**EVOLUTION OF HARMONIC MAPS WITH DIRICHLET BOUNDARY CONDITIONS**

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**INTRODUCTION**

In this paper we shall study a left over problem concerning the heat flow of harmonic maps on manifolds with boundary. Let \((M, g)\) be a compact smooth \(m\)-dimensional Reimannian manifold with nonempty smooth boundary \(\partial M\), and let \((N, h)\) be a compact smooth \(n\)-dimensional Reimannian manifold without boundary. We denote \(M \cup \partial M\) by \(\overline{M}\). Since \((N, h)\) can be isometrically embedded into an Euclidean space \(\mathbb{R}^k\), for some \(k > n\), we may view \(N\) as a submanifold of \(\mathbb{R}^k\).

In local coordinates on \(M\), the energy of a map \(u : M \to N \hookrightarrow \mathbb{R}^k\) is given by

\[
E(u) = \frac{1}{2} \int_M g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \sqrt{g} \, dx ,
\]

here and here after \((g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}\), \(g = \det(g_{\alpha\beta})\), \(1 \leq \alpha, \beta \leq m\) and a summation convention is employed.

The Euler-Lagrange equation associated with the functional (0.1) is

\[
\Delta u = A(u)(du, du) ,
\]

where \(\Delta\) denotes the Laplace-Beltrami operator on \(M\) and \(A(u)\) is the second fundamental form of \(N\) in \(\mathbb{R}^k\) at \(u\).

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We shall concern with the following evolution problem for a map $u : M \times \mathbb{R}_+ \to N$:

$$\frac{\partial u}{\partial t} = \Delta u - A(u)(du, du), \quad \text{for } (x, t) \in M \times \mathbb{R}_+, \tag{0.3}$$

$$u(x, 0) = u_0(x), \quad \text{for } x \in M, \quad \text{and} \quad u(\cdot, t)|_{\partial M} = u_0(\cdot)|_{\partial M}. \tag{0.4}$$

For simplicity, we assume also that $u_0$ is smooth on $\overline{M}$. It will be clear later on in the paper that the $C^{2,\alpha}$ smoothness of $M, \partial M, u_0$ and $N$ are sufficient for all purposes.

It is well known that (0.3)-(0.4) admits a unique smooth solution locally. The global existence of a smooth solution to (0.3)-(0.4) can be shown in the case that the Reimannian curvature of $N$ is nonpositive. (see, e.g., [H] and references therein). Without such curvature hypothesis on $N$, one can, in general, construct examples of finite-time blow-up solutions of (0.3)-(0.4) even in the case that $m = 2$; see [CDY].

On the other hand, Chen and Struwe [CS] established the global existence and partial regularity of weak solutions of (0.3) and (0.4), under an additional hypothesis that $\partial M = \phi$ (cf. also [S]). Here we have the following generalization of their result to the case that $\partial M$ is nonempty.

**Theorem.** There is a global weak solution $u : M \times \mathbb{R}_+ \to N$ to (0.3)-(0.4) with $\partial_t u \in L^2(M \times \mathbb{R}_+)$ and $\nabla u \in L^\infty(\mathbb{R}_+, L^2(M))$ which is smooth off a singular set $\Sigma$. Set $\Sigma$ is closed in $\overline{M} \times \mathbb{R}_+$ and has a locally finite $m$-dimensional Hausdorff measure with respect to the parabolic metric $(\delta((x, t), (y, s)) = |x - y| + \sqrt{|t - s|})$.

Moreover, as $t \to +\infty$ suitably, $u(\cdot, t)$ converges weakly in $H^1(M, N)$ to a harmonic map $u_\infty : M \to N$ with $u_\infty|_{\partial M} = u_0|_{\partial M}$, which is smooth off a set $\Sigma_\infty \subset \overline{M}$ whose $(m - 2)$-dimensional Hausdorff measure can be bounded in terms of $c^2$-norm of $u_0$ and $E(u_0)$.

As in [Ch], one can show that $\Sigma_t = \{(x, t) \in \Sigma\}$ has finite $(m - 2)$-dimensional Hausdorff measure for each $t \in \mathbb{R}_+$.

The proof of the above theorem follows from the same line of argument as that in [S] and [CS]. There are two principal difficulties. The first one is to establish the monotonicity inequality near the boundary $\partial M \times \mathbb{R}_+$. Here we
use, besides the integration by parts trick from [C], some careful estimates on approximate solutions. The second difficulty is to prove the small energy regularity theorem; see [S]. In order to use the Bochner-type inequality for the energy density of the map and mean-value inequality for subsolutions of the heat equations to derive $L^\infty$-estimates on the gradient of maps at those points near the boundary $\partial M \times \mathbb{R}_+$, we go back to the original equations for approximate solutions and obtain first the gradient estimates at boundary $\partial M \times \mathbb{R}_+$.

To simplify the presentation, we consider first the case $N$ is a standard sphere in an Euclidean space. The monotonicity inequality and the small energy regularity theorem are proven in Section 2 and Section 3, respectively. The general $N$ can be handled after some necessary modifications, and this is done in the final section.

1. Monotonicity Inequality

When $N$ is the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we consider, as in [CS], the following approximate solutions: $u = u^k, k = 1, 2, \ldots$,

$$
\begin{align*}
&u_t - \Delta u + k(|u|^2 - 1)u = 0 \quad (x, t) \in M \times \mathbb{R}_+ \\
&u(\cdot, t)|_{\partial M} = u_0|_{\partial M} \quad t \in \mathbb{R}_+ \\
&u(x, 0) = u_0(x) \quad x \in M
\end{align*}
$$

(1.1)

For any fixed $k = 1, 2, \ldots$, problem (1.1) has a unique smooth solutions $u = u^k$ with $\partial_t u^k, \nabla^2 u^k \in L^p((0, \infty) \times M)$ for all $1 < p < \infty$. Note that since $|u_0|(x) = 1$, then $|u^k|(x, t) \leq 1$ by the maximum principle for parabolic equations; see [LSU]. But we do not need such precise estimates. In general, any uniform $L^\infty$-bound on $u^k$ is sufficient for our purpose.

