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MOMENT PROBLEMS FOR BOUNDED FUNCTIONS

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CONTENTS

1. INTRODUCTION

Given a measure space (X, μ), let $\{\Phi_j\}$ be a sequence of functions ("kernels") in $L^1(X, \mu)$ normalized by

$$\| \Phi_j \|_1 \overset{\text{def}}{=} \int_X |\Phi_j| d\mu = 1. \tag{1.1}$$

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Consider the *moment problem* (or *interpolation problem*)

$$\int_X f\Phi_j d\mu = a_j \qquad (j = 1, 2, \cdots), \tag{1.2}$$

where $\{a_j\}$ is a prescribed *data sequence* and f is a (measurable) function on X to be found. (Throughout, all the functions and number sequences involved are assumed to be complex valued). We will be concerned with bounded solutions f to (1.2), so of course we have to make the obvious restriction that $\{a_j\} \in \ell^\infty$.

Now a natural question is: Which sequences of kernels $\{\Phi_j\}$ have the property that, for any $\{a_j\} \in \ell^\infty$, the moment problem (1.2) has a solution $f \in L^\infty(X, \mu)$?

Although there is little hope for a complete answer to be provided in such a general setting, we nonetheless point out a simple sufficient condition, which is in many cases close to necessary. Namely, we show that if there exist pairwise disjoint subsets $E_j (j = 1, 2, \cdots)$ of X such that essentially more than 50% of Φ_j is supported on E_j, in the sense that

$$\inf_j \int_{E_j} |\Phi_j| d\mu > \frac{1}{2}, \tag{1.3}$$

then there always is a bounded solution f to the moment problem (1.2) whenever $\{a_j\} \in \ell^\infty$.

A slightly more precise version of this result is stated, as Theorem 1, and proved in Section 2 below. Our method is, perhaps, somewhat crude; it hinges on the elementary fact that a bounded linear operator T satisfying $\| I - T \| < 1$ has to be invertible. However, unexpectedly enough, the arising condition (1.3) turns out to be sharp; examples will be furnished to show that the constant 1/2 in (1.3) is best possible!

Further, because of its simplicity and generality, our "abstract interpolation theorem" (see Theorem 1 below) has a wide range of applications. It often yields satisfactory results in some obscure cases where more special methods fail. In particular, it enables us to develop a unified approach to such miscellaneous topic as

- interpolating sequences for bounded harmonic functions in higher dimensions and/or in multiply connected domains,
- nondifferentiability of integrals in $\mathbb{R}^n$,
- growth of entire functions represented by gap series,
- interpolation problems for analytic functions of bounded mean oscillation and embedding theorems for coinvariant subspaces in H^1,
- Fourier series of L^∞ functions.

These applications are collected in Section 3. Finally, Section 4 contains an open problem concerning translates of L^1 functions.

2. The Abstract Interpolation Theorem

Theorem 1. *Let* $\{\Phi_j\}_{j=1}^\infty$ *be a sequence of functions in* $L^1(X,\mu)$ *normalized by (1.1). suppose there is a collection* $\{E_j\}_{j=1}^\infty$ *of pairwise disjoint measurable sets* $E_j \subset X$, *for which* (1.3) *holds true. Then, for any data sequence* $\{a_j\} \in \ell^\infty$, *the moment problem* (1.2) *has a solution* $f \in L^\infty(X,\mu)$ *satisfying*

$$\| f \|_\infty \stackrel{\text{def}}{=} \mu\text{-ess}\ \sup_X |f| \le (2\sigma - 1)^{-1} \sup_j |a_j|, \tag{2.1}$$

where σ *is the value of the infimum in* (1.3) *(so that* $\sigma > 1/2$*).*

Proof. We construct the desired solution in the form

$$f = \sum_{k=1}^\infty c_k \frac{|\Phi_k|}{\Phi_k} \chi_{E_k}, \tag{2.2}$$

where $\{c_k\} \in \ell^\infty$ and χ_{E_k} stands for the characteristic function of E_k (at those points where $\Phi_k = 0$, set $|\Phi_k|/\Phi_k = 1$). For such an f one has

$$\| f \|_\infty = \sup_k |c_k| \tag{2.3}$$

and

$$\int_X f \Phi_j d\mu = \sum_{k=1}^\infty b_{jk} c_k, \tag{2.4}$$

where $b_{jk} \stackrel{\text{def}}{=} \int_{E_k} (|\Phi_k|/\Phi_k)\, \Phi_j d\mu$. Thus we are led to consider the linear operator $T : \ell^\infty \to \ell^\infty$ defined by

$$T\{c_k\}_{k=1}^\infty = \left\{\sum_{k=1}^\infty b_{jk}c_k\right\}_{j=1}^\infty .$$

Clearly, for $j = 1, 2, \cdots$

$$\sum_k |b_{jk}| \le \sum_k \int_{E_k} |\Phi_j| d\mu \le \int_X |\Phi_j| d\mu = 1, \tag{2.5}$$

and so T is bounded with $\| T \|_{\ell^\infty \to \ell^\infty} \le 1$.

To prove that $T\ell^\infty = \ell^\infty$, we verify that $\| I - T \|_{\ell^\infty \to \ell^\infty} < 1$, where $I = id_{\ell^\infty}$. The operator $I - T$ is generated by the matrix $\{\delta_{jk} - b_{jk}\}_{j,k=1}^\infty$. Since $0 \le b_{jj} \le 1$, we have

$$\begin{aligned} \| I - T \|_{\ell^\infty \to \ell^\infty} &= \sup_j \sum_k |\delta_{jk} - b_{jk}| \\ &= \sup_j \left(1 - b_{jj} + \sum_{k:k\ne j} |b_{jk}|\right) \\ &= \sup_j \left(1 - 2b_{jj} + \sum_k |b_{jk}|\right) \\ &\le 2 \sup_j (1 - b_{jj}) : \end{aligned} \tag{2.6}$$

here the last inequality relies on (2.5). Because by (1.3) $b_{jj} = \int_{E_j} |\Phi_j| d\mu \ge \sigma > 1/2$, the right-hand side in (2.6) is bounded by $2(1-\sigma) < 1$. It follows that T is invertible and $T^{-1} = \sum_{k=0}^\infty (I - T)^k$, whence

$$\begin{aligned} \| T^{-1} \|_{\ell^\infty \to \ell^\infty} &\le \sum_{k=0}^\infty \| I - T \|_{\ell^\infty \to \ell^\infty}^k \\ &\le \sum_{k=0}^\infty 2^k (1-\sigma)^k \\ &= (2\sigma - 1)^{-1}. \end{aligned}$$

Finally, given $\{a_j\} \in \ell^\infty$, let $\{c_k\} = T^{-1}\{a_j\}$ so that

$$\sum_{k=1}^\infty b_{jk}c_k = a_j \qquad (j = 1, 2, \cdots)$$

and

$$\sup_k |c_k| \le (2\sigma - 1)^{-1} \sup_j |a_j|.$$

In view of (2.3) and (2.4), the function f defined by (2.2) satisfies (1.2) and (2.1), as required. □

3. Applications

3.1. Interpolation by bounded harmonic functions. Let

$$\mathbb{R}_+^{n+1} \stackrel{\text{def}}{=} \{(x, y) : x \in \mathbb{R}^n, y > 0\}.$$

The unsolved problem is to characterize, in the case $n > 1$, the sequences $\{z_j\} \subset \mathbb{R}_+^{n+1}$ such that every interpolation problem $u(z_j) = a_j \quad (j = 1, 2, \cdots)$ with $\{a_j\} \in \ell^\infty$ has a solution u in the class of bounded harmonic functions on $\mathbb{R}_+^{n+1}$. (If $\{z_j\}$ has this property, it will be called a *harmonic interpolating sequence*). See [CG] and [Am] for some partial results to this end; see also [G2, chapter vii] for a complete solution in the case $n = 1$.

For a fixed point $z \in \mathbb{R}_+^{n+1}$, denote by P_z the corresponding *Poisson kernel* and by $\omega(z, E, \mathbb{R}_+^{n+1})$ the *harmonic measure*, evaluated at z, of a Borel set $E \subset \mathbb{R}^n$ with respect to $\mathbb{R}_+^{n+1}$; thus $\omega(z, E, \mathbb{R}_+^{n+1}) = \int_E P_z(t)dt$.

Theorem 2. *Given a sequence $\{z_j\} \subset \mathbb{R}_+^{n+1}$, suppose there are pairwise disjoint Borel subsets $E_j \quad (j = 1, 2, \cdots)$ of $\mathbb{R}^n$ such that*

$$\inf_j \omega(z_j, E_j, \mathbb{R}_+^{n+1}) > \frac{1}{2}. \tag{3.1}$$

Then $\{z_j\}$ is a harmonic interpolating sequence.

