,

$$|(e^{u_{t_0}}\cdot)|_{r-2,r-2} \le C_2, \tag{7.77}$$

i.e.

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}}\cdot))^{-1}(e^{u_{t_0}}\cdot)|_{r-2,r} \le C_3, \tag{7.78}$$

$C_3 = C_3(g, g_0)$ independent of t. The final estimate concerns

$$|(1 - e^{v-u_{t_0}+\vartheta(u-u_{t_0}-(v-u_{t_0}))})\cdot|_{r,r-2}, \tag{7.79}$$

where as usual the point indicates that the corresponding expression acts by multiplication. We write

$$1 - e^{v-u_{t_0}+\vartheta(u-u_{t_0}-(v-u_{t_0}))} =$$
$$= -\sum_{i=1}^{\infty}[v - u_{t_0} + \vartheta(u - u_{t_0} - (v - u_{t_0})]^i/i!$$

As above, this series converges in Ω^r and for $|v-u_{t_0}+\vartheta(u-u_{t_0}-(v-u_{t_0}))|_r$ sufficiently small $|\sum_{i=1}^{\infty}[v - u_{t_0} + \vartheta(u - u_{t_0} - (v - u_{t_0}))]^i/i!|_r$ becomes arbitrary small.

For any $f \in \Omega^r$, the operator norm of $(f\cdot) : \Omega^r \to \Omega^{r-2}, (f\cdot)w = f\cdot w$, can be estimated by $C(r)\cdot|f|_r$. This yields

LEMMA 7.21. *For any $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that*

$$|(1 - e^{v-u_{t_0}+\vartheta(u-u_{t_0}-(v-u_{t_0}))})\cdot|_{r,r-2} \le \varepsilon_1$$

for all u, v with $|u - u_{t_0}|_r, |v - u_{t_0}|_r \le \delta_1$.

Proof. Given $\varepsilon_1 > 0$, there exists δ_1' such that for $|v-u_{t_0}+\vartheta(u-u_{t_0}-(v-u_{t_0}))|_r <$ δ_1'

$$C(r)\cdot|\sum_{i=1}^{\infty}[v - u_{t_0} + \vartheta(u - u_{t_0} - (v - u_{t_0}))]^i/i!|_r \le \varepsilon_1.$$

Set $\delta_1 = \delta_1'/4$. Then

$$|v - u_{t_0} + \vartheta(u - u_{t_0} - (v - u_{t_0}))|_r < |v - u_{t_0}|_r + |u - u_{t_0}|_r + |v - u_{t_0}|_r =$$
$$= |u - u_{t_0}|r + 2|v - u_{t_0}|_r < 2(|u - u_{t_0}|_r + |v - u_{t_0}|_r) \le 4\delta_1 = \delta_1'.$$

□

COROLLARY 7.22 *There exists $\delta_1 > 0$ such that $|u - u_{t_0}|_r \le \delta_1, |v - u_{t_0}|_r \le \delta_1$ implies*

$$|(\Delta + (e^{u_{t_0}}\cdot))^{-1}(e^{u_{t_0}}\cdot|_{r-2,r}\cdot$$
$$\cdot|(1 - e^{v-u_{t_0}+\vartheta(u-u_{t_0}-(v-u_{t_0}))}\cdot|_{r,r-2} \le \frac{1}{4}. \tag{7.80}$$

Proof. Set in (7.21) $\varepsilon_1 = \frac{1}{4}\cdot\frac{1}{C_3}$, C_3 from (7.78). □

COROLLARY 7.23 *There exists* $\delta_1 > 0$ *such that for* $|u - u_{t_0}|_r \leq \delta_1, |v - u_{t_0}|_r \leq \delta_1$

$$|T_t u - T_t v|_r \leq (C \cdot |t - t_0| + \frac{1}{4})|u - v|_r, \tag{7.81}$$

where C *comes from 7.9.*

Proof. This follows immediately from (7.12), (7.13), (7.14), (7.15) and (7.80). □

If we would choose $|t_0 - t|$ sufficiently small, then the map T_t would be contractive. But this does not make sense since until now we did not define a complete metric space on which T_t acts. This will be the next and last step in our approach. But we will use the inequality (7.81) in this step.

PROPOSITION 7.24. *Suppose* $u_{t_0} \in \Omega^r, r > 3, \Delta_{g_{t_0}} u_{t_0} + K(g_{t_0}) + e^{u_{t_0}} = 0$. *There exist* $\delta, \delta_1 > 0$ *independent of* t_0 *such that* T_t *maps* $M_{t_0,\delta_1} = \{u \in \Omega^r | \, |u - u_{t_0}|_r \leq \delta_1\}$ *into itself for* $|t - t_0| \leq \delta$. *Moreover* T_t *is contracting.*

Proof. We start estimating $T_t u - u_{t_0}$:

$$\begin{aligned} |T_t u - u_{t_0}|_r &= |T_t u - T_{t_0} u_{t_0}|_r \leq \\ &\leq |T_t u - T_t u_{t_0}|_r + |T_t u_{t_0} - T_{t_0} u_{t_0}|_r. \end{aligned} \tag{7.82}$$

For $|u - u_{t_0}|_r \leq \delta_1, \delta_1$ from 7.23,

$$|T_t u - T_t u_{t_0}|_r \leq (C \cdot |t - t_0| + \frac{1}{4})|u - u_{t_0}|_r.$$

Hence for $|t - t_0| \leq \delta', |u - u_{t_0}|_r \leq \delta_1$

$$(C \cdot |t - t_0| + \frac{1}{4}) \leq \frac{1}{2}$$

and

$$|T_t u - T_t u_{t_0}|_r \leq \frac{1}{2}|u - u_{t_0}|_r \leq \frac{1}{2}\delta_1. \tag{7.83}$$

It remains to estimate $|T_t u_{t_0} - T_{t_0} u_{t_0}|_r$. But by an easy calculation

$$\begin{aligned} T_t u_{t_0} - T_{t_0} u_{t_0} = -(\Delta_{g_{t_0}} + (e^{u_{t_0}} \cdot))^{-1}((\Delta_{g_t} - \Delta_{g\,t_0})u_{t_0} + \\ + K(g_t) - K(g_{t_0})). \end{aligned}$$

We are done if for $|t - t_0| \leq \delta''$

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}} \cdot))^{-1}(\Delta_{g_{t_0}} - \Delta_{g_t})u_{t_0}|_r < \delta_1/4. \tag{7.84}$$

$$|(\Delta_{g_{t_0}} + (e^{u_{t_0}} \cdot))^{-1}(K(g_{t_0}) - K(g_t))|_r < \delta_1/4. \tag{7.85}$$

The existence of such a δ'' follows immediately from 7.9, 7.15, (7.74) for (7.84) and from 7.15, 7.19 for (7.85). Let now $\delta = \min\{\delta', \delta''\}$. Then we infer from (7.82)-(7.85)

$$|T_t u - u_{t_0}|_r \leq \delta_1,$$

i.e. $T_t : M_{t_0,\delta_1} \to M_{t_0,\delta_1}$. T_t is contractive according to (7.81) since for $|t - t_0| \leq \delta$

$$(C \cdot |t - t_0| + \frac{1}{4}) \leq \frac{1}{2}.$$

This finishes the existence proof of theorem 7.7 and yields uniqueness in a moving ball $M_{t,\delta_1}, 0 \leq t \leq 1$. We prove now the uniqueness in all of Ω^r.