For fixed $k$, $u = u^k$ satisfies also the following energy estimate:

**Lemma 1.1.** Let $u_0 \in H^1(M, N)$. Then

$$
(1.2) \quad \int_M |\nabla u|^2 \, dM + \int_M \frac{k}{2} (|u|^2 - 1)^2 \, dM + \int_0^t \int_M |u_t|^2 \, dM \, dt
$$

$$
= \int_M |\nabla u_0|^2 \, dM \, , \quad \text{for all } t > 0 .
$$
Let \( \rho_0 \) be a suitably small positive constant such that for any \( p_0 \in \partial M \), one can choose a coordinate system \( \{x_a\} \) in such a way that the set \( B^{M}(p_0) = \{ p \in \overline{M} : \text{dist}_M(p, p_0) < \rho_0 \} \) corresponds to the half ball \( B^+(\rho_0) = \{ x \in \mathbb{R}^m, |x| < \rho_0, x_m > 0 \} \). For a regular solution \( u = u^k \) of (1.1), we define

\[
e_k(u) = \frac{1}{2} g^{\alpha \beta} u_{x_\alpha} u_{x_\beta} + \frac{k}{4} (|u|^2 - 1)^2 ;
\]

\[
G_{x_0}(x, t) = \{4\pi(t_0 - t)\}^{-m/2} \exp \left\{ -\frac{|x - x_0|^2}{4(t_0 - t)} \right\} ,
\]

where \( t < t_0, z_0 = (x_0, t_0) \in \overline{M} \times (0, \infty) ; \)

\[
G(x, t) = G_{x_0}(x, t) ;
\]

\[
T^+_R = \{(x, t) : x \in \mathbb{R}^m, -4R^2 < t < -R^2 \} ;
\]

\[
\Psi^+(R) = \int_{T^+_R} e_k(u) G \phi^2(x) g(x) \, dx \, dt ,
\]

here \( \phi \in C^{\infty}(B_{\rho_0}), 0 \leq \phi \leq 1, \phi(x) \equiv 1 \) for \( |x| \leq \rho_0/2 \). Thus \( \phi \) may be chosen so that \( ||\phi||_{C^2} \leq C(M) . \)

**Theorem 1.2 (Monotonicity Inequality).** Suppose that

\[
u = u^k : B^+_{\rho_0}(0) \times [-T, 0] \to \mathbb{R}^{n+1}
\]

is a regular solution of (1.1) (we may assume also that \( T \leq \rho_0^2 \) ). Then, for any \( 0 < R < R_0 \leq \sqrt{T}/2 \), we have

\[
(1.3) \quad \Psi^+(R) \leq \exp[c_*(R_0^{1-\varepsilon} - R^{1-\varepsilon})] \Psi^+(R_0)
\]

\[
+ c_*(R_0^{1-\varepsilon} - R^{1-\varepsilon})(E_0 + 1) , \quad \text{for any } \varepsilon \in \left(0, \frac{1}{2}\right) .
\]

where \( E_0 = E(u_0) \), and \( c_* \) is a constant depending only on \( M, N \) and \( C^2 \)-norm of \( u_0 \) on \( \partial M \). Here \( c_* \) may depend also on \( C^1 \)-norm of \( \phi \) which, after suitable choices of \( \phi \), is a constant depending only on \( M \).

**Proof.** For simplicity we present the proof for the case that \( M = \mathbb{R}^m_+ = \{ x \in \mathbb{R}^m : x_m > 0 \} \). In this case, we may choose \( \phi \) to be identically equal to 1. As in [CS], the general case follows easily.
Let \( u_R(x,t) = u(Rx, R^2t) \) and \( h_R(x) = h_R(x') = u_0(Rx') \), where \( x' = (x_1, \ldots, x_{m-1}) \). Denote \( V_R = \frac{d}{dR} u_R = \frac{(x \cdot \nabla u_R + 2t \partial_t u_R)}{R} \). Then,

\[
\Psi^+(R) = \frac{1}{2} \int_{T_1^+} \left\{ |\nabla u_R|^2 + \frac{k}{2} (|u_R|^2 - 1)^2 \right\} G \, dx \, dt
\]
\[
= \frac{1}{2} \int_{T_1^+} \left\{ |\nabla u_R|^2 + \frac{kR^2}{2} (|u_R|^2 - 1)^2 \right\} G \, dx \, dt
\]

\((\phi \equiv 1 \text{ in this case}). Thus

\[
\frac{d}{dR} \Psi^+(R) = \int_{T_1^+} \nabla V_R \nabla u_R G \, dx \, dt + \int_{T_1^+} kR^2 (|u_R|^2 - 1) u_R V_R G \, dx \, dt
\]

\[(1.4) \]

\[
\Delta = I + II + III .
\]

It is obvious that \( III \geq 0 \). For the first term, we have

\[(1.5) \]

\[
I = \int_{T_1^+} \nabla u_R \nabla \left( V_R - \frac{x'}{R} (\nabla x \cdot h_R) \right) G \, dx \, dt + \int_{T_1^+} \nabla u_R \cdot \nabla \left( \frac{x'}{R} \nabla x \cdot h_R \right) \nabla G \, dx \, dt
\]

\[
= \int_{T_1^+} \nabla u_R \cdot \nabla \left( \frac{x'}{R} \nabla x \cdot h_R \right) G \, dx \, dt - \int_{T_1^+} \Delta u_R \left( V_R - \frac{x'}{R} \nabla x \cdot h_R \right) G \, dx \, dt
\]

\[- \int_{T_1^+} \frac{x \cdot \nabla u_R}{2t} \left( V_R - \frac{x'}{R} \nabla x \cdot h_R \right) G \, dx \, dt .
\]

Here we have used the fact that \( \nabla G = \frac{x}{2t} G \). Hence by equation (1.1), one has

\[
I + II = - \int_{T_1^+} \left( \partial_t u_R + \frac{x \cdot \nabla u_R}{2t} \right) \left( V_R - \frac{x'}{R} \nabla x \cdot h_R \right) G \, dx \, dt
\]

\[
+ \int_{T_1^+} \nabla u_R \cdot \nabla \left( \frac{x'}{R} \nabla x \cdot h_R \right) G \, dx \, dt
\]

\[
+ \int_{T_1^+} kR^2 (|u_R|^2 - 1) u_R \left( \frac{x'}{R} \nabla x \cdot h_R \right) G \, dx \, dt
\]
We have

\[
A = \int_{T_1^+} \frac{R}{2t} V_R^2 G \, dx \, dt + \int_{T_1^+} \frac{R}{2t} V_R \left( \frac{x'}{R} \cdot \nabla x' h_R \right) G \, dx \, dt \\
+ \int_{T_1^+} k R^2 (|u_R|^2 - 1) u_R \left( \frac{x'}{R} \cdot \nabla x' h_R \right) G \, dx \, dt \\
+ \int_{T_1^+} \nabla u_R \cdot \nabla \left( \frac{x'}{R} \cdot \nabla x' h_R \right) G \, dx \, dt \\
\Delta = A + B + C + D.
\]

