A DESCRIPTION OF THE DISCRETE SPECTRUM OF \((SL(2), E_{7(-25)})\)∗

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1. Introduction. Let \(E_{8,4}\) be the simply connected quaternionic \(E_8\). It is the unique simply connected simple Lie group of exceptional type \(E_8\) and real rank 4. In [GrW] Gross and Wallach constructed the minimal representation \(\pi_{min}\) of \(E_{8,4}\). It is an irreducible unitary representation with minimal Gelfand-Kirillov dimension, and its annihilator in the (complexified) universal enveloping algebra is the Joseph ideal.

In this paper we will give a description of the discrete spectrum of the restriction of \(\pi_{min}\) to the symmetric subgroup \(E_{7,3} \times SU(1,1)\). Here \(E_{7,3}\) is a connected simple Lie group of type \(E_7\) that gives rise to the hermitian tube domain of Cartan type \(E_7\). For an integer \(k \geq 2\) let \(\pi_k\) be the holomorphic discrete series representation of \(SU(1,1) = SL(2)\) of lowest weight \(k\). Let \(\pi_{-k}\) be the contragredient of \(\pi_k\). Write the unitary decomposition of the restriction of \(\pi_{min}\) as

\[
\pi_{min}|_{E_{7,3} \times SU(1,1)} = (\bigoplus_{|k| \geq 2} \theta_k \otimes \pi_k) \bigoplus \text{(continuous spectrum)}
\]

Since \(\pi_{min}\) is self-contragredient, we see \(\theta_{-k}\) is contragredient to \(\theta_k\). So it suffices to describe \(\theta_k\) for \(k \geq 2\). It turns out that

\[
\theta_k = \sigma_k \oplus \sigma'_k
\]

is the sum of two representations, where \(\sigma_k\) is an irreducible highest weight unitary representation which belongs to the discrete series when \(k \geq 10\). The representation \(\sigma'_k\) is admissible (in fact it has multiplicity free \(K\)-types) and non-tempered. If \(k \geq 4\) the \(K\)-type structure of \(\sigma'_k\) is identical to that of a derived functor module. For \(k \geq 10\), \(\sigma'_k\) is an irreducible unitary representation with non-zero cohomology at bidegree \((10,1)\) (and so \(\sigma'_{-k}\) has cohomology at bi-degree \((1,10)\)). Thus when \(k \geq 10\), \(\theta_k\) is the sum of two irreducible representations, one of them belongs to the discrete series while the other is very far from being tempered. This is the rough description of the discrete spectrum. For details see §5-6.

The determination of \(\theta_k\) depends heavily on the fact that the groups \(E_{7,3}\) and \(SU(1,1)\) (essentially) form a reductive dual pair in \(E_{8,4}\). As such they fit into the following seesaw diagram

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Here $E_{6,0}$ is the semi-simple part of the maximal compact subgroup of $E_{7,3}$, a compact simple Lie group of type $E_6$. With this diagram in mind, we first determine the decomposition of $\pi_{min}$ restricted to $E_{6,0} \times SU(2,1)$ in §3. This is not difficult since $E_{6,0}$ is compact. The decomposition takes the form

$$\pi_{min}|_{E_{6,0} \times SU(2,1)} = \bigoplus_{p,q \geq 0} \pi(p\omega_6 + q\omega_1) \otimes \sigma_{p,q}$$

where $\omega_j$ denotes the $j$-th fundamental weight for $E_{6,0}$ and $\pi(\lambda)$ is the irreducible finite dimensional representation with highest weight $\lambda$. It turns out that each representation $\sigma_{p,q}$ is irreducible and belongs to the generic discrete series of $SU(2,1)$.

Next in §4 we study the restriction of $\sigma_{p,q}$ to $SU(1,1)$. We consider $U(2,1)$ and $U(1,1)$ instead, extending $\sigma_{p,q}$ to $U(2,1)$. Then there is another seesaw diagram for dual pairs in the rank 6 symplectic group $Sp_{12}(\mathbb{R})$:

$$U(2,1) \quad U(1,1) \times U(1,1) \quad U(1) \times U(1,1) \quad U(1,1)$$

This diagram is analyzed using our earlier results in [Lib] and results of Repka [Rep] on tensor products of holomorphic and anti-holomorphic representations. This leads to the explicit determination of the $K$-type structure of $\theta_k$. Upon inspection of this structure we find that $\theta_k$ contains a highest weight module, namely $\sigma_k$. Finally, we know a priori that $\theta_k$ is quasi-simple with infinitesimal character given by Rallis and Schiffmann [RaS] (see also [Lia]). Together we have enough information to give the description of $\theta_k$ outlined above.

Wee Teck Gan [Gan] has shown that the minimal representation of $E_{8,4}$ is automorphic. It follows that the non-tempered representations $\sigma'_k$ described here are also automorphic representations.

