$p$-adic Berglund-Hübisch duality

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Berglund-Hübisch duality is an example of mirror symmetry between orbifold Landau-Ginzburg models. In this paper we study a D-module-theoretic variant of Borisov’s proof of Berglund-Hübisch duality. In the $p$-adic case, the D-module approach makes it possible to endow the orbifold chiral rings with the action of a non-trivial Frobenius endomorphism. Our main result is that the Frobenius endomorphism commutes with Berglund-Hübisch duality up to an explicit diagonal operator.

1. Introduction

Berglund-Hübisch duality was originally introduced [3] as a generalization of the Greene-Plesser construction [4] of mirror pairs. Let $W(x) \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ be an invertible polynomial defining a Calabi-Yau hypersurface $X$ and let $G \subset (\mathbb{C}^*)^n$ be a group fixing $W$. Then the Berglund-Hübisch dual of the orbifold of $X$ by $G$ is the hypersurface $X^T$, defined by the “transpose” invertible polynomial $W^T(x) \in \mathbb{C}[x]$, orbifolded by an explicitly constructed group $G^T \subset (\mathbb{C}^*)^n$ fixing $W^T$. As shown in [6] and [8], the Berglund-Hübisch construction can be further generalized to Landau-Ginzburg models with invertible potentials (not necessarily of Calabi-Yau type) as follows. For any invertible polynomial $W(x)$ the bigraded chiral ring of the orbifold Landau-Ginzburg model $(W(x), G)$ is isomorphic to the (twisted) chiral ring of the orbifold Landau-Ginzburg model $(W^T(x), G^T)$.

In the context of the vertex algebra approach to mirror symmetry [1], Borisov [2] has shown that, as an isomorphism of bigraded vector spaces (that is, disregarding the multiplicative structure), Berglund-Hübisch duality can be lifted to the level of chains. Let $\mathbb{C}[x, y]_0$ be the quotient of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by the ideal $\langle x_1y_1, \ldots, x_ny_n \rangle$ and let $\wedge(\mathbb{C}^n)$ be the standard exterior representation of the Clifford algebra with generators $e_i$, 1115
$e_i^\vee$ and relations $e_i e_j^\vee + e_j^\vee e_i = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Borisov’s construction hinges on the differential

$$\delta_\infty = \sum_{i=1}^{n} x_i \partial_{x_i} W(x) \otimes e_i + \sum_{i=1}^{n} y_i \otimes e_i^\vee$$

acting on $\mathbb{C}[x, y]_0 \otimes \Lambda(\mathbb{C}^n)$. As shown in [2], $(\mathbb{C}[x, y]_0 \otimes \Lambda(\mathbb{C}^n), \delta_\infty)$ contains a copy of the standard Koszul resolution of the Milnor ring $\mathbb{C}[x]/dW$ in such a way that the inclusion is a quasi-isomorphism. The starting point for this paper is to deform $\delta_\infty$ to

$$\delta_\pi = \sum_{i=1}^{n} (x_i \partial_{x_i} + \pi x_i \partial_{x_i} W(x)) \otimes e_i + \sum_{i=1}^{n} (y_i \partial_{y_i} + \pi y_i) \otimes e_i^\vee,$$

where $\pi \in \mathbb{C}^*$ is an arbitrary constant. As it turns out, the complex $(\mathbb{C}[x, y]_0 \otimes \Lambda(\mathbb{C}^n), \delta_\pi)$ contains a copy of the de Rham complex of the D-module $\mathbb{C}[x] e^{\pi W(x)}$. The quasi-isomorphism (see e.g. [9]) between the latter and the Milnor ring allows us to provide an alternate chain-level realization of Berglund-Hübsch duality. More precisely, our method yields a chain-level proof of the “total unprojected” (in the terminology of [8]) Berglund-Hübsch duality, from which the usual “projected” duality of [2] can be obtained by restricting to the invariant sectors as in [8].

The key difference between our construction and [2] emerges if one replaces $\mathbb{C}[x]$ with the ring $\mathbb{C}_p^\dagger(x)$ of $p$-adic overconvergent power series. While the de Rham cohomology of the D-module $\mathbb{C}_p^\dagger(x) e^{\pi W(x)}$ (where now $\pi$ is a fixed $(p-1)$-th root of $-p$) is still isomorphic to the $p$-adic Milnor ring, the de Rham chain model has extra structure: a non-trivial Frobenius endomorphism which descends to cohomology. In this paper we show that the Frobenius endomorphisms extends naturally to a chain map $\text{Fr}$ acting on the full chain complex $\mathbb{C}_p^\dagger(x, y)_0 \otimes \Lambda(\mathbb{C}_p^n)$. It is then natural to ask how the Frobenius endomorphism interacts with the Berglund-Hübsch duality quasi-isomorphism $\Delta$. Our main result is that, at the level of cohomology, $\Delta$ and $\text{Fr}$ commute up to an explicit diagonal operator whose entries are non-negative integer powers of $p$.

The interplay between the cohomological Frobenius and Berglund-Hübsch duality was first noticed in [10] and used to explore some arithmetic consequences of Berglund-Hübsch duality in the spirit of [12]. The present work originated as an attempt to understand the results of [10] at the level of chains. We hope to further investigate the arithmetic implications of our construction in future work.
This paper is organized as follows. In Section 2 we review some basic facts about invertible polynomials defined over a field $F$. In Section 3 and Section 4 we introduce our “de Rham” version of Borisov’s complex attached to a suitable matrix $A$, which we denote by $B_A(F)$. In Section 5 we point out that $B_A(F)$ is the total complex of a $\mathbb{Z} \times \mathbb{Z}$-bigraded bicomplex. In Section 6 we show that $B_A(F)$ is quasi isomorphic to the de Rham cohomology of a certain D-module. To do this we follow the analogous argument given by Borisov in [2]. However, the bigrading of [2] is no longer preserved by our differentials and this is why we need the bigrading introduced in Section 3 instead. In Section 7 we prove that $B_A(F)$ is quasi-isomorphic to a subcomplex $C_A(F)$ which is in turn canonically isomorphic to $C_A^T(F)$. Together with the results of Section 5, this proves unprojected Berglund-Hübsch duality. In Sections 8 and 9 we specialize to the $p$-adic case and observe that the constructions of the previous sections can be extended by replacing polynomials with overconvergent $p$-adic power series. While not changing cohomology, this allows for the extra room needed in order to define a natural chain-level Frobenius endomorphism Fr à la Dwork (see e.g. [9], [11]) whose compatibility with Berglund-Hübsch duality is then addressed. Finally, in Section 10 we illustrate our constructions by working out two simple examples.

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2. Invertible polynomials

Let $F$ be a field and consider the map

$$W: \text{GL}_n(\mathbb{Z}_{\geq 0}) \to F[x] = F[x_1, \ldots, x_n]$$

defined by

$$A \mapsto W_A(x) = \sum_{i=1}^n x^{e_i \cdot A},$$

where $\{e_i\}_{1 \leq i \leq n}$ is the standard basis of $\mathbb{Z}^n$, and for $v = (v_1, \ldots, v_n) \in \mathbb{Z}_{\geq 0}^n$ we write $x^v = x_1^{v_1} \cdots x_n^{v_n}$. For simplicity, we assume that $\text{char} F = 0$.
or \( \text{char } \mathbb{F} > \det A \). A matrix \( A \in \text{GL}_n(\mathbb{Z}_{\geq 0}) \) is Berglund-Hübsch over \( \mathbb{F} \) if \( W_A(x) \) is an invertible polynomial, i.e. if \( W_A(x) \) is quasi-homogeneous and 
\( (\partial_1 W_A(x), \ldots, \partial_n W_A(x)) \) is a regular sequence in \( \mathbb{F}[x] \). For each \( n \in \mathbb{Z}_{\geq 0} \) we let

\[
\text{BH}(\mathbb{F}) = \bigcup_n \text{BH}_n(\mathbb{F}) ,
\]

where

\[
\text{BH}_n(\mathbb{F}) = \{ A \in \text{GL}_n(\mathbb{Z}_{\geq 0}) \mid A \text{ is Berglund-Hübsch over } \mathbb{F} \} .
\]

**Remark 2.1.** Berglund-Hübsch matrices satisfy the following properties.

1) If \( A \in \text{BH}_n(\mathbb{F}) \) and \( B \in \text{BH}_m(\mathbb{F}) \), then \( A \oplus B \in \text{BH}_{n+m}(\mathbb{F}) \).