Fix $x_0 \in M^2$ and denote by $d(x) = d(x, x_0)$ the Riemannian distance. Let $u, v \in \Omega^r, r > 3$, be solutions of

$$\Delta_g u + K(g) + e^u = 0.$$

We obtain $u, v, u - v$ bounded, C^2 and

$$\Delta_g(u - v) = -(e^u - e^v).$$

There are two cases.
1. $u - v$ achieves its supremum in $U_1(x_0) = \{x | d(x) \leq 1\}$. e.g. in x_1. Then $\Delta(u - v)(x_1) \geq 0, -(e^{u(x_1)} - e^{v(x_1)}) \geq 0, e^{u(x_1)} \leq e^{v(x_1)}, (u-v)(x_1) \leq 0$ of the supreme point x_1, hence $(u - v)(x) \leq 0$ everywhere, $u(x) \leq v(x)$.
2. Or we apply Yau's generalized maximum principle: $f \in C^2$,

$$\limsup_{d(x) \to \infty} \frac{f(x) - f(x_0)}{d(x)} \leq 0$$

and

$$\lim_{\substack{d(x) \to \infty \\ f(x) \geq f(x_0)}} \frac{K(x)(f(x) - f(x_0))}{d(x)} = 0.$$

Then there are points $(x_k)_k \subset M$ such that $\lim\limits_{k \to \infty} f(x_k) = \sup f$, $\lim\limits_{k \to \infty} \nabla f(x_k) = 0$ and $\limsup\limits_{k \to \infty} \Delta f(x_k) \geq 0$. See [31] for the proof.

In our case $f = u - v$. Then we have $(x_k)_k$ such that $\lim\limits_{k \to \infty} (u-v)(x_k) = \sup(u-v)$, $\lim\limits_{k \to \infty} \nabla(u - v)(x_k) = 0$, $\limsup \Delta(u - v)(x_k) \geq 0$, hence $\limsup(e^v - e^u)(x_k) \geq 0$, $\limsup(v - u)(x_k) \geq 0$, $\limsup(u - v)(x_k) \leq 0$, $\sup(u - v) \leq 0, u \leq v$ everywhere.

Quite similar $v \leq u$, i.e. $u = v$. This finishes uniqueness and the proof of theorem 7.7. □

REMARKS. 1. We had several versions of the proof. But the particular useful proposal to work with the equation $u = (\Delta + \frac{e^u - 1}{u})^{-1}(-(K + 1))$ has been made by Gorm Salomonsen.
2. A seemingly more direct approach proving $\mathcal{S} = [0, 1]$ would amount to prove the following assertion. Assume $t_1 < t_2 < \ldots < t_0, t_\nu \to t_0$, $\Delta_{g_{t_0}} u_{t_\nu} + e^{u_{t_\nu}} = 0$. Then

a. $(u_{t_\nu})_\nu$ is a Cauchy sequence with respect to $|\ \ |_r$.
b. $u_{t_\nu} \to u_{t_0}$
c. $\Delta_{g_{t_0}} u_{t_0} + K(g_{t_0}) + e^{u_{t_0}} = 0$.

But writing down a straightforward approach proving a., c. leads immediately to the key estimates performed by us.
3. We assumed $\inf \sigma_e(\Delta_{g_0}) > 0$. This implied $\inf \sigma(\Delta_{g_t}) \geq c > 0, 0 \leq t \leq 1$, which was of essential meaning for all of our t independent a priori estimates. The assumption

$\inf \sigma_e(\Delta_{g_0}) > 0$ would be redundant if we would know that $u_t(x) \geq a$ for all t and $x \in M$. We even proved this fact but in the proof we essentially used $\inf \sigma_e(\Delta_{g_0}) > 0$. From $u_t \in \Omega^r, r > 3$, follows $u_t(x) \geq \inf u_t$ for all $x \in M$ but it could be that $\inf u_t$ with growing t becomes smaller and smaller. Then, if $\inf \sigma_e(\Delta_{g_0}) = 0$, the norm of $(\Delta_{g_t} + (e^{u_t}\cdot))^{-1}$ grows and grows. This would destroy the existence proof for the δ in (7.10), (7.11). If $\inf \sigma_e(\Delta_{g_0}) = 0$ then $\inf \sigma_e(\Delta_{g_t} + e^{u_t}) = 1$ but this insight would not help immediately. We could conclude that below 1 there are only isolated eigenvalues of finite multiplicity. They are > 0 for all t. But we are not able - at least until now - to prove the existence of a $c > 0$ such that $\lambda_{\min}(\Delta_{g_t} + e^{u_t}) \geq c$, $0 \leq t \leq 1$. The proof of 7.10 does not work since there we used the convergence $\Delta_{g_t} \to \Delta_{g_{t^*}}$ for $t \to t^*$. If we replace Δ by $\Delta + e^u$ then we must prove $u_t \to u_{t^*}$ for $t \to t^*$ in a certain sense. But this is more or less equivalent to theorem 7.7 and the natural proof of $u_t \to u_{t^*}$ would just use $\inf \sigma_e(\Delta_{g_0}) > 0$. Nevertheless it could be possible to drop this assumption. But then we would have to study very carefully the intimate relation between $\inf u_{t_0}$ and

$$|(\Delta_{g_t} + e^{u_t})^{-1}|_{r-2,r}, t \in]t_0 - \varepsilon, t_0 + \varepsilon[\cap [0,1].$$

4. Now there arises the natural question, do there exist metrics g_0 with $K(g_0) = -1, r_{inj}(g_0) > 0$ and $\inf \sigma_e(\Delta_{g_0}) > 0$? The answer is yes. Consider Y-pieces Y_k where the lengths of the boundary geodesics grow exponentially with k, roughly spoken, and build an infinite ladder out of them. More precisely we take for $k = 1, 3, 5 \ldots$ Y_k to be the Y-piece with boundary geodesics of length $3^k L$ ($L > 0$ is a fixed constant) and Y_{k+1} the Y-piece with boundary geodesics of lengths $3^k L$, $3^k L$, $3^{k+1} L$. The two are pasted together. Built up all ladder ends by each metrically dilated Y-pieces. Then, using Cheegers constant, one can show that in this case $\inf \sigma_e(\Delta_{g_0}) > 0$ in addition to $K \equiv -1$ and $r_{inj}(g_0) > 0$. We shortly explain this. If K is any smooth, compact submanifold of $M^2, \dim K = 2$, we set

$$h^k(\varepsilon) = \inf \frac{vol(\partial N)}{vol(N)},$$

where $N \subset M \setminus K$ is a neighbourhood of the isolated end of $\varepsilon, \partial N$ dividing ε into a compact and noncompact part (which is an element of ε). Denote $h^{ess}(\varepsilon) = \sup_K h^K$. Then

$$\frac{1}{4}(h^{ess})^2 \leq \inf \sigma_e(\Delta_{g_0}(\varepsilon)).$$

See [4] for details. If we construct g_0 as above then $h^{ess} > 0$. We refer to [7]. □.