**NOTATIONS.** We use $E_{n,r}$ to denote a connected simple Lie group of exceptional type $E_n$ and split rank $r$. This will be made more precise when the group is actually introduced. Let $\epsilon_{n,r}$ be the corresponding complexified Lie algebra. Up to isomorphisms the latter is of course independent of the second subscript $r$. But we keep it
in our notations to reminder ourselves which real Lie group the complex Lie algebra comes from. The root subgroup corresponding to a compact root is denoted SU(2) and one corresponding to a non-compact root will be SU(1, 1). Their complexified Lie algebras will be su(2) and su(1, 1) respectively. We will denote by \( V_\mu \) or \( V(\mu) \) the irreducible module with highest weight \( \mu \), of whatever Lie group or Lie algebra in question.

2. Subalgebras of \( \mathfrak{e}_{8,4} \). Consider \( S = E_{8,4} \) (simply connected). The maximal compact subgroup of \( S \) is \( SU(2) \times E_{7,0} \) which contains a Cartan subgroup \( H \). We introduce coordinates so that the complexified Lie algebra of \( H \) is \( \mathfrak{h} \cong \mathbb{C}^8 \), and that the restriction of the Cartan-Killing form is the standard inner product given by

\[
< x, y > = x_1 y_1 + \cdots + x_8 y_8
\]

Let \( e_j \) denotes evaluation on the \( j \)-th coordinate. We may assume that the roots of \( \mathfrak{h} \) in \( \mathfrak{e}_{8,4} \) are as enumerated in [Hel, Ch. X], namely

\[
\pm e_i \pm e_j \quad (1 \leq i < j \leq 8)
\]

and

\[
\frac{1}{2} (\pm e_1 \pm \cdots \pm e_8)
\]

with an even number of minus signs. The simple roots are

\[
\alpha_1 = \frac{1}{2} (e_1 + e_8 - e_2 - \cdots - e_7), \quad \alpha_2 = e_1 + e_2
\]

and

\[
\alpha_j = e_{j-1} - e_{j-2}, \quad (3 \leq j \leq 8)
\]

We assume that \( SU(2) \) is the root subgroup corresponding to the root \( e_7 + e_8 \). Consequently roots in \( \mathfrak{e}_{7,0} \) are precisely those perpendicular to \( e_7 + e_8 \). Write the Cartan decomposition of \( \mathfrak{e}_{8,4} \) as

\[
(2.1) \quad \mathfrak{e}_{8,4} = \mathfrak{su}(2) \oplus \mathfrak{e}_{7,0} \oplus \mathfrak{p}
\]

As a module for \( SU(2) \times E_{7,0} \), \( \mathfrak{p} \) has highest weight \( e_6 + e_8 \). It can also be written as

\[
(2.2) \quad \mathfrak{p} = \mathbb{C}^2 \otimes U(\lambda)
\]

Here \( U(\lambda) \) is the miniscule module of \( E_{7,0} \) of dimension 56, and \( \lambda \) is the highest weight which is also the 7-th fundamental weight for \( E_{7,0} \).

Next let \( \mathfrak{e}_{6,0} \subset \mathfrak{e}_{7,0} \) be the (simple) subalgebra generated by all the roots which are orthogonal to both \( e_6 + e_8 \) and \( e_7 + e_8 \). The centralizer of \( \mathfrak{e}_{6,0} \) in \( \mathfrak{e}_{7,0} \) is the one-dimensional torus \( \mathbb{C} \cdot h \) where

\[
h = (0, 0, 0, 0, 2, -1, 1) \in \mathfrak{h}
\]

Let \( \mathbb{C}(k) \) be the one-dimensional space on which the element \( h \) acts via the scalar \( k \). As a module for \( \mathfrak{e}_{6,0} + \mathbb{C} h \) one has

\[
(2.3) \quad U(\lambda) = \mathbb{C}(3) + \mathbb{C}(-3) + V \otimes \mathbb{C}(1) + V^* \otimes \mathbb{C}(-1)
\]
where $V$ is an irreducible representation of $\mathfrak{e}_6$ of dimension 27. More precisely we take the first six simple roots of $\mathfrak{e}_{8,4}$ as simple roots for $\mathfrak{e}_{6,0}$ and let $\omega_j$ be the $j$-th fundamental weight. Then the highest weights of $V$ and $V^*$ are $\omega_6$ and $\omega_1$ respectively.

Let $E_{6,0}$ and $T_h$ be the compact connected subgroups of $E_{8,4}$ corresponding to the Lie algebras $\mathfrak{e}_{6,0}$ and $\mathfrak{c}_h$ respectively. The centralizer of $E_{6,0}$ in $E_{8,4}$ is of type $A_2$ (see [Rub]) with maximal compact subgroup $SU(2) \times T_h$. Thus the centralizer is just $SU(2,1)$, which contains the root subgroup $SU(1,1)$ corresponding to the root $e_6 + e_8$. In this way $T_h$ is identified with the center of the maximal compact subgroup of $SU(2,1)$. We write 

$$T(SU(2,1)) = T_h = E_{7,0} \cap SU(2,1)$$ 

Note that the roots in $\mathfrak{su}(2,1)$ are 

$$\pm(e_7 + e_8), \quad \pm(e_6 + e_8), \quad \pm(e_6 - e_7)$$ 

We take $-e_7 - e_8, e_6 + e_8$ as the simple roots.