2) If

\[
\begin{bmatrix}
A \\
0 \\
C
\end{bmatrix} \in \text{BH}(\mathbb{F}),
\]

then \( C \in \text{BH}(\mathbb{F}) \). We call \( A \in \text{BH}_n(\mathbb{F}) \) *irreducible* if it cannot be written as \( B \oplus C \) with \( B, C \in \bigcup_{m \leq n} \text{BH}_m(\mathbb{F}) \).

3) Let \( \mathcal{W}_n \subseteq \text{GL}_n(\mathbb{Z}_{\geq 0}) \) be the Weyl group. Given \( S \in \mathcal{W}_n \) and \( A \in \text{BH}_n(\mathbb{F}) \), then \( SA, AS \in \text{BH}_n(\mathbb{F}) \). Moreover,

\[
W_{SA}(x) = W_A(x) \quad \text{and} \quad W_{AS}(x) = W_A(x) \cdot S ,
\]

where \( \cdot \) denotes the right action of \( \mathcal{W}_n \) on \( \mathbb{F}[x] \) by permutation of the variables.

**Remark 2.2.** Let \( A \in \text{BH}_n(\mathbb{F}) \) and suppose that \( \mathbb{F} \) contains a primitive \( (\det A) \)-th root of unity \( \zeta \). We define the *group of scaling symmetries* of \( A \in \text{BH}_n(\mathbb{F}) \) to be \( G_A = \mathbb{Z}^n / (\mathbb{Z}^n A^\top) \). The terminology is justified by the following observation. The group \( \mathbb{Z}^n / (\det(A)\mathbb{Z}^n) \) acts on \( \mathbb{F}[x] \) by

\[
\mu \cdot x^\gamma = \zeta^{\mu^\top} x^\gamma .
\]

Under this action \( \mu \cdot W_A(x) = W_A(x) \) if and only if \( \mu A^\top = (\det A) \lambda \) for some \( \lambda \in \mathbb{Z}^n \). Let \( \nu : G_A \to \mathbb{Z}^n / (\det(A)\mathbb{Z}^n) \) be such that \( \nu(\lambda) = \lambda \det(A) A^{-\top} \) for every \( \lambda \in G_A \). Then \( \nu \) provides a canonical identification between \( G_A \) and the stabilizer of \( W_A(x) \) under the action defined by (2.1). In the rest of the paper we slightly abuse notation and identify each equivalence class \( \lambda + \mathbb{Z}^n A^\top \in G_A \) with its unique representative \( \lambda \in \mathbb{Z}^n \) such that \( 0 \leq (\lambda A^{-\top})_i < 1 \) for all \( i = 1, \ldots, n \). Using this identification, to \( \lambda \in G_A \) we attach the
subset $J_\lambda \subseteq \{1, \ldots, n\}$ such that $i \in J_\lambda$ if and only if $(\lambda A^{-T})_i \in \mathbb{Q} \setminus \mathbb{Z}$. We define the submatrix $A^\lambda$ of $A$ such that $W_{A^\lambda}(x)$ is obtained from $W_A(x)$ by setting $x_i = 0$ whenever $i \in J_\lambda$.

**Proposition 2.3 ([6]).** Let $A \in \text{BH}_n(\mathbb{F})$ be irreducible. Then there exists $S \in \mathcal{W}_n$ such that $W_{AS}(x)$ is in one of the following canonical forms:

1) a loop,
\[x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n},\]

2) a chain,
\[x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}.\]

**Corollary 2.4.** Let $A \in \text{BH}_n(\mathbb{F})$. Then

1) $A^T \in \text{BH}_n(\mathbb{F})$,

2) for each $\lambda \in G_A$, we have $A^\lambda \in \text{BH}_{n-|J_\lambda|}(\mathbb{F})$, and

3) the matrix defined by
\[A^\text{orb} := \bigoplus_{\lambda \in G_A} A^\lambda\]

is in $\text{BH}_{n|G_A| - \sum |J_\lambda|}(\mathbb{F})$.

**Corollary 2.5.** Let $A \in \text{BH}_n(\mathbb{F})$ and let $\beta \in \mathbb{Z}^n$ such that $(\beta A^{-1})_i \in \mathbb{Q} \setminus \mathbb{Z}$.

1) If $A$ is a chain, then $(\beta A^{-1})_j, (\beta A^{-T})_k \in \mathbb{Q} \setminus \mathbb{Z}$ for all $1 \leq j \leq i \leq k \leq n$.

2) If $A$ is a loop, then $(\beta A^{-1})_j, (\beta A^{-1})_k \in \mathbb{Q} \setminus \mathbb{Z}$ for all $1 \leq j, k \leq n$.

**Proof.** Both statements follow from
\[A^T_{ii}(\beta A^{-T})_i + (\beta A^{-T})_{i+1} = \beta_i = (\beta A^{-1})_{i-1} + A_{ii}(\beta A^{-1})_i,\]
where $i$ is considered modulo $n$ in the case of loops. \qed

**3. Exterior operators**

Let $e_1, \ldots, e_n$ be the standard generators of $\mathbb{F}^n$. We denote by $\bigwedge(\mathbb{F}^n)$ the exterior algebra $\bigwedge(\mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_n)$ viewed as a representation of the Clifford algebra $\text{Cl}_n(\mathbb{F})$ with generators $e_i$ (multiplication) and $e_i^\vee$ (contraction),
and (odd) commutators \([e_i, e_j^\vee] = \delta_{ij}\) for all \(1 \leq i, j \leq n\). As an \(F\)-module, 
\(\bigwedge(F^n)\) is generated by monomials \(e^I = e^{I_1}_1 \cdots e^{I_n}_n\), where \(I = (I_1, \ldots, I_n) \in \mathbb{Z}_{\geq 0}^n\). In particular, \(e^I = 0\) if and only if \(I_i \geq 2\) for some \(i\). Given \(A \in BH_n(F)\) and \(\pi \in F^*\), for \(1 \leq i \leq n\) we also consider

\[
E_{A,i} = \pi \sum_{j=1}^n e_j A_{ji}^T \quad \text{and} \quad E_{A,i}^\vee = \frac{1}{\pi} \sum_{j=1}^n e_j^\vee (A^{-1})_{ji},
\]

so that

\[
[E_{A,i} E_{A,j}^\vee] = \sum_{k,m} A_{ki}^T (A^{-1})_{mj} [e_k, e_m^\vee] = \sum_k A_{ik} (A^{-1})_{kj} = \delta_{ij}.
\]

**Lemma 3.1.** If \(*^A \in GL (\bigwedge(F^n))\) is defined by

\[
*^A (e_{i_1} \cdots e_{i_k}) = E_{A^T,i_1}^\vee E_{A^T,i_2}^\vee \cdots E_{A^T,i_k}^\vee (E_{A^T,1} E_{A^T,2} \cdots E_{A^T,n}),
\]

then

1) \(*^A E_{A,i} = e_i^\vee *^A, \ *^A E_{A,i}^\vee = e_i *^A,\) and

2) \(*^{A^T} *^A\) commutes with the action of \(Cl_n(F)\) on \(\bigwedge(F^n)\).

**Proof.** By definition,

\[
*^A e_i = E_{A^T,i}^\vee *^A \quad \text{and} \quad *^A e_i^\vee = E_{A^T,i} *^A.
\]

Therefore,

\[
*^A E_{A,i} = *^A \pi \sum_j e_j A_{ji}^T = \pi \sum_j E_{A^T,j}^\vee A_{ji}^T *^A = \sum_{k,j} e_k^\vee (A^{-T})_{kj} A_{ji}^T *^A e_i^\vee *^A.
\]

Similarly, \(*^A E_{A,i}^\vee = e_i *^A\). This proves part (1). Part (2) follows from

\[
*^{A^T} *^A e_i = *^{A^T} E_{A^T,i}^\vee *^A = e_i *^{A^T} *^A
\]

and

\[
*^{A^T} *^A e_i^\vee = *^{A^T} E_{A,i} *^A = e_i^\vee *^{A^T} *^A.
\]

\(\square\)
Remark 3.2. The operator

\[ \text{ext} = \sum_{i=1}^{n} e_i e_i^\vee = \sum_{i=1}^{n} E_{A,i} E_{A,i}^\vee \]

is diagonal on \( \bigwedge (\mathbb{F}^n) \). If \( \text{char} \mathbb{F} = 0 \), its eigenvalues count the total exterior degree. Moreover,

\[ *^A \text{ext} = \sum_{i=1}^{n} e_i^\vee e_i *^A = (n \text{ Id} - \text{ext}) *^A . \]

4. The basic complex

Given a graded vector space \( V \) endowed with a differential \( d \) of degree 1, we denote by \( (V, d) \) the corresponding chain complex and by \( H(V, d) \) its cohomology. If \( V \) is bigraded and \( d, d' \) are differentials of bidegree \((1,0)\) and \((0,1)\) respectively, we denote the corresponding bicomplex by \( (V, d, d') \) and by \( H(V, d, d') \) its total cohomology. If \( V \) is a vector space acted upon by a collection of commuting endomorphisms \( \phi_1, \ldots, \phi_n \), we denote the corresponding Koszul complex by \( \text{Kos}(V, \phi_1, \ldots, \phi_n) \).