We have shown in theorem 7.4 and corollary 3.6 that

$$comp(g_0)_{-1} \quad \text{and} \quad comp(g_0)/comp(1)$$

have the structure of Hilbert manifolds. Now we are able to state

THEOREM 7.25. *Assume $g_0 \in \mathcal{M}(I, B_\infty)$ with $K(g_0) \equiv -1$, $\inf \sigma_e(\Delta_{g_0}) > 0, r > 3$. Then $comp(g_0)_{-1} \subset \mathcal{M}^r(I, B_\infty)$ and $comp(g_0)/comp(1), comp(1) \subset \mathcal{P}^r_\infty(g_0)$, are diffeomorphic manifolds.*

Proof. Consider $\pi : comp(g_0) \to comp(g_0)/comp(1)$ and $\pi_{-1} = \pi|_{comp(g_0)_{-1}}$. The latter map is bijective according to theorem 7.7. We are done if we can show that the differential $d\pi_{-1}$ is well defined and an isomorphism at any point. Now

$$T_{[g]}comp(g_0)/comp(1) \cong T_g\, comp(g_0)/T_g(comp(1)\cdot g) =$$
$$= \{[h] | h \in \Omega^r(S^2T^*, g)\}, [h] = \{h + \lambda g | \lambda \in \Omega^r(M)\}.$$

Then, by an easy consideration, $d\pi_{-1}|_g$ is given by $h \to [h]$. $d\pi_{-1}$ is surjective at g if for any $[h]$ we find a representative $h + \lambda g \in T_g\, comp(g_0)_{-1} = \ker d(K+1) = \ker dK$, i.e. $d(K(g)+1)(h+\lambda g) = 0$. By suitable choice of λ, we can assume w.l.o.g. $tr_g h = 0$. Then we have to solve

$$2\Delta_g\lambda + \delta_g\delta_g h - \Delta_g\lambda + \lambda = 0$$
$$\Delta_g\lambda + \lambda = -\delta_g\delta_g h,$$

but $\Delta_g + 1$ is bijective, as we already know. □

If we assume for a moment that $\mathcal{D}_0^{r+1}(g_0)$ acts on $comp(g_0)$, then we can sharpen 7.25 as follows.

LEMMA 7.26. *The diffeomorphism* $\pi_{-1} : comp(g_0)_{-1} \to comp(g_0)/comp(1)$ *is* $\mathcal{D}_0^{r+1}$ *equivariant.*

Proof. If $\mathcal{D}_0^{r+1}$ acts on $comp(g_0)$ then on $comp(g_0)_{-1}$ too: $K(f^*) = f^*K(g) = K(g)\circ f$, i.e. $K(g) \equiv -1$ implies $K(f^*g) \equiv -1$. Furthermore $\pi_{-1}(f^*g) = [f^*g] = f^*\pi_{-1}(g)$. □

This allows to establish at least formally an isomorphism between

$$comp(g_0)_{-1}/\mathcal{D}_0^{r+1} \quad \text{and} \quad (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1}.$$

We discuss this in sections 9 and 10.

8. The spaces of almost complex and complex structures for $n = 2$. In this section we develop the approach, sketched in section 4 for arbitrary $n = 2m$, for $n = 2$. First we start with arbitrary $n = 2m, M^n$ oriented. Fix any metric g and $r \geq 1$. Then

$$\mathcal{A}^r = \mathcal{A}^r(g) = \sum_{i\in I} comp(J_i)$$

is well defined. Here

$$comp(J) = \{J' \in A^r |\, |J - J'|_{g,r} < \infty\} \tag{8.1}$$

is a Hilbert manifold. The Hilbert manifold structure can be seen as follows. There is a real representation $GL(m,\mathbb{C}) \to GL^+(2m,\mathbb{R})$ given by

$$(A + iB) \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

which gives the coset space $GL^+(2m,\mathbb{R})/GL(m,\mathbb{C})$. $GL(m,\mathbb{C})$ is just the isotropy group of the canonical almost complex structure $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ on $\mathbb{R}^{2m}$. Let L be the $GL^+(2m,\mathbb{R})$ principal bundle of frames lying in the fixed orientation. Then the space $\mathcal{A}$ of all almost complex structures is given by

$$\mathcal{A} = C^\infty(L \times_{GL^+(2m,\mathbb{R})} GL^+(2m,\mathbb{R})/GL(m,\mathbb{C})) \subset C^\infty(T_1^1(M)).$$

On open manifolds with infinite volume it does not make sense to speak of square integrable (together with derivatives) sections in $\mathcal{A}$, since such sections do not exist because $\det J = 1$. $C^\infty(T_1^1(M))$ is endowed with a canonical uniform structure $\mathfrak{U}^r$ generated by the basis $\mathfrak{L} = \{V_\delta\}_{\delta>0}$,

$$V_\delta = \{(t,t') \in C^\infty(T_1^1(M)) \,|\, |t-t'|_{g,r} < \delta\}$$

which induces the uniform structure of lemma 4.1 on $\mathcal{A}$ thus giving $\mathcal{A}^r = \mathcal{A}^r(g)$. For later applications we do not consider $\mathcal{A} \subset C^\infty(T_1^1(M))$ but restrict ourselves to ${}^b_\infty\mathcal{A}(g) = \{J \in \mathcal{A} \,|\, |(\nabla^g)^i J|_g \le C_i \text{ for all } i\}$. Then ${}^b_\infty\mathcal{A}(g) \subset {}^b_\infty\Omega(T_1^1, g) = \bigcap_m {}^b_m\Omega(T_1^1, g)$. The elements of ${}^b_\infty\mathcal{A}(g)$ are the almost complex structures of "bounded geometry".