On the other hand we can realize $\mathfrak{su}(2,1)$ as follows. Let $I$ be the $3 \times 3$ diagonal matrix with $1,1,-1$ on the diagonal. Then $SU(2,1)$ can be identified with the group of all complex matrices $A$ with determinant 1, such that $A^t IA = I$. We take the space of diagonal matrices in $\mathfrak{su}(2,1)$ as a Cartan subalgebra. Let $\varepsilon_j$ denote evaluation on the $j$-th diagonal element and take $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3$ as the simple roots. Thus on the Cartan algebra of $\mathfrak{su}(2,1)$ we have

$$(2.4) \quad -e_7 - e_8 = \varepsilon_1 - \varepsilon_2, \quad e_6 + e_8 = \varepsilon_2 - \varepsilon_3$$ 

The element $h$ is then identified with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \in \mathfrak{su}(2,1)$$

Next let $\mathfrak{e}_{7,3}$ be the centralizer of $\mathfrak{su}(1,1)$ in $\mathfrak{e}_{8,4}$. Let $E_{7,3} \subset E_{8,4}$ be the corresponding connected subgroup. The group $E_{7,3}$ has maximal compact subgroup 

$$K = E_{6,0} \times T(E_{7,3}),$$

where $T(E_{7,3})$ is a one dimensional torus which is the same as the centralizer of $SU(1,1)$ in $SU(2,1)$. Thus

$$(2.5) \quad T(E_{7,3}) = \{ t(a) = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in U(1) \}$$

For an integer $\mu$ let $(\mu)_{7,3}$ denote the character of $T(E_{7,3})$ taking $t(a)$ to $a^\mu$. We fix an orientation of the circle $T(E_{7,3})$ by choosing the element

$$(2.6) \quad t_{7,3} = (0,0,0,0,0,-1,2,1) \in t(\mathfrak{e}_{7,3}) \subseteq \mathfrak{h}$$

In view of (2.4), we see that $t_{7,3}$ is identified with the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathfrak{su}(2,1)$$
The Cartan decomposition of $\mathfrak{e}_{7,3}$ is

$$\mathfrak{e}_{7,3} = (\mathfrak{e}_{6,0} \oplus \mathfrak{t}(\mathfrak{e}_{7,3})) \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where as $(\mathfrak{e}_{6,0} \oplus \mathfrak{t}(\mathfrak{e}_{7,3}))$-modules

$$\mathfrak{p}^+ = V(\omega_1) \otimes (2)_{7,3}, \quad \mathfrak{p}^- = V(\omega_6) \otimes (-2)_{7,3}$$

Thus $\mathfrak{p}^\pm$ are the $\pm 2$ eigenspaces for $\text{ad}(t_{7,3})$. The highest weights for $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are $-e_6 + e_8$ and $e_5 - e_7$ respectively.

### 3. Restriction to $E_{6,0} \times SU(2,1)$

Let $\pi_{\text{min}}$ be the minimal representation of $E_{8,4}$ constructed by Gross and Wallach [GrW]. A great deal of information about $\pi_{\text{min}}$ is available from that paper. But here all we need is its $(SU(2) \times E_{7,0})$-type structure given by

$$\pi_{\text{min}}|_{SU(2) \times E_{7,0}} = \bigoplus_{n=0}^{\infty} S^{n+8}(\mathbb{C}^2) \otimes U(n\lambda)$$

Here $S^k(\mathbb{C})$ denotes the $k$-th symmetric power of the standard action of $SU(2)$ on $\mathbb{C}^2$. The weight $\lambda$ is as in §2, and $U(n\lambda)$ is the irreducible module with highest weight $n\lambda$. The reader will realize that the following lemma is contained in Proposition 3.1 of [HPS].

**Lemma 3.1.** When restricted to $\mathfrak{e}_{6,0} \oplus \mathfrak{ch}$ the irreducible $\mathfrak{e}_{7,0}$ module $U(n\lambda)$ decomposes as

$$U(n\lambda) = \bigoplus V(p\omega_6 + q\omega_1) \otimes \mathbb{C}(p - q + 3r - 3s)$$

where the sum is over all non-negative integers $p, q, r, s$ with

$$p + q + r + s = n$$

**Proof.** See [HPS], section 3.

The spectrum decomposition of $\pi_{\text{min}}$ restricted to $E_{6,0} \times SU(2,1)$ can now be described as