We have \( A = -\int_{T_1^+} \frac{R}{2t} V_R^2 G \, dx \, dt \geq 0, \)

\[
B = \int_{T_1^+} \frac{R}{2t} V_R \left( \frac{x'}{R} \cdot \nabla x' h_R \right) G \, dx \, dt \\
\geq -\frac{A}{4} + R \| \nabla u_0 \|_{L^\infty(M)}^2 \int_{T_1^+} \frac{|x'|^2}{2t} G \, dx \, dt \\
\Delta = -\frac{A}{4} - c_1,
\]

and \( D \geq -\int_{T_1^+} |\nabla u_R|^2 G \, dx \, dt - \int_{T_1^+} \left| \nabla \left( \frac{x'}{R} \cdot \nabla x' h_R \right) \right|^2 G \, dx \, dt \)

\[
\geq -\Psi^+(R) - c_2.
\]

where \( c_1 \leq c(m) R \| \nabla u_0 \|_{L^\infty(M)}^2 \leq c(m) \| \nabla u_0 \|_{L^\infty(M)}^2 \) (we shall assume also that \( R \leq 1 \)), and

\[
c_2 \leq c(m) \left( \| \nabla u_0 \|_{L^\infty(M)}^2 + R^2 \| \nabla^2 u_0 \|_{L^\infty(M)}^2 \right).
\]

To prove Theorem 1.2, it suffices to show

\[
(1.7) \quad C \geq -\frac{A}{4} - \frac{c_3}{R^\varepsilon} \left( \Psi^+(R) + 1 + E_0 \right), \quad \text{for } \varepsilon \in \left( 0, \frac{1}{2} \right),
\]

and for some constant \( c_3 \) depending on \( M, N, \) and \( u_0 \). In fact, (1.7) and above calculations imply that

\[
(1.8) \quad \frac{d}{dR} \Psi^+(R) \geq -\frac{c_4}{R^\varepsilon} \left( \Psi^+(R) + 1 + E_0 \right), \quad \text{for } \varepsilon \in \left( 0, \frac{1}{2} \right)
\]
and with \( c_4 = \max\{c_1 + c_3, c_3 + 1\} \). The conclusion of Theorem 1.2 follows from (1.8) by a simple integration. \( \square \)

The remainder of this section is devoted to showing (1.7) or equivalently the following estimate:

\[
(1.9) \quad \int_{T_t^+} kR^2 (|u_R|^2 - 1) u_R \left( \frac{x'}{R} \cdot \nabla x' h_R \right) G \, dx \, dt \leq \frac{A}{4} + \frac{c_3}{R^\epsilon} (\Psi^+(R) + 1 + E_0) .
\]

**Lemma 1.3.** There is a constant \( c_5 \) depending only on \( M, N \) and \( u_0 \) such that, for any \( \lambda \in (0, 1) \),

\[
(1.10) \quad \int_{T_t^+} kR^2 |u_R|^2 - 1 |u_R|^2 G \, dx \, dt \leq \frac{\lambda}{4} A + \frac{c_5}{\lambda} (\Psi^+(R) + 1) .
\]

**Proof.** Multiplying the equation (1.1) by \( u_R \phi(|u_R|^2 - 1)G \), where \( \phi \in C^\infty(\mathbb{R}) \), \( \phi(0) = 0 \) and

\[
\phi(s) = \begin{cases} 1 & \text{if } s \geq \frac{1}{k} \\ -1 & \text{if } s \leq -\frac{1}{k} \end{cases}, \quad \phi'(s) \geq 0 ,
\]

we obtain that

\[
\int_{T_t^+} \partial_t u_R \cdot u_R \phi(|u_R|^2 - 1)G \, dx \, dt + \int_{T_t^+} \nabla u_R \nabla (u_R \phi(|u_R|^2 - 1)G) \, dx \, dt
\]

\[
+ \int_{T_t^+} kR^2 (|u_R|^2 - 1)|u_R|^2 \phi(|u_R|^2 - 1)G \, dx \, dt = 0 .
\]

(Note that \( \phi(|u_R|^2 - 1) = 0 \) on the boundary \( x_m = 0 \).)

Also we have

\[
\int_{T_t^+} \nabla u_R \nabla (u_R \phi(|u_R|^2 - 1)G) \, dx \, dt
\]

\[
= \int_{T_t^+} |\nabla u_R|^2 \phi(|u_R|^2 - 1)G \, dx \, dt + 2 \int_{T_t^+} |u_R \nabla u_R|^2 \phi(|u_R|^2 - 1)G \, dx \, dt
\]

\[
+ \int_{T_t^+} \frac{x \cdot \nabla u_R}{2t} \cdot u_R \phi(|u_R|^2 - 1)G \, dx \, dt .
\]
Thus
\[
\int_{T^+_t} kR^2 |u_R|^2 - 1 |u_R|^2 G \, dx \, dt = \int_{T^+_t \cap \{||u_R|^2 - 1| \leq \frac{1}{4}\}} + \int_{T^+_t \cap \{||u_R|^2 - 1| > \frac{1}{4}\}} \leq \int_{T^+_t} |u_R|^2 R^2 G \, dx \, dt + \int_{T^+_t} kR^2 (|u_R|^2 - 1) \phi (|u_R|^2 - 1) |u_R|^2 G .
\]

Since $|u_R| \leq 1$ (bounded by a constant will be sufficient), the first term on the right-hand side is bounded by $c(m)R^2 \leq c(m)$.

The second term is, by above calculations, given by
\[
- \int_{T^+_t} \frac{2t}{2t} \partial_t u_R + x \cdot \nabla u_R u_R \phi (|u_R|^2 - 1)G \, dx \, dt
\]
\[
- \int_{T^+_t} |\nabla u_R|^2 \phi (|u_R|^2 - 1)G \, dx \, dt - 2 \int_{T^+_t} |u_R \cdot \nabla u_R|^2 \phi' (|u_R|^2 - 1)G \, dx \, dt
\]
\[
\leq - \frac{R}{2t} V_R \cdot u_R \phi (|u_R|^2 - 1)G \, dx \, dt - \int_{T^+_t} |\nabla u_R|^2 \phi (|u_R|^2 - 1)G \, dx \, dt ,
\]
and hence, it is bounded by, for any $\lambda \in (0, 1),$
\[
\Psi^+(R) + \frac{\lambda}{4} \int_{T^+_t} \frac{RV_R^2}{-2t} \, G + \frac{1}{\lambda} \int_{T^+_t} \frac{R}{-2t} |u_R|^2 G \, dx \, dt = \frac{\lambda}{4} A + \Psi^+(R) + \frac{1}{\lambda} c(m)R .
\]
This proves (1.10). $\square$

**Lemma 1.4.** There is a constant $c_6$ depending only on $\overline{M}, N$ and $u_0$ such that

(1.11)
\[
\int_{T^+_t} kR^2 | |u_R|^2 - 1 | G \, dx \, dt \leq \frac{\lambda A}{4} + c_6 \lambda^{-1}(1 + \Psi^+(R)) , \quad \text{for any } \lambda \in (0, 1). 
\]