Proof. Apply Theorem 1 with $X = \mathbb{R}^n, \mu =$ the Lebesgue measure on $\mathbb{R}^n$, and $\Phi_j = P_{z_j}$. □

Since the harmonic measure $\omega(z, E, \mathbb{R}_+^{n+1})$ has a nice geometrical meaning (it equals the normalized angle at which E is seen from z), condition (3.1) makes it an easy matter to construct numerous examples of harmonic interpolating sequences.

We now show that the constant 1/2 in (3.1), and hence also in (1.3), is sharp. Moreover, we will see that in (3.1) one cannot even replace $>$ by $=$. To this end, we construct a sequence $\{z_j\}$ of points in the upper half-plane

$\mathbb{C}_+ \stackrel{\text{def}}{=} \{\text{Im} \quad z > 0\} \cong \mathbb{R}^2_+$ and a collection $\{E_j\}$ of pairwise disjoint subintervals of $\mathbb{R}$, so that $\inf_j \omega(z_j, E_j, \mathbb{C}_+) = 1/2$, and yet $\{z_j\}$ is not a (harmonic) interpolating sequence. A similar construction can be carried out in higher dimensions.

For k a fixed positive integer, set $p_k^+ = k^{-1} + i$ and $p_k^- = -k^{-1} + i$, where $i = \sqrt{-1}$. For the half-axis $\mathbb{R}_+ \stackrel{\text{def}}{=} (0, +\infty)$ we have $\omega(p_k^+, \mathbb{R}_+, \mathbb{C}_+) > 1/2$; let $\ell_k > 0$ be so large that the interval $(0, \ell_k) \stackrel{\text{def}}{=} I_k$ still satisfies $\omega_k \stackrel{\text{def}}{=} \omega(p_k^+, I_k, \mathbb{C}_+) > 1/2$. We also have then $\omega(p_k^-, -I_k, \mathbb{C}_+) = \omega_k > 1/2$, where $-I_k = (-\ell_k, 0)$. As $k \to +\infty$, one has $\ell_k \to +\infty$ and $\omega_k \to 1/2$. The desired sequences $\{z_j\}$ and $\{E_j\}$ are now obtained by shifting the already chosen points and intervals to the right. For $k = 1, 2, \cdots$ set

$$\begin{aligned} z_{2k-1} &= p_k^- + \tau_k, \qquad & z_{2k} &= p_k^+ + \tau_k, \\ E_{2k-1} &= -I_k + \tau_k, \qquad & E_{2k} &= I_k + \tau_k, \end{aligned}$$

where $0 = \tau_1 < \tau_2 < \cdots$ and τ_{k+1} is chosen inductively so as to ensure that the left endpoint of E_{2k+1} be greater than the right endpoint of E_{2k}. The arising intervals E_j are thus pairwise disjoint.

Clearly, for all $k = 1, 2, \cdots$

$$\omega(z_{2k-1}, E_{2k-1}, \mathbb{C}_+) = \omega(z_{2k}, E_{2k}, \mathbb{C}_+) = \omega_k > 1/2,$$

and so $\inf_j \omega(z_j, E_j, \mathbb{C}_+) = 1/2$. On the other hand,

$$z_{2k} - z_{2k-1} - p_k^+ - p_k^- = 2k^{-1} \underset{k\to\infty}{\longrightarrow} 0.$$

Since $\text{Im}\, z_j = 1$, this means that the points $\{z_j\}$ are not *separated.* In other words, they do not satisfy

$$\inf_{j\neq\ell} |z_j - z_\ell| / \text{Im}\, z_j > 0, \tag{3.2}$$

whereas it is easily shown that (3.2) is a necessary condition for $\{z_j\} \subset \mathbb{C}_+$ to be a (harmonic) interpolating sequence (see [G2, chapter vii]). We are done.

Theorem 3. *Let* $\{z_j\} \subset \mathbb{R}^{n+1}_+$*, so that* $z_j = (x_j, y_j)$*, where* $x_j \in \mathbb{R}^n$ *and* $y_j > 0$*. There is a constant* $c = c_n > 0$ *making the following statement true:*

If

$$|z_j - z_k| \geq c y_j \qquad \textit{whenever } j \neq k, \tag{3.3}$$

then $\{z_j\}$ *is a harmonic interpolating sequence.*

Before proceeding with the proof, we remark that the mere existence of some constant $c > 0$ for which (3.3) holds is exactly the *separation condition*, reducing to (3.2) in the planar case. It is well known that, in any dimension n, this separation condition is necessary but not sufficient for harmonic interpolation. Another important necessary condition is that the measure $\sum_j y_j^n \delta_{z_j}$, where δ_{z_j} is the unit point mass at z_j, must be a *Carleson measure* (for a more detailed discussion of these matters, see [CG] and the references therein). However, Theorem 3 says that, for c large enough, the harmonic interpolation property is ensured by (3.3) alone.

Next, we remark that the Poisson kernels P_{z_j} can be written in the form

$$P_{z_j}(t) = y_j^{-n} P\left(\frac{t - x_j}{y_j}\right), \quad t \in \mathbb{R}^n,$$

where $P(\xi) \stackrel{\text{def}}{=} \gamma_n (1 + |\xi|^2)^{-(n+1)/2}$ and γ_n is the normalizing constant factor. Thus, Theorem 3 is a special case of the following result.

Theorem 4. *Let* $\{z_j = (x_j, y_j)\} \subset \mathbb{R}_+^{n+1}$. *Suppose that* $K \in L^1(\mathbb{R}^n)$ *satisfies* $\int_{\mathbb{R}^n} |K(x)| dx = 1$ *and* K_j *is defined by*

$$K_j(t) = y_j^{-n} K\left(\frac{t - x_j}{y_j}\right), \qquad t \in \mathbb{R}^n.$$

There is a constant $c(K) > 0$ *making the following statement true: If* (3.3) *holds with* $c > c(K)$ *then, for any* $\{a_j\} \in \ell^\infty$, *the moment problem*

$$\int_{\mathbb{R}^n} f(x) K_j(x) dx = a_j \qquad (j = 1, 2, \cdots)$$

has a solution $f \in L^\infty(\mathbb{R}^n)$.

Proof. Let B denote the ball $\{x \in \mathbb{R}^n : |x| < R\}$, where R is large enough, so that $\int_B |K(x)| dx > 1/2$. For $j = 1, 2, \cdots$ set

$$B_j \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : |x - x_j| < y_j R\}.$$

A change of variable yields

$$\int_{B_j} |K_j(x)|dx = \int_B |K(x)|dx > \frac{1}{2},$$

and so condition (1.3) is fulfilled with $\Phi_j = K_j, E_j = B_j$ and $d\mu(x) = dx$. To apply Theorem 1, it remains to make sure that the balls B_j be disjoint.

Assume that $x \in B_j \cap B_k, j \neq k$. We may also assume that $y_j \geq y_k$, so we have

$$\begin{aligned} |z_j - z_k| &\leq |x_j - x_k| + y_j - y_k \\ &\leq |x_j - x| + |x - x_k| + y_j - y_k \\ &\leq y_j R + y_k R + y_j - y_k \qquad (3.4) \\ &= y_j(R+1) + y_k(R-1) \\ &\leq c_R y_j, \end{aligned}$$

where $c_R \overset{\text{def}}{=} \max(2R, R+1)$. On the other hand, by (3.3), $|z_j - z_k| \geq cy_j$. For $c > c_R$, this obviously contradicts (3.4), whence $B_j \cap B_k = \emptyset$. Thus, c_R is eligible as $c(K)$, and the proof is complete. □

For one-dimensional Poisson kernel, $P(t) = \pi^{-1}(1+t^2)^{-1}$, one can take R to be any number which is > 1. Therefore, the above proof yields the following

Corollary. *Suppose that the points $\{z_j = x_j + iy_j\} \subset \mathbb{C}_+$ satisfy (3.3) with a fixed $c > 2$. Then $\{z_j\}$ is an interpolating sequence.*

(Recall that, by a theorem of Garnett, "harmonic interpolating sequences" in $\mathbb{C}_+$ coincide with "analytic interpolating sequences", defined in terms of H^∞ functions; both are characterized by the *Carleson condition*

$$\inf_j \prod_{k:k\neq j} \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| > 0.$$

See [G1] and [G2, chapter vii].)

We conclude this section with a few remarks.

1) In Theorem 2, one can safely replace $\mathbb{R}^{n+1}_+$ by an arbitrary domain Ω (planar or spatial) that is regular in the potential-theoretic sense. Namely, *if $\{z_j\} \subset \Omega$ and E_j are pairwise disjoint Borel subsets of $\partial\Omega$ such that* $\inf_j \omega(z_j, E_j, \Omega) > 1/2$ (here $\omega(\cdot, \cdot, \Omega)$ is the harmonic

measure with respect to Ω) *then, given any values* $\{a_j\} \in \ell^\infty$, *one can find a bounded harmonic function* u *on* Ω *satisfying* $u(z_j) = a_j \quad (j = 1, 2, \cdots)$.