**Proposition 3.2.** We have

$$\pi_{\text{min}}|_{E_6 \times SU(2,1)} = \bigoplus \pi(p\omega_6 + q\omega_1) \otimes \sigma_{p,q}$$

where $\sigma_{p,q}$ is a generic discrete series representation. More precisely, take

$$\alpha_1 = \epsilon_1 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_2$$

to be the positive roots, with simple roots $\alpha_1 = \epsilon_1 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_2$. Let $\omega'_1, \omega'_2$ be the corresponding fundamental weights. Then $\sigma_{p,q}$ has Harish-Chandra parameter

$$\lambda = (p + 4)\omega'_1 + (q + 4)\omega'_2.$$

**Proof.** From (3.1) and (3.2) we immediately conclude

$$\pi_{\text{min}}|_{E_6 \times SU(2,1)} = \bigoplus V(p\omega_6 + q\omega_1) \otimes \sigma_{p,q}$$

with

$$\sigma_{p,q}|_{\mathfrak{su}(2)+\mathfrak{ch}} = \bigoplus_{r,s \geq 0} S^{p+q+r+s+8}(\mathbb{C}^2) \otimes \mathbb{C}(p - q + 3r - 3s)$$
Let \( \sigma'_{p,q} \) be the discrete series representation of \( SU(2, 1) \) with Harish-Chandra parameter \( (p + 4)\omega'_1 + (q + 4)\omega'_2 \). Up to equivalence there is only one representation with the same \( (\mathfrak{su}(2) \times \mathfrak{ch}) \)-type structure as \( \sigma'_{p,q} \). But the \( (\mathfrak{su}(2) \times \mathfrak{ch}) \)-types of \( \sigma'_{p,q} \) are given by the Blattner formula [HeS] and those of \( \sigma_{p,q} \) are given by (3.4). One checks that \( \sigma_{p,q} \) and \( \sigma'_{p,q} \) have exactly the same \( (\mathfrak{su}(2) \times \mathfrak{ch}) \)-types. Therefore \( \sigma_{p,q} \simeq \sigma'_{p,q} \), as claimed.

4. Restriction from \( SU(2, 1) \) to \( S(U(1) \times U(1, 1)) \). We need to understand the restriction of \( \sigma_{p,q} \) to the subgroup \( SU(1, 1) \). This is more or less the same as understanding restriction from \( U(2, 1) \) to the symmetric subgroup \( U(1) \times U(1, 1) \), and will be done using suitable seesaw dual pairs in the symplectic group \( Sp_{12}(\mathbb{R}) \).

Let \( \tilde{U}(1, 1) \) be the two-fold cover of \( U(1, 1) \) determined by \( det(\cdot)^{1/2} \). Local theta correspondence gives rise to a bijection between certain discrete series of \( \tilde{U}(1, 1) \) and of \( U(2, 1) \). We shall describe the correspondence for the cases we need here. The relevant Harish-Chandra parameters of \( \tilde{U}(1, 1) \) will be of the form \( \lambda = (a, b) \), where \( a, b \) are integers, and either \( a > b > 0 \), or \( 0 > a > b \). Let \( \pi(\lambda) \) be the corresponding discrete series representation of \( \tilde{U}(1, 1) \). To each such \( \lambda \) we associate a Harish-Chandra parameter \( \Lambda = \theta(\lambda) \) by the formula

\[
\theta(\lambda) = \begin{cases} (a, 0; b) & \text{if } a > b > 0 \\ (0, b; a) & \text{if } 0 > a > b \end{cases}
\]

Let \( \tau(\Lambda) \) be the corresponding discrete series representation of \( U(2, 1) \). The following can be read off from [Lib, §6].

**Lemma 4.1.** Under the local theta correspondence we have \( \pi(\lambda) \leftrightarrow \tau(\Lambda) \), where \( \Lambda = \theta(\lambda) \) is as given above.

Henceforward we shall only consider the first case: \( a > b > 0 \). This will be sufficient for our purposes here. It is straightforward to verify

**Lemma 4.2.** We have \( \tau(a, 0; b)|_{SU(2, 1)} = \sigma_{p,q} \) if and only if

\[
a = p + q + 8, \quad b = q + 4
\]

Fix \( a, b \) as above. A representation of \( SU(1, 1) \) occurs in the restriction of \( \sigma_{p,q} \) if and only if it occurs in the restriction of \( \tau(a, 0; b) \). Let \( \pi_k \) be the holomorphic discrete series representation of \( SU(1, 1) \) with lowest weight \( k \geq 2 \). Suppose \( \pi_k \) occurs in the restriction of \( \tau(a, 0; b) \). Then some extension of \( \pi_k \) to \( U(1) \times U(1, 1) \) must also occur in the restriction of \( \tau(a, 0; b) \). We write such an extension as \( \alpha \otimes \pi(\lambda_1, \lambda_2) \). Here \( \alpha \) is an integer, identified with the character \( t \mapsto t^a \) of \( U(1) \), and \( (\lambda_1, \lambda_2) \) is a Harish-Chandra parameter. We must have

\[
\alpha + \lambda_1 + \lambda_2 = a + b \tag{4.1}
\]

\[
\lambda_1 - \lambda_2 = k - 1 \tag{4.2}
\]

We shall make use of the seesaw diagram
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\[
\begin{array}{cc}
U(2, 1) & U(1, 1) \\
\times & \times \\
U(1) & U(1, 1) \\
\times U(1, 1) & U(1, 1)
\end{array}
\]

(4.3)

There is a corresponding diagram for representations:

\[
\begin{array}{cc}
\tau(a, 0; b) & \theta(\alpha) \otimes \pi(\lambda_1', \lambda_2') \\
\otimes \pi(\lambda_1, \lambda_2) & \pi(a, b)
\end{array}
\]

