ON PROJECTIVELY FLAT HERMITIAN MANIFOLDS

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Let \((M^n, g)\) be a \(n\)-dimensional compact hermitian manifold, with \(n \geq 2\). \((M, g)\) will be called projectively flat, if its curvature matrix is of the form \(\Theta = \alpha I_n\), where \(\alpha\) is a \((1, 1)\)-form. Note that any metric conformal to \(g\) would also be projectively flat. In §1, we shall classify such manifolds, and in §2, we will give an application which may be considered as a generalization to higher dimensions of the Bogomolov’s Theorem on \(\text{VII}_0\) surfaces.

First of all, let us correct an error in our previous paper [L-Y-Z]. We found this error after the paper was in print. On page 220, the vanishing of \(c^2\) and \(c_2\) does not imply that the Hermitian-Einstein metric \(h\) is flat, but only projectively flat, i.e., \(\Theta = \alpha I_2\). So the argument there is incomplete. However, this gap can be easily fixed by applying the results of P. Gauduchon ([G]) and D. Fried ([F]). Start from the projectively flat compact hermitian surface \((S, h)\). By [G], \(h\) is locally conformally Kähler. That is, there exists a covering \(\{U_\alpha\}\), and \(f_\alpha \in C^\infty(U_\alpha, \mathbb{R})\), such that each \(e^{f_\alpha}g\) is a Kähler metric in \(U_\alpha\). Note \(e^{f_\alpha}g\) is also projectively flat, hence flat, and in \(U_\alpha \cap U_\beta\), \(f_\alpha - f_\beta\) is a constant.

Therefore, \(S\) is a complex similarity manifold. By Theorem 2 of [F], it is either covered by a complex 2-torus or a Hopf surface. This completes the proof of Bogomolov’s theorem.  

1. PROJECTIVELY FLAT MANIFOLDS

Now let us consider the projectively flat manifolds in general dimensions. First let us fix some notations. On a hermitian manifold \((M^n, g)\), let \(e = \)
(\(e_1, \ldots, e_n\)) be a local unitary frame and \(\varphi = (\varphi_1, \ldots, \varphi_n)\) its dual coframe. Let \(\theta, \Theta\) the \(n \times n\) matrices of connection and curvature under \(e\), and \(\tau = (\tau_1, \ldots, \tau_n)\) the torsion forms under \(e\). Each \(\tau_i\) is a \((2,0)\)-form. The structure equations and the first Bianchi identity are:

\[
d\varphi = \varphi \land \theta + \tau, \quad d\theta - \theta \land \theta = \Theta, \quad d\tau = \varphi \land \Theta - \tau \land \theta
\]

Write \(\tau_i = \frac{1}{2} \sum_{j,k=1}^n T^i_{jk} \varphi_j \land \varphi_k\), where \(T^i_{jk} = -T^i_{kj}\), and denote by \(\omega = \varphi \land \varphi^*\) the Kähler form of \(g\) (we omit the factor \(\sqrt{-1}\)).

Consider the Gauduchon torsion 1-form \(\eta ([G])\) defined by

\[
\eta = \frac{1}{n-1} \sum_{j,k=1}^n T^k_{jk} \varphi_j.
\]

It is easy to check that \(\partial(\omega^{n-1}) = (n-1)\eta \land \omega^{n-1}\), hence is uniquely determined and globally defined. First of all, one has:

**Lemma 1.** If \(\Theta = \alpha I_n\), then \(\overline{\partial} \eta = \alpha\).

*Proof.* Write the \((0,1)\) part of \(\theta^{ii} = \sum_{l=1}^n A_{ij,l} \overline{\varphi}_l\) and \(\alpha = \sum_{i,j=1}^n \alpha_{ij} \varphi_i \land \overline{\varphi}_j\).

By the structure equation and the first Bianchi identity,

\[
\overline{\partial} \varphi = \varphi \land \theta''; \quad \overline{\partial} \tau = \varphi \land \alpha - \tau \land \theta''
\]

Hence for each \(i, j, k,\) and \(l\), one has:

\[
\nabla_i T^i_{jk} = \delta_{ik} \alpha_{j\overline{l}} - \delta_{ik} \alpha_{j\overline{l}} + \sum_{r=1}^n T^i_{rk} A_{jr,\overline{l}} - \sum_{r=1}^n T^i_{rj} A_{kr,\overline{l}} - \sum_{r=1}^n T^i_{rk} A_{r,ij}
\]

Therefore

\[
\sum_{k=1}^n \nabla_i T^k_{jk} = -(n-1)\alpha_{j\overline{i}} + \sum_{r,k=1}^n T^k_{rk} A_{jr,\overline{i}}
\]

This leads to \(\overline{\partial} \eta = \alpha\), and the lemma is proved. \(\square\)