Given \( A \in \text{BH}_n(\mathbb{F}) \), consider the subring \( R_A(\mathbb{F}) \) of \( \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n] \) generated by monomials \( x^\gamma y^\lambda \) such that \( (\lambda A^T)^i \geq 0 \) for all \( 1 \leq i \leq n \). We define \( R_A(\mathbb{F}) \) to be the quotient of \( \widehat{R_A(\mathbb{F})} \) by the ideal generated by monomials \( x^\gamma y^\lambda \) for which \( \gamma A^{-1} \lambda^T > 0 \). Given \( \pi \in \mathbb{F}^* \), we define \( \theta_{A,i}, T_{A,i}^\gamma, \psi_{A,i}^\gamma, \varphi_{A,i} \in \text{End}_\mathbb{F}(R_A(\mathbb{F})) \) by the formulas

\[
\begin{align*}
\theta_{A,i}(x^\gamma y^\lambda) &= \gamma_i x^\gamma y^\lambda; \\
T_{A,i}^\gamma(x^\gamma y^\lambda) &= \pi^{-1} (\lambda A^{-T})_i x^\gamma y^\lambda; \\
\psi_{A,i}^\gamma(x^\gamma y^\lambda) &= x^\gamma y^\lambda + e_i A^T; \\
\varphi_{A,i}(x^\gamma y^\lambda) &= \pi (\theta_{A,i} W(x)) x^\gamma y^\lambda = \pi \sum_{j=1}^{n} A_{ji} x^{\gamma + e_i A^T} y^\lambda.
\end{align*}
\]

We also define the odd linear endomorphisms of \( R_A(\mathbb{F}) \otimes \bigwedge (\mathbb{F}^n) \)

\[ d_{A,i} = (\theta_{A,i} + \varphi_{A,i}) e_i, \quad d_A = \sum_{i=1}^{n} d_{A,i} \]
Lemma 4.1. $B_A(\mathbb{F}) = (\mathcal{R}_A(\mathbb{F}) \otimes \wedge(\mathbb{F}^n), d_A + d_A^\vee)$ is a chain complex.

Proof. The morphism $d_A$ is the Koszul differential for the sequence

$$(\theta_{A,1} + \varphi_{A,1}, \theta_{A,2} + \varphi_{A,2}, \ldots, \theta_{A,n} + \varphi_{A,n})$$

of commuting operators acting on $\mathcal{R}_A(\mathbb{F})$. Therefore $[d_A, d_A] = 0$ and similarly $[d_A^\vee, d_A^\vee] = 0$. Moreover, since $(\theta_{A,i} + \varphi_{A,i})$ and $(T_{A,j}^\vee + \psi_{A,j}^\vee)$ commute,

$$[d_{A,i}, d_{A,j}^\vee] = [(\theta_{A,i} + \varphi_{A,i}) e_i, (T_{A,j}^\vee + \psi_{A,j}^\vee) e_j^\vee]$$

$$= (\theta_{A,i} + \varphi_{A,i}) (T_{A,j}^\vee + \psi_{A,j}^\vee) [e_i, e_j^\vee]$$

$$= (\theta_{A,i} + \varphi_{A,i}) (T_{A,j}^\vee + \psi_{A,j}^\vee) \delta_{ij}.$$

If

$$0 \neq (\theta_{A,i} T_{A,i}^\vee) (x^\gamma y^\lambda) = \gamma_i (A^{-1} \lambda^T)_i x^\gamma y^\lambda,$$

then $x^\gamma y^\lambda = 0$ in $\mathcal{R}_A(\mathbb{F})$ and thus $(\theta_{A,i} + \varphi_{A,i}) (T_{A,i}^\vee + \psi_{A,i}^\vee) = 0$. For

$$(\varphi_{A,i} T_{A,i}^\vee) (x^\gamma y^\lambda) = \sum_{j=1}^n A_{ji}(\lambda A^{-T})_i x^{\gamma + e_j A} y^\lambda$$

we note that if for some $j$ we have $A_{ji}, (\lambda A^{-T})_i > 0$, then

$$(\gamma + e_j A) A^{-1} \lambda^T \geq (e_j A) (A^{-1} \lambda^T) = \sum_{m=1}^n A_{jm}(A^{-1} \lambda^T)_m \geq A_{ji}(A^{-1} \lambda^T)_i > 0$$

and conclude as before that $x^{\gamma + e_j A} y^\lambda = 0$ in $\mathcal{R}_A(\mathbb{F})$. It is similarly shown that $\varphi_{A,i} \psi_{A,i}^\vee = 0$ and $\theta_{A,i} \psi_{A,i}^\vee = 0$. Therefore, $[d_A, d_A^\vee] = 0$. □

Remark 4.2. Since for $\lambda \in G_A$ we take $0 \leq (\lambda A^{-T})_i < 1$ for each $i$ by Remark 2.2, the condition $\gamma A^{-1} \lambda^T = 0$ imposed on a monomial $x^\gamma y^\lambda$ means that $\gamma_i = 0$ if $\lambda$ acts non-trivially on $x_i$. 

and

$$d_{A,i}^\vee = (T_{A,i}^\vee + \psi_{A,i}^\vee) e_i^\vee, \quad d_A^\vee = \sum_{i=1}^n d_{A,i}^\vee.$$
5. Bigrading

Let $P_{A,i}^\vee \in \text{End}_\mathbb{F} (\mathcal{R}_A(\mathbb{F}))$ be given by

$$P_{A,i}^\vee (x^\gamma y^\lambda) = \begin{cases} 0, & \text{if } (\lambda A^{-T})_i = 0; \\ x^\gamma y^\lambda, & \text{otherwise} \end{cases}.$$  

Lemma 5.1. Let $Q_{A,i}, Q_{A,i}^\vee, Q_A$ and $Q_A^\vee$ be linear endomorphisms of $\mathcal{R}_A(\mathbb{F}) \otimes \wedge(\mathbb{F}^n)$ defined by

$$Q_{A,i}^\vee = P_{A,i}^\vee e_i^\vee e_i \quad \text{and} \quad Q_{A,i} = e_i^\vee e_i + Q_{A,i}^\vee$$

as well as

$$Q_A = \sum_{i=1}^n Q_{A,i} \quad \text{and} \quad Q_A^\vee = \sum_{i=1}^n Q_{A,i}^\vee.$$  

Then for each $1 \leq i, j \leq n$,

1) $[Q_{A,i}, Q_{A,j}^\vee] = 0$, 

2) $[Q_{A,i}^\vee, d_{A,j}] = 0$ and $[Q_{A,i}, d_{A,j}] = \delta_{ij} d_{A,j}$, 

3) $[Q_{A,i}, d_{A,j}^\vee] = 0$ and $[Q_{A,i}^\vee, d_{A,j}] = \delta_{ij} d_{A,j}^\vee$.

Proof. The operators $Q_{A,i}$ and $Q_{A,j}^\vee$ commute because they have monomials of the form $x^\gamma y^\lambda e^I$ as a common basis of eigenvectors, which proves (1). For (2),

$$[Q_{A,i}^\vee, d_{A,j}] = [P_{A,i}^\vee e_i^\vee e_i, (\theta_{A,j} + \varphi_{A,j}) e_j] = P_{A,i}^\vee (\theta_{A,j} + \varphi_{A,j}) [e_i^\vee e_i, e_j] = \delta_{ij} P_{A,i}^\vee (\theta_{A,j} + \varphi_{A,j}) e_j^\vee.$$  

The proof of Lemma 4.1 shows that $P_{A,i}^\vee \varphi_{A,i} = 0$. Similarly, $P_{A,i}^\vee \theta_{A,i}(x^\gamma y^\lambda) \neq 0$ implies that $\gamma_i (A^{-1} \lambda T)_i > 0$ so that the corresponding term is 0 in $\mathcal{R}_A(\mathbb{F})$. Therefore, $[Q_{A,i}^\vee, d_{A,j}] = 0$, which in turn implies that

$$[Q_{A,i}, d_{A,j}] = (\theta_{A,j} + \varphi_{A,j}) [e_i^\vee e_i^\vee, e_j] = \delta_{ij} d_{A,j}.$$  