(4.4)

Here

\[
\theta(\alpha) = \begin{cases} 
\pi(\alpha, 0), & \alpha \geq 0 \\
\pi(0, \alpha), & \alpha \leq 0
\end{cases}
\]

When \(\alpha = 0\) we understand \(\theta(0)\) to be the limit of holomorphic discrete series representation with infinitesimal character \((0, 0)\). In any case, the restriction of \(\theta(\alpha)\) to \(SU(1, 1)\) is holomorphic of lowest weight \(|\alpha| + 1\). The parameters \(\lambda_1, \lambda_2\) are in \(\mathbb{Z} + 1/2\) (since they parameterize discrete series of the linear group \(U(1, 1)\)). For \(\pi(\lambda_1, \lambda_2)\) to occur in the correspondence with \(U(1, 1)\), \(\lambda_1, \lambda_2\) must be both positive or both negative (see [Pau]). In the first case, \((\lambda_1', \lambda_2') = (\lambda_1, \lambda_2)\); while in the second case \((\lambda_1', \lambda_2') = (\lambda_2, \lambda_1)\). The following lemma can be easily deduced from results of Repka [Rep].

**Lemma 4.3.** The representation \(\pi(a, b)\) occurs in \(\theta(\alpha) \otimes \pi(\lambda_1', \lambda_2')\) if and only if

\[
\alpha = p + k \pmod{2}
\]

and either \(\lambda_1, \lambda_2 < 0\), or \(\lambda_1, \lambda_2 > 0\) and \(|\alpha| + k \leq p + 4\).

Now assume that the restriction of \(\alpha \otimes \pi(\lambda_1, \lambda_2)\) to \(T(E_{7,3})\) contains \((\mu)_{7,3}\) (see §2). This is just the condition

\[
\lambda_1 + \lambda_2 - 2\alpha = \mu
\]

which together with (4.1)-(4.2) determines \(\lambda_1, \lambda_2\) and \(\alpha\) completely, namely

\[
\begin{cases} 
\alpha = \frac{1}{2}(p + 2q + 12 - \mu) \\
\lambda_1 = \frac{1}{6}(2p + 4q + 24 + \mu) + \frac{1}{2}(k - 1) \\
\lambda_2 = \frac{1}{6}(2p + 4q + 24 + \mu) - \frac{1}{2}(k - 1)
\end{cases}
\]

(4.6)
The condition (4.5) now translates to
\[ \mu \equiv 3k - 2p + 2q \quad \text{(mod 6)} \]
Accordingly we write
\[ \mu = 3k - 2p + 2q + 6r \]
Then \( r \in \mathbb{Z} \) and we have
\[
\begin{align*}
\alpha &= -k + p + 4 - 2r \\
\lambda_1 &= k + q + 4 + r - \frac{1}{2} \\
\lambda_2 &= q + 4 + r + \frac{1}{2}
\end{align*}
\]
In terms of the parameters \( k, p, q, r \) we can formulate
**Proposition 4.4.** The restriction of \( \sigma_{p,q} \) to \( T(E_{7,3}) \times SU(1,1) \) contains

\[(3k - 2p + 2q + 6r)_{7,3} \otimes \pi_k\]
if and only if either

\[0 \leq r \leq p + 4 - k\]
or

\[r \leq -k - q - 4\]

5. Description of \( \theta_k \). We write the decomposition of the restriction of \( \pi_{\min} \) to \( E_{7,3} \times SU(1,1) \) as

\[ \pi_{\min}|_{E_{7,3} \times SU(1,1)} = \bigoplus_{|k| \geq 2} \theta_k \otimes \pi_k \bigoplus \text{ (continuous spectrum)} \]

Here for \( k \geq 2 \), \( \pi_{-k} \) is the anti-holomorphic discrete series representation of \( SU(1,1) \) of highest weight \(-k\). Since \( \pi_{\min} \) is self-contragredient, we see that \( \theta_{-k} \) is the contragredient of \( \theta_k \). Therefore it suffices to describe \( \theta_k \) for \( k \geq 2 \).

We have shown

**Lemma 5.1.** The \( \mathfrak{k} \)-type structure of \( \theta_k \) is given by

\[ \theta_k|_{E_6 \times T(E_{7,3})} = \bigoplus \pi(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3} \]

with

\[ \mu = 3k - 2p + 2q + 6r \]

and either

\[0 \leq r \leq p + 4 - k\]
or

\[r \leq -k - q - 4\]
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**Remark.** With respect to the action of \(\mathfrak{t}\) (and not just \(e_{6,0}\)) the highest weights of \(p^\pm\) are \(e_8 - e_6\) and \(e_5 - e_7\) respectively. If \(U(\lambda)\) denotes a representation of \(\mathfrak{t}\) with highest weight \(\lambda\) then the typical \(\mathfrak{t}\)-type described in the above lemma can also be written as

\[
U(p(e_5 - e_7) + q(e_8 - e_6)) \otimes (3k + 6r)_{7,3}
\]

It follows from the lemma that \(\theta_k\) is admissible. By Rallis and Schiffman [RaS] (see also [Lia]), we know that it is also quasi-simple. Its infinitesimal character is given as follows. Choose any positive root system. Let \(\rho\) be the half-sum of positive roots. Suppose \(\alpha\) is the simple root such that one obtains a Dynkin diagram of type \(E_6\) by taking out \(\alpha\) from the Dynkin diagram of type \(E_7\). Let \(\omega\) be the fundamental weight dual to \(\alpha\). Then the infinitesimal character of \(\theta_k\) is

\[
(5.6) \quad (k - 10)\omega + \rho
\]

If (5.4) (resp. (5.5)) is satisfied we shall say that the corresponding \(K\)-type is of type (5.4) (resp. (5.5)).

