HARMONIC RADIAL COMBINATIONS AND DUAL MIXED VOLUMES

Y. D. CHAI† AND YOUNG SOO LEE‡

Abstract. For star bodies, p-harmonic radial combinations were introduced and studied in several papers. In this paper we study the relations of the dual quermassintegrals of p-harmonic radial combinations of star bodies to their dual quermassintegrals and obtain the upper bound for the dual quermassintegrals of p-harmonic radial combinations of star bodies.

For star bodies, p-harmonic radial combinations were introduced and studied in several papers. The aim of this article is to study them further, that is, we investigate the relations of the dual quermassintegrals of p-harmonic radial combinations of star bodies to their dual quermassintegrals and obtain the upper bound for the dual quermassintegrals of p-harmonic radial combinations of star bodies.

1. Preliminaries. By a convex body in $E^n, n \geq 2$, we mean a compact convex subset of $E^n$ with nonempty interior. Let $S^{n-1}$ denote the unit sphere centered at the origin in $E^n$, and write $O_{n-1}$ for the $(n-1)$-dimensional volume of $S^{n-1}$. Let $B$ be the closed unit ball in $E^n$, write $\omega_n$ for the $n$-dimensional volume of $B$. Note that

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(1 + \frac{1}{n})}, \quad \text{and} \quad O_{n-1} = n\omega_n.$$

For each direction $u \in S^{n-1}$, we define the support function $h(K,u)$ on $S^{n-1}$ of the convex body $K$ by

$$h(K,u) = \sup\{u \cdot x | x \in K\}$$

and the radial function $\rho(K,u)$ on $S^{n-1}$ of the convex body $K$ is

$$\rho(K,u) = \sup\{\lambda > 0 | \lambda u \in K\}.$$

If $\rho(K,u)$ is positive and continuous, call $K$ a star body (about the origin), and write $S$ for the set of star bodies (about the origin) of $E^n$. Sets $A, B$ are called homothetic if $A = \lambda B + t$ with $t \in E^n$ and $\lambda > 0$ or one of them is a singleton (a one-point set).

The polar body of a convex body $K$, denoted by $K^*$, is another convex body defined by

$$K^* = \{y | x \cdot y \leq 1 \text{ for all } x \in K\}.$$ 

The polar body has the well known property that

$$h(K^*, u) = 1/\rho(K, u) \quad \text{and} \quad \rho(K^*, u) = 1/h(K, u),$$

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†Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea (ydc@yuri.skku.ac.kr). This research was supported by the Brain Korea 21 Project and 98 SKKU Faculty Research Fund.
‡School of Electrical and Computer Engineering, Sungkyunkwan University, Suwon 440-746, South Korea. Current address: Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea (yslee@math.skku.ac.kr). This research is financially supported by the BK21 Project.
Let $K_j$ be a star body in $E^n$ with $o \in K_j$, $1 \leq j \leq n$. Then we define the dual mixed volumes $\hat{V}(K_1, \cdots, K_n)$ by

$$\hat{V}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) du,$$

where $du$ signifies the area element on $S^{n-1}$. Let

$$\hat{V}_i(K_1, K_2) = \hat{V}(\underbrace{K_1, \cdots, K_1}_{n-i}, \underbrace{K_2, \cdots, K_2}_i).$$

The dual quermassintegrals are the special dual mixed volumes defined by

$$\hat{W}_i(K) = \hat{V}_i(K, B).$$

Note that $\hat{W}_0(K) = V(K)$ is the volume of $K$, while $\hat{W}_n(K) = \omega_n$ does not depend on $K$.

2. Main Results. Fix a real $p \geq 1$. For $K, L \in S$, and $\lambda, \mu \geq 0$ (not both zero), the $p$-harmonic radial combination $\lambda \cdot K +_p \mu \cdot L \in S$ is defined by

$$\rho(\lambda \cdot K +_p \mu \cdot L, u)^{-p} = \lambda \rho(K, u)^{-p} + \mu \rho(L, u)^{-p}.$$

We obtain easily from the definition the following.

**Theorem 1 (Positive multisublinear).** Let $K, L \in S$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero). Then, for any $M_2, \cdots, M_n \in S$,

$$\hat{V}(\lambda \cdot K +_p \mu \cdot L, M_2, \cdots, M_n) \leq \lambda \hat{V}(K, M_2, \cdots, M_n) + \mu \hat{V}(L, M_2, \cdots, M_n)$$

where $\lambda + \mu = 1$.