**Proof.** Since $(|u_R|^2 - 1)^2 = (|u_R|^2 - 1)|u_R|^2 - (|u_R|^2 - 1)$, thus
\[
| |u_R|^2 - 1 | \leq (|u_R|^2 - 1)^2 + |u_R|^2 - 1 | u_R |^2.
\]
Therefore (1.11) follows from (1.10) and the definition of $\Psi^+(R)$. $\square$
Lemma 1.5. For any $\varepsilon \in (0, \frac{1}{2})$, there is a constant $c_\varepsilon$ depending on $M, N$ and $u_0$ such that

$$
(1.12) \quad \int_{T_1^+} k R^2 |u_R|^2 \left| |u_R|^2 - 1 \right| |x|^2 G \, dx \, dt \leq \frac{\lambda A}{4} + \frac{c_\varepsilon}{\lambda} (1 + E_0 + \Psi^+(R))/R^\varepsilon, \quad \text{for } \lambda \in (0, 1).
$$

Proof. We follow the same line of proof as that for Lemma 1.3. Multiplying (1.1) by $u_R \phi(|u_R|^2 - 1)|x|^2 G$, to obtain

$$
\int_{T_1^+} k R^2 \left| |u_R|^2 - 1 \right| |u_R|^2 |x|^2 G \, dx \, dt = \int_{T_1^+ \cap \{||u_R|^2 - 1| \leq \frac{1}{4} \}} \int_{T_1^+ \cap \{||u_R|^2 - 1| > \frac{1}{4} \}} + \int_{T_1^+} R^2 |u_R|^2 |x|^2 G \, dx \, dt + \int_{T_1^+} k R^2 (|u_R|^2 - 1) \phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt.
$$

The first term is again bounded by $c(m) R^2 \leq c(m)$, and the second term is now given by

$$
\begin{align*}
&\int_{T_1^+} - \frac{R}{2} V_R \cdot u_R \phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt - \int_{T_1^+} |\nabla u_R|^2 \phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt \\
&- 2 \int_{T_1^+} \nabla u_R \cdot u_R \phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt - 2 \int_{T_1^+} x \cdot \nabla u_R \cdot u_R \phi(|u_R|^2 - 1) G \, dx \, dt \\
&\leq \int_{T_1^+} 2 |\nabla u_R|^2 |x|^2 G \, dx \, dt + \int_{T_1^+} |u_R|^2 G \, dx \, dt \\
&+ \frac{\lambda}{4} \int_{T_1^+} \frac{R}{-2} V_R^2 G \, dx \, dt + \frac{1}{\lambda} \int_{T_1^+} \frac{R}{-2} |u_R|^2 |x|^4 G \, dx \, dt \\
&\leq \frac{\lambda}{4} A + \frac{c(m)}{\lambda} + 2 \int_{T_1^+} |\nabla u_R|^2 |x|^2 G \, dx \, dt.
\end{align*}
$$

Finally we estimate the last term $\int_{T_1^+} |x|^2 |\nabla u_R|^2 G \, dx \, dt$ as follows:

$$
\int_{T_1^+} |x|^2 |\nabla u_R|^2 G \, dx \, dt = \int_{-4}^{-1} \int_{|x|^2 \geq \frac{1}{R^2}, x_m > 0} |x|^2 |\nabla u_R|^2 G \, dx \, dt
$$
The first term is clearly bounded by $R^{-\varepsilon}\Psi^+(R)$ (see the definition of $\Psi^+(R)$). The second term is bounded by

$$c(m)R^{-\varepsilon}e^{-\frac{mR}{R^{16}}}{\int_{-4}^{-1}} |\nabla u_R|^2 \, dx \, dt \leq c(m)R^{-\varepsilon-m}e^{-\frac{u_{10}^{-\varepsilon}}{10}}{\int_{-4}^{-R^2}} |\nabla u|^2 \, dx \, dt$$

$$\leq c(m)R^{-\varepsilon-m+2}e^{-\frac{u_{10}^{-\varepsilon}}{16}} E(u_0).$$

Since $0 < R \leq 1$, the right-hand side of the above inequality is bounded by $c(m, \varepsilon)E(u_0)$. The conclusion (1.12) follows. □

Proof of (1.9).

$$\left| \int_{T^+_1} kR^2(|u_R|^2 - 1)u_R \left( \frac{x'}{R} \cdot \nabla_{x'} h_R \right) G \, dx \, dt \right|$$

$$\leq (||\nabla u_0||_{L^\infty(M)} + 1) \int_{T^+_1} kR^2|u_R| ||u_R|^2 - 1| |x|G \, dx \, dt$$

$$\leq (||\nabla u_0||_{L^\infty(M)} + 1) \int_{T^+_1} kR^2 |u_R|^2 - 1 |1 + |u_R|^2|x|^2|G \, dx \, dt.$$

Applying Lemma 1.4 and Lemma 1.5, one has the right-hand side of the above inequality is bounded by

$$(||\nabla u_0||_{L^\infty(M)} + 1) \left[ \frac{\lambda}{2} A + \frac{c_6 + c_7}{\lambda} (1 + E_0 + \Psi^+(R))/R^\varepsilon \right], \text{ for all } \lambda \in (0, 1),$$

$$\leq \frac{A}{4} + c_8 (1 + E_0 + \Psi^+(R))/R^\varepsilon.$$

The latter inequality follows by letting $\lambda = \frac{1}{2}(1 + ||\nabla u_0||_{L^\infty(M)})^{-1}$, and $c_8 = c_8(M, N, u_0)$. This completes the proof of (1.9) and hence the proof of Theorem 1.2. We note that one may take $\varepsilon = \frac{1}{4}$ in Theorem 1.2. □

2. SMALL ENERGY REGULARITY THEOREM

Having established the monotonicity inequality for all points on $\overline{M}$ (for points inside $M$ we refer to [S] and [CS]), we now want to prove the small
energy regularity theorem for solutions of (1.1) on $\overline{M}$. We shall consider only those points at the boundary $\partial M \times R_+$. If a point $p_0$ is at the interior of $M$ and the ball $B_{\rho_0}^M(p_0) = \{ p \in \overline{M} : \text{dist}_M(p,p_0) < \rho_0 \}$ is cut by $\partial M$, the result can be proved in the same way as that for the boundary points. We shall also refer to [S] and [CS] for the case that the ball $B_{\rho_0}^M(p_0)$ is contained entirely in $M$.

Denote $P_R(x_0) = \{(x,t) : |x - x_0| < R, |t - t_0| < R^2 \}$, $P_R(0)$, and $P_R^+ = P_R \cap \{x_m \geq 0 \}$.