For part (3), if $i \neq j$

$$[P_{A,i}^\vee, T_{A,j}^\vee + \psi_{A,j}^\vee] = 0 = [e_i^\vee e_i^\vee, e_j^\vee].$$
If $i = j$, then $e_i \bar{e}_i e_i = 0$ and $e_i \bar{e}_i e_i \bar{e}_i = e_i \bar{e}_i$, which means that

$$[Q_{A,i}, d_{A,j}^\vee] = \delta_{ij} P_{A,j}^\vee (T_{A,j}^\vee + \psi_{A,j}^\vee) e_i^\vee = \delta_{ij} d_{A,j}^\vee.$$

Similarly,

$$[Q_{A,j}, d_{A,j}^\vee] = [e_i e_i^\vee, (T_{A,j}^\vee + \psi_{A,j}^\vee) e_j^\vee] + \delta_{ij} d_{A,j}^\vee = \delta_{ij} (-(T_{A,j}^\vee + \psi_{A,j}^\vee) e_j^\vee e_j^\vee + d_{A,j}^\vee) = 0. \quad \square$$

**Remark 5.2.** In particular, with respect to the Spec($Q_A$) × Spec($Q_A^\vee$) bigrading, $B_A(F)$ is the total complex of the bicomplex $(R_A(F) \otimes \Lambda(F^n), d_A, d_{A}^\vee)$. 

### 6. Unprojected orbifold de Rham cohomology

Given $A \in BH_n(F)$ and $\pi \in F^*$, let

$$M_A(F) = F[x]e^{\pi W_A(x)}$$

be the module over the Weyl algebra $F[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ on which $x_i$ acts by multiplication and $\partial_i$ acts according to the formula

$$(6.1) \quad \partial_i \cdot P(x) = \partial_i P(x) + \pi(\partial_i W_A(x)) P(x)$$

for each $1 \leq i \leq n$ and $P(x) \in M_A(F)$. Note that $e^{\pi W_A(x)}$ is a formal symbol serving as a reminder of the Weyl algebra action. We denote by DR$_A(F)$ the *de Rham complex* of $M_A(F)$, which is by definition the Koszul complex

$$\text{Kos} (M_A(F), \partial_1, \partial_2, \ldots, \partial_n),$$

where each $\partial_i$ acts as in Equation (6.1). Given $\lambda \in \mathbb{Z}_{\geq 0}^n$ such that $(\lambda A^{-T})_i \geq 0$ for all $1 \leq i \leq n$, let $R_A^\lambda(F) \subseteq R_A(F)$ be generated by monomials of the form $x^\gamma y^{\lambda+\mu} A^T$ for some $\gamma, \mu \in \mathbb{Z}_{>0}^n$. Then $R_A^\lambda(F) \otimes \Lambda(F^n)$ is closed under $d_A + d_{A}^\vee$ and we denote by $B_{A}^\lambda(F) \subseteq B_A(F)$ the corresponding subcomplex.

**Lemma 6.1.** If $A \in BH_n(F)$, then

1) $B_A(F) \cong \bigoplus_{\lambda \in G_A} B_{A}^\lambda(F)$, and

2) $B_{A}^\lambda(F)$ is quasi-isomorphic to $B_{A}^0(F)$. 


Proof. Part (1) holds because $G_A = \mathbb{Z}^n / \mathbb{Z}^n A^T$ and $(\lambda A^{-T})_i \geq 0$ for $1 \leq i \leq n$. For (2), note that since $x_i y^\lambda = 0$ in $R^\lambda_A(\mathbb{F})$ for any $i \in J_\lambda$, we have

$$R^\lambda_A(\mathbb{F}) \cong R^0_A(\mathbb{F}) \otimes \mathbb{F}[\psi_{A,i}^{\vee}]_{i \in J_\lambda} y^\lambda,$$

and thus

$$B^\lambda_A(\mathbb{F}) \cong B^0_A(\mathbb{F}) \otimes \text{Kos} \left( \mathbb{F}[\psi_{A,i}^{\vee}]_{i \in J_\lambda} y^\lambda, (T_{A,i}^{\vee} + \psi_{A,i}^{\vee})_{i \in J_\lambda} \right).$$

The cohomology of the second factor is isomorphic to $\mathbb{F} y^\lambda$, making the inclusion $B^0_A(\mathbb{F}) \hookrightarrow B^\lambda_A(\mathbb{F})$ a quasi-isomorphism. □

**Proposition 6.2.** The complex $B^0_A(\mathbb{F})$ is canonically quasi-isomorphic to $\text{DR}_A(\mathbb{F})$.

Proof. The map $\Theta : M_A(\mathbb{F}) \otimes \wedge (\mathbb{F}^n) \rightarrow R^0_A(\mathbb{F}) \otimes \wedge (\mathbb{F}^n)$ defined by $\Theta(x^\gamma e^I) = x^\gamma + I e^I$ gives rise to an embedding

$$\text{DR}_A(\mathbb{F}) \hookrightarrow B^0_A(\mathbb{F})$$

of complexes. For each $\gamma \in \mathbb{Z}^n_{\geq 0}$, let $\wedge_\gamma = \wedge \left( \bigoplus_{i=0} e_i \right)$. Then

$$\left( R^0_A(\mathbb{F}) \otimes \wedge (\mathbb{F}^n), d_A^{\vee} \right) = \bigoplus_{\gamma,I} C_{\gamma,I},$$

where

$$C_{\gamma,I} = \left( x^\gamma + I [y^{e_i A^T}]_{(\gamma+I)_i = 0} \otimes e^I \wedge_{(\gamma+I)_i = 0} \sum_{(\gamma+I)_i = 0} d_A^{\vee} \right)$$

is a Koszul complex with cohomology $\mathbb{F} x^\gamma + I e^I$. This implies that

$$H \left( \frac{R^0_A(\mathbb{F}) \otimes \wedge (\mathbb{F}^n)}{\text{Im} \Theta}, d_A^{\vee} \right) = 0,$$

and using the spectral sequence of first quadrant bicomplexes we conclude that

$$H \left( B^0_A(\mathbb{F}) / \text{DR}_A(\mathbb{F}) \right) = 0.$$

Therefore, the inclusion $\text{DR}_A(\mathbb{F}) \hookrightarrow B^0_A(\mathbb{F})$ is a quasi-isomorphism. □
**Corollary 6.3.** Let $A \in \text{BH}_n(F)$ and let $S(A)$ be the collection of monomials $x^{\gamma}e^I$ such that $|I| = n$ and $1 \leq \gamma_i \leq a_i = A_{ii}$ for all $i = 1, \ldots, n$. Then

1) $H(B^0_A(F))$ is isomorphic to the Milnor ring $F[x]/dW_A(x)$.

2) If $W_A(x)$ is a loop, then $H(B^0_A(F))$ is generated by monomials in $S(A)$.

3) If $W_A(x)$ is a chain, then $H(B^0_A(F))$ is generated by those monomials in $S(A)$ of the form

$$x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_{2m-1}^{a_{2m-1}}x_{2m}^{a_{2m}}x_{2m+1}^{\gamma_{2m+1}}\cdots x_n^{\gamma_n}$$

where $m \geq 0$ is such that $\gamma_{2m+1} < a_{2m+1}$.

**Proof.** Part (1) follows from Proposition 6.2 and the fact (see e.g. [9]) that there is a linear map from $F[x]/dW_A(x)$ to $H(\text{DR}_A(F))$ sending monomials to monomials. Comparison with the standard monomial basis for the Milnor ring of chains and loops (see e.g. [6]) establishes (2) and (3). \[\Box\]

**Proposition 6.4.** The natural inclusion of $\text{DR}_{A_{\text{orb}}}(F)$ into $B_A(F)$ is a quasi-isomorphism.

**Proof.** The proposition follows from Lemma 6.1 and Proposition 6.2. \[\Box\]

## 7. Unprojected duality

Given $A \in \text{BH}_n(F)$ and $\pi \in F^*$, let $\psi_{A,i}, T_{A,i} \in \text{End}_F(\mathcal{R}_A(F))$ for $1 \leq i \leq n$ be defined by

$$\psi_{A,i}(x^{\gamma}y^\lambda) = x^{\gamma}e_i A y^\lambda$$

and

$$T_{A,i}(x^{\gamma}y^\lambda) = \pi^{-1}(\gamma A^{-1})_i x^{\gamma}y^\lambda,$$

so that $d_A = \sum_{i=1}^n \hat{d}_{A,i}$, where

$$\hat{d}_{A,i} = (T_{A,i} + \psi_{A,i})E_{A,i}.$$

**Remark 7.1.** Since we are using logarithmic differentials, $e_i$ can be naturally interpreted as $dx_i/x_i$. One motivation for the change of basis to $E_{A,i}$ is the Shioda map $x^{\gamma} \mapsto z^{\gamma A^{-1} \det(A)}$ which sends $W_A$ to $z^{e_1 \det(A)} + \cdots + z^{e_n \det(A)}$. If we interpret $E_{A,i}$ as $dz_i/z_i$, its definition is simply the chain rule.
Let \( S_A(F) \subseteq R_A(F) \) be generated by monomials \( x^\gamma y^\lambda \) such that 
\((\gamma A^{-1})_i \geq 0 \) for all \( 1 \leq i \leq n \). Then \( S_A(F) \otimes \bigwedge F^n \) is closed under \( d_A + d'_A \). Let \( C_A(F) \subseteq B_A(F) \) denote the corresponding subcomplex.