Now we restrict for our purposes to $n = 2$, $m = 1$. Then $J^2 = -1$ if and only if $tr\, J = 0$ and $\det J = 1$. Denote by $\overline{{}^b_\infty\Omega(T_1^1, g)}^r$ the completion of ${}^b_\infty\Omega(T_1^1, g)$ with respect to $\mathfrak{U}^r$. Let $t \in {}^b_\infty\Omega(T_1^1, g)$ and $comp(t) \subset \overline{{}^b_\infty\Omega(T_1^1, g)}^r$ its component in $\overline{{}^b_\infty\Omega(T_1^1, g)}^r$. Then $comp(t) = t + \Omega^r(T_1^1, g)$ is an affine space with $\Omega^r(T_1^1, g)$ as vector space. If $tr\, t \notin \Omega^r(M, g)$ then $comp(A)$ does not contain a tensor field s with $tr\, s \equiv 0$. Such a component does not contain any almost complex structure. If $tr\, t \in \Omega^r(M, g)$ then $tr(t + t') = 0$ if and only if $tr\, t' = -tr\, t$ and for $tr : comp(t) \to \Omega^r(M, g)$, $tr^{-1}(0) \cong -tr(t) g^i_j + (\Omega^r(T_1^1, g) \cap \{tr = 0\}) \cong \Omega^r(T_1^1, g) \cap \{tr = 0\}$ which is a closed linear subspace $\mathcal{N}$ of $\Omega^r(T_1^1, g)$ with tangent space $\Omega^r(T_1^1, g) \cap \{tr = 0\}$. Similarly, if $1 \notin \det(comp(t))$ then $comp(t)$ does not contain any almost complex structure. In the other case $\mathcal{M} = \det^{-1}(1)$ is a submanifold of $comp(t)$ with $T_J\mathcal{M} = \{H \in \Omega^r(T_1^1, g) | tr(JH) = 0\}$. Hence if $tr\, t \in \Omega^r(M, g)$ and $1 \in \det(comp(t))$, then $comp(t)$ contains a component $comp(J) = \mathcal{N} \cap \mathcal{M} \subset comp(t)$, $\mathcal{N}$ and $\mathcal{M}$ intersect transversally. Moreover, $tr\, H = 0$ and $tr\, JH = 0$ if and only if $JH + HJ = 0$. The topology of $comp(J)$ is that induced from $comp(t)$, i.e. we have (8.1).

Since we consider in the sequel only $\overline{{}^b_\infty\mathcal{A}(g)}^r$ we denote this for the sake of brevity once again with $\mathcal{A}^r(g)$ but always keeping in mind that we completed a space of bounded almost complex structures. Then

$$\mathcal{A}^r(g) = \sum_{i \in I} comp\,(J_i).$$

Forming $\bigcap_r \mathcal{A}^r(g)$, we obtain back all ∞ -bounded smooth almost complex structures. It is an absolutely standard fact that a smooth almost complex structure J is integrable, i.e. induced from a complex structure $c = \{(U_i, \varphi_i)\}_i$ if and only if the Nijenhius tensor $N(J)$ equals to zero, $N(J) = 0$,

$$N(J)(X,Y) = 2\{[JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]\}.$$

Denote for general $n = 2m$ by $\mathcal{C}^r$ all elements $J \in \mathcal{A}^r$ such that $N(J) = 0$. As well known, for $n = 2m = 2$, $N(J) = 0$ for all J.

9. The action of $\mathcal{D}_0^{r+1}$. We consider (M^n, g_0), $g_0 \in \mathcal{M}(I, B_k)$, $k \ge r+1 > \frac{n}{2}+2$, $comp(g_0) \subset \mathcal{M}^r(I, B_k)$. Then $\mathcal{D}_0^{r+1}(g_0) = \mathcal{D}_0^{\infty, r+1}(g_0) = \mathcal{D}_0^{r+1}(comp(g_0))$ is well defined. We want to show that $\mathcal{D}_0^{r+1}$ acts on $comp(g_0)$, i.e. if $g \in comp(g_0)$, $f \in \mathcal{D}_0^{r+1}$, then $f^*g \in comp(g_0)$. If $f \in \mathcal{D}_0^{r+1}$, then there exist vector fields $X_1, \ldots, X_n, X_i \in \Omega^{r+1}(TM, g_0)$ such that

$$f = \exp X_u \circ \ldots \circ \exp X_1.$$

More carefully, $X_2 \in \Omega^{r+1}((\exp X_1)^*TM, (\exp X_1)^*\nabla^{g_0})$ and so on, but if $f_1 \sim f_2$, then $\Omega^i(f_1^*T, f_1^*\nabla) \cong \Omega^i(f_2^*T, f_2^*\nabla)$ as equivalent Hilbert spaces, which will be discussed below. We start with a simple special case, $f = \exp X, X \in \Omega^{r+1}(TM, g_0)$. According to (2.3), there exists a sequence $X_\nu \in C_0^\infty(TM), X_\nu \underset{|\ |_{g_0,r}}{\longrightarrow} X$. This implies $\exp X_\nu \to \exp X = f$ in our topology of $\mathcal{D}_0^{r+1}$. Moreover $\exp X_\nu \in C^{\infty,\infty}(M,M) \cap \mathcal{D}_0^{r+1}$. Hence $(\exp X_\nu)^*g'$ satisfies (I) and (B_k) for any $g' \in comp(g_0) \cap \mathcal{M}(I, B_k)$.

We want to estimate $(\exp X_\nu)^*g' - g'$ which needs some explanations.

If $E \to M$ is a vector bundle, $f = (f_E, f_M)$ a bundle map, $c : M \to E$ a section, then it is for $f_M \neq id$ impossible to compare c and f^*c since they live in different bundles, c is a section of $E \to M, f^*c$ a section of $f^*E \to M$. If we must or want to compare them we must use a canonical equivalence between E and f^*E - if such an equivalence exists. Consider g' as a section of S^2T^*, f^*g' as a section of $f^*S^2T^*$. If $f = \exp X, X \in \Omega_{r+1}(TM, g_0) \cap^{b,k} \Omega(TM, g_0)$, then we have a canonical bundle equivalence, the parallel displacement of the fibre over $\exp X$ along $\exp sX$ to $\exp 0$. If g_0 has bounded geometry up to order k then this equivalence is also bounded up to order k. Having this construction in mind, it makes sense to consider for a section $c : M \to T_v^u$

$$f^*c - c = (f^* - id)c$$

or the pointwise operator norm

$$|f^*c - c|_x.$$

Our considerations generalize to the case where we replace id by some f and $\exp X$ is now defined for $X \in \Omega^r(f^*TM)$. We proved in [14], p. 284, (4.95) and p. 292, (5.16) the following key

PROPOSITION 9.1. *Assume $(M^n, g), (N^{n'}, h)$ with (I) and $(B_k), k \geq r+1 > \frac{n}{2} + 2, f \in \Omega^{2,r+1}(M,N)$, $f' = \exp Y, Y \in \Omega^{r+1}(f^*TN)$. Then there exist polynomials $R_\mu(|Y|, |\nabla Y|, \ldots, |\nabla^{n+1} Y|)$ such that*

$$|\nabla^\mu(f^* - f'^*)|_x \leq R_\mu, \mu \leq r. \tag{9.1}$$

Moreover, the R_μ are square integrable, $\int |R_\mu|^2 \leq R'_\mu(|Y|_{g_0,r+1})$, where R'_μ is a polynomial without constant term. In particular

$$|f^* - f'^*|_{g_0,r} < \infty \tag{9.2}$$

and $|f^* - f'^*|_{g_0,r} \to 0$ if

$$|Y|_{g_0,r+1} \to 0. \tag{9.3}$$

□

COROLLARY 9.2 *Under the assumptions of 9.1,*

$$\Omega^{r+1}(f^*TN) \cong \Omega^{r+1}(f'^*TN) \tag{9.4}$$

as equivalent Hilbert spaces. □

After this preparations we are ready to state

THEOREM 9.3. *Assume* $g_0 \in \mathcal{M}(I, B_k), k \geq r+1 > \frac{n}{2}+2$. *Then* $\mathcal{D}_0^{r+1}(g_0)$ *acts on* $comp(g_0) \subset \mathcal{M}^r(I, B_k)$.