As in the previous section, various constants should depend only on $M$, $\partial M$, $N$ and possibly $E_0$ and $u_0|_{\partial M}$. We have the following

**Theorem 2.1.** Let $u = u^k : B_{\rho_0}^+ \times [-T,T] \to N$ be a regular solution of (1.1) and assume that $T \leq \rho_0^2 \leq 1$. There exist constants $\varepsilon_0, \delta \in (0, \frac{1}{2})$ and $c$ such that if for some $0 < R \leq \min(\varepsilon_0, \sqrt{T}/2)$, the inequality

$$\Psi^+(R) \leq \varepsilon_0$$

is satisfied, then there hold

$$\sup_{P_R^+} e(u_k) \leq c \left[(\delta R)^{-2} + \|u_0\|_{L^2(\partial M)} \right].$$

**Proof.** We prove (2.2) by a contradiction argument. Suppose Theorem 2.1 is not true, after various normalizations as those in [CS] and [C], one is lead to the existence of a sequence of solutions $u_i$ of (1.1) in $P^+_1$ with the following properties:

(i) $\frac{\partial}{\partial t} u_i - \Delta u_i + k_i(|u_i|^2 - 1)u_i = 0$ in $P^+_1$,
(ii) $e_k(u_i) = \frac{1}{2} |\Delta u_i|^2 + \frac{k}{4} (|u_i|^2 - 1)^2 \leq 4$ in $P^+_1$,
(iii) $e_k(u_i)(x_i,0) = 1$, with $x_i \to 0$ as $i \to \infty$,
(iv) $u_{i|x_m=0} = h_i(x')$ with $|\nabla^2 h_i| \leq \varepsilon_i^2 \| \nabla^2 u_0 \|_{L^\infty(\partial M)} \to 0$, $|\nabla h_i| \leq \varepsilon_i \| \nabla u_0 \|_{L^\infty(\partial M)}$,
(v) $\int_{P^+_1} e_k(u_i) \, dx \, dt \leq \varepsilon_i \to 0^+$ as $i \to \infty$,
(vi) $|h_i| = 1$ (cf. [CS] and [C]).

Moreover, via the calculation of [CS], we have the following Bochner-type inequality for $e_k(u_i)$:

$$\partial_t e_k(u_i) - \Delta e_k(u_i) \leq c_0 e_k(u_i) \in P^+_1.$$
We now would like to obtain a contradiction from (i)--(vi) and (2.3).

To do so we may also assume that $k_i \geq 400$ in (ii). For, otherwise we would obtain from (i) and (iv) $W^{2,p}$-estimates for $u_i$, that is,

\[(2.4) \quad \|\nabla^2 u_i\|_{L^p(P_{i/2}^+)} + \|\nabla u_i\|_{L^p(P_{i/2}^+)} \leq c(p), \]

for $1 < p < \infty$ (see [LSU]).

Moreover, by (v) and standard estimates for semilinear heat equations (cf. [LSU]), one has

\[(2.5) \quad \sup_{P_{i/2}^+} e_k(u_i) \leq c \left( \varepsilon_i + \|\nabla h_i\|^2_{L^2(\partial M)} \right) \leq c \varepsilon_i. \]

The latter inequality contradicts to (iii).

Now since $k_i \geq 400$, (ii) implies in particular that

\[|u_i|^2 - 1 \leq \frac{1}{5}. \]

We introduce a decomposition (polar decomposition) for $u_i = R_i W_i, R_i = |u_i|, W_i = \frac{u_i}{|u_i|}$ both are now well-defined. Moreover

\[(2.6) \quad |\nabla u_i|^2 = R_i^2 |\nabla W_i|^2 + |\nabla R_i|^2 \leq 4 \in P_{1}^+ \quad \text{and} \quad R_i \in \left[ \frac{4}{5}, 1 \right]. \]

From (i) we also derive the following equations

\[(2.7) \quad \frac{\partial W_i}{\partial t} - \Delta W_i - |\nabla W_i|^2 W_i - 2 \frac{\nabla R_i}{R_i} \cdot \nabla W_i = 0 , \]

and

\[(2.8) \quad \frac{\partial R_i}{\partial t} - \Delta R_i + k_i (R_i^2 - 1) R_i + |\nabla W_i|^2 R_i = 0 . \]

Since $|\nabla W_i| \leq 7, |\frac{2\nabla R_i}{R_i}| \leq 5$ by (2.6), we obtain from (2.7) that (cf.[LSU])

\[(2.9) \quad \sup_{P_{2i/3}^+} |\nabla W_i|^2 \leq c_p \left[ \int_{P_{i+1}} |\nabla W_i|^2 + \|\nabla W_i\|^2_{L^p(P_{i}^+)} + |\nabla h_i|^2_{L^2(\partial M)} \right], \quad \text{for} \quad p = 2m \]

\[\leq c \varepsilon_i^\frac{1}{m} \to 0 \quad \text{as} \quad i \to \infty . \]

In particular, we have $\|\nabla W_i\|_{L^\infty(\partial M \times [-\frac{3}{2}, 0])} \leq c \varepsilon_i^\frac{1}{m} \to 0$. 

Next we look at the equation for $\rho_i = 1 - R_i$:

\begin{equation}
\frac{\partial \rho_i}{\partial t} - \Delta \rho_i = R_i |\nabla W_i|^2 - k_i \rho_i R_i(1 + R_i) .
\end{equation}

Since $0 \leq \rho_i \leq 1$ in $P_i^+$ and $\rho_i = 0$ on $\{x_m = 0\}$, we have

\begin{equation}
\begin{cases}
\frac{\partial \rho_i}{\partial t} - \Delta \rho_i \leq c \varepsilon_i^m \in P_{2/3}^+ \\
\rho_i \big|_{x_m=0} = 0.
\end{cases}
\end{equation}

Hence, for $x \in P_{1/2}^+$,

$$\rho_i(x) \leq c \left( \int_{P_{2/3}^+} \rho_i + \varepsilon_i^\frac{1}{m} \right) x_m , \quad (\text{cf. [LSU]})$$

$$\leq c x_m \varepsilon_i^\frac{1}{m} , \quad \text{by (v)}.$$

Therefore, we also have

\begin{equation}
\| \nabla \rho_i \|_{L^\infty(\partial M \times [-\frac{1}{2}, 0])} = \| \nabla R_i \|_{L^\infty(\partial M \times [-\frac{1}{2}, 0])} \leq c \varepsilon_i^\frac{1}{m} .
\end{equation}

Let $\tilde{e} = \max\{0, e_k(u_i) - 2c \varepsilon_i^\frac{1}{m}\}$, then (2.3) implies that

$$\partial_t \tilde{e} - \Delta \tilde{e} \leq c_0 \tilde{e} , \quad \text{in } P_{1/2}^+ .$$

Moreover, above arguments show $\tilde{e} \big|_{x_m=0} = 0$. Thus the Moser's estimate for the linear heat equations implies that

\begin{equation}
\sup_{P_{1/4}^+} \tilde{e} \leq c \int_{P_{1/2}^+} \tilde{e} \leq c \int_{P_{1/2}^+} e_k(u_i) \leq c \varepsilon_i ,
\end{equation}

which goes to zero as $i \to \infty$.