**Lemma 7.2.** The inclusion \( C_A(F) \hookrightarrow B_A(F) \) is a quasi-isomorphism.

**Proof.** Consider the filtration
\[
S_A(F) \subseteq F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^1 = R_A(F),
\]
where \( F^i \) is spanned by monomials \( x^\gamma y^\lambda \) such that \((\gamma A^{-1})_j \geq 0 \) for all \( j < i \). In particular, \( F^i/F^{i+1} \) is canonically identified with the space of monomials \( x^\gamma y^\lambda \) such that \((\gamma A^{-1})_i < 0 \). Consider the filtration \( G^\bullet(F) = F^\bullet \otimes \bigwedge F^n \) of \( R_A(F) \otimes \bigwedge F^n \). Notice that
\[
\left( \frac{G^i(F)}{G^{i+1}(F)}; d_A, d'_A \right)
\]
is a bicomplex with respect to the \( \text{Spec}(Q_A) \times \text{Spec}(Q'_A) \) bigrading, while
\[
\left( \frac{G^i(F)}{G^{i+1}(F)}; \hat{d}_A, d_A - \hat{d}_A \right)
\]
is a bicomplex with respect to the \( \text{Spec}(E_{A,i} \bar{E}'_{A,i}) \times (\text{ext} - E_{A,i} \bar{E}'_{A,i}) \) bigrading. Therefore, in order to prove that
\[
H \left( \frac{G^i(F)}{G^{i+1}(F)}, d_A + d'_A \right) = 0,
\]
it is sufficient to show that
\[
H \left( \frac{G^i(F)}{G^{i+1}(F)}, \hat{d}_A \right) = 0.
\]
If this is the case, the result then follows from the spectral sequence of the filtered complex \( (B_A(F), G^\bullet(F)) \). To prove (7.2), we distinguish the following two cases.

First, suppose that char \( F = 0 \). In this case, \( T_{A,i} \) acts by nonzero eigenvalues on \( F^i/F^{i+1} \). By looking at the filtration of \( F^i/F^{i+1} \) by \( \text{Spec}(T_{A,i}) \),
we conclude that $T_{A,i} + \psi_{A,i}$ is injective. Therefore, $H \left( \frac{G^i(F)}{G^{i+1}(F)}, \hat{d}_{A,i} \right)$ is concentrated in top $\text{Spec}(E_{A,i}E'_{A,i})$-degree and isomorphic to the quotient

$$F^i / \left( F^{i+1} + \text{Im}(T_{A,i} + \psi_{A,i}) \right).$$

On the other hand, for each $f \in F^i$ there exists $N \in \mathbb{N}$ such that $\psi_{A,i}^n f \in F^{i+1}$, which implies (7.2). See Figure 1 for an illustration.

Second, suppose that $\text{char} \ F = p > \det A$. Let $\mathbb{K}$ be a field such that $\text{char} \mathbb{K} = 0$ and $F = \mathbb{K}/p\mathbb{K}$. Consider the short exact sequence of complexes

$$0 \longrightarrow \left( \frac{G^i(K)}{G^{i+1}(K)}, \hat{d}_{A,i} \right) \overset{p}{\longrightarrow} \left( \frac{G^i(K)}{G^{i+1}(K)}, \hat{d}_{A,i} \right) \longrightarrow \left( \frac{G^i(F)}{G^{i+1}(F)}, \hat{d}_{A,i} \right) \longrightarrow 0.$$

Taking the long exact sequence and using the characteristic 0 case established above yields (7.2).

**Proposition 7.3.** Let $D^A: S_A(F) \to S_{A^T}(F)$ be defined by $D^A(x^\gamma y^\lambda) = x^\lambda y^\gamma$. Then,

$$\Delta^A = D^A \otimes \ast^A: S_A(F) \otimes \bigwedge(F^n) \to S_{A^T}(F) \otimes \bigwedge(F^n).$$
induces an isomorphism of complexes
\[ C_A(\mathbb{F}) \xrightarrow{\cong} C_{A^T}(\mathbb{F}). \]

**Proof.** Since \( D^A D^A = \text{Id} \) and \( \ast^A \in \text{GL}(\wedge^m(\mathbb{F}^n)) \), we only need to prove that \( \Delta^A \) is a chain map. Using
\[ D^A \psi_A, i = \psi_{A^T}, i D^A \quad \text{and} \quad D^A T_A, i = T_{A^T}, i D^A, \]
it follows that
\[ \Delta^A ((T_A, i + \psi_A, i)E_A, i) = ((T_{A^T}, i + \psi_{A^T}, i)e_i^\vee) \Delta^A \]
and
\[ \Delta^A ((T_{A^T}, i + \psi_{A^T}, i)e_i^\vee) = ((T_{A^T}, i + \psi_{A^T}, i)E_{A^T}, i) \Delta^A. \] \( \square \)

**Theorem 7.4 (Unprojected Berglund-Hübsch Duality).** The complexes \( DR_{A, orb}(\mathbb{F}) \) and \( DR_{(A^T), orb}(\mathbb{F}) \) are canonically quasi-isomorphic.

**Proof.** The theorem follows from Proposition 7.3, Lemma 7.2, and Proposition 6.4. \( \square \)

### 8. Overconvergent power series

Let \( p \in \mathbb{Z}_{\geq 0} \) be a prime, \( \overline{\mathbb{K}} = \mathbb{C}_p \), \( \mathbb{F} = \overline{\mathbb{K}}/p\overline{\mathbb{K}} \), \( A \in \text{BH}_n(\mathbb{F}) \), and \( \pi \in \overline{\mathbb{K}} \) such that \( \pi^{p-1} = -p \). Let \( \overline{\mathcal{R}}_A(\overline{\mathbb{K}}) \) be the ring of overconvergent power series
\[ \sum_{\gamma, \lambda \in \mathbb{Z}^n_{\geq 0}} a_{\gamma, \lambda} x^\gamma y^\lambda \]
such that \((\lambda A^{-T})_i \geq 0\) for all \( 1 \leq i < n \), and such that there exists \( M > 0 \) for which
\[ \text{ord}_p(a_{\gamma, \lambda}) \geq M(|\gamma| + |\lambda|) \quad \text{(8.1)} \]
for all but finitely many \( \gamma, \lambda \). Similarly, define \( \overline{\mathcal{R}}_A(\overline{\mathbb{K}}), S_A^l(\overline{\mathbb{K}}), B_A^l(\overline{\mathbb{K}}), \) and \( \overline{C}_A(\overline{\mathbb{K}}) \) as before, by replacing polynomials with overconvergent power series.
Lemma 8.1. 1) The inclusions
\[ \begin{align*}
B_A(K) & \longrightarrow B_A^\dagger(K) \\
\uparrow & \uparrow \\
C_A(K) & \longrightarrow C_A^\dagger(K)
\end{align*} \]
are quasi-isomorphisms.

2) \( \Delta^A \) extends to an isomorphism of complexes
\[ C_A^\dagger(K) \cong C_A^\dagger(F). \]

Proof. To prove (1), let \( f = \sum a_{\gamma, \lambda} x^\gamma y^\lambda \) be an overconvergent power series. Since \( \text{ord}_p(a_{\gamma, \lambda}) \geq 1 \) for all but finitely many \( \gamma, \lambda \), by reducing modulo \( p \) we obtain a polynomial \( \bar{f} \in \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n] \). Therefore, \( R_A(K) \) and \( R_A^\dagger(K) \) both reduce modulo \( p \) to \( R_A(F) \). Since \( B_A(F) \) and \( C_A(F) \) decompose into subcomplexes with cohomology concentrated in top degree, the statement follows from the long exact sequences of the diagram
\[ \begin{array}{ccccccc}
0 & \longrightarrow & B_A^\dagger(K) & \longrightarrow & B_A^\dagger(F) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B_A(K) & \longrightarrow & B_A(F) & \longrightarrow & 0
\end{array} \]
as in [9, Theorem 8.5]. Therefore \( B_A(K) \hookrightarrow B_A^\dagger(K) \) is quasi-isomorphism. Similarly, \( C_A(K) \hookrightarrow C_A^\dagger(K) \) is a quasi-isomorphism. The rest of the statement follows from Lemma 7.2.