**Proof.** In the definition of the $p$-harmonic radial combination, if we use the fact that $f(x) = x^{-\frac{1}{p}} (p \geq 1)$ is convex, then

$$\rho(\lambda \cdot K +_p \mu \cdot L, u) \leq \lambda \rho(K, u) + \mu \rho(L, u), \quad \lambda + \mu = 1.$$ 

So using the definition of the dual mixed volume, we easily obtain

$$\hat{V}(\lambda \cdot K +_p \mu \cdot L, M_2, \cdots, M_n) = \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K +_p \mu \cdot L, u) \rho(M_2, u) \cdots \rho(M_n, u) du$$

$$\leq \frac{1}{n} \int_{S^{n-1}} \left( \lambda \rho(K, u) + \mu \rho(L, u) \right) \rho(M_2, u) \cdots \rho(M_n, u) du$$

$$= \lambda \hat{V}(K, M_2, \cdots, M_n) + \mu \hat{V}(L, M_2, \cdots, M_n). \quad \Box$$

Now we consider the dual quermassintegrals of the $p$-harmonic radial combinations.

In the following theorem we obtain the upper bound for the dual quermassintegrals of the $p$-harmonic radial combinations of star bodies.

**Theorem 2.** Let $K, L \in S$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero). Then, for the $p$-harmonic radial combination $\lambda \cdot K +_p \mu \cdot L$

$$\hat{W}_i(\lambda \cdot K +_p \mu \cdot L) \leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \frac{1}{n} \left( \int_{S^{n-1}} \rho(K, u)^{(\frac{n-i}{2})} du \right)^{\frac{1}{2}} \left( \int_{S^{n-1}} \rho(L, u)^{(\frac{n-i}{2})} du \right)^{\frac{1}{2}}.$$
for \( i = 0, 1, \ldots, n \) and \( s > 1 \) and \( \frac{1}{s} + \frac{1}{t} = 1 \). Equality holds if and only if both \( K \) and \( L \) are balls such that \( \lambda \frac{1}{s} K = \mu \frac{1}{t} L \).

**Proof** From the definition of the \( p \)-harmonic radial combination, we have

\[
\frac{1}{\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^p} = \frac{1}{\lambda \rho(K, u)^p} + \mu \frac{1}{\rho(L, u)^p}
\]

\[
= \frac{\lambda \rho(L, u)^p + \mu \rho(K, u)^p}{\rho(K, u)^p \rho(L, u)^p}.
\]

It follows from the inequality between arithmetic and geometric means that

\[
\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^p \geq \frac{\rho(K, u)^p \rho(L, u)^p}{2 \sqrt{\lambda \rho(K, u)^p \rho(L, u)^p}}
\]

\[
= \frac{1}{2 \sqrt{\lambda \mu}} (\rho(K, u) \rho(L, u))^\frac{p}{2}.
\]

It also follows that

\[
\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} \leq \left( \frac{1}{2 \sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \left( \rho(K, u) \rho(L, u) \right)^\frac{n-i}{2}
\]

and, using Hölder's inequality for integrals,

\[
\bar{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) = \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} du
\]

\[
\leq \left( \frac{1}{2 \sqrt{\lambda \mu}} \right)^\frac{n-i}{p} \frac{1}{n} \int_{S^{n-1}} \left( \rho(K, u) \rho(L, u) \right)^\frac{n-i}{2} du
\]

\[
\leq \left( \frac{1}{2 \sqrt{\lambda \mu}} \right)^\frac{n-i}{p} \frac{1}{n} \left( \int_{S^{n-1}} \rho(K, u)^{\frac{(n-i)s}{2}} du \right)^\frac{1}{t}
\]

\[
\times \left( \int_{S^{n-1}} \rho(L, u)^{\frac{(n-i)t}{2}} du \right)^\frac{1}{t}
\]

where \( s > 1 \) and \( \frac{1}{s} + \frac{1}{t} = 1 \). From the equality conditions of the arithmetic-geometric mean inequality and Hölder's inequality for integrals in last two inequality of (2.2), equality holds if and only if both \( K \) and \( L \) are balls such that \( \lambda \frac{1}{s} K = \mu \frac{1}{t} L \). □

We obtain the relations of the dual quermassintegrals of \( p \)-harmonic radial combinations of star bodies to their dual quermassintegrals.

**Corollary 1.** Let \( K, L \in S \), a real \( p \geq 1 \), and \( \lambda, \mu \geq 0 \) (not both zero). Then, for the \( p \)-harmonic radial combination \( \lambda \cdot K \hat{+}_p \mu \cdot L \),

\[
\bar{W}_i^2(\lambda \cdot K \hat{+}_p \mu \cdot L) \leq \left( \frac{1}{4 \lambda \mu} \right)^\frac{n-i}{p} \bar{W}_i(K) \bar{W}_i(L).
\]