(2.13) is an obvious contradiction to (iii), and thus we complete the proof of Theorem 2.1. □

Remark 2.1. The proof of the main theorem (stated in the introduction) is now identical to that in [S], [C], and [CS], and therefore we omit the details here.
3. General Target Manifolds

Here we shall consider the target manifold $N$ being a compact smooth Reimannian submanifold of $\mathbb{R}^{n+\ell}$ without boundary. Instead of (1.1), we consider approximate solutions, $u = u_k, k = 1, 2, \ldots$, to following equations: (cf. [C] or [CS])

\begin{equation}
\partial_t u - \Delta u + k\chi'(\text{dist}^2(u, N)) \frac{d}{du} \left( \frac{\text{dist}^2(u, N)}{2} \right) = 0,
\end{equation}

for $(x, t) \in M \times \mathbb{R}_+$, and

\begin{equation}
u(x, 0) = u_0(x), \quad u(\cdot, t)|_{\partial M} = u_0(\cdot)|_{\partial M}, \quad \text{for } t \in \mathbb{R}_+
\end{equation}

where $\chi$ is smooth monotone function on $\mathbb{R}_+$ with $\chi(x) = s$ for $0 \leq s \leq \delta_N^2, \chi(x) \equiv 2\delta_N^2$, for $s \geq 2\delta_N^2$. Here $\delta_N \in (0, 1/2)$ is a positive constant so that the nearest neighbor projection $\pi_N : \mathbb{R}^{n+\ell} \to N$ is well-defined and smooth in a $2\delta_N$-neighborhood of $N$. Moreover, we may also assume that $\|D\pi_N(u) - P_N(u)\| \leq 1/4$, for $u \in \mathbb{R}^{n+\ell}, \text{dist}(u, N) \leq 2\delta_N$. Here $P_N(u)$ is orthonormal projection of $\mathbb{R}^{n+\ell}$ onto $T_{\pi_N(u)}N$, the tangent space of $N$ at $\pi_N(u)$.

For each fixed $k = 1, 2, \ldots$, it is again standard to show (cf. [LSU]) that there is a unique smooth solution of (3.1)–(3.2). Moreover, it satisfies the energy identity (1.2) (with the term $\int_M (|u|^2 - 1)^2 dM$ replacing by $\int_M \chi(\text{dist}^2(u, N)) dM$).

As in [C] and [CS], we define

\[ e_k(u) = \frac{1}{2} g^{\alpha\beta} u_\alpha \cdot u_\beta + \frac{k}{4} \chi(\text{dist}^2(u, N)) \]

and $\Psi^+(R)$ as before, etc... We claim $\Psi^+(R)$ satisfies the monotonicity inequality (1.3).

To see this, we follow the proof of Theorem 1.2. As in (1.4), we have (for $M = \mathbb{R}_+^M$ case)

\begin{equation}
\frac{d}{dR} \Psi^+(R) \geq \int_{T^+_1} \nabla V_R \cdot \nabla u_R G \, dx \, dt \\
+ \int_{T^+_1} \frac{1}{2} k R^2 \chi'(\text{dist}^2(u_R, N)) \left( \frac{d}{du} \text{dist}^2(u_R, N) \right) V_R G \, dx \, dt.
\end{equation}
Applying integration by parts as in (1.5) and (1.6), we then obtain

\[ \frac{d}{dR} \Psi^+(R) \geq A + B + C + D, \]

where

\[ A = - \int_{T_1^+} \frac{R}{2t} V_R^2 G \, dx \, dt \geq 0, \]

\[ B \geq \frac{A}{4} - c_1, \]

\[ D \geq -\Psi^+(R) - c_2. \]

Here \( A, B, D \) are as in (1.6) before, and the absolute value of \( C \) is given by the left-hand side of (3.6) below.

Hence the issue is to verify

\[ C \geq -\frac{A}{4} - \frac{c_3}{R^\varepsilon} (\Psi^+(R) + E_0 + 1), \quad \varepsilon \in \left(0, \frac{1}{2}\right) \]

where \( c_1, c_2 \) and \( c_3 \) are constants as before.

(3.5) is equivalent to

\[ \left| \int_{T_1^+} k R^2 \chi'(\text{dist}^2(u_R, N)) \left( \frac{d}{du} \text{dist}^2(u_R, N) \right) \frac{x'}{R} \nabla \chi' h_R G \, dx \, dt \right| \]

\[ \leq \| \nabla u_0 \|_{L^\infty(\partial M)} \int_{T_1} k R^2 \chi'(\cdot) \text{dist}(u_R, N) |x| G \, dx \, dt \]

\[ \leq \frac{A}{4} + \frac{c_3}{R^\varepsilon} (\Psi^+(R) + E_0 + 1). \]

For \( \text{dist}(u_R, N) < 2\delta_N \), we let \( \frac{d}{du} \text{dist}^2(u_R, N) = 2\nu(u_R) \text{dist}(u_R, N) \). Then \( \nu(u_R) \) is a well-defined unit vector as long as \( \text{dist}(u_R, N) > 0 \). Moreover, \( \nu(u_R) \text{dist}(u_R, N) \) is a smooth function of \( u_R \), for \( u_R \) in \( 2\delta_N \)-neighborhood of \( N \).

We let \( \phi(s) \) be a monotone increasing, smooth function on \( R_+ \) with \( \phi(s) \equiv 0 \) for \( s \leq \frac{1}{4k^2} \) and \( \phi(s) \equiv 1 \) for \( s \geq \frac{1}{k^2} \). As in Section 2, we would like to multiply equation (3.1) by \( \phi(\text{dist}^2(u_R, N)) \nu(u_R) \chi'(\text{dist}^2(u_R, N)) G \). Since this is a smooth function of \( u_R \) and since it is supported on \( \{ u_R \in \mathbb{R}^{n+\ell} : \text{dist}(u_R, N) \in \} \)
\[ \left[ \frac{1}{2t}, \sqrt{2\delta_N} \right] \} \) we should find an equation for \( \text{dist}(u_R, N) \) in \( \Omega = \{(x, t) \in T_1^+ : 0 < \text{dist}(u_R, N) \leq \sqrt{2\delta_N} \}. \]

Let \( u_R = v_R + (u_R - v_R), v_R = \pi_N(u_R) \). Then \( (u_R - v_R) = \nu(u_R)\text{dist}(u_R, N) \), for \( (x, t) \in \Omega \). Denote \( d = \text{dist}(u_R, N) \), then \( d \) satisfies

\[
(3.7) \quad d_t - \Delta d - d(\Delta \nu(u_R), \nu(u_R)) + R^2 k \chi'(d^2)d - \langle \Delta \pi_N(u_R), \nu(u_R) \rangle = 0 \quad \text{in} \quad \Omega.
\]

(Note that (3.7) is simply the component of (3.1) in \( \nu(u_R) \) direction.)