**Proposition 5.2.** (a) The representation \(\theta_k\) contains the irreducible highest weight module \(\sigma_k\) with highest weight

\[
(\frac{k}{2} + 4)(e_6 - 2e_7 - e_8)
\]

Note that this corresponds to the one-dimensional \(K\)-type \(V(0) \otimes (-3k - 24)_{7,3}\) with \(p = q = 0, r = -k - 4\). If \(k \geq 10\) then \(\sigma_k\) is an anti-holomorphic discrete series representations.

(b) The representation \(\sigma_k\) contains all \(K\)-types of type (5.5), but none of type (5.4).

**Proof.** Consider the \(K\)-type with \(p = q = 0\) and \(r = -k - 4\). Let \(v_0\) be any non-zero vector belonging to this one dimensional space. Since \(e_{6,0}\) annihilates \(v_0\), we see that the map

\[
X \mapsto X \cdot v_0 \quad (X \in p^+)
\]

is an \(e_{6,0}\)-homomorphism from \(p^+\) into \(\theta_k\). Thus if \(p^+v_0\) is non-zero it must be of type \(V(\omega_1) \otimes (-3k - 22)\). But clearly this is not any one of the \(K\)-types described in Proposition 4.4. Thus \(p^+\) must annihilate \(v_0\). This proves (a).

Since \(\sigma_k\) is a highest weight representation, any of its \(K\)-types must have highest weight of the form

\[
(\frac{k}{2} + 4)(e_6 - 2e_7 - e_8) + \sum n_j\beta_j
\]

where the \(\beta_j\)'s are roots in \(p^-\), and the \(n_j\)'s are non-negative integers. Suppose this is a \(K\)-type \(V(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3}\) of type (5.4). Then

\[
\mu = 3k - 2p + 2q + 6r = -3k - 24 - 2\sum n_j
\]

so

\[
p = 3k + q + 3r + 12 + \sum n_j \geq 3k + q + 12 + \sum n_j
\]
On the other hand by taking inner product with $\alpha_6$ we see

$$p \leq \text{coefficient of } e_5 - e_7 \leq \sum n_j$$

Thus we have a contradiction which shows $\sigma_k$ can not contain any $K$-type of type (5.4).

Now write

$$\theta_k = \sigma_k \oplus \sigma'_k$$

To finish the proof of (b) we need to show that $\sigma'_k$ does not contain any $K$-type satisfying (5.5). Suppose it does. We choose such a $K$-type satisfying

$$W = V(p\omega_6 + q\omega_1) \otimes (\mu)_{\tau,3}$$

with $\mu$ maximal, where $\mu = 3k - 2p + 2q + 6r$ with $r \leq -k - q - 4$. Consider the map

$$p^+ \otimes W \rightarrow \theta_k, \quad X \otimes v \mapsto X \cdot v$$

This is a $K$-homomorphism. Suppose that the image of this map is non-zero (i.e. $p^+$ does not annihilate $W$). Then $p^+ \otimes W$ must contain an irreducible constituent of the form described by Lemma 5.1. But the highest weights in $p^+ \otimes W$ are all of the form

$$(\text{highest weight of } W) + (a \text{ weight in } p^+)$$

From this we conclude that the only irreducible representation contained in $p^+ \otimes W$ and of the kind described in Lemma 5.1 is

$$V(p\omega_6 + (q + 1)\omega_1) \otimes (\mu + 2)_{\tau,3}.$$ 

Since $\mu$ was assumed to be maximal, this $K$-type must be of the form (5.4). But

$$\mu + 2 = 3k - 2p + 2(q + 1) + 6r.$$ 

So we must have $r \geq 0$ which is a contradiction. The contradiction shows that $p^+$ annihilates $W$. But then $W$ generates an irreducible highest weight module contained in $\theta_k$. The infinitesimal character of a highest weight module is easily obtained from the highest weight. Since $W$ is different from $V(0) \otimes (-3k - 24)$, one can easily check that the infinitesimal character can not be that given by (5.6). So we conclude that in fact $W$ does not exist. This proves the proposition.

6. Description of $\sigma'_k$.

**Lemma 6.1.** The representation $\sigma'_k$ has a unique lowest $K$-type

(6.1) $$V(p\omega_6) \otimes (3k - 2p)_{\tau,3}$$

where $p = \max(k - 4, 0)$.