Part (2) follows as in Proposition 7.3 after noticing that the overconvergence property is preserved by \( D^A \). \qed

9. The Frobenius endomorphism

Let \( p^{Q_A}, p^{Q_A^\vee} \in \text{End} \left( R_A^\dagger(K) \otimes \bigwedge (K^n) \right) \) be defined by
\[ p^{Q_A}(x^\gamma y^\lambda e^I) = p^\xi x^\gamma y^\lambda e^I \quad \text{and} \quad p^{Q_A^\vee}(x^\gamma y^\lambda e^I) = p^{\xi^\vee} x^\gamma y^\lambda e^I, \]
where \( Q_A(x^\gamma y^\lambda e^I) = \xi x^\gamma y^\lambda e^I \) and \( Q_A^\vee(x^\gamma y^\lambda e^I) = \xi^\vee x^\gamma y^\lambda e^I. \)
Lemma 9.1. If $\Theta'_A: \mathcal{R}_{\Lambda}^\dagger(\mathbb{K}) \to \mathcal{R}_{pA}^\dagger(\mathbb{K})$ is defined by $\Theta'_A(x^\gamma y^\lambda) = x^{p^\gamma} y^{p^\lambda}$, then

$$ Fr'_A = (\Theta'_A \otimes \text{Id}_{\Lambda(\mathbb{K})^n}) p^{Q_A}. $$

defines a chain map $\mathcal{B}_{\Lambda}^\dagger(\mathbb{K}) \to \mathcal{B}_{pA}^\dagger(\mathbb{K})$.

Proof. This follows from

$$ Fr'_A d_{A,i} = \Theta'_A(\theta_{A,i} + \varphi_{A,i}) e_i p^{Q_A+1} = d_{pA,i} Fr'_A; $$

$$ Fr'_A d_{A,i} = \Theta'_A(T_{A,i}^\gamma + \psi_{A,i}^\gamma) e_i p^{Q_A} = d_{pA,i} Fr'_A. \quad \square $$

Lemma 9.2. If $\Theta''_A: \mathcal{R}_{pA}^\dagger(\mathbb{K}) \to \mathcal{R}_{\Lambda}^\dagger(\mathbb{K})$ is defined by

$$ \Theta''_A(x^\gamma y^\lambda) = Z_A(x) Z^\tau_A(y) x^\gamma y^\lambda, $$

where $Z_A(x) = e^{\pi(W_{pA}(x) - W_A(x))}$, then

$$ Fr''_A = (\Theta''_A \otimes \text{Id}_{\Lambda(\mathbb{K})^n}) p^{Q_A^\tau}. $$

defines a chain map $\mathcal{B}_{pA}^\dagger(\mathbb{K}) \to \mathcal{B}_{\Lambda}^\dagger(\mathbb{K})$.

Proof. It is well known (see e.g. [9]) that $Z_A(x)$ satisfies (8.1). Therefore, $\Theta''_A$ is well defined. We compute

$$ (\theta_{A,i} + \varphi_{A,i}) \Theta''_A = \theta_{A,i} (W_{pA}(x) - W_A(x)) \Theta''_A + \Theta''_A \theta_{pA,i} + \varphi_{A,i} \Theta''_A $$

$$ = \Theta''_A \varphi_{pA,i} - \varphi_{A,i} \Theta''_A + \Theta''_A \theta_{pA,i} + \varphi_{A,i} \Theta''_A $$

$$ = \Theta''_A (\theta_{pA,i} + \varphi_{pA,i}), $$

from which we see that

$$ Fr''_A d_{pA,i} = \Theta''_A (\theta_{pA,i} + \varphi_{pA,i}) e_i p^{Q_A^\tau} = d_{A,i} Fr''_A. $$

To see that $Fr''_A d_{pA,i} = d_{A,i} Fr''_A$, note that for each $1 \leq i \leq n$, $T_{A,i}^\gamma$ satisfies

$$ T_{A,i}^\gamma \left( y^\lambda Z^\tau_A(y) \right) $$

$$ = T_{A,i}^\gamma \left( y^\lambda e^{\pi(W_{pA}(y) - W_A(y))} \right) $$

$$ = \pi \left( T_{A,i}^\gamma W_{pA}(y) - T_{A,i}^\gamma W_A(y) \right) y^\lambda Z^\tau_A(y) + Z^\tau_A(y) T_{A,i}^\gamma y^\lambda $$

$$ = y^\lambda Z^\tau_A(y) \left( py_{pA,i}^\gamma - y_{A,i}^\gamma \right) + Z^\tau_A(y) p^{\gamma-1}(\frac{1}{p} A^{-T})_i y^\lambda $$

$$ = y^\lambda Z^\tau_A(y) \left( p\psi_{pA,i}^\gamma - \psi_{A,i}^\gamma \right) + Z^\tau_A(y) p T_{pA,i}^\gamma y^\lambda. $$
We therefore have that
\[(T_{A,i}^\vee + \psi_{A,i}^\vee) \Theta''_A = p \Theta'' \psi_{pA,i} + p \Theta'' T_{pA,i} + \Theta''_A \psi_{A,i} = p \Theta'' (T_{pA,i} + \psi_{pA,i}),\]
and so
\[\text{Fr}_A d_{pA,i} \psi_{A,i} = \Theta''_A (T_{pA,i} + \psi_{pA,i} + \psi_{A,i}) e_i p^{Q_A^{-1}} = d_{A,i} d_{A,i}'' \qed \]

**Lemma 9.3.** Let \( \hat{P}_{A,i} \in \text{End}_K \left( \mathcal{S}_A^t(K) \right) \) be defined by
\[
\hat{P}_{A,i}(x^\gamma y^\lambda) = \begin{cases} 0 & \text{if } (\gamma A^{-1})_i = 0; \\ x^\gamma y^\lambda & \text{otherwise.} \end{cases}
\]

1) If we define
\[
\hat{Q}_{A,i} = \hat{P}_{A,i} E_{A,i} E_{A,i}^\vee; \quad \hat{Q}_{A,i}^\vee = E_{A,i}^\vee E_{A,i} + \hat{Q}_{A,i},
\]
then
\[
\Delta^A Q_{A,i}^\vee = \hat{Q}_{A,i} \Delta^A \quad \text{and} \quad \Delta^A Q_{A,i} = \hat{Q}_{A,i}^\vee \Delta^A.
\]

2) If we define
\[
\hat{d}_{A,i} = (T_{A,i} + \psi_{A,i}) E_{A,i} \quad \text{and} \quad \hat{D}_{A,i}^\vee = (\theta_{A,i}^\vee + \psi_{A,i}^\vee) E_{A,i}^\vee,
\]
then
\[
[\hat{Q}_{A,i}, \hat{d}_{A,j}] = 0 = [\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}] \quad \text{and} \quad [\hat{Q}_{A,i}, \hat{d}_{A,j}] = \delta_{ij} \hat{d}_{A,j} \quad [\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}] = \delta_{ij} \hat{D}_{A,j}^\vee.
\]