We note that \(-\langle \Delta \nu(u_R), \nu(u_R) \rangle = |\nabla \nu(u_R)|^2 \) and that

\[
|\nabla \nu| \leq \|D\pi_N\|_{L^\infty}|\nabla u_R| \leq C|\nabla u_R|.
\]

Hence \( |\nabla (u_R - v_R)| \leq C|\nabla u_R| \), and \( |\langle \Delta \pi_N(u_R), \nu(u_R) \rangle| \leq C|\nabla u_R|^2 \). The last inequality follows from a direct computation, see e.g., (3.15) below.

We therefore have

\[
(3.8) \quad d_t - \Delta d + |\Delta \nu(u_R)|^2 + \chi'(d^2)kR^2 d \leq C|\nabla u_R|^2, \quad \text{for} \quad (x, t) \in \Omega.
\]

Now let us estimate first the quantity

\[
\int_{T_1^+} kR^2 \text{dist}(u_R, N) \chi' \left( \text{dist}^2(u_R, N) \right) G \, dx \, dt
\]

\[
\leq \int_{T_1^+ \cap \{(\text{dist}(u_R, N) \leq \frac{1}{2}\}} R^2 \|\chi^\prime\|_{L^\infty} G \, dx \, dt
\]

\[
+ \int_{T_1^+ \cap \{\frac{1}{2} < \text{dist}(u_R, N) \leq \delta_N\}} kR^2 \text{dist}(u_R, N) \phi(\text{dist}^2(u_R, N)) |\chi'(\text{dist}^2(u_R, N))|^2 G \, dx \, dt
\]

\[
+ \int_{T_1^+} kR^2 \chi(\text{dist}^2(u_R, N))G \, dx \, dt \cdot \frac{\|\chi^\prime\|_{L^\infty}}{\delta_N}
\]

\[
\leq \frac{c_0}{\delta_N} \Psi^+(R) + c_0 + \int_{T_1^+} kR^2 \text{dist}(u_R, N) \phi(\text{dist}^2(u_R, N)) \left[\chi'(\text{dist}^2(u_R, N))\right]^2 G \, dx \, dt
\]

\[
\Delta I + c_0 \left(1 + \frac{\Psi^+(R)}{\delta_N}\right).
\]

Therefore, via \( \chi' \geq 0 \),

\[
\int_{T_1^+} kR^2 \text{dist}(u_R, N) G \, dx \leq I + 2c_0 \left(1 + \frac{\Psi^+(R)}{\delta_N}\right).
\]
To estimate $I$, we multiply equation (3.8) by $\phi(d^2)\chi'(d^2)G$ to obtain

\begin{equation}
\begin{aligned}
(3.9) \quad & \int_{T^+_1}^1 (d_t - \Delta d)\phi(\cdot)\chi'(\cdot)G \, dx \, dt + \int_{T^+_1}^1 \phi(\cdot)(\chi'(\cdot))^2kR^2dG \, dx \, dt \\
& \leq \int_{T^+_1}^1 c|\nabla u|^2\phi(\cdot)\chi'(\cdot)G \, dx \, dt \leq c\Psi^+(R)
\end{aligned}
\end{equation}

But

\begin{align*}
\int_{T^+_1}^1 -\Delta d \phi(\cdot)\chi'(\cdot)G \, dx \, dt \\
= \int_{T^+_1}^1 |\nabla d|^2(2\chi''(\cdot)\phi + 2\phi(\cdot)\chi')dG \, dx \, dt + \int_{T^+_1}^1 \frac{x}{2t} \cdot \nabla d\phi(\cdot)\chi'(\cdot)G .
\end{align*}

Since $|\nabla d|^2 \leq |\nabla (u - \pi_N u)|^2 \leq c|\nabla u|^2$ we obtain, from (3.9), that

\begin{equation}
I \leq c\Psi^+(R) - \int_{T^+_1}^1 \left( d_t + \frac{x}{2t} \nabla d \right) \phi(\cdot)\chi'(\cdot)G
\end{equation}

\begin{equation}
\leq c\Psi^+(R) + \frac{\lambda}{4} A + \frac{c(m)}{\lambda} , \quad \forall \lambda \in (0,1) .
\end{equation}

Here we have used the fact that

\begin{align*}
u_R &= d(u_R)\nu(u_R) + \pi_N(u_R) \quad \text{and} \quad \left| d_t + \frac{x}{2t} \nabla d \right|^2 \leq \frac{R}{2|t|} |x \cdot \nabla u_R + 2t \partial_t u_R|^2 / R^2 .
\end{align*}

Similarly, if we multiply the equation (3.1) and (3.8) by $|x|^2G\phi(d^2)\chi'(d^2)$, then we obtain, as in Section 1, that

\begin{equation}
\int kR^2\text{dist}(u_R, N)|x|^2G \, dx \leq \frac{\lambda}{4} A + \frac{c(m)}{\lambda} + \frac{c}{R^2} (\Psi^+(R) + 1 + E_0) .
\end{equation}

This completes the proof of the monotonicity inequality.

Finally, to the end of the paper, we outline the modification for the proof of Theorem 2.1 for general $N$. As in the proof of Theorem 2.1, it reduces to show the following is impossible (cf. also [C] and [CS]): there is a sequence of $u^i$ solutions of (3.1) such that

\begin{enumerate}
\item[(i)] $\frac{\partial u^i}{\partial t} - \Delta u^i + k_i\chi'(\text{dist}^2(u^i, N)) \frac{d}{du} \left( \text{dist}^2(u^i, N) \right) = 0$ in $P^+_1$,
\item[(ii)] $e_k(u^i) = \frac{1}{2} |\nabla u^i|^2 + \frac{k_i}{4} \chi(\text{dist}^2(u^i, N)) \leq 4$, in $P^+_1$,
\item[(iii)] $e_k(u^i)(x_i, 0) = 1$, with $(x_i, 0) \in P^+_1$ and $x_i \to 0$, as $i \to \infty$.
\end{enumerate}
(iv) \[ u^i|_{x_m=0} = h_i(x'), \| \nabla h_i \|_{L^\infty(\partial M)} + \| \nabla^2 h_i \|_{L^\infty(\partial M)} \leq \delta_i \to 0, \]

(v) \[ \int_{P_{\delta}^+} e_k(u^i) \, dx \, dt = \varepsilon_i \to 0 \text{ as } i \to +\infty \]

(vi) \[ h_i(x') \in N. \]

Moreover,

\[
\frac{\partial}{\partial t} e_k(u^i) - \Delta e_k(u^i) \leq c_0 e_k(u^i) \in P_{\delta}^+. \]

Also \( k_i \to +\infty \) as \( i \to \infty \).