Next let \(\sigma = (\tau - \eta \land \varphi) \otimes e\). Then one has \(\overline{\partial} \sigma = (\alpha - \overline{\partial} \eta) \varphi \otimes e = 0\). So \(\sigma\) is a holomorphic section of \(E = \Omega_M \otimes \Omega_M \otimes T_M\), where \(T_M, \Omega_M\) denotes the holomorphic tangent and cotangent bundle of \(M\). Let \(h\) be the hermitian metric on \(E\) induced from \(g\) on \(T_M\). Fix a point \(x \in M\), choose holomorphic
frame \( v_1, \ldots, v_N \) of \( E \) near \( x \), so that at \( x \), \( h_{ij} = \delta_{ij} \), \( dh_{ij} = 0 \). Write \( \sigma = \sum_{i=1}^N \sigma_i v_i \), then at \( x \):

\[
\partial \bar{\partial} \|\sigma\|^2 = \sum_{i=1}^N \partial \sigma_i \wedge \partial \bar{\sigma}_i - \Theta_{\sigma \bar{\sigma}}(h) \geq \|\sigma\|^2 \alpha
\]

Here we used the fact that \( \Theta(h) = -\alpha I_N \). Note that if \( \partial \bar{\partial} \omega^{n-1} = 0 \), then \( \bar{\eta} \wedge \omega^{n-1} = \eta \wedge \bar{\eta} \wedge \omega^{n-1} \). When \( M \) is compact,

\[
0 = \int \partial \bar{\partial} \|\sigma\|^2 \wedge \omega^{n-1} \geq \int \|\sigma\|^2 \eta \wedge \bar{\eta} \wedge \omega^{n-1}
\]

therefore we have

**Lemma 2.** If \( M \) is compact, \( \Theta = \alpha I_n \), and \( \partial \bar{\partial} \omega^{n-1} = 0 \), then either \( \eta = 0 \) or \( \tau = \eta \wedge \varphi \).

When \( \eta = 0 \), \((M, g)\) is called balanced, the first Ricci form \( r \) equals to the third Ricci form \( s \):

\[
r_{ij} - s_{ij} = \sum_{k=1}^n R_{kkij} - R_{ikkj} = \nabla_j \sum_{k=1}^n T_{ik}^k = 0
\]

But for projectively flat metric, \( r - s = (n - 1)\alpha \), so \( \eta = 0 \) implies \( \Theta = 0 \) in this case.

While when \( \tau = \eta \wedge \varphi \), \( \partial \omega = \tau \wedge \varphi^* = \eta \wedge \omega \). Hence \( \partial \eta \wedge \omega = 0 \). When \( n \geq 3 \), this gives \( \partial \eta = 0 \), while when \( n = 2 \), since \( \partial \eta = \alpha \) is closed, \( 0 = \int \partial \eta \wedge \partial \bar{\eta} \) implies \( \partial \eta = 0 \). Therefore \( d(\eta + \bar{\eta}) = \alpha + \bar{\alpha} = 0 \), so locally \( g \) will be conformal to some Kähler metric, which is necessarily flat. So \( M \) is a complex similarity manifold. By [F], \((M, g)\) is a finite undercover of either a flat complex torus, or a Hopf manifold of the form \((C^n \setminus 0)/\mathbb{Z}^\phi\), where \( \phi(z) = azA \) is a complex expansion: \( A \in U(n) \), \( a > 1 \) and \( z = (z_1, \ldots, z_n) \).

In conclusion, one has:

**Theorem 1.** Let \((M^n, g)\) be a compact projectively flat hermitian manifold, and suppose its Kähler form \( \omega \) satisfies \( \partial \bar{\partial} \omega^{n-1} = 0 \). Then either \((M, g)\) is flat and balanced \((d\omega^{n-1} = 0)\), or \( M \) is a finite undercover of a quotient \( C^n \setminus 0/\mathbb{Z}^\phi \) with \( \phi \) a complex expansion.
Note that by [G1], for any compact hermitian manifold $(M, g)$, there exists an unique (up to homothety) metric $h$ in the conformal class of $g$ such that $\partial \bar{\partial} \omega_h^{n-1} = 0$; and if $g$ is projectively flat, so is $h$. Therefore, any projectively flat metric on $M$ is conformal to one of the metrics in Theorem 1.

For compact hermitian flat manifold $(M, g)$, the torsion tensor is parallel. So the first Bianchi identity gives exactly the Poisson identities, and the universal covering space is a complex Lie group $G$ equipped with an left invariant flat metric. That is, $M = \Gamma \backslash G$, where $\Gamma \subset G \cdot C$ is a discrete subgroup of the semidirect product of $G$ with a compact subgroup $C \subset \text{Aut}(G)$. See [Go] or [K-T] for example.

As a byproduct, we get the following

**Corollary 2.** Any compact hermitian flat manifold $(M^n, g)$ is balanced, i.e., $d\omega_g^{n-1} = 0$.

This is because we can first conformally deform $g$ to get a balanced and projectively flat metric $h = e^f g$. Since any connected Lie group $G$ is either a $K(\pi, 1)$ or has $\pi_3(G) \neq 0$, so $G$ can not be homotopic to $S^{2n-1}$ if $n \geq 3$; while when $n = 2$, there are only two simply-connected complex Lie groups, both biholomorphic to $C^2$, so $M$ can not be Hopf. Therefore, by Theorem 1, we know $h$ is again flat, so $f$ is pluriharmonic, hence a constant, and $g$ is balanced.