**Proof.** Recall that $\mathfrak{h} \simeq \mathfrak{h}^* \simeq \mathbb{C}^6$ with standard inner product

$$< x, y > = x_1 y_1 + \cdots + x_8 y_8$$

The roots and weights for various subalgebras of $\mathfrak{e}_{8,4}$ are all embedded in $\mathfrak{h}^*$, and the restriction of $<,>$ can be used as the norm that comes into the definition of the lowest $K$-type [Vog, Chapter 5]. With this understanding we have

$$\omega = \omega_6 = (0, 0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$$
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\[ \omega^* = \omega_1 = (0,0,0,0,-\frac{2}{3},-\frac{2}{3},\frac{2}{3}) \]

and \((\mu)_{7,3}\) corresponds to the weight

\[ (0,0,0,0,-\frac{\mu}{6},\frac{\mu}{3},\frac{\mu}{6}) \]

Furthermore the half-sum of positive roots of \(\epsilon_{6,0}\) is

\[ \rho_c = (0,1,2,3,4,-4,-4,4) \]

Therefore if \(\lambda\) denote the highest weight of \(V(p\omega_6 + q\omega_1) \otimes (\mu)_{7,3}\) a simple calculation shows

\[ <\lambda + 2\rho_c,\lambda + 2\rho_c> = (p+8)^2 + \frac{1}{3}(p+2q+24)^2 + \frac{1}{6}\mu^2 + 56 \]

This is a quadratic form and it is elementary to verify that, subject to the conditions (5.3)-(5.4), the minimum is reached precisely when

\[ p = \max(k-4,0), \quad q = 0, \quad r = 0 \]

This is what the lemma says.

Note that when \(k \geq 4\) the lowest \(K\)-type specified in the above lemma is

\[ V((k-4)\omega_6) \otimes (k+8)_{7,3} \]

with highest weight

\[ (6.2) \quad (0,0,0,0,k-4,-\frac{k}{2},4,\frac{k}{2}) \]

Let

\[ (6.3) \quad x_0 = (0,0,0,0,2,-1,0,1) \in \mathfrak{h} \cap \epsilon_{7,3} \]

Let \(q = q(x_0)\) be the parabolic subalgebra of \(\epsilon_{7,3}\) defined as the sum of eigenspaces for \(ad(x_0)\) with non-negative eigenvalues. Set

\[ (6.4) \quad \gamma = (0,0,0,0,1,-\frac{1}{2},0,\frac{1}{2}) \in \mathfrak{h}^* \]

For \(\lambda = (k-10)\gamma\) define the representations \(R^j_q(\lambda) = R^j_q((k-10)\gamma)\) as in [Vog]. It turns out that the parameter \((k-10)\gamma\) always fulfills that condition. So \(R^j_q((k-10)\gamma) = 0\) unless \(j = S = \dim(u \cap \mathfrak{t})\), where \(u\) is the nilpotent radical of \(q\). We will see that \(S = 16\) here. If \(k \geq 10\) then \(R^S_q((k-10)\gamma) = A_q((k-10)\gamma)\) is a unitary representation with non-zero cohomology [VoZ].

**Lemma 6.2.** For \(k \geq 4\) the representations \(\sigma_k^j\) and \(R^S_q((k-10)\gamma)\) have exactly the same infinitesimal character and \(K\)-type structure.

**Proof.** Let \(I\) be the centralizer of \(x_0\). Let \(u\) be the sum of eigenspaces for \(ad(x_0)\) with positive eigenvalues. We have the Levi decomposition \(q = I + u\). It is easy to see that \([l,l]\) is of type \(E_6\), and in fact it comes from a real form of Cartan type \(EIII\). This is the real form that gives rise to the exceptional hermitian domain of type \(E_6\). Thus we may write \([l,l] = \epsilon_{6(-14)}\).
We construct a set of simple roots for $e_{7,3}$ such that all roots in $u$ are positive for the system determined by the simple roots. Take the first 5 simple roots $\alpha_1, \cdots, \alpha_5$ of $e_{8,4}$ listed in §2. Then we add $\alpha_6 = e_7 - e_4$. These form a system of simple roots for $e_{6(-14)}$. Next we add $\alpha_7 = e_5 - e_7$. These together form a system of simple roots for $e_{7,3}$. With respect to this system, we see that $\gamma$ is nothing but the 7-th fundamental weight. Let $\rho$ be the half sum of positive roots with respect to the positive system we just defined. The representation $R^S_\omega((k-10)\gamma)$ has infinitesimal character $(k-10)\gamma + \rho$, which agrees with the infinitesimal character of $\theta_k$ (and hence of $\sigma_k^+$) given by (5.6).

It is easy to check that

$$\delta_k \equiv (k-10)\gamma + 2\rho(u \cap p) = (0,0,0,0,k-4,-\frac{k}{2},4,\frac{k}{2})$$

which is the highest weight of the lowest $K$-type specified in Lemma 6.1.