As (2.7) and (2.8), we consider the equations satisfied by \( \pi_N(u^i)(x,t) \in N \) and \( u^i - \pi_N(u^i) \perp T_{\pi_N(u^i)}N \). To do so we choose a point \( (x_0, t_0) \in P_{\delta}^+ \) and coordinate systems of \( \mathbb{R}^{n+\ell} \) so that near \( \pi_N u(x_0, t_0) = 0 \in \mathbb{R}^{n+\ell} \), \( N \) can be represented by a graph \( G : B_{4\delta}^n(0) \to \mathbb{R}^{\ell} \) where \( \delta \in (0, \delta_{N/4}) \) is a constant depending only on \( N \). Moreover, \( G \) satisfies

\[
N \cap (B_{4\delta}^n(0) \times [-4\delta, 4\delta]) = \text{graph}(G). \]

and

\[
|\nabla^2 G| + |\nabla^3 G| \leq c_N, \text{ on } B_{\delta}^n(0), \text{ with } c_N 4\delta < \frac{1}{10}. \]

Since (ii), we may assume that \( u(P_{\delta}^+) \subset B_{3\delta}^n(0) \times [-3\delta, 3\delta] \).

We also choose a smooth orthonormal from \( \{e_1, \ldots, e_{n+\ell}\} \) along graph \( G \) so that \( e_i(0) = (0, \ldots, 1, 0, \ldots, 0) \), and that \( \{e_1, \ldots, e_n(p)\} \) span \( T_p N \).

Let us define a diffeomorphism \( F : u \in \mathbb{R}^{n+\ell} \to V \in \mathbb{R}^{n+\ell} \) near \( 0 \in \mathbb{R}^{n+\ell} \) as follows:

\[
\begin{cases} 
V_j = e_j(0) \cdot \pi_N(u), & \text{for } j = 1, \ldots, n, \text{ and} \\
V_j = e_j(\pi_N u) \cdot (u - \pi_N u), & \text{for } j = n+1, \ldots, n+\ell.
\end{cases}
\]

(Note that \( \pi_N(u) = (V_1, \ldots, V_n, G(V_1, \ldots, V_n)). \))

Equivalently, one has

\[
(3.14) \quad u = (V_1, \ldots, V_n, G(V_1, \ldots, V_n)) + \sum_{j=n+1}^{n+\ell} V_j e_j(V),
\]

where \( e_j(V) = e_j(V_1, \ldots, V_n, G(V_1, \ldots, V_n)). \)
Now we calculate the equation for \( u^i - \pi_N u^i \) and \( \pi_N u^i \) at the point \((x_0, t_0) \in P^{i+}_d\). We may also assume

\[
u_i(x_0, t_0) = (0, \ldots, 0, d_i), \quad d_i = |u_i(x_0, t_0) - \pi_N u_i(x_0, t_0)| \geq 0
\]

for simplicity. At \((x_0, t_0)\), (i) reduces to

\[
(3.15) \quad (I_n + d_i M_1) \frac{\partial V^T}{\partial t} - (I_n + d_i M_2) \Delta V^T = M_3 (\nabla V^T, \nabla V^T) + d_i M_4 (\nabla V^T, \nabla V^T),
\]

and

\[
(3.16) \quad \frac{\partial V^\perp}{\partial t} - \Delta V^\perp = M_5 (\nabla V^T, \nabla V^T) + d_i M_6 (\nabla V^T, \nabla V^T) - k \chi'(d_i^2) d_i e_{n+\ell}(0)
\]

where \( V^T = (V_i, \ldots, V_n), V^\perp = (V_{n+\ell}, \ldots, V_{n+\ell}), d_i M_1, d_i M_2, d_i M_4, d_i M_6 \) are smooth matrix-valued functions of \( V \) and is bounded by \( c d_i \) at \((x_0, t_0)\), \( d_i = |\pi_N u^i - u^i| \), \( M_3 \) and \( M_5 \) are also smooth matrix-valued functions of \( V \). Note that all \( M_j \)'s depend only on \( N \), hence the function \( G \) definition \( N \), and bounded by \( \|\nabla^2 G\|_{L^\infty} + \|\nabla^2 G\|_{L^\infty} \), and also that \( |\nabla V| \leq c|\nabla u^i| \).

From (3.15) one thus concludes that \( \pi_N u^i \) satisfies an inequality of the form

\[
(3.17) \quad \left| A \frac{d(\pi_N u^i)}{dt} - B \Delta(\pi_N u^i) \right| \leq c_N |\nabla \pi_N u^i|, \quad (x, t) \in P^{i+}_d \quad \text{with}
\]

\[
\|A - I_n\|_{L^\infty} + \|B - I_n\|_{L^\infty} \leq d_N |u^i - \pi_N u^i| \to 0 \quad \text{as} \quad i \to \infty.
\]

Moreover, \( \pi_N u^i|_{x_m=0} = h_i(x') \). Hence the \( W^{2,p} \) estimate for linear parabolic equations and (3.17) imply that

\[
\|\nabla \pi_N (u^i)\|^2_{L^\infty(P^{i+}_d)} \leq c_\delta \left( \int_{P^{i+}_d} |\nabla u^i|^2 + \|\nabla h_i\|_{L^2}^2 \right) \leq c_\delta (\varepsilon_i + \delta_i) \to 0 \quad \text{as} \quad i \to \infty.
\]

On the other hand, if we take the component of equation (3.16) in \( e_{n+\ell}(0) \) direction, we obtain

\[
(3.18) \quad \frac{\partial d}{\partial t} - \Delta d \leq -k \chi'(d^2) d + c|\nabla u|^2
\]
whenever \( d(x, t) > 0 \), (cf. also (2.10)) and \((x, t) \in P^+_\delta \) since \( d\big|_{x_m=0} = 0 \), we obtain from (3.18) that

\[
\begin{align*}
d(x, t) & \leq \tilde{d}(x, t) \quad \text{in } P^+_\delta \\
\frac{\partial}{\partial t} \tilde{d} - \Delta \tilde{d} & = c|\nabla u|^2 \quad \text{in } P^+_\delta \\
\tilde{d} & = d \quad \text{on } \partial P^+_\delta - \{t = 0\}.
\end{align*}
\]

It is easy to see, from (ii) and (v) that

\[
\tilde{d} \leq c\varepsilon_i^n x_m \to 0, \quad \text{in } P^+_{\delta/2}, \quad \text{as } i \to \infty,
\]

and therefore

\[
e_k(u^i)\big|_{x_m=0} \to 0, \quad \text{for } -\frac{\delta}{2} \leq t \leq 0.
\]

The desired contradiction follows as before.

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