In particular, for $n = 2$, we get the well-known fact that any compact hermitian flat surface has to be Kähler, namely, a complex 2-torus or a hyperelliptic surface.

### 2. AN COROLLARY

By the proof of [L-Y-Z], the theorem of Bogomolov on VII\textsubscript{0} surfaces ([B], [B1]) can now be stated in a slightly more general way, namely, if $M^2$ is a compact complex surface with stable tangent bundle $T_M$ (with respect to a hermitian metric) and with $c_1^2 = c_2 = 0$, then $M$ must be either flat or similarity Hopf.

In this section, we want to generalize this into higher dimensions by apply Theorem 1 in §1. First let us recall the definition of **refined Chern classes** by
Bott and Chern ([B-C]). Suppose $E$ is a holomorphic vector bundle over a compact complex manifold $M^n$. Then for any two hermitian metrics $h$, $h'$ on $E$, there always exists smooth functions $f_k$ such that $\sqrt{-1}\partial \bar{\partial} f_k = C_k(h) - C_k(h')$, where $C_k$ denotes the Chern forms. Hence the Chern forms define the refined Chern classes $\hat{c}(E)$ in $\hat{H}^{k,k}(M) = \text{Ker}(d) \cap A^{k,k}/\text{Im}(\sqrt{-1}\partial \bar{\partial})$. Here $A^{k,k}$ is the space of all smooth real $(k,k)$ forms on $M$.

Next, let us recall the definition of astheno-Kähler from [J-Y]. A hermitian metric $g$ on a compact complex $n$-manifold $M$ is called astheno-Kähler, if its metric form $\omega_g$ satisfies $\partial \bar{\partial} \omega_g = 0$. Any product manifold of curves and surfaces is astheno-Kähler. However, it would be more interesting to construct some “non-trivial” examples.

A necessary condition for the existence of such metrics is that, any semipositive $(2,2)$ current can not be $\partial \bar{\partial}$-exact (unless it is trivial). Note that for $n = 3$, this is also a sufficient condition. (More generally, on a compact complex manifold, the non-existence of (non-trivial) $\partial \bar{\partial}$-exact positive $(n-1,n-1)$ current (acting on $(1,1)$ forms) always implies the existence of a hermitian metric $g$ with $\partial \bar{\partial} \omega_g = 0$. Following the work of Harvey and Lawson ([H-L]), this is not hard to show.)

In particular, any global holomorphic 1-form $\varphi$ on $M$ must be closed, as $\partial \bar{\partial}(\varphi \wedge \bar{\varphi}) = - \partial \varphi \wedge \bar{\partial} \varphi \geq 0$. So, for example, a compact complex parallelizable manifold $M$ (i.e., $T_M$ holomorphically trivial) can not be astheno-Kähler unless it is a complex torus. However, we believe that this definition has its potential in the future study of the non-Kähler geometry.

Now if we start off with a compact astheno-Kähler manifold $(M^n, g)$. Let $E$ be a $g$-polystable holomorphic vector bundle of rank $r$ on $M$ (i.e., $E$ is the direct sum of $g$-stable bundles with the same $g$-slope ($g$-degree divided by rank)). Then by [L-Y] (when $n = 2$, also by Buchdahl [Bu]), $E$ admits a hermitian metric $h$ which is $g$-Einstein: $tr_g(\Theta_h) = \mu I_r$ for some constant $\mu$. By the Lübke-Kobayashi inequality, $(C_1^2(E, h) - 2^{r-1} C_2(E, h)) \wedge \omega_g^{n-2} \leq 0$ pointwisely. So if $\hat{c}^2(E) - 2^{r-1} \hat{c}_2(E) = 0$ in $\hat{H}^{2,2}(M)$ (or $\geq 0$ in the obvious sense), then $(E, h)$ is projectively flat: $\Theta_h = \alpha I_r$. In particular, when $E = T_M$, by Theorem 1, we get the following:
Corollary 3. Let $(M^n, g)$ be a compact hermitian manifold which is astheno-Kähler (i.e., $\partial \bar{\partial} \omega^{n-2}_g = 0$). Suppose that $T_M$ is $g$-polystable and the refined Chern classes satisfy $c_1^2 = c_2 = 0$ in $\bar{H}^{2,2}(M)$. Then either $M$ is similarity Hopf, or it admits a flat hermitian metric $h$.

Obviously the condition on the refined Chern classes can be replaced by $c_1^2 - \frac{2n}{n-1} c_2 \geq 0$ in the sense that it can be represented by a pointwisely nonnegative $(2, 2)$ form, or that its product with any $[\Omega]$ is nonnegative, for any $\partial \bar{\partial}$-closed nonnegative $(n-2, n-2)$ form $\Omega$ on $M$.

We also conjecture that the non-Kähler flat manifolds or similarity Hopf manifolds of dimension $\geq 3$ do not admit astheno-Kähler metrics. This is true in some special cases, but at this moment we are unable to prove it in general. After this, the conclusion of Corollary 3 could be replaced by: “$M$ is covered by a complex torus”.

Acknowledgement. We would like to thank the referee of the article for several helpful suggestions.

References

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Received July 12, 1993