In particular, this is true for infinitely connected planar domains, in which case harmonic interpolating sequences need not coincide with analytic ones (see [G1] for an example).

2) A similar theorem remains valid for bounded harmonic functions with values in a Banach space. This can be derived from the appropriate vectorial of Theorem 1.
3) In addition, we have a nice linear operator of interpolation at our disposal (it appears in the proof of Theorem 1).
4) Theorem 2 was announced, without any proof, in [D1].

3.2. Interpolation by mean values and nondifferentiability of integrals in $\mathbb{R}^n$. .

Let A be a (Lebesgue) measurable subset of $\mathbb{R}^n$ with $0 < |A| < +\infty$, where $|A| \stackrel{\text{def}}{=} \int_A dx$. Given a function $f \in L^1(A) = L^1(A, dx)$, denote by $\mathfrak{M}(f, A)$ the *mean value* of f over A, i.e.,

$$\mathfrak{M}(f, A) \stackrel{\text{def}}{=} \frac{1}{|A|} \int_A f(x)dx.$$

This section deals with interpolation problems of the form

$$\mathfrak{M}(f, A_j) = a_j \qquad (j = 1, 2, \cdots), \tag{3.5}$$

where A_j are given (distinct) subsets of $\mathbb{R}^n$, $\{a_j\} \in \ell^\infty$ is a prescribed sequence, and $f \in L^\infty(\mathbb{R}^n)$ is the function we are looking for. Of course, if the A_j's are disjoint, the problem becomes trivial (to solve (3.5), put $f = \sum_j a_j \chi_{A_j}$). The next theorem shows that, if the A_j's are now allowed to overlap to a limited extent, then it is still possible to find a bounded solution f to (3.5) whenever $\{a_j\} \in \ell^\infty$.

Theorem 5. *Let* $\{A_j\}$ *be a family of measurable subsets of* $\mathbb{R}^n$ *with* $0 < |A_j| < +\infty$. *Suppose there are pairwise disjoint measurable sets* $E_j \quad (j = 1, 2, \cdots)$

such that $E_j \subset A_j$ *and*

$$\inf_j |E_j|/|A_j| > 1/2. \tag{3.6}$$

Then, for every $\{a_j\} \in \ell^\infty$, *the interpolation problem* (3.5) *has a solution* $f \in L^\infty(\mathbb{R}^n)$.

Proof. Apply Theorem 1 with $X = \mathbb{R}^n, d\mu(x) = dx$ and $\Phi_j = |A_j|^{-1}\chi_{A_j}$. □

EXAMPLE 3.1. Consider the intervals $A_1 = (-1,0), A_2 = (0,1)$ and $A_3 = (-1,1)$ together with their subintervals $E_1 = (-1,12/2), E_2 = (1/2,1)$ and $E_3 = (-1/2,1/2)$. Obviously, for $j = 1,2,3$ one has $E_j \subset A_j$ and $|E_j|/|A_j| = 1/2$; besides, the E_j's are disjoint. On the other hand, the three kernels $|A_j|^{-1}\chi_{A_j}$ are linearly dependent. For this stupid reason, the interpolation (3.5) is not possible for arbitrary $a_1, a_2, a_3 \in \mathbb{C}$ with a function $f \in L^\infty(\mathbb{R})$. Thus, a weaker version of (3.6), with $>$ replaced by $=$, is no longer sufficient for the "mean value interpolation property" of $\{A_j\}$.

EXAMPLE 3.2. Let $x_0 \in \mathbb{R}^n$ and let $\{Q_j\}$ be a sequence of open cubes[1] containing x_0 and such that $\sup_j |Q_{j+1}|/|Q_j| \le 1/4$. Letting $E_j \overset{\text{def}}{=} Q_j \setminus \bigcup_{k=1}^\infty Q_k$, we have $E_j \subset Q_j$ and $E_j \cap E_\ell = \emptyset$ for $j \ne \ell$. Furthermore,

$$\begin{aligned} |\bigcup_{k=j+1}^\infty Q_k| &\le \sum_{k=j+1}^\infty |Q_k| \\ &\le \sum_{k=1}^\infty \left(\frac{1}{4}\right)^k |Q_j| \\ &= \frac{1}{3}|Q_j|, \end{aligned}$$

hence $|E_j| \ge \frac{2}{3}|Q_j|$. By Theorem 5 we conclude that every interpolation problem

$$\mathfrak{M}(f, Q_j) = a_j \quad (j = 1, 2, \cdots), \quad \{a_j\} \in \ell^\infty,$$

can be solved with a function f in $L^\infty(\mathbb{R}^n)$. In particular, choosing an appropriate sequence $\{a_j\}$, we obtain a function $f \in L^\infty(\mathbb{R}^n)$ whose integral is *not differentiable* at x_0, in the sense that $\mathfrak{M}(f, Q_j)$ has no limit as $j \to \infty$.

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[1]Throughout, a "cube" means "a cube with edges parallel to the axes"

Now a classical theorem of Lebesgue says that, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for almost all $x \in \mathbb{R}^n$, one has the *differentiation property*

$$\lim_{j\to\infty} \mathfrak{M}\left(f, Q_j^{(x)}\right) = f(x), \tag{3.7}$$

whenever $\left\{Q_j^{(x)}\right\}$ is a sequence of open cubes containing x with $\left|Q_j^{(x)}\right| \to 0$. (See [Gu] for a proof, as well as for an exhaustive discussion of what happens when the cubes are replaced by other families of contracting neighborhoods of x).

Our next theorem can be regarded as a refined version of Example 2. In particular, it implies that Lebesgue's differentiation property (3.7) can be violated, in a very drastic way, for a bounded function f at every point x of a prescribed compact set of measure zero.

Theorem 6. *Let F be a compact subset of $\mathbb{R}^n$ with $|F| = 0$. To each point $x \in F$ one can assign a family of open cubes $\left\{R_j^{(x)}\right\}_{j=0}^{\infty}$ in such a way that the following statements hold true:*

(a) *The family $\bigcup_{x\in F}\left\{R_j^{(x)}\right\}$ is countable.*

(b) *For every $x \in F$, one has $x \in \bigcap_{j=1}^{\infty} R_j^{(x)}$ and $\lim_{j\to\infty} |R_j^{(x)}| = 0$.* *item[(c)] For every $\{a_j\} \in \ell^\infty$, there is a function $f \in L^\infty(\mathbb{R}^n)$ such that $\mathfrak{M}\left(f, R_j^{(x)}\right) = a_j$ for $j = 1, 2, \cdots$ and for all $x \in F$.*

Proof. Let Ω_1 be an open set such that $F \subset \Omega_1$. There is a partition $\Omega_1 = \bigcup_{k=1}^{\infty} Q_1^k$ where Q_1^k are pairwise disjoint half-open cubes. (A "half-open cube" means the Cartesian product of n half-open intervals of the form $[\alpha_1, \alpha_1 + \ell), \cdots, [\alpha_n, \alpha_n + \ell)$.) Let $\tilde{Q}_1^k$ denote the open cube with the same center as Q_1^k, for which $|\tilde{Q}_1^k| = \frac{4}{3}|Q_1^k|$. Clearly, $Q_1^k \subset \tilde{Q}_1^k$ and so $F \subset \Omega_1 \subset$ $bigcup_{k=1}^{\infty}\tilde{Q}_1^k$. Since F is compact, one can find a finite subcollection $\{\tilde{Q}_1^k\}_{k=1}^{N_1}$ covering F, so that $F \subset \bigcup_{k=1}^{N_1} \tilde{Q}_1^k$.

Now let Ω_2 be an open set such that $F \subset \Omega_2 \subset \Omega_1$ and

$$|\Omega_2| < \frac{1}{4}\min\left\{|Q_1^k| : 1 \le k \le N_1\right\}.$$

As above, we write $\Omega_2 = \bigcup_{k=1}^{\infty} Q_2^k$, where Q_2^k are appropriate disjoint half-open cubes. For each k, we consider the corresponding open cube $\tilde{Q}_2^k$, concentric with Q_2^k and satisfying $|\tilde{Q}_2^k| = \frac{4}{3}|Q_2^k|$. Finally, we choose a finite subfamily $\{\tilde{Q}_2^k\}_{k=1}^{N_2}$ with $F \subset \cup_{k=1}^{N_2}\tilde{Q}_2^k$.