Proof. We have to show, $g \in comp(g_0), f \in \mathcal{D}_0^{r+1}$ imply $f^*g \in comp(g_0)$. The other properties of an action are trivially satisfied. We start with the simplest case $f = \exp X, X \in \Omega^{r+1}(TM, g_0)$. We know from $g \in comp(g_0)$ that there exists a sequence $(g_\nu)_\nu, g_\nu \in \mathcal{M}(I, B_k) \cap comp(g_0)$, $g_\nu \underset{|\ |_{g_0,r}}{\longrightarrow} g$. In particular

$$|g_\nu - g_0|_{g_0,r} \leq |g_\nu - g|_{g_0,r} + |g - g_0|_{g_0,r} \leq C \tag{9.5}$$

for all ν. Moreover, according to (2.3), there exists a sequence $(X_\mu)_\mu, X_\mu \in C_0^\infty(TM)$, $X_\mu \underset{|\ |_{g_0,r+1}}{\longrightarrow} X$. If we define $f_\mu := \exp X_\mu$, then $f_\mu \to f$ in $\mathcal{D}_0^{r+1}$. Consider the diagonal sequence $f_\nu^* g_\nu$. Clearly, $f_\nu^* g_\nu \in \mathcal{M}(I, B_k)$. $f_\nu^* g_\nu \in comp(g_0)$ since

$$\begin{aligned}|f_\nu^* g_\nu - g_\nu|_{g_0,r} = |(f_\nu^* - id)g_\nu|_{g_0,r} \leq |(f_\nu^* - id)(g_0 + g_\nu - g_0)|_{g_0,r} \leq \\ \leq |(f_\nu^* - id)g_0|_{g_0,r} + |(f_\nu^* - id)(g_\nu - g_0)|_{g_0,r} < \infty.\end{aligned}$$

The latter follows from

$$|f_\nu^* - id|_{g_0,r} \leq R_r'(|X_\nu|_{r+1}),$$

(2.20) for $|\alpha| = 0$ and $\nabla^{g_0} g_0 = 0$, (9.5) and the module structure theorem. We would be done if we could show $(\exp X_\nu)^* g_\nu \to (\exp X)^* g$, i.e. $|f_\nu^* g_\nu - f^* g|_{g_0,r} \underset{\nu \to \infty}{\longrightarrow} 0$. But

$$\begin{aligned}|f_\nu^* g_\nu - f^* g|_{g_0,r} \leq |(f_\nu^* - f^*)g_\nu|_{g_0,r} + |f^*(g_\nu - g)|_{g_0,r} \leq \\ \leq |(f_\nu^* - f^*)g_0|_{g_0,r} + |(f_\nu^* - f^*)(g_\nu - g_0)|_{g_0,r} + \\ + |(f^* - id)(g_\nu - g)|_{g_0,r} + |g_\nu - g|_{g_0,r}.\end{aligned} \tag{9.6}$$

All terms on the right hand side of (9.6) converge to zero for $\nu \to \infty$. Now we consider the general case $f \in \mathcal{D}_0^{r+1}$, $f = \exp X_u \circ \ldots \circ \exp X_1$ and write

$$\begin{aligned}f^* - id = (\exp X_u \circ \ldots \circ \exp X_1)^* - (\exp X_{u-1} \circ \ldots \circ \exp X_1)^* + \\ + (\exp X_{u-1} \circ \ldots \circ \exp X_1)^* - (\exp X_{u-2} \circ \ldots \circ \exp X_1)^* + \ldots \\ + (\exp X_2 \exp X_1)^* - (\exp X_1)^* + (\exp X_1)^* - id.\end{aligned} \tag{9.7}$$

We approximate as above $X_{i\nu} \underset{|\ |_{g_0,r}}{\longrightarrow} X_\nu, X_{i\nu} \in C_0^\infty(TM)$. Then $f_\nu = \exp X_{u\nu} \circ \ldots \circ \exp X_{1\nu} \to \exp X_u \circ \ldots \circ \exp X_1 = f$. Applying the triangle inequality to (9.7) and the general version (9.1) and its integration we conclude quite similar as in the case $f = \exp X$. □

As we have already seen, the action of $\mathcal{D}_0^{r+1}(g_0)$ on $comp(g_0)$ induces an action of $\mathcal{D}_0^{r+1}$ on $comp(g_0)_{-1}$. Now we state a very nice property of this action.

THEOREM 9.4. *The action of* $\mathcal{D}_0^{r+1}$ *on* $comp(g_0)_{-1}$ *is free.*

Proof. Assume $f \in \mathcal{D}_0^{r+1}, f^*g = g$ for some $g \in comp(g_0)_{-1}$. We must show $f = id_{M^2}$. $f \in \mathcal{D}_0^{r+1}$ implies the existence of a homotopy $h_t, 0 \leq t \leq 1, h_1 = f, h_0 = id, h_t \in \mathcal{D}_0^{r+1}$. Let $\pi : (\tilde{M}^2, \tilde{g}) \to (M^2, g)$ be the universal metric covering. Then there are liftings $\tilde{h}_0 = id, \tilde{h}_t$ of h_t and $\tilde{h}_1 = \tilde{f}$ covers f. $\tilde{f}$ commutes with the deck-transformations and hence $dist(\tilde{x}, \tilde{f}(\tilde{x}))$ depends only on $x = \pi(\tilde{x})$.

LEMMA 9.5. *Assume* (M^n, g) *with nonpositive sectional curvature and with negative definite Ricci tensor,* f *as above. If* $dist(\tilde{x}, \tilde{f}(\tilde{x}))$ *obtains an absolute maximum at* $x_0 \in M$ *then* $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$*, i.e.* $\tilde{f} = id, f = id$.

See [25], p. 57-59 for a proof. □

But in our case $f = \exp X_u \circ \ldots \circ \exp X_1$, $h_t = \exp tX_u \circ \ldots \exp tX_1, X_1 \in \Omega^{r+1}(M, g_0)$, $|X_i|_{g,x} \leq r_{inj}(M^2, g), r+1 > 4$, for every $\varepsilon > 0$ there exist a compact set K such that ${}^{b,2}|X_i| < \varepsilon$ outside of K. Hence $dist(\tilde{x}, f(\tilde{x}))$ attains a maximum at some $x_0 \in M$. If $dist(\tilde{x}_0, \tilde{f}(x_0)) = 0$, we are done. In the other case we conclude once again from 9.5 $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$, i.e. in any case $\tilde{f} = id, f = id$. In our case g must not be smooth, but it is C^3 and into all calculations and considerations of [25], p. 57-59, enter only second derivatives of g. □

COROLLARY 9.6 $\mathcal{D}_0^{r+1}$ *acts freely on* $comp(g_0)/comp(1)$.