Let $\tau$ be an irreducible representation of $K$ which acts on the space $Z$. The weight $\delta_k$ defines a one dimensional representation of $L \cap K$. Let

$$S(u \cap p) = \sum_{m=0}^{\infty} S^m(u \cap p)$$

be the symmetric algebra of $u \cap p$. The proof of the generalized Blattner formula in [Vog] shows that the multiplicity of $\tau$ in $R^S_\omega((k-4)\omega)$ is equal to the dimension of the space

$$\text{Hom}_{L \cap K}(H^0(u \cap t, Z), S(u \cap p) \otimes \delta_k)$$

(Combine (6.3.15) and (6.3.20) of [Vog]). Since $H^0(u \cap t, Z)$ is just the space of $u \cap t$ invariants in $Z$, the highest weight vectors of the $L \cap K$ module $H^0(u \cap t, Z)$ are precisely the highest weight vectors of the $K$ module $Z$. Thus it suffice to decompose $S^m(u \cap p)$, $m = 0, 1, 2, \cdots$ as $L \cap K$ modules.

It is easy to check that the semi-simple part of $L \cap t$ is $so(10)$ generated by the first 5 simple root $\alpha_1, \cdots, \alpha_5$ list in §2. Let $t$ be the center of $L \cap t$. It acts on $u \cap p^+$ by $(e_8 - e_6)|_t$ and on the one dimensional space $u \cap p^-$ by $(e_5 - e_7)|_t$. Since $e_5 - e_7$ is orthogonal to all roots in $so(10)$ it defines a one-dimensional representation of all of $L \cap t$. With respect to the action of $so(10)$, $u \cap p^+$ is isomorphic to the standard module $C^{10}$. Thus as a module for $L \cap t = so(10) + t$ we have

$$u \cap p = (C^{10} \otimes (e_8 - e_6)|_t) \bigoplus (e_5 - e_7)|_t$$

Therefore

$$S^m(u \cap p) = \sum_{l=0}^{m} S^l(C^{10}) \otimes [l(e_8 - e_6) + (m-l)(e_5 - e_7)]|_t$$

The restriction of $e_8 - e_6$ to $so(10) \cap h$ defines the first fundamental weight for $so(10)$ which we denote by $\omega'_1$. Now

$$S^l(C^{10}) = \mu_l \oplus \mu_{l-2} \cdots$$

where $\mu_j$ is the irreducible representation of $so(10)$ with highest weight $j\omega'_1$. We obtain

$$(6.5) \quad S(u \cap p) \otimes \delta_k = \sum (S^l(C^{10}) \otimes [l(e_8 - e_6) + (m-l)(e_5 - e_7)]|_t) \otimes \delta_k$$
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where the sum is over all integers \(m, l, j\) with
\[
0 \leq j \leq l \leq m, \quad j \equiv l \pmod{2}
\]

Set
\[
r = \frac{l - j}{2}, \quad q = j, \quad p = r + (m - l) + k - 4
\]

Then \(0 \leq r \leq p + 4 - k\) and one checks easily that the highest weight of the typical summand in (6.5) is precisely the highest weight of the representation \(V(p \omega_1 + q \omega_1) \otimes (\mu)_{7,3}\) with \(\mu = 3k - 2p + 2q + 6r\), that appears in (5.2) and satisfies (5.4). This proves the lemma.

With respect to the positive root system for \(e_{7,3}\) introduced during the proof of the above lemma, we have
\[
\rho = (0, 1, 2, 3, 5, -\frac{17}{2}, 4, \frac{17}{2})
\]

The restriction of \(\rho\) to the center of \(I\) is equal to \(9\gamma\). It is then easy to see that \((k - 10)\gamma\) is always in the fair range. For \(k \geq 9\) it is in the weakly good range, so \(R^S_q((k - 10)\gamma)\) is irreducible. Finally if \(k \geq 10\) then \(R^S_q((k - 10)\gamma) = A_q((k - 10)\gamma)\) is a unitary representation with non-zero cohomology. In this case the infinitesimal character and full \(K\)-type structure is more than enough to determine the isomorphism class of the representation in question [VoZ, Proposition 6.1]. So Lemma 6.2 implies

**Theorem 6.3.** For \(k \geq 10\) we have \(\sigma'_k \simeq A_q((k - 10)\gamma)\).

Note that
\[
(6.6) \quad \dim(u \cap p^+) = 10, \quad \dim(u \cap p^-) = 1
\]

So the representation \(A_q((k - 10)\gamma)\) has non-zero cohomology in bi-degree \((10, 1)\). In this regard we observe that up to conjugation by \(K\), there are exactly two \(\theta\)-stable parabolics satisfying (6.6). If \(x'_0\) is the parameter (cf. (6.3)) that defines the parabolic \(q'\) different from \(q\) and satisfies (6.6) then \(x_0\) and \(x'_0\) are conjugate to each other via an outer automorphism of \(E_{6,9}\). Also \(q, q'\) and their complex conjugates together constitute all parabolics satisfying \(u \cap p = 11\). Thus there are exactly 4 families of unitary representations with non-zero cohomology at the minimal degree \(r_\gamma = 11\) (cf. [VoZ, p. 38]), and that they have cohomology only at bi-degrees \((10, 1)\) and \((1, 10)\).

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**References**