Continuing by induction, we construct for each $j = 3, 4, \cdots$ an open set Ω_j such that $F \subset \Omega_j \subset \Omega_{j-1}$ and

$$|\Omega_j| < \frac{1}{4}\min\left\{|Q_{j-1}^k| : k = 1, \cdots, N_{j-1}\right\} \tag{3.8}$$

(this is always possible because $|F| = 0$); then we obtain the families of cubes $\left\{Q_j^k\right\}_{k=1}^{\infty}, \left\{\tilde{Q}_j^k\right\}_{k=1}^{\infty}$ and $\left\{\tilde{Q}_j^k\right\}_{k=1}^{N_j}$ exactly as before.

Since $F \subset \bigcup_{k=1}^{N_j}\tilde{Q}_j^k \quad (j = 1, 2, \cdots)$, for each $x \in F$ and for each j there is a $k_j = k_j(x), 1 \le k_j \le N_j$, such that $x \in \tilde{Q}_j^{k_j}$. Set $R_j^{(x)} \overset{\text{def}}{=} \tilde{Q}_j^{k_j(x)}$. This done, conditions (a) and (b) are easily verified (in particular, (3.8) implies that $|R_j^{(x)}| \to 0$).

To prove (c), we are going to apply Theorem 5 to the cubes $\tilde{Q}_j^k$ $(j = 1, 2, \cdots; k = 1, \cdots, N_j)$, the corresponding subsets E_j^k defined by $E_j^k \overset{\text{def}}{=} Q_j^k \setminus \Omega_{j+1}$. Obviously, $E_j^k \subset \tilde{Q}_j^k$. Next we notice that $E_j^k \cap E_\ell^m = \emptyset$, provided that either $j \ne \ell$ or $k \ne m$. Indeed, assuming $j < \ell$, one has

$$E_\ell^m \subset Q_\ell^m \subset \Omega_\ell \subset \Omega_{j=1},$$

whereas $E_j^k \cap \Omega_{j+1} = \emptyset$. Otherwise, if $j = \ell$ but $k \ne m$, we have $Q_j^k \cap Q_j^m = \emptyset$ and so $E_j^k \cap E_j^m = \emptyset$.

In order to estimate the ratio $|E_j^k|/|\tilde{Q}_j^k|$ from below, we write

$$\begin{aligned} |Q_j^k| &= |E_j^k| + |Q_j^k \cap \Omega_{j+1}| \\ &\le |E_j^k| + |\Omega_{j+1}| \\ &\le |E_j^k| + \frac{1}{4}|Q_j^k|, \end{aligned}$$

where the last inequality relies on (3.8). Hence

$$|E_j^k| \ge \frac{3}{4}|Q_j^k| = \left(\frac{3}{4}\right)^2 |\tilde{Q}_j^k|.$$

Fortunately $\left(\frac{3}{4}\right)^2 = \frac{9}{16} > \frac{1}{2}$, and so the hypotheses of Theorem 5 are fulfilled, when applied to $\left\{\tilde{Q}_j^k\right\}$ and $\left\{E_j^k\right\}$ $(j = 1, 2, \cdots; \quad k = 1, \cdots, N_j)$.

Consequently, given $\{a_j\} \in \ell^\infty$, Theorem 5 provides a function $f \in L^\infty(\mathbb{R}^n)$ satisfying, for every fixed j,

$$\mathfrak{M}\left(f, \tilde{Q}_j^k\right) = a_j \qquad (k = 1, \cdots, N_j).$$

Recalling that, for each $x \in F, R_j^{(x)}$ is contained among $\tilde{Q}_j^k \quad (k = 1, \cdots, N_j)$, we arrive at (c). □

3.3. Systems of powers on a half-axis and the radial behavior of entire functions represented by gap series. In this section Theorem 1 will be applied to the following special case: $X = \mathbb{R}_+ = (0, +\infty), d\mu(x) = \exp(-\beta x^\rho)dx$, where $\beta > 0$ and $\rho > 0$, and

$$\Phi_k(x) = \gamma_k x^k, \quad \text{where} \quad \gamma_k = \frac{\rho\beta^{\frac{(k+1)}{\rho}}}{\Gamma\left(\frac{k+1}{\rho}\right)}, \tag{3.9}$$

so that $\int_0^\infty \Phi_k(x)d\mu(x) = 1$. It turns out that if k ranges over a certain sparse subset $\{k_j\}$ of $\mathbb{N}$ (here $\mathbb{N}$ denotes the positive integers) then condition (1.3) holds true for the corresponding family $\{\Phi_{k_j}\}$.

We begin with some technical preparations. Once $\beta > 0$ and $\rho > 0$ are fixed, to each integer $k > \rho - 1$ we attach a new positive number $k' \stackrel{\text{def}}{=} \frac{(k+1)}{\rho - 1}$; further, for $\delta \in (0, \sqrt{k'})$ we let

$$a_k(\delta) \stackrel{\text{def}}{=} \left(\frac{k' - \delta\sqrt{k'}}{\beta}\right)^{\frac{1}{\rho}} \quad \text{and} \quad b_k(\delta) \stackrel{\text{def}}{=} \left(\frac{k' + \delta\sqrt{k'}}{\beta}\right)^{\frac{1}{\rho}}.$$

Lemma. *There are absolute constants $\delta > 0, \sigma \in \left(\frac{1}{2}, 1\right)$ and $k_0 > 0$ making the following statement true: If $\Phi_k(x)$ is defined by (3.9) for $k \in \mathbb{N}$ and $d\mu(x) = \exp(-\beta x^\rho)dx$, then*

$$\int_{a_k(\delta)}^{b_k(\delta)} \Phi_k(x)d\mu(x) > \sigma$$

whenever $k \geq k_0\rho$.

Proof. Denote the integral in question by $I_k(\delta)$, so that

$$I_k(\delta) = \frac{\rho\beta^{k'+1}}{\Gamma(k'+1)} \int_{a_k(\delta)}^{b_k(\delta)} x^k \exp(-\beta x^\rho)dx.$$

A change of variable, $u = \frac{(\beta x^\rho - k')}{\sqrt{k'}}$, yields

$$I_k(\delta) = \frac{\sqrt{k'}}{\Gamma(k'+1)} \left(\frac{k'}{e}\right)^{k'} \int_{-\delta}^{\delta} \left(1 + \frac{u}{\sqrt{k'}}\right)^{k'} \exp(-u\sqrt{k'})du. \tag{3.10}$$

When $k \to \infty$ (or, equivalently, $k' \to \infty$), the factor in front of the last integral tends to $(2\pi)^{-\frac{1}{2}}$ by Stirling's formula, whereas the integrand tends to $\exp\left(\frac{-u^2}{2}\right)$. Thus,

$$\lim_{k\to\infty} I_k(\delta) = (2\pi)^{-\frac{1}{2}} \int_{-\delta}^{\delta} \exp\left(\frac{-u^2}{2}\right) du. \tag{3.11}$$

Since $(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(\frac{-u^2}{2}\right) du = 1$, we can fix any number $\sigma \in \left(\frac{1}{2}, 1\right)$ and find a $\delta > 0$ for which the right-hand side in (3.11) is $> \sigma$. This done, (3.10) shows that $I_k(\delta)$ becomes $> \sigma$ as soon as k' is larger than suitable $M > 0$. Thus $I_k(\delta) > \sigma$ if $\frac{k}{\rho} \geq M + 1 \stackrel{\text{def}}{=} k_0$. □

Remark. In what follows, it will be desirable to make δ as small as possible. The tables for the normal distribution, contained in most textbooks on probability theory, tell us that one can take e.g. $\delta = 0.68$ and $\sigma = 0.503$.

Now we consider an increasing sequence $\{k_j\}_{j=1}^{\infty} \subset \mathbb{N}$ and look at the kernels Φ_{k_j}, defined as above. In connection with the arising moment problem, our plan is to apply Theorem 1, where the corresponding subsets E_j are chosen as intervals $(a_{k_j}(\delta), b_{k_j}(\delta))$. To do that, we have to make sure that our intervals are disjoint. Clearly, this last requirement means that

$$b_{k_j}(\delta) \leq a_{k_{j+1}}(\delta), \quad j = 1, 2, \cdots,$$

which in turn can be rewritten as

$$k_{j+1} - k_j \geq \delta\sqrt{\rho}\left(\sqrt{k_{j+1} + 1 - \rho} + \sqrt{k_j + 1 - \rho}\right). \tag{3.12}$$

We arrive at the following result.

Theorem 7. *Let $\Phi_k(x)$ be defined by (3.9) and $d\mu(x) = \exp(-\beta x^\rho)dx$, where $\beta > 0$ and $\rho > 0$. There exist absolute constants $\eta > 0$ and $k_0 > 0$ making the following statement true: If a sequence $\{k_j\}_{j=1}^{\infty} \subset \mathbb{N}$ satisfies*

$$k_0\rho \leq k_1 < k_2 < \cdots \tag{3.13}$$

and

$$k_{j+1} - k_j \geq \eta\sqrt{\rho k_{j+1}} \quad (j = 1, 2, \cdots), \tag{3.14}$$

then every moment problem

$$\int_0^\infty f(x)\Phi_{k_j}(x)d\mu(x) = \alpha_j \quad (j = 1, 2, \cdots), \quad \{\alpha_j\} \in \ell^\infty,$$

can be solved with a function $f \in L^\infty(\mathbb{R}_+)$ *whose norm* $\|f\|_\infty$ *is bonded by another absolute constant time* $\sup_j |\alpha_j|$.