This follows immediately from 7.25 and 9.4. □

10. The connection between hyperbolic metrics and almost complex structures. Start with a metric $g_0 \in \mathcal{M}(I,, B_\infty), K(g_0) \equiv -1$, as in the sections above. Define an almost complex structure $J_0 = J(g_0)$ as follows. Write the volume form of g_0 in local coordinates as

$$\mu(g_0)_{kj} dx^k \wedge dx^j.$$

Then

$$J_{0j}^i = J(g_0)_j^i := -g_0^{ik} \mu(g_0)_{kj},$$

or in a more invariant form,

$$J_0 = J(g_0) = -g_0^{-1} \mu(g_0)$$

or

$$g_0(X, J(g_0)Y) = -\mu(g_0)(X, Y).$$

An easy calculation shows

$$J_{0i}^k J_{0k}^j = -\delta_i^j, \quad \text{i.e.} \quad J_0^2 = -id,$$
$$(\nabla^{g_0})^i J(g_0) = 0 \quad \text{for all} \quad i > 0$$

and $\sup_x |J(g_0)|_{g_0,x} \leq C$, i.e. $J(g_0) \in {}^b_\infty\mathcal{A}(g_0)$. Consider now $comp(J_0) \subset \mathcal{A}^r(g_0)$ and define for $g \in comp(g_0)$

$$\phi(g) := J(g) := g^{-1}\mu(g),$$

i.e.

$$J(g)_j^i := -g^{ik}\mu(g)_{kj}.$$

PROPOSITION 10.1. ϕ *has the following properties.*

1. ϕ *maps* $comp(g_0) \subset \mathcal{M}^r(I, B_\infty)$ *into* $comp(J_0) \subset \mathcal{A}^r(g_0)$.
2. g *is Hermitian with respect to* $J(g)$*, i.e.* $g(J(g)X, J(g)Y) = g(X, Y)$.
3. $\phi(e^u \cdot g) = \phi(g)$
4. $\phi(g_1) = \phi(g_2)$ *implies* $g_1 = e^u \cdot g_2, e^u \in comp(1)$.

5. ϕ maps comp(g_0) onto comp(J_0).
6. ϕ : comp$(g_0) \to$ comp(J_0) is a submersion with $\ker D\phi = \Omega^{r,c}(S^2T^*, g) = \{h \in \Omega^r(S^2T^*, g) | h(x) = p(x) \cdot g(x), p \in \Omega^r\}$.

Proof. 1. There exists a sequence (g_ν) in $comp(g_0) \cap \mathcal{M}(I, B_\infty)$, $g_\nu \underset{|\ |_{g_0,r}}{\longrightarrow} g$. This implies $J(g_\nu) = g_\nu^{-1}\mu(g_\nu) \underset{|\ |_{g_0,r}}{\longrightarrow} g^{-1}\mu(g) = J(g)$, i.e. if $g \in comp(g_0)$ then $J(g) \in comp(J(g_0))$.
2. This has been proved in [29].
3. $\phi(e^u \cdot g) = (e^u g)^{-1}\mu(e^u \cdot g) = e^{-u}g^{-1}(e^u)^{2/2}\mu(g) = g^{-1}\mu(g) = \phi(g)$
4. Assume $\phi(g_1) = \phi(g_2)$, $g_1^{-1}\mu(g_1) = g_2^{-1}\mu(g_2)$. Moreover $\mu(g_2) = e^u \cdot \mu(g_1)$. Hence $e^u \cdot g_2^{-1} = g_1^{-1}$, $g_2 = e^u \cdot g_1$. By assumption $|g_2 - g_1|_{g_1,r} < \infty$, i.e. $|e^u g_1 - g_1|_{g_1,r} = |(e^u - 1)g_1|_{g_1,r} < \infty$ which is equivalent to $|e^u - 1|_{g_1,r} < \infty, |e^u - 1|_{g_0,r} < \infty$. The other condition for $e^u \in comp(1)$ can be similarly easy proven.
5. Let $J \in comp(J_0)$. We have to show that there exists $g \in comp(g_0)$ such that $\phi(g) = J$. There exists a sequence $J_\nu \in comp(J_0), J_\nu \in {}^b_\infty\mathcal{A}(g_0)$, $J_\nu \underset{|\ |_{g_0,r}}{\longrightarrow} J$. Define g_ν by

$$g_\nu(X,Y) := \frac{1}{2}(g_0(X,Y) + g_0(J_\nu X, J_\nu Y)). \tag{10.1}$$

Then g_ν and g_0 are quasi isometric. $g_\nu \in \mathcal{M}(I, B_\infty)$ follows from $J_\nu \in {}^b_\infty\mathcal{A}(g_0)$. Moreover,

$$\begin{aligned} g_\nu - g_0 &= \frac{1}{2}(g_0(J_\nu\cdot, J_\nu\cdot) - g_0(\cdot,\cdot)) = \\ &= \frac{1}{2}(g_0(J_\nu\cdot, J_\nu\cdot) - g_0(J_0\cdot, J_0\cdot)) = \\ &= \frac{1}{2}(g_0((J_\nu - J_0)\cdot, (J_\nu - J_0)\cdot) + \\ &\quad + 2\, g_0(J_0\cdot, (J_\nu - J_0)\cdot)). \end{aligned} \tag{10.2}$$

Now $(\nabla^{g_0})^i g_0 = 0$, $|J_\nu - J_0|_{g_0,r} < \infty$ imply $|g_\nu - g_0|_{g_0,r} < \infty$, i.e. $g_\nu \in comp(g_0)$. We additionally infer from (10.2) that $(g_\nu)_\nu$ is a Cauchy sequence, $g_\nu \to g \in comp(g_0)$. Forming the limit $\nu \to \infty$ in (10.1), we conclude

$$g(X,Y) = \frac{1}{2}(g_0(X,Y) + g_0(J_\nu X, J_\nu Y)). \tag{10.3}$$

The fact that (10.3) implies $\phi(g) = J$ has been proven in [29].
6. Let $h \in T_g\, comp(g_0)$ with local components h_{ij}. It has been shown in [29], p. 23, that

$$(D\phi(g)(h))^i_j = -[(H - \frac{1}{2}(tr\, H)I)J]^i_j, H = (h^i_j). \tag{10.4}$$

We conclude from the invertibility of J and (10.4)

$$\ker D\phi(g) = \Omega^{r,c}(S^2T^*, g)$$

which is a closed subspace.
For $J \in comp(J_0)$

$$HJ = -JH \quad \text{if and only if} \quad tr\, H = 0 \quad \text{and} \quad H \quad \text{is } g\text{-symmetric.}$$

Hence $(H - \frac{1}{2}(tr\, H) \cdot I)J$ runs through all of $T_J\, comp(J_0) = \{K | KJ + JK = 0\}$ if H runs through all of $\{H | tr\, H = 0\}$, i.e. $D\phi$ is surjective, ϕ an submersion. □

According to 10.1, 3. and 4., ϕ induces a map $\phi : comp(g_0)/comp(1)$, and we just proved