Equality holds if and only if \( K \) is homothetic to \( L \) such that \( \lambda \frac{1}{s} K = \mu \frac{1}{t} L \).
Proof. In Theorem 2 take \( s = t = 2 \). Then

\[
\tilde{W}_i(\lambda \cdot K^{+p} \mu \cdot L) \leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \frac{1}{n} \left( \int_{S^{n-1}} \rho(K,u)^{n-i} du \right)^{\frac{1}{2}} \left( \int_{S^{n-1}} \rho(L,u)^{n-i} du \right)^{\frac{i}{2}}
\]

which implies that

\[
\tilde{W}^2_i(\lambda \cdot K^{+p} \mu \cdot L) \leq \left( \frac{1}{4\lambda \mu} \right)^{\frac{n-i}{p}} \frac{1}{n^2} \left( \int_{S^{n-1}} \rho(K,u)^{n-i} du \right) \left( \int_{S^{n-1}} \rho(L,u)^{n-i} du \right) = \left( \frac{1}{4\lambda \mu} \right)^{\frac{n-i}{p}} \tilde{W}_i(K)\tilde{W}_i(L).
\]

The equality condition follows from the equality cases of the arithmetic-geometric mean inequality and Hölder’s inequality for \( s = t = 2 \). \( \square \)

From Theorem 2 we also obtain the following.

**Corollary 2.** Let \( K, L \in S, \) a real \( p \geq 1, \) and \( \lambda, \mu \geq 0 \) (not both zero). Then

\[
\tilde{W}_i(\lambda \cdot K^{+p} \mu \cdot L) \leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} \left( \rho(K,u)\rho(L,u) \right)^{\frac{n-i}{2}} du.
\]

Equality holds if and only if \( K \) is homothetic to \( L \) such that \( \lambda^{\frac{1}{p}} L = \mu^{\frac{1}{p}} K \).

**Proof.** Take \( i = 0 \) in (2.2). The equality condition follows from the equality case of the arithmetic-geometric mean inequality. \( \square \)

We also obtain the upper bound for the dual quermassintegrals of any star body and its polar body. \( K \) is homothetic to \( K^* \) if and only if \( K \) is a ball. So we obtain the following.

**Corollary 3.** Let \( K \in S, \) a real \( p \geq 1, \) and \( \lambda, \mu \geq 0 \) (not both zero) and \( K^* \) the polar body of \( K \). Then

\[
\tilde{W}_i(\lambda \cdot K^{+p} \mu \cdot K^*) \leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \omega_n
\]

for \( i = 0, 1, \ldots, n, \) with equality if and only if \( K \) is a ball.

**Proof.** If we take \( L = K^* \) in (2.2) and use \( \rho(K^*, u) = 1/h(K,u) \), then we obtain

\[
\tilde{W}_i(\lambda \cdot K^{+p} \mu \cdot K^*) \leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} \left( \rho(K,u)\rho(K^*, u) \right)^{\frac{n-i}{2}} du \\
= \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho h(K,u)}{h(K,u)} \right)^{\frac{n-i}{2}} du \\
\leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \frac{1}{n} \int_{S^{n-1}} du \\
= \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^{\frac{n-i}{p}} \omega_n.
\]

The second inequality follows since \( h(K,u) \geq \rho(K,u) \). \( \square \)
COROLLARY 4. Let $K \in S$, a real $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero). Then

$$V(\lambda \cdot K \hat{+}_p \mu \cdot K^*) \leq \left( \frac{1}{2\sqrt{\lambda \mu}} \right)^n \omega_n,$$

with equality if and only if $K$ is a ball.

Proof. Take $i = 0$ in Corollary 3. \(\square\)

We obtain the relations of the dual quermasintegrals of $p$-harmonic radial combinations of star bodies to their dual quermasintegrals.

THEOREM 3. Let $K, L \in S$ and $\lambda, \mu \geq 0$ (not both zero). Then, for the $p$-harmonic radial combination $\lambda \cdot K \hat{+}_p \mu \cdot L$,

$$\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) \leq \lambda \tilde{W}_i(K) + \mu \tilde{W}_i(L), \quad \lambda + \mu = 1$$

for $p > 0$ and $0 < i < n$, with equality if and only if $K = L$.

Proof. From the definition of the $p$-harmonic radial combination,

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{-p} = \lambda \rho(K, u)^{-p} + \mu \rho(L, u)^{-p}.$$

It follows that

$$\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} \leq \lambda \rho(K, u)^{n-i} + \mu \rho(L, u)^{n-i}, \quad \lambda + \mu = 1.$$

The inequality above follows from the H"{o}lder inequality.

Therefore

$$\tilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L) = \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} du$$

$$\leq \frac{1}{n} \int_{S^{n-1}} \left( \lambda \rho(K, u)^{n-i} + \mu \rho(L, u)^{n-i} \right) du$$

$$= \frac{\lambda}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} du + \frac{\mu}{n} \int_{S^{n-1}} \rho(L, u)^{n-i} du$$

$$= \lambda \tilde{W}_i(K) + \mu \tilde{W}_i(L). \quad \square$$

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