Proof. Let δ, σ and k_0 be chosen as in Lemma¿ An obvious estimate yields

$$\sqrt{k_{j+1} + 1 - \rho} + \sqrt{k_j + 1 - \rho} \leq 2\sqrt{2}\sqrt{k_{j+1}}.$$

It follows that, for $\eta = 2\sqrt{2}\delta$, (3.14) implies (3.12), and so the intervals $E_j \stackrel{\text{def}}{=} (a_{k_j}(\delta), b_{k_j}(\delta))$ are pairwise disjoint. By Lemma, we have

$$\int_{E_j} \Phi_{k_j}(x)d\mu(x) > \sigma > \frac{1}{2} \quad (j = 1, 2, \cdots),$$

since $k_j \geq K_0\rho$. Applying Theorem 1 (with Φ_j replaced by Φ_{k_j}) completes the proof. □

Now we point out man amusing restatement of Theorem 7. Of concern will be the growth of a "lacunary" entire function

$$f(z) = \sum_{j=1}^\infty c_j z^{k_j}, \quad z \in \mathbb{C}$$

with coefficient c_j decreasing in a prescribed way, along various rays $\{\arg z = \varphi\}$. Roughly speaking, it turns out that for every fixed $\varphi, -\pi < \varphi \leq \pi, f(re^{i\varphi})$ enjoys the maximal possible growth rate as $r \to +\infty$, the growth being measured in terms of weighted L^1 means

$$\mathcal{I}(f, \varphi) \stackrel{\text{def}}{=} \int_0^\infty |f(re^{i\varphi})| \exp(-\beta r^\rho)dz.$$

Some related results and the open problems can be found in [M].

Theorem 8. *Let $\beta > 0$ and $\rho > 0$. Assume that $\{k_j\}_{j=1}^{\infty} \subset \mathbb{N}$ satisfies* (3.13) *and* (3.14)*, the constants k_0 and η being the same as in Theorem 7. Suppose that f is an entire function with the power series expansion (3.15) for which*

$$\mathcal{S}(\{c_j\}) \stackrel{\text{def}}{=} \rho^{-1} \sum_{j=1}^{\infty} |c_j| \beta^{-\frac{(k_j+1)}{\rho}} \Gamma\left(\frac{k_j+1}{\rho}\right) < +\infty$$

Then, for every $\varphi \in (-\pi, \pi]$, one has

$$\text{const} \cdot \mathcal{S}\{c_j\}) \leq \mathcal{I}(f, \varphi) \leq \mathcal{S}(\{c_j\}) \tag{3.16}$$

with an absolute positive constant on the left.

Proof. The right-hand inequality in (3.16) is immediate. Indeed, since

$$\int_0^{\infty} x^k d\mu(x) = \gamma_k^{-1}$$

where $d\mu(x) = \exp(-\beta x^{\rho})dx$ and γ_k is defined in (3.9), we have

$$\mathcal{I}(f, \varphi) \leq \sum_{j=1}^{\infty} |c_j| \int_0^{\infty} r^{k_j} d\mu(r) = \sum_{j=1}^{\infty} |c_j| \gamma_{k_j}^{-1} = \mathcal{S}(\{c_j\}).$$

Now we turn to the left-hand inequality in (3.16). By Theorem 7, under the stated hypotheses on $\{k_j\}$, the operator

$$g \mapsto \left\{ \gamma_{k_j} \int_0^{\infty} g(x) x^{k_j} d\mu(x) \right\}_{j=1}^{\infty}$$

maps $L^{\infty}(\mathbb{R}_+)$ onto ℓ^{∞}. A standard duality argument yields

$$\int_0^{\infty} \left| \sum_{j=1}^{N} \lambda_j \gamma_{k_j} x^{k_j} \right| d\mu(x) \geq \text{const} \sum_{j=1}^{N} |\lambda_j|$$

with some absolute positive constant, whenever $N \in \mathbb{N}$ and $\lambda_1, \cdots, \lambda_N \in \mathbb{C}$. Setting $\lambda_j = c_j \gamma_{k_j}^{-1}$ we get

$$\int_0^{\infty} \left| \sum_{j=1}^{N} c_j x^{k_j} \right| d\mu(x) \geq \text{const} \sum_{j=1}^{N} |c_j| \gamma_{k_j}^{-1}. \tag{3.17}$$

The assumption $\mathcal{S}(\{c_j\}) < +\infty$ ensures that the polynomials $\sum_{j=1}^{N} c_j x^{k_j}$ converge to $f(x)$ in $L^1(\mathbb{R}_+, d\mu)$ as $N \to \infty$, and so (3.17) implies

$$\int_0^{\infty} |f(x)| d\mu(x) \geq \text{const} \sum_{j=1}^{\infty} |c_j| \gamma_{k_j}^{-1}.$$

Thus $\mathcal{I}(f,0) \geq \text{const} \cdot \mathcal{S}(\{c_j\})$. Replacing f by

$$f_\varphi(z) \overset{\text{def}}{=} f(ze^{i\varphi}) = \sum_{j=1}^{\infty} c_j e^{ik_j\varphi} z^{k_j},$$

we obtain $\mathcal{I}(f,\varphi) \geq \text{const} \cdot \mathcal{S}(\{c_j\})$, as required. □

We proceed with the following observation: The smaller is ρ (i.e., the milder is the decay rate of the weight $\exp(-\beta x^\rho)$ at infinity), the weaker become gap conditions (3.13) and (3.14) imposed on $\{k_j\}$ in Theorems 7 and 8. Our present purpose is to show that, for suitable "mild" weights, one can do without any gaps at all.

We restrict ourselves to one typical example. Namely, we consider the weight

$$w(x) \overset{\text{def}}{=} \exp\left(-\frac{1}{4}\log^2 x\right), \quad 0 < x < +\infty. \tag{3.18}$$

A direct computation shows that

$$\int_0^\infty x^k w(x)dx = 2\sqrt{\pi}e^{(k+1)^2},$$

so the kernels involved are the normalized powers

$$\Phi_k(x) \overset{\text{def}}{=} (2\sqrt{\pi})^{-1} e^{-(k+1)^2} x^k \quad (k = 0, 1, 2, \cdots). \tag{3.19}$$

Theorem 9. *Let $w(x)$ and $\Phi_k(x)$ be defined by (3.18) and (3.19) respectively. Then every moment problem*

$$\int_0^\infty f(x)\Phi_k(x)w(x)dx = a_k \quad (k = 0, 1, \cdots), \quad \{a_k\} \in \ell^\infty,$$

can be solved with a function $f \in L^\infty(\mathbb{R}_+)$, whose norm is bounded by an absolute constant times $\sup_k |a_k|$.

Proof. For $k = 0, 1, \cdots$ consider the intervals $E_k \overset{\text{def}}{=} (e^{2k+1}, e^{2k+3})$. Obviously, they are pairwise disjoint subsets of $\mathbb{R}_+$. Further,

$$\int_{E_k} \Phi_k(x)w(x)dx - \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-u^2} du$$

(here the two variables are related by $\log x = 2(u + k + 1)$), and this last integral, while independent of k, turns out to be $> \frac{1}{2}$. (Once again, the normal distribution enters in.) Applying Theorem 1 completes the proof. □

The next theorem on entire functions is but a restatement of Theorem 9. To derive it, one employs standard duality considerations, similar to those used in the passage from Theorem 7 to Theorem 8. So we merely state the result.

Theorem 10. *Let* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *be an entire function such that*

$$\mathcal{R}(\{c_k\}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} |c_k| e^{(k+1)^2} < +\infty.$$

Set

$$\mathcal{J}(f,\varphi) \stackrel{\text{def}}{=} \int_0^{\infty} |f(re^{i\varphi})| \exp\left(-\frac{1}{4}\log^2 r\right) dr.$$

Then, for every $\varphi \in (-\pi, \pi]$, *one has*

$$\text{const} \cdot \mathcal{R}(\{c_k\}) \le \mathcal{J}(f,\varphi) \le 2\sqrt{\pi}\mathcal{R}(\{c_k\})$$

with an absolute positive constant on the left.

Remarks. 1) It is well known that entire functions of slow growth behave, in many respects, like polynomials. Theorem 10 provides one more result to this end, showing that such functions enjoy the maximal possible growth rate along every ray $\{\arg z = \varphi\}$.