**Proof.** We compute
\[
[\hat{Q}_{A,i}, \hat{d}_{A,k}] = [\hat{P}_{A,i} E_{A,i} E_{A,i}^\vee, (T_{A,j} + \psi_{A,j}) E_{A,j}] = \delta_{ij} \hat{P}_{A,i} (T_{A,j} + \psi_{A,j}) E_{A,j} = \delta_{ij} \hat{d}_{A,j},
\]
from which we see that
\[
[\hat{Q}_{A,i}^\vee, \hat{d}_{A,j}] = [E_{A,i}^\vee E_{A,i}, \hat{d}_{A,j}] + \delta_{ij} \hat{d}_{A,j} = \delta_{ij} (-\hat{d}_{A,j} + \hat{d}_{A,j}) = 0.
\]
Similarly,
\[
[\hat{Q}_{A,i}, \hat{d}^\vee_{A,j}] = \hat{P}_{A,i} \left( \theta^\vee_{A,j} + \psi^\vee_{A,j} \right) [E_{A,i} E^\vee_{A,j} E^\vee_{A,j}]
\]
\[
= -\delta_{ij} \hat{P}_{A,i} \left( \theta^\vee_{A,i} + \varphi^\vee_{A,i} \right) E^\vee_{A,i}.
\]
Since \( \hat{P}_{A,i} \theta^\vee_{A,i} (x^\gamma y^\lambda) \neq 0 \) implies \( (\gamma A^{-1}) (\lambda A^{-T})_i > 0 \), then
\[
(\gamma A^{-1})_i A_{ii} (\lambda A^{-T})_i > 0
\]
and thus \( \gamma A^{-1} \lambda^T > 0 \). Therefore, \( [\hat{Q}_{A,i}, \hat{d}^\vee_{A,j}] = 0 \). As a consequence,
\[
[\hat{Q}^\vee_{A,i}, \hat{d}^\vee_{A,j}] = [E^\vee_{A,i} E_{A,i}, \hat{d}^\vee_{A,j}] = \delta_{ij} \hat{d}^\vee_{A,j},
\]
which concludes the proof of (2). For (1), we compute
\[
\Delta^A Q^\vee_{A,i} = D^A P^\vee_{A,i} \otimes A^e_i e_i = \hat{P}_{A^T,i} D^A \otimes E_{A^T,i} E^\vee_{A^T,i} A^e_i = \hat{Q}_{A^T,i} \Delta^A;
\]
\[
\Delta^A Q_{A,i} = \Delta (e_i e_i^\vee + Q^\vee_{A,i}) = \left( E^\vee_{A^T, i} E_{A^T, i} + \hat{Q}_{A^T, i} \right) \Delta^A = \hat{Q}^\vee_{A^T, i} \Delta^A.
\]

**Proposition 9.4.** For each \( A \) in BH\(_n\)(\( \mathbb{F} \)) the Frobenius endomorphism defined by
\[
\text{Fr}_A = \left( (\Theta''_A \Theta'_{A}) \otimes \text{Id}_{\Lambda(\mathbb{F}^n)} \right) p^{Q_A + Q^\vee_A}
\]
is a chain map and
\[
\Delta^A \text{Fr}_A = \text{Fr}_{A^T} \Delta^A p^{2\text{ext} - n} p^{-2\hat{Q}_A} p^{2Q^\vee_A}.
\]

**Proof.** Since \( p^{Q^\vee_A} (\Theta'_{A} \otimes \text{Id}) = (\Theta'_{A} \otimes \text{Id}) p^{Q_A} \), then \( \text{Fr}_A = \text{Fr}_A'' \text{Fr}_A' \) is a chain map. For the second statement, \( D^A \Theta''_{A} \Theta'_{A} = \Theta''_{A^T} \Theta'_{A^T} D^A \) implies
\[
\Delta^A \text{Fr}_A = \Delta^A \left( \Theta''_{A} \Theta'_{A} \otimes \text{Id} \right) p^{Q_A + Q^\vee_A}
\]
\[
= \text{Fr}_{A^T} p^{-Q_{A^T} - Q^\vee_{A^T}} \Delta^A p^{Q_A + Q^\vee_A}
\]
\[
= \text{Fr}_{A^T} \Delta^A p^{-\hat{Q}_A - \hat{Q}^\vee_A} p^{Q_A + Q^\vee_A}
\]
\[
= \text{Fr}_{A^T} \Delta^A p^{2\text{ext} - n} p^{-2\hat{Q}_A} p^{2Q^\vee_A}.
\]

**Theorem 9.5.** Let \( \#_A \) (respectively \( \#^\vee_A \)) be the operator on \( S_A(\mathbb{K}) \) diagonalized by monomials and such that the eigenvalue of \( x^\gamma y^\lambda \) is the number of non-integer entries of \( \gamma A^{-1} \) (respectively \( \lambda A^{-T} \)). If \( \kappa \) is such that
\((\kappa\pi)^{(p-1)} = p\), then the twisted Frobenius endomorphism

\[
\text{TFr}_A = \text{Fr}_A(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}
\]

is a chain map, and

\[
H(\Delta^A)H(\text{TFr}_A) = H(\text{TFr}_{A^T})H(\Delta^A).
\]

**Proof.** Since \((\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}\) is diagonalized by monomials and acts trivially on \(\Lambda(\mathbb{K}^n)\), it commutes with \(d_A + d_A^\vee\). Therefore, \(\text{TFr}_A\) is a chain map. Using Proposition 9.4, we calculate

\[
\Delta^A \text{TFr}_A = \Delta^A \text{Fr}_A(\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}
\]

\[
= \text{Fr}_{A^T} \Delta^A p^{2 \text{ext} - n} p^{-2\widehat{Q}_A} p^{2Q_A^\vee} (\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}
\]

\[
= \text{TFr}_{A^T} \Delta^A p^{2 \text{ext} - n} p^{-2\widehat{Q}_A} p^{2Q_A^\vee} (\kappa\pi)p^{(p-1)(\#_A - \#_A^\vee)},
\]

where the last step follows from

\[
(\kappa\pi)^{-(p-1)(\#_A^\vee - \#_A^\vee^T)/2} \Delta^A = \Delta^A (\kappa\pi)^{(p-1)(\#_A - \#_A^\vee)/2}.
\]

Therefore, the theorem is proven if the eigenvalues of

\[
\text{(9.1)} \quad 2 \text{ext} - n - 2\widehat{Q}_A + 2Q_A^\vee \quad \text{and} \quad -(\#_A - \#_A^\vee)
\]

agree on a monomial basis \(x^{\gamma}y^{\lambda}e^I\) for \(H(B_A^\lambda(\mathbb{K}))\) for each \(\lambda \in G_A\). By Lemma 6.1 and Corollary 6.3 one can choose generators of the form \(x^{\gamma+I}y^{\lambda}e^I\), where \(|I| = n - |J_\lambda|\) and \(0 \leq (\lambda A^{-1})_i < 1\) for all \(i = 1, \ldots, n\). In particular, the eigenvalue of \(2 \text{ext} - n + 2Q_A^\vee - \#_A^\vee\) on \(x^{\gamma+I}y^{\lambda}e^I \in S(A)\) is \(|I|\). On the other hand, inspection of the bases for the cohomology of chains and loops given in Corollary 6.3 shows that \((2\widehat{Q}_A - \#_A) = \text{ext} \) on \(S(A)\), which concludes the proof. \(\square\)

10. Examples

**Example 10.1.** Let \(n = 1\) and \(A_{11} = 2\). Then \(W_A(x) = W_A^T(x) = x^2\) and \(G_A = G_{A^T} = \mathbb{Z}/2\mathbb{Z}\). The exterior operators are \(E_{A,1} = 2\pi e_1\) and \(E_{A,1}^\vee = \ldots\)
\[\frac{1}{2\pi}e^V_1.\] Moreover, \(\mathcal{R}_A^0(\mathbb{F}) = \mathbb{F}[x_1] \oplus y_1^2\mathbb{F}[y_1^2]\) and \(\mathcal{R}_A^1(\mathbb{F}) = y_1\mathbb{F}[y_1^2].\) The differentials are
\[
d(x_1^{\gamma_i}) = \gamma_1 x_1^{\gamma_i} e_1 + 2\pi x_1^{\gamma_i+2} e_1;
\]
\[
d^\vee(y_1^{\lambda_1} e_1) = \frac{1}{2\pi} \lambda_1 y_1^{\lambda_1} + y_1^{\lambda_1+2}.
\]
It follows that \(H(B_0^1(\mathbb{F})) = \mathbb{F}x_1 e_1\) and \(H(B_1^1(\mathbb{F})) = \mathbb{F}y_1\) are mapped one into the other by \(\Delta_A.\) The relations in cohomology are
\[
x_1^{2k+1} e_1 = (-2\pi)^{-1} (2k - 1) x_1^{2(k-1)+1} e_1 = \cdots = (-2\pi)^{-k} (2k - 1)!! x_1 e_1;
\]
\[
y_1^{2k+1} = (-2\pi)^{-1} (2k - 1) y_1^{2(k-1)+1} = \cdots = (-2\pi)^{-k} (2k - 1)!! y_1.
\]
Let \((c_m)\) be the sequence of rational numbers defined by
\[e^{\pi(t^p-t)} = \sum_{m \geq 0} c_m (-\pi)^m t^m.
\]
The action of the twisted Frobenius map in cohomology is thus
\[
H(TFr_A)(x_1 e_1)
= p(\kappa \pi)^{(p-1)/2} e^{\pi(x_1^{p^2}-x_1^2)} x_1^p e_1
= p(\kappa \pi)^{(p-1)/2} \sum_{m \geq 0} c_m (-\pi)^m x_1^{2(m+\frac{p-1}{2})+1} e_1
= p(\kappa \pi)^{(p-1)/2} \left( \sum_{m \geq 0} c_m (-\pi)^{-\frac{p-1}{2}-2-(m+\frac{p-1}{2})} (2(m-1)+p)!! \right) x_1 e_1
= p\kappa^{(p-1)/2} \left( \left( \frac{p-1}{2} \right)! + O(p) \right) x_1 e_1.
\]
Similarly,
\[
H(TFr_A)(y_1)
= p^2(\kappa \pi)^{-(p-1)/2} \left( \sum_{m \geq 0} c_m (-\pi)^{-\frac{p-1}{2}} 2^{-(m+\frac{p-1}{2})} (2(m-1)+p)!! \right) y_1
= p\kappa^{(p-1)/2} \left( \left( \frac{p-1}{2} \right)! + O(p) \right) y_1.
\]
Comparison with the non-commutative Weil conjectures of Kontsevich [5] seems to suggest a further overall rescaling of TFr_A. This is likely to be
relevant for arithmetic applications. We hope to come back to this point in future work.