THEOREM 10.2. *The induced map*

$$\phi : comp(g_0)/comp(1) \to comp(J_0),$$
$$[g] \to -g^{-1}\mu(g),$$

is an isomorphism of Hilbert manifolds. □

THEOREM 10.3. $\mathcal{D}_0^{r+1}$ *acts on* $comp(J_0)$ *from the right as follows:*

$$J \cdot f := f^* J := f_*^{-1} J f_*.$$

Proof. It is absolutely trivial that $(J \cdot f)^2 = -id$, $J \cdot (f_1 \cdot f_2) = (J \cdot f_1) \cdot f_2$. The nontrivial fact we must show is that $f^* J \in comp(J_0)$. We indicate how to do this but omit the details. There exists a sequence $J_\nu \in comp(J_0)$, $J_\nu \underset{|\ |_{g_0,r}}{\longrightarrow} J$, $J_\nu \in {}^b_\infty\mathcal{A}(g_0)$. First we consider the simpler case for $f, f = \exp X$, $X \in \Omega^{r+1}(TM, g_0)$. Then $X = \lim X_\nu, X_\nu \in C_0^\infty(TM)$. Set $f_\nu = \exp X_\nu$. $f_\nu^* J_\nu \in {}^b_\infty\mathcal{A}(g_0)$ and $f_\nu^* J_\nu \in comp(J_0)$ since $|f_\nu^* J_\nu - J_\nu|_{g_0,r} < \infty$. It remains to show

$$f_\nu^* J_\nu \underset{|\ |_{g_0,r}}{\longrightarrow} f^* J = J \cdot f.$$

But

$$\begin{aligned} f_\nu^* J_\nu - f^* J &= \\ &= f_{\nu *}^{-1}(J_\nu - J) f_{\nu *} + f_{\nu *}^{-1} J (f_{\nu *} - f_*) + (f_{\nu *}^{-1} - f_*^{-1}) J f_* \end{aligned} \qquad (10.5)$$

We get from [14] estimates that $|\nabla^i f_{\nu *}^{-1}|_{g_0,x}$, $|\nabla^i f_{\nu *}|_{g_0,x}$, $|\nabla^i f_*|_{g_0,x}$ are bounded by integrable polynomials, and $|id|$ for $i \le r$ ($f_* = f_* - id + id$). Thereafter we use $(\nabla^{g_0})^i J = (\nabla^{g_0})^i (J - J_0)$, $|J_\nu - J|_{g_0,r} \to 0$,

$$|f_{\nu *} - f_*|_{g_0,r} \to 0, |f_{\nu *}^{-1} - f_*^{-1}|_{g_0,r} \to 0 \qquad (10.6)$$

and the module structure theorem thus obtaining $|f_\nu^* J_\nu - f^* J|_{g_0,r} \to 0$. If $f = \exp X_u \circ \ldots \circ \exp X_1$ then we apply the decomposition (9.7) and proceed in the same manner. (10.6) is a highly nontrivial result in [14] related to the topology = uniform structure of $\mathcal{D}_0^{r+1}$. □

LEMMA 10.4. *The diffeomorphism*

$$\phi : comp(g_0)/comp(1) \to comp(J_0)$$

is $\mathcal{D}_0^{r+1}$ *-equivariant.*

Proof.

$$\phi(f^*[g]) = \phi[f^*g] = (f^*g)^{-1}\mu(f^*g) =$$
$$= (f^*g)^{-1}(f^*\mu(g)) = f^*(g^{-1}\mu(g)) = f^*\phi([g]).$$

□

This yields

THEOREM 10.5. *Suppose* $g_0 \in \mathcal{M}(I, B_\infty), K(g_0) \equiv -1$, $inf \sigma_e(\Delta_{g_0}) > 0, r > 3$. *Then for* $comp(g_0) \subset \mathcal{M}^r(I, B_\infty)$, $comp(1) \subset \mathcal{P}^r_\infty(g_0)$ *and* $comp(J_0) \subset \mathcal{A}^r(g_0)$

$$comp(g_0)_{-1}/\mathcal{D}_0^{r+1}, (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1}, comp(J_0)/\mathcal{D}_0^{r+1}$$

are isomorphic topological spaces. □

This justifies the following preliminary

Definition. Each of the spaces

$$comp(g_0)_{-1}/\mathcal{D}_0^{r+1}, (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1}, comp(J_0)/\mathcal{D}_0^{r+1}$$

is called the Teichmüller space

$$\mathcal{T}^r(comp(g_0))$$

of $comp(g_0)$.

The main task of Teichmüller theory consists of describing the topology and geometry of the Teichmüller space.

REMARKS. 1. If M^2 is closed then $\mathcal{M}^r(I, B_\infty)$, $\mathcal{T}^r, \mathcal{A}^r$ consist of one component and

$$\mathcal{T}^r(M^2) = \mathcal{M}^r_{-1}/\mathcal{D}_0^{r+1} \cong (\mathcal{M}^r/\mathcal{P}^r)/\mathcal{D}_0^{r+1} \cong \mathcal{A}^r/\mathcal{D}_0^{r+1}.$$

In the open case $\mathcal{M}^r(I, B_\infty)$ consists of uncountably many components. To each component $comp(g_0)$ we can attach $comp(1) \subset \mathcal{P}^r_\infty(g_0)$ and $comp(J(g_0)) \subset \mathcal{A}^r(g_0)$. Each component has its own Teichmüller space and theory.

$$\mathcal{T}^r(comp(g_0)) = (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1} \cong comp(J_0)/\mathcal{D}_0^{r+1}$$

is defined for any component. But in the compact case a nice manifold structure and explicit charts can be established easily and transparently by means of $\mathcal{M}_{-1}/\mathcal{D}_0^{r+1}$. Having this in mind, we considered $comp(g_0)_{-1}$. But only such components with $comp(g_0)_{-1} \neq \phi$ are interesting. Therefore we started with a metric g_0 with $K(g_0) \equiv -1$. Then $comp(g_0)_{-1} \subset comp(g_0)$ is a Hilbert submanifold as expressed by 7.4. The isomorphism of $comp(g_0)_{-1}/\mathcal{D}_0^{r+1}$ to $comp(J_0)/\mathcal{D}_0^{r+1}$, i.e. to a moduli space of complex structures could be established only under the additional assumption $inf \sigma_e(\Delta_{g_0}) > 0$. This is in a certain sense natural, at least not strange.

$(comp(g_0)/comp(1))/\mathcal{D}_0^{r+1}$ is defined without any hint to partial differential equations. $comp(g_0)_{-1}/\mathcal{D}_0^{r+1} \cong (comp(g_0)/comp(1))/\mathcal{D}_0^{r+1}$ refers to the moduli space of a family of partial differential equations, $\Delta_g u + K(g) + e^u = 0, g \in comp(g_0)$. This family must be "good", which means in our case $\inf \sigma_e(\Delta_{g_0}) > 0$.