2) It seems amazing that one can prove theorems on entire functions by purely "real variable" means, as above, without even being aware of Cauchy's theorem!

3) Similar strategy can be used in the case where rays are replaced by certain curves ending at ∞. Also, it enables one to handle analytic functions on the disk. Further generalizations (e.g., in several complex variables) are possible as well.

3.4. Interpolation by BMOA functions and embedding theorems for coinvariant subspaces in H^1. Let $\mathbb{D}$ denote the disk $\{|z| < 1\}$ and $\mathbb{T}$ its boundary. Further, let m stand for the normalized Lebesgue measure on $\mathbb{T}$, so that $m(\mathbb{T}) = 1$. A function g, analytic in $\mathbb{D}$, is said to belong to the class BMOA (=*bounded mean oscillation* + *analyticity*) if there exists an $f \in L^\infty = L^\infty(\mathbb{T}, m)$ such that

$$g(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} dm(\zeta), \quad z \in \mathbb{D}. \tag{3.20}$$

The BMOA norm is then introduced by

$$\| g \|_{\mathrm{BMOA}} \stackrel{\text{def}}{=} \inf \{ \| f \|_\infty : f \in L^\infty \text{ and (3.20) holds} \} .$$

See [G2, chapter vi] for alternative definitions and for a systematic treatment of the spaces BMO and BMOA. The reader is also refered to [G2] for some standard facts about Hardy spaces H^P and Blaschke products, occurring in this section.

It is not hard to see that every $g \in$ BMOA satisfies

$$g(z) = O\left(\log \frac{1}{1-|z|}\right), \quad |z| \to 1.$$

So a natural question arises: Which sequences $\{z_j\} \subset \mathbb{D}$ have the property that, for any $\{a_j\} \in \ell^\infty$, the interpolation problem

$$g(z_j) = a_j \log \frac{1}{1-|z_j|} \qquad (j = 1, 2, \cdots) \tag{3.21}$$

can be solved with a function $g \in$ BMOA?

Here we provide a simple sufficient condition.

Theorem 11. *Suppose that the points $\{z_j\} \subset \mathbb{D}$ satisfy*

$$|z_j - z_k| \geq \cdot c \left(1 - |z_j|\right)^s, \quad j \neq k, \tag{3.22}$$

for some fixed $c > 0$ and $s \in \left(0, \frac{1}{2}\right)$. Then every interpolation problem (3.21) *has a solution $g \in BMOA$ whenever $\{a_j\} \in \ell^\infty$.*

Proof. As easily verified,

$$\int_{\mathbb{T}} \frac{dm(\zeta)}{|1 - z\zeta|} \sim \frac{1}{\pi} \log \frac{1}{1-|z|}, \qquad z \to 1.$$

Consequently, there are positive numbers γ_j tending to 1 and such that the kernels

$$\Phi_j(\zeta) \stackrel{\text{def}}{=} \pi \gamma_j \left(\log \frac{1}{1-|z_j|}\right)^{-1} \frac{1}{1 - z_j \bar{\zeta}}, \qquad \zeta \in \mathbb{T},$$

(we may assume $0 \notin \{z_j\}$) become normalized, i.e. $\int_{\mathbb{T}} |\Phi_j| dm = 1$.

Further, for $j = 1, 2, \cdots$ we consider the subarcs

$$E_j \stackrel{\text{def}}{=} \left\{ \zeta \in \mathbb{T} : |\zeta - z_j| < \frac{c}{2} \left(1 - |z_j|\right)^s \right\},$$

where c and s are the same as in (3.22). We claim that the E_j's are disjoint. Indeed, assuming that $\zeta \in E_j \cap E_k \quad (j \neq k)$ and $|z_j| \leq |z_k|$, we have

$$\begin{aligned} |z_j - z_k| &\leq |z_j - \zeta| + |\zeta - z_k| \\ &< \frac{c}{2}(1 - |z_j|)^s + \frac{c}{2}(1 - |z_k|)^s \\ &\leq c(1 - |z_j|)^s, \end{aligned}$$

which contradicts (3.22).

It remains to estimate $\int_{E_j} |\Phi_j| dm$ from below, so as to make Theorem 1 applicable. Let $\alpha_j = \frac{z_j}{|z_j|}$ be the midpoint of the arc E_j, and let β_j be one of its endpoints, so that $|\alpha_j - z_j| = 1 - |z_j|$ and $|\beta_j - z_j| = \frac{c(1-|z_j|)^s}{2}$. Now if I_j is the subarc with endpoints α_j and β_j, one has

$$\begin{aligned} \int_{E_j} \frac{dm(\zeta)}{|1 - z_j\bar{\zeta}|} &= 2\int_{I_j} \frac{dm(\zeta)}{|1 - z_j\bar{\zeta}|} \\ &= \frac{1}{\pi}\int_{I_j} \frac{|d\zeta|}{|\zeta - z_j|} \\ &\geq \frac{1}{\pi}\left|\int_{\alpha_j}^{\beta_j} \frac{d\zeta}{\zeta - z_j}\right| \\ &= \frac{1}{\pi}\left|\log(\beta_j - z_j) - \log(\alpha_j - z_j)\right|, \end{aligned}$$

where log is a suitable branch of the logarithm. The last quantity is

$$\begin{aligned} &\geq \frac{1}{\pi}\left(\log|\beta_j - z_j| - \log|\alpha_j - z_j|\right) \\ &= \frac{1}{\pi}\left\{(1 - s)\log\frac{1}{1 - |z_j|} - \log\frac{2}{c}\right\}. \end{aligned}$$

Eventually, we have

$$\int_{E_j} \frac{dm(\zeta)}{|1 - z_j\bar{\zeta}|} \geq \frac{1}{\pi}\left\{(1 - s)\log\frac{1}{1 - |z_j|} - \log\frac{2}{c}\right\}.$$

Multiplying both sides by $\pi\alpha_j\left(\log\frac{1}{(1-|z_j|)}\right)^{-1}$ gives

$$\int_{E_j} |\Phi_j(\zeta)| dm(\zeta) \geq \alpha_j\left\{1 - s - \left(\log\frac{1}{1 - |z_j|}\right)^{-1}\log\frac{2}{c}\right\}.$$

As $j \to \infty$, this last expression on the right tends to $1 - s > \frac{1}{2}$. Consequently, for a fixed $\sigma \in \left(\frac{1}{2}, 1-s\right)$ and a suitable $N + N_{(\sigma)} \in \mathbb{N}$ we have

$$\inf_{j>N} \int_{E_j} |\Phi_j| dm > \sigma > \frac{1}{2}.$$

Now, given a sequence $\{a_j\} \in \ell^\infty$, Theorem 1 ensures the existence of a function $f \in L^\infty$ for which

$$\int_{\mathbb{T}} f(\zeta)\Phi_j(\zeta)dm(\zeta) = \pi\gamma_j a_j, \quad j > N,$$

or equivalently,

$$g(z_j) = a_j \log \frac{1}{1-|z_j|}, \quad j > N,$$

where g is obtained from f as in (3.20).

Finally, let B_N be the Blaschke product with zeroes $\{z_j : j > N\}$ (it does exist because (3.22) implies $\sum_j (1-|z_j|) < +\infty$), and let $h \in H^\infty$ be a solution of the finite interpolation problem

$$h(z_j) = \frac{1}{B_N(z_j)} \left(a_j \log \frac{1}{1-|z_j|} - g(z_j)\right), \quad j = 1, \cdots, N.$$

Clearly, the function $G \stackrel{\text{def}}{=} g + B_N h$ is in BMOA and

$$G(z_j) = a_j \log \frac{1}{1-|z_j|} \text{for all} \quad j \in \mathbb{N}$$

so we are done. □

Given a sequence $\{z_j\} \subset \mathbb{D}$ of pairwise distinct points with $\sum_j (1-|z_j|) < +\infty$, let $B = B_{\{z_j\}}$ denote the Blaschke product with zero sequence $\{z_j\}$. Define the subspace K_B^1 of H^1 as follows: $K_B^1 \stackrel{\text{def}}{=} H^1 \cap B\bar{H}_0^1$, where H^1 is the classical Hardy space, $H_0^1 \stackrel{\text{def}}{=} \{f \in H^1 : f(0) = 0\}$ and the bar denotes complex conjugation. One easily verifies that K_B^1 is closed and invariant under the *backward shift operator* $f \mapsto \frac{(f-f(0))}{z}$ acting on H^1. (Equivalently, K_B^1 is a *coinvariant subspace* of the *forward shift* $f \mapsto zf$.) Also, it can be shown that K_B^1 equals the L^1-closed linear hull of the family of rational fractions

$$r_j(\zeta) \stackrel{\text{def}}{=} \frac{1}{1-\bar{z}_j\zeta}, \qquad j = 1, 2, \cdots$$

This enables us to restate Theorem 11 in terms of K_B^1.