**Example 10.2.** Consider the dual chains $W_A(x) = x_1^2x_2 + x_3^2$ and $W_{A^T}(x) = x_1^2 + x_1x_2^3$. The elements of $G_A = \mathbb{Z}^2/\mathbb{Z}^2A^T$ and $G_{A^T} = \mathbb{Z}^2/\mathbb{Z}^2A$ are given in Table 1. We can find basis elements $x^\gamma y^\lambda e^I$ of $C_A$ and $C_{A^T}$ as described in the proof of Theorem 9.5. Each row of Table 2 contains a pair of elements dual under $\Delta^A$ (up to constants), as well as the eigenvalues of $Q_A + Q_A^\vee$ and $(\#_A - \#_A^\vee)/2$ applied to $x^\gamma y^\lambda e^I$. Here we are using $*^A(e_1e_2) = 1$, $*^A(e_2) = -E_{A^T,1} = -2\pi e_1$ and $*^A(1) = E_{A^T,1}E_{A^T,2} = (2\pi e_1)(\pi e_1 + 3\pi e_2) = 6\pi^2 e_1 e_2$.

Note also that

$$\Delta^A(x_1^2x_2e_1e_2) = y_1^2y_2 \equiv 3\pi x_1x_2^3e_1e_2,$$

since $(d_{A^T} + d_{A^T}'(e_1) = 3\pi x_1x_2^3e_1e_2 + y_1^2y_2$.

We now turn to writing $\text{TFr}_A(x^\gamma y^\lambda e^I)$ in terms of this basis for a few elements. Since for any $x^\gamma$,

$$(\theta_{A,1} + \varphi_{A,1})(x^{\gamma e_1}) = \gamma_1 x^\gamma + \pi (2x^{\gamma e_1}) ;$$

$$(\theta_{A,2} + \varphi_{A,2})(x^{\gamma e_2}) = \gamma_2 x^\gamma + \pi (x^{\gamma e_1} + 3x^{\gamma e_2}) ,$$

in $H(B_A^1(\mathbb{F}))$ we have the relation

$$\gamma x^\gamma y^\lambda e^I = (-\pi)(x^{\gamma e_1} y^\lambda e^I, x^{\gamma e_2} y^\lambda e^I) A,$$
Table 2: Duality between $C_A$ and $C_{A^T}$ for $W_A(x) = x_1^2x_2 + x_3^3$.

<table>
<thead>
<tr>
<th>$C_A$</th>
<th>$Q_A + Q_A^\vee$</th>
<th>$(#<em>A - #</em>{A^T})/2$</th>
<th>$C_{A^T}$</th>
<th>$Q_{A^T} + Q_{A^T}^\vee$</th>
<th>$(#<em>{A^T} - #</em>{A^T}^\vee)/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1x_2e_1e_2$</td>
<td>2</td>
<td>1</td>
<td>$y_1y_2$</td>
<td>4</td>
<td>$-1$</td>
</tr>
<tr>
<td>$x_1x_2^2e_1e_2$</td>
<td>2</td>
<td>1</td>
<td>$y_1y_2^2$</td>
<td>4</td>
<td>$-1$</td>
</tr>
<tr>
<td>$x_1x_3e_1e_2$</td>
<td>2</td>
<td>1</td>
<td>$y_1y_2^3$</td>
<td>4</td>
<td>$-1$</td>
</tr>
<tr>
<td>$x_1x_2e_1e_2$</td>
<td>2</td>
<td>0</td>
<td>$x_1x_2^3e_1e_2$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$x_2y_1e_2$</td>
<td>3</td>
<td>0</td>
<td>$x_1y_2e_1$</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$x_2^2y_1e_2$</td>
<td>3</td>
<td>0</td>
<td>$x_1y_2^2e_1$</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$y_1y_2$</td>
<td>4</td>
<td>$-1$</td>
<td>$x_1x_2e_1e_2$</td>
<td>2</td>
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<tr>
<td>$y_1y_2^2$</td>
<td>4</td>
<td>$-1$</td>
<td>$x_1x_2^3e_1e_2$</td>
<td>2</td>
<td>1</td>
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<td>$y_1^2y_2$</td>
<td>4</td>
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<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$y_1^2y_2^2$</td>
<td>4</td>
<td>$-1$</td>
<td>$x_1^2x_2e_1e_2$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

which implies for $i = 1, 2$ that

$$x^{\gamma + e_i A} y^i e^I = (-\pi)^{-1} (\gamma A^{-1})_i x^\gamma y^i e^I.$$

Therefore, for $i = 1, 2$,

$$x^{\gamma + k_i e_i A} y^i e^I = (-\pi)^{-1} ((\gamma + (k_i - 1)e_i A) A^{-1})_i x^{\gamma + (k_i - 1)e_i A} y^i e^I$$

$$= (-\pi)^{-2} ((\gamma A^{-1})_i + (k_i - 1)) ((\gamma A^{-1})_i + (k_i - 2)) x^{\gamma + (k_i - 2)e_i A} y^i e^I$$

$$= (-\pi)^{-k_i} ((\gamma A^{-1})_i)_{(k_i)} x^\gamma y^i e^I.$$

Take $x_1x_2e_1e_2$ so that $\gamma = (1, 1)$ and $\gamma A^{-1} = (\frac{1}{2}, \frac{1}{6})$. Suppose that $p$ is a prime such that $6 | (p - 1)$. Then we can write

$$(p, p) = (1, 1) + \left( \frac{p - 1}{2}, \frac{p - 1}{6} \right) A,$$
which using Equation (10.1) gives

\[
\begin{align*}
\text{TF}_{\mathcal{A}}(x_1 x_2 e_1 e_2) \\
= p^2 (\kappa \pi)^{p-1} x_1^p x_2^p Z_{\mathcal{A}}(x) e_1 e_2 \\
= p^3 x_1^p x_2^p \left( \sum_{k_1 \geq 0} (-\pi)^{k_1} c_{k_1} x_1^{k_1} \right) \left( \sum_{k_2 \geq 0} (-\pi)^{k_2} c_{k_2} x_2^{k_2} \right) e_1 e_2 \\
= p^3 (-\pi)^{-\frac{2(p-1)}{3}} \left( \sum_{k_1, k_2 \geq 0} c_{k_1} c_{k_2} \left( \frac{1}{2} \right) \left( \frac{1}{6} \right) \right) x_1^{1/2} x_2^{1/2} e_1 e_2,
\end{align*}
\]

where we have used the fact that \( Z_{\mathcal{A}}(y) = 1 + \mathcal{O}(y_1, y_2) \). Next, consider

\[
\begin{align*}
\text{TF}_{\mathcal{A}}(x_2^2 y_1 e_2) \\
= p^3 e^{\pi (x_1^p - x_1)} e^{\pi (y_1^p - y_1)} x_2^{2p} y_1^p e_2.
\end{align*}
\]

By Equation (6.2), in cohomology we have the relation

\[
y^{\lambda + k_1 e_1 A^T} = (-\pi)^{-k_1} \left( (\lambda A^{-T})_1 \right) \lambda = (-\pi)^{k_1} \left( \frac{3\lambda_1 - \lambda_2}{6} \right) \lambda,
\]

which if \( 6 \mid (p - 1) \) implies that

\[
\begin{align*}
\text{TF}_{\mathcal{A}}(x_2^2 y_2 e_2) \\
= p^3 (-\pi)^{-\frac{7(p-1)}{6}} \left( \sum_{k_1, k_2 \geq 0} c_{k_1} c_{k_2} \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) \right) x_2^{2} y_1 e_2.
\end{align*}
\]

References


p-adic Berglund-Hübsch duality


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