2. It is very easy to give examples of components $comp(g) \subset \mathcal{M}^r(I, B_\infty)$ such that $comp(g)_{-1} = \phi$. Consider the infinite ladder $L^2 = \mathop{\sharp}\limits_{-\infty}^{+\infty} T^2, T^2$ the 2-torus,

straightly embedded into $\mathbb{R}^3$ with periodic curvature $K(g)$. If there would be a metric $g' \in comp(g)$ with $K(g') \equiv -1$ then $\int |K(g) - K(g'|^2 = \infty$ in contradiction to $\int |K(g) - K(g')|^2 < \infty$ for g, g' in the same component. Nevertheless (L^2, g) has a canonical conformal = complex structure and is, according to the general uniformization theorem, pointwise conformally equivalent to a metric g_0 with $K(g_0) \equiv 1$. But $g_0 \notin comp(g)$, i.e. the conformal factor is not contained in $comp(1)$. This supports our procedure: not counting $g's \in \mathcal{M}^r(I, B_\infty)$ and associated conformal structures but counting the components $comp(g_0)$ with $comp(g_0)_{-1} \neq \phi$ and counting the metrics with $K \equiv -1$ inside such components. Moreover, in this way we get manifold structures for $comp(g_0)_{-1}, comp(g_0)/comp(1)$, $comp(J_0)$, and, if things are going well, even for the Teichmüller spaces. □

11. Topology and geometry of the Teichmüller space. An outlook. The further procedure concerning topology and geometry of Teichmüller spaces is indicated by the compact case and the usual approach to moduli spaces in geometry and global analysis. The steps are as follows.

1. To show that the orbits under the action of $\mathcal{D}_0^{r+1}$ are submanifolds.
2. To prove the existence of a slice.
3. The slice produces charts and a manifold structure.
4. The dimension of this manifold coincides with the dimension of the tangent space to the slice and is given in the compact case by the index theorem. In the open case it will be infinite.
5. The geometry of Teichmüller spaces with respect to the Weil-Petersson metric can be similarly calculated as in the compact case. In the compact case, the solution of steps 1-3 is more or less standard, it uses well known theorems of Ebin, Palais and others and has been successfully been performed by Tromba in [29]. In the open case, 1-3 are totally unclear since the applied theorems of Ebin, Palais are not available. Hence we have to reestablish some versions of them for our noncompact case.

1. has been already solved by us, the solution is highly nontrivial.
2. The existence of a slice has not yet been completely established. The standard proofs use the properness of the action of $\mathcal{D}^{r+1}$ on $\mathcal{M}^r$ in the compact case. This is definitely wrong for open manifolds. But our situation in Teichmüller theory is much better. We have to consider only the action of $\mathcal{D}_0^{r+1}$ on $comp(g_0)_{-1}$. The nonexistence of a slice would imply the following.

1. The existence of $g_\nu \to g$, $g, g_\nu \in comp(g_0)_{-1}$
2. The existence of $f_\nu \notin U_\varepsilon(id) \subset \mathcal{D}_0^{r+1}$, $f_\nu \in \mathcal{D}_0^{r+1}$, such that $f_\nu^* g_\nu \to g$.

We would be done if we could derive from 1. and 2. a contradiction. Helpful for such a contradiction would be the following

THEOREM 11.1. *Assume $g \in comp(g_0)_{-1}$. Then $\mathcal{D}_0^{r+1}$ does not contain any isometry of g different from id.* □

1. and 2. would imply (by a small effort) that $f_\nu^* g - g$ becomes arbitrarily small. then, according to 11.1, we do not have any isometry $\neq id$ in $\mathcal{D}_0^{r+1}$, on the other hand we would have outside of $U_\varepsilon(id)$ in $\mathcal{D}_0^{r+1}$ arbitrary good almost isometries, $|f_\nu^* g - g|_r \to 0$ for $\nu \to \infty$. If we could sharpen this "almost" contradiction to a contradiction, we would be done. This problem is under investigation. The assumption $\inf \sigma_e(\Delta_{g_0}) > 0$ will play once again an essential role.

Theorem 11.1 is already completely proved.

In classical Teichmüller theory only smooth metrics and smooth diffeomorphisms have been considered and

$$\mathcal{T}(M^2) := \mathcal{M}_{-1}/\mathcal{D}_0 \quad \text{or} \quad (\mathcal{M}/\mathcal{P})/\mathcal{D}_0 \quad \text{or} \quad \mathcal{A}/\mathcal{D}_0.$$

But in the strong language of global analysis one needs good topologies in $\mathcal{M}, \mathcal{T}, \mathcal{D}_0$, $\mathcal{A}, \mathcal{M}_{-1}$, good properties of the actions and the implicit function theorem. $\mathcal{M}^r, \mathcal{T}^r$, $\mathcal{D}_0^{r+1}, \mathcal{A}^r, \mathcal{M}_{-1}^r$ have this properties but they contain many nonsmooth elements. For this reason one would like to apply ILH-theory. This assumes smooth Hilbert manifolds, i.e. (B_∞). But we started with $g_0 \in \mathcal{M}(I, B_\infty)$ hence 6.1 - 6.4 are applicable and we set as in section 6

$$comp^\infty(g_0) = \varprojlim_r comp^r(g_0), comp^r(g_0) = comp(g_0) \subset \mathcal{M}^r(I, B_\infty),$$
$$\mathcal{D}_0^\infty = \varprojlim_r \mathcal{D}_0^{r+1}, comp^\infty(J_0) = \varprojlim comp^r(J_0),$$
$$comp^\infty(g_0)_{-1} = \varprojlim_r comp^r(g_0)_{-1}.$$

Then the isomorphisms

$$comp^r(g_0)_{-1}/\mathcal{D}_0^{r+1} \xrightarrow{\cong} (comp^r(g_0)/comp^r(1))/\mathcal{D}_0^{r+1} \xrightarrow{\cong}$$
$$\xrightarrow{\cong} comp^r(J_0)/\mathcal{D}_0^{r+1}$$

pass into isomorphisms for $r = \infty$

$$comp^\infty(g_0)_{-1}/\mathcal{D}_0^\infty \xrightarrow{\cong} (comp^\infty(g_0)/comp^\infty(1))/\mathcal{D}_0^\infty \xrightarrow{\cong}$$
$$\xrightarrow{\cong} comp^\infty(J_0)/\mathcal{D}_0^\infty.$$

These are spaces of smooth elements with an ILH-topology. One now would like to define

$$\mathcal{T}(comp(g_0)) := \mathcal{T}^\infty(comp(g_0)) := comp^\infty(g_0)_{-1}/\mathcal{D}_0^\infty$$
$$\cong (comp^\infty(g_0)/comp^\infty(1))/\mathcal{D}_0^\infty \cong comp^\infty(J_0)/\mathcal{D}_0^\infty.$$

Hence knowledge of all $\mathcal{T}^r(comp(g_0))$ would imply knowledge of $\mathcal{T}^\infty(comp(g_0))$. We study the topology and geometry of $\mathcal{T}^r(comp(g_0))$ in the second part of this paper.

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