Theorem 12. *Suppose that* $\{z_j\}$ *is a sequence of points in* $\mathbb{D}\backslash\{0\}$ *satisfying* (3.22) *with* $0 < s < \frac{1}{2}$. *Let* B, K_B^1 *and* r_j *be defined as above. The following assertions hold true:*

(a) *The family* $\{r_j\}$ *forms an unconditional basis in* K_B^1. *More precisely, there are constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

(3.23)
$$c_1 \sum_{j=1}^{N} |\lambda_j| \log \frac{1}{1-|z_j|} \leq \| \sum_{j=1}^{N} \lambda_j r_j \|_{H_1} \leq c_2 \sum_{j=1}^{N} |\lambda_j| \log \frac{1}{1-|z_j|}$$

whenever $N \in \mathbb{N}$ *and* $\lambda_1, \cdots, \lambda_N \in \mathbb{C}$.

(b) *Given a linear operator* T, *defined originally on the (non-closed) linear hull of* $\{r_j\}$ *and taking values in a Banach space* Y, *the existence of a bounded linear extension* $T : K_B^1 \to Y$ *is equivalent to the condition*

$$\| Tr_j \|_Y = O\left(\log \frac{1}{1-|z_j|}\right) \quad \text{as} \quad j \to \infty. \tag{3.24}$$

Proof. (a) The right-hand inequality in (3.23) is immediate. The other one is readily derived from Theorem 11 by a duality argument. Perhaps, the best way to do it is to construct a function $g \in \text{BMOA}$ with

$$g(z_j) = \frac{\lambda_j}{|\lambda_j|} \log \frac{1}{1-|z_j|}, \quad \| g \|_{\text{BMOA}} \leq \text{const},$$

and to consider the integral $I \stackrel{\text{def}}{=} \int_g \sum_1^N \bar{\lambda}_j \bar{r}_j dm$. On the one hand, Cauchy's formula says that

$$I = \sum_1^N \bar{\lambda}_j g(z_j) = \sum_1^N |\lambda_j| \log \frac{1}{1-|z_j|}.$$

On the other hand,

$$|I| \leq \| g \|_{\text{BMOA}} \left\| \sum_1^N \lambda_j r_j \right\|_{H^1} \leq \text{const} \left\| \sum_1^N \lambda_j r_j \right\|_{H^1}$$

in view of the (H^1, BMOA) duality.

(b) The necessity of condition (3.24) is obvious. Conversely, if (3.24) holds then, for any finite linear combination $\sum_j \lambda_j r_j$, we have

$$\begin{aligned}\Big\|T\Big(\sum_j \lambda_j r_j\Big)\Big\|_Y &\le \sum_j |\lambda_j| \, \| Tr_j \|_Y \\ &\le \text{const} \sum_j |\lambda_j| \log \frac{1}{1-|z_j|} \\ &\le \text{const} \Big\| \sum_j \lambda_j r_j \Big\|_{H^1},\end{aligned}$$

where the last inequality is contained in (3.23). Because the finite sums $\sum_j \lambda_j r_j$ span K_B^1, there exists a bounded linear extension of T going from K_B^1 to Y. □

Remarks. 1) The situation in Theorem 12 is reminiscent of the so-called atomic decomposition of H^1 (see e.g. [G2], chapter vi, Exercise 11).

2) Theorem 12 remains valid if one drops the assumption $0 \notin \{z_j\}$ and replaces $\log \frac{1}{(1-|z_j|)}$ by $\log \frac{2}{(1-|z_j|)}$ on the right-hand side of (3.23).

The rest of this section is devoted to some embedding theorems for the subspace K_B^1 and to the study of tangential limits of functions in K_B^1.

Theorem 13. *Suppose that a sequence $\{z_j\} \subset \mathbb{D}$ satisfies (3.22) with some $c > 0$ and $s \in (0, \frac{1}{2})$. Let μ be a positive Borel measure on* $\operatorname{clos}\mathbb{D}$ *such that $\mu(\mathbb{T} \cap \operatorname{clos}\{z_j\}) = 0$. Necessary and sufficient that*

$$K_B^1 \subset L^p(\mu), \tag{3.25}$$

where $B = B_{\{z_j\}}$ and $p \in [1, +\infty)$, is the condition

$$\left(\int \frac{d\mu(\zeta)}{|1-\bar{z}_j\zeta|^p}\right)^{\frac{1}{p}} = O\left(\log \frac{1}{1-|z_j|}\right) \quad as \quad j \to \infty$$

Proof. Apply Theorem 12, part (b), to the inclusion map that sends each finite linear combination $\sum_j \lambda_j r_j$ to itself, regarded as an element of $L^p(\mu)$. Note that, by the closed graph theorem, (3.25) is equivalent to the fact that the map in question possesses a bounded linear extension going from K_B^1 to $L^p(\mu)$. □

Theorem 14. *Let Ω be an open subset of $\mathbb{D}$, and let $\{z_j\}$ be a sequence of points in $\mathbb{D}$ satisfying* (3.22) *with some $c > 0$ and $s \in (0, \frac{1}{2})$. The following are equivalent.*

(i) *$K_B^1 \subset H^\infty(\Omega)$, where $H^\infty(\Omega)$ stands for the space of bounded analytic functions on Ω.*

(ii) *For every function $f \in K_B^1$ and every point $\zeta_0 \in \mathbb{T} \cap \operatorname{clos} \Omega$, there exists the limit $\lim\limits_{z \to \zeta_0, z \in \Omega} f(z)$.*

(iii) $\liminf\limits_{j \to \infty} \left\{ \operatorname{dist}(z_j, \Omega^*) \cdot \log \frac{1}{1-|z_j|} \right\} > 0$, *where $\Omega^* \stackrel{\text{def}}{=} \{1/\bar{z} : z \in \Omega\}$ and $\operatorname{dist}(\cdot, \cdot)$ denotes the usual Euclidean distance.*

In the proof below we assume, without any loss of generality, that $o \notin \{z_j\}$ and $\Omega \subset \{\frac{1}{2} < |z| < 1\}$. This last assumption yields

$$(\operatorname{dist}(z_j, \Omega^*))^{-1} \le \sup_{z \in \Omega} |r_j(z)| \le 2\,(\operatorname{dist}(z_j, \Omega^*))^{-1}. \tag{3.26}$$

Proof. (i)$\Rightarrow$(iii). If (i) holds then the arising inclusion map has to be continuous, hence

$$\sup_{z \in \Omega} |r_j(z)| = 0 \left(\log \frac{1}{1 - |z_j|} \right) \quad \text{as} \quad j \to \infty.$$

In view of (3.26), we arrive at (iii).

(iii)$\Rightarrow$(ii). Given a function $f \in K_B^1$, write

$$f(z) = \sum_{j=1}^{\infty} \lambda_j r_j(z) \qquad (\lambda_j \in \mathbb{C}),$$

where the series converges in H^1 (and hence also pointwise in $\mathbb{D}$). By Theorem 12, part (a), it follows that

$$\sum_{j=1}^{\infty} |\lambda_j| \log \frac{1}{1 - |z_j|} < +\infty.$$

Combining (3.26) and (iii), we get

$$\sup_{z \in \Omega} |r_j(z)| \le 2\,(\operatorname{dist}(z_j, \Omega^*))^{-1} \le \text{const} \cdot \log \frac{1}{1 - |z_j|}.$$

Consequently, for $z \in \Omega$ one has

$$|\lambda_j r_j(z)| \le \text{const} \cdot |\lambda_j| \log \frac{1}{1 - |z_j|}.$$

Now if $\zeta_0 \in \mathbb{T} \cap \mathit{clos}\Omega$ we conclude that

$$\lim_{z\to\zeta_0, z\in\Omega} f(z) = \sum_{j=1}^{\infty} \lambda_j r_j(\zeta_0)$$

by the dominated convergence theorem.

(ii)⇒(i). This is obvious. □

Remarks. 1) Let $\psi : (0,2) \to (0,1)$ be an increasing continuous function with $\lim_{t\to 0^+} \psi(t) = 0$ such that $t \mapsto t - \psi(t)$ is also an increasing function for t small enough. Consider the domain

$$\Omega = \Omega_\psi \stackrel{\text{def}}{=} \{z \in \mathbb{D} : 1 - |z| > \psi(|1-z|)\},$$

so that $\operatorname{clos}\Omega \cap \mathbb{T} = \{1\}$ and ψ is responsible for the order of contact between $\partial\Omega$ and $\mathbb{T}$ at 1. It is not hard to see that for $\Omega = \Omega_\psi$ condition (iii) in Theorem 14 can be rewritten in the form

$$\lim_{j\to\infty} \inf \psi(|1 - z_j|) \log \frac{1}{1 - |z_j|} > 0.$$

In particular, letting $\psi(t) = ct, 0 < c < 1$, we get

$$\lim_{j\to\infty} \inf |1 - z_j| \log \frac{1}{1 - |z_j|} > 0,$$

which is necessary and sufficient (once (3.22) holds) that all functions in K_B^1 have nontangential limits at 1. Related results for coinvariant subspaces in H^P, with $p > 1$, can be found in [AC] and [C1].

2) In convention with Theorem 13 above, we cite [C2], [C3], [D2], and [D3], where similar embedding theorems are established for the coinvariant subspaces $K_\theta^P \stackrel{\text{def}}{=} H^P \cap \theta \bar{H}_0^P$ generated by various inner functions θ.

3) In connection with Theorem 11 above, we mention the paper [S], where the values of BMOA functions on generic interpolating sequences are characterized. However, the author does not see how that characterization might be possibly used to derive Theorem 11.

3.5. Prescribing partial sums of Fourier series for bounded functions. Given a function $f \in L^\infty(-\pi, \pi)$, denote its k-th Fourier coefficient by $\hat{f}(k)$, so that

$$\hat{f}(k) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \qquad (k \in \mathbb{Z})$$

and consider the *partial sums* $S_n f$ of its *Fourier series*,

$$(S_n f)(x) \stackrel{\text{def}}{=} \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} \qquad (n = 0, 1, \cdots).$$

It well know that

$$(S_n f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \mathcal{D}_n(x - t) dt,$$

where $\mathcal{D}_n$ is the *Dirichlet kernel* given by

$$\mathcal{D}_n(t) = \frac{\sin\left(n + \frac{1}{2}\right) t}{\sin \frac{1}{2}}.$$

Because f is bounded and

$$L_n \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathcal{D}_n(t)| dt \sim \frac{4}{\pi^2} \log n \tag{3.27}$$

as $n \to \infty$ (see [Z], chapter ii, section 12), it follows that $\| S_n f \|_\infty = O(\log n)$.

In this section we deal with moment problems of the form

$$(S_{n_j} f)(x_j) = a_j \log n_j, \tag{3.28}$$

where n_j are positive integers, x_j are fixed points in $(-\pi, \pi)$, $\{a_j\}$ is a prescribed sequence in ℓ^∞ and, finally, $f \in L^\infty(-\pi, \pi)$ is a function to be found.

Theorem 15. *Let $\{n_j\}$ be an increasing subsequence on $\mathbb{N}$ such that $\sum_{j=1}^{\infty} n_j^{-2}$ $< +\infty$ for some $\alpha \in (0, \frac{1}{2})$. Then there exist a sequence $\{x_j\} \subset (-\pi, \pi)$ and a number $N \in \mathbb{N}$ with the following property: Whenever $\{a_j\} \in \ell^\infty$, one can find a function $f \in L^\infty(-\pi, \pi)$ satisfying* (3.28) *for all $j > N$.*

Proof. It can be shown that

$$\frac{1}{2\pi} \int_{-\pi/n^\alpha}^{\pi/n^\alpha} |\mathcal{D}_n(t)| dt = \frac{4}{\pi^2} (1 - \alpha) \log n + 0(1) \tag{3.29}$$

as $n \to \infty$. The computations involved are almost identical to those used in [Z] to derive (3.27), so we do not include them here. Dividing both sides in (3.29) by L_n and taking (3.27) into account, we get

$$\lim_{n\to\infty} \frac{1}{L_n} \int_{-\pi/n^\alpha}^{\pi/n^\alpha} |\mathcal{D}_n(t)| \frac{dt}{2\pi} = 1 - \alpha.$$

Consequently, for a fixed $\sigma \in \left(\frac{1}{2}, 1-\alpha\right)$ and a suitable $N \in \mathbb{N}$ we have

$$\text{(3.30)} \qquad \frac{1}{L_n} \int_{-\pi/n^\alpha}^{\pi/n^\alpha} |\mathcal{D}_n(t)| \frac{dt}{2\pi} > \sigma \quad \text{whenever } n > N.$$

Making N still larger, if necessary, we can arrange it so that $\sum\limits_{j=N+1}^{\infty} n_j^{-\alpha} < 1$. This enables us to choose a family of non-overlaping intervals I_j $(j > N)$, contained in $(-\pi, \pi)$, of length $|I_j| = 2\pi n_j^{-\alpha}$. This done, let x_j be the midpoint of I_j, so that $I_j = \left(x_j - \frac{-\pi}{n_j^\alpha}, x_j + \frac{\pi}{n_j^\alpha}\right)$. Now if $j > N$ then also $n_j > N$, and (3.30) yields

$$\frac{1}{L_{n_j}} \int\limits_{I_j} |\mathcal{D}_{n_j}(x_j - t)| \frac{dt}{2\pi} = \frac{1}{L_{n_j}} \int_{-\pi/n_j^\alpha}^{\pi/n_j^\alpha} |\mathcal{D}_{n_j}(t)| \frac{dt}{2\pi} > \sigma > \frac{1}{2}.$$

Finally, we apply Theorem 1 with $X = (-\pi, \pi), d\mu(t) = \frac{dt}{2\pi}, \Phi(t) = \mathcal{D}_{n_j}(x_j - t)/L_{n_j}$ and $E_j = I_j (j > N)$. Condition (1.3) has already been verified; thus, given $\{a_j\} \in \ell^\infty$, Theorem 1 provides a function $f \in L^\infty(-\pi, \pi)$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \Phi_j(t) dt = a_j \frac{\log n_j}{L_{n_j}}, \qquad j > N.$$

This condition coincides with (3.28), and the proof is therefore complete. □

Remark. One might wish to prescribe the values of $\mathcal{S}_{n_j} f$ at a fixed point x_0 (say, $x_0 = 0$). In this case the above method yields the following: If the sequence $\{n_j\}$ is so sparse that $\inf\limits_j \left(\frac{\log n_{j+1}}{\log n_j}\right) > 2$, then every moment problem

$$(\mathcal{S}_{n_j} f)(0) = a_j \log n_j, \qquad \{a_j\} \in \ell^\infty,$$

has a solution $f \in L^\infty(-\pi, \pi)$. However, a more precise result can be found in [A1].

4. A question on translates of an L^1 function

Suppose that $K \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} |K(x)|dx = 1$. Consider the family of *translates* $K(\cdot - x_j)$, where $\{x_j\}$ is a certain sequence of real numbers. The question we are going to discuss is: What are the conditions on K and $\{x_j\}$ under which every moment problem

$$\int_{-\infty}^{\infty} f(t)K(t-x_j)dt = a_j \quad (j = 1, 2, \cdots), \quad \{a_j\} \in \ell^\infty,$$

has a solution $f \in L^\infty(\mathbb{R})$? Equivalently, when is it true that

$$\int_{-\infty}^{\infty} \Big|\sum_j \lambda_j K(t-x_j)\Big| dt \geq \text{const} \sum_j |\lambda_j| \tag{4.1}$$

with const > 0, uniformly for all finite sequences $\{\lambda_j\}$ of complex numbers?

It is easily shown that a necessary condition is

$$\inf\{|x_j - x_\ell| : j \neq \ell\} > 0 \tag{4.2}$$

(cf. [G2], chapter vii, section 4). On the other hand, Theorem 1 (see also the proof of Theorem 4) provides a nice sufficient condition, which can be viewed as a refined version of (4.2). Namely, if there exists an interval $I \subset \mathbb{R}$ of length $|I|$ such that

$$\int_I |K(x)|dx > \frac{1}{2} \quad \text{and} \quad \inf\{|x_j - x_\ell| : j \neq \ell\} \geq |I|, \tag{4.3}$$

then we have (4.1).

To see that (4.3) is sharp, consider the case where $K = \frac{1}{2}\chi_{(-1,1)}, x_j = j$ and $I = (0, 1)$. Obviously, we have

$$\int_I |K(x)|dx = \frac{1}{2} \quad \text{and} \quad \inf\{|x_j - x_\ell| : j \neq \ell\} = |I| = 1,$$

whereas

$$\begin{aligned} &\int_{-\infty}^{\infty} |\sum_{j=0}^{N} (-1)^j K(t-j)| dt \\ &= \int_{-\infty}^{\infty} \left| \frac{1}{2}\chi_{(-1,0)}(t) + \frac{1}{2}(-1)^N \chi_{(N,N+1)}(t) \right| dt \\ &= 1 \end{aligned}$$

and so (4.1) fails for $\lambda_j = (-1)^j$ ($j = 0, \cdots, N$ with N large enough).

It should be mentioned, however, that sometimes condition (4.2) alone is sufficient for (4.1) to hold. In particular, this happens [CG] when K is a rational function. Thus, dealing with generic K's one has to integrate, as it were, between (4.2) and (4.3).

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