Local entropy rigidity for hyperbolic manifolds

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We study deformations of compact hyperbolic manifolds of a given total volume. We show that along any non-trivial deformation the topological entropy and the difference between topological entropy and Liouville entropy are locally strictly convex functions of the deformation parameter, thus providing a partial positive answer to a conjecture of A. Katok.

§ 1. Introduction.

1.1. Statement of the problem.

1.1.1. Notation. Throughout this article, $M$ denotes a compact smooth manifold of dimension $n$. Given a $C^2$ Riemannian metric $g$ on $M$, the geodesic flow determined by $g$ on the unit tangent bundle $S_g M$ is denoted by $T_g$ or $T_g^t$, or simply by $g^t$ when this abuse of notation will not be confusing.

1.1.2. Geodesic flows on the unit tangent bundles $S_g M$ of compact Riemannian manifolds $(M, g)$ of negative curvature are the chief examples of transitive Anosov flows.

We recall that a continuous flow $T : (v, t) \in N \times \mathbb{R} \mapsto T^t v \in N$ is transitive if it has a dense orbit. A flow $T$ on a compact manifold $N$ is an Anosov flow [Ano67] if it is $C^1$ and the tangent bundle $TN$ of $N$ splits continuously in $T$-invariant subbundles $TN = E^0 \oplus E^u \oplus E^s$ satisfying the conditions:

1. $E^0$ is the tangent space to the orbits of flow.

2. There exists positive constants $\lambda$ and $C$ such that $\|dT^t|E^s\| < Ce^{-\lambda t}$ and $\|dT^{-t}|E^u\| < Ce^{\lambda t}$ for all $t > 0$.

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1.1.3. Anosov flows have many invariant measures. Given a $T^t$-invariant probability measure $m$, we denote by $h(T,m)$ or $h(T^t,m)$ the measure theoretic entropy of the time 1 transformation $T^1$ with respect to the measure $m$ [Pet83]. The supremum

$$h_{\text{top}}(T) = \sup \{ h(T,m) \mid T^t\text{-invariant probability measure } m \}$$

of all measure theoretic entropies is attained, for Anosov flows, by a unique measure, the Margulis measure $m_0$, also called the measure of maximal entropy [Mar70, Bow72a, Bow74, BR75].

In the case of a geodesic flow $T^t_g$, we also have a smooth $T^t_g$-invariant measure on $S_gM$, the Liouville measure $\text{Liouv}(g)$, which arises from the contact structure on $S_gM$.

1.1.4. For locally symmetric manifolds of negative curvature, it is not difficult to see that the Liouville measure and the Margulis measure coincide. In other words, for geodesic flows on locally symmetric manifolds of negative curvature, the measure theoretical entropy of Liouville measure coincides with the topological entropy. The converse of this statement is the content of the following conjecture of A. Katok.

1.1.5. Conjecture [Kat82, BK85]. The topological entropy and the entropy of the Liouville measure for a geodesic flow on a negatively curved manifold coincide, (if and) only if the manifold is locally symmetric.

A. Katok himself showed in [Kat82] that the conjecture is true if one considers metrics conformally equivalent to a locally symmetric metric. In particular, the conjecture is true for surfaces of genus greater than 1.

The topological entropy has also another geometric interpretation [Man79]. Denote by $\tilde{M}$ the universal cover of $M$. For a negatively curved metric $g$ on $M$, let $B_g,r(p)$ be the ball of radius $r$ centred at $p \in \tilde{M}$ for the lift to $\tilde{M}$ of metric $g$. Then

$$h_{\text{top}}(T_g) = \lim_{r \to \infty} \frac{1}{r} \log \text{Vol}_g(B_{g,r}(p)).$$

In words, the topological entropy of the geodesic flow coincides with the volume growth of balls in the universal cover of $M$.

A related conjecture of Gromov, formulated at about the same time as Katok's conjecture, states:
1.1.6. Conjecture ([Gro83]). Among all metrics of volume equal to the volume of a locally symmetric metric $g_0$ on a manifold $M$, the volume growth of balls in the universal cover $\tilde{M}$ of $M$ is minimized at the metrics isometric to $g_0$.

Besson, Courtois and Gallot ([BCG94a] and [BCG94b]) have recently announced a proof of Gromov's conjecture for rank-one neg. curved locally symmetric manifolds which builds up on their 1992 theorem:

1.1.7. Theorem ([BCG]). Let $g_0$ be a metric of constant negative curvature on a compact manifold $M$ of dimension $n$. Then, in the space of $H^s$ metrics with volume equal to the volume of $g_0$, there exists a neighbourhood $U$ of $g_0$ in the $H^s$ topology ($s > n/2 + 2$), such that the minimum of the topological entropy in $U$ is attained only by metrics isometric to $g_0$.

1.1.8. In consideration of the fact that the method of Besson, Courtois and Gallot applies so far only to topological entropy, in order to shade some light on Katok’s entropy conjecture 1.1.5, it is very interesting to study the functions

$$\text{Ent}_{\text{top}} : g \in M^r(M) \mapsto h_{\text{top}}(T_g)$$

$$\text{Ent}_{\text{Liouv}} : g \in M^r(M) \mapsto h(T_g, \text{Liouv}(g))$$

in a neighbourhood of the locally symmetric metric $g_0$. Here we have denoted by $(M, g_0)$ a compact locally symmetric manifold of negative curvature and set

$$M^r(M) = \{C^r \text{ metrics } g \text{ on } M \mid \text{Vol}_g(M) = \text{Vol}_{g_0}(M)\}.$$

1.1.9. We remark that the above functions $\text{Ent}_{\text{top}}$ and $\text{Ent}_{\text{Liouv}}$ are known to be smooth by previous works respectively of Katok, Knieper, Pollicott and Weiss [KKPW89] (for the topological entropy) and of Contreras (for the Liouville entropy) [Con92], provided we restrict ourselves to negatively curved metrics. It is also known [KKW91] that a locally symmetric metric $g_0$ is a critical point of $\text{Ent}_{\text{top}}$ and $\text{Ent}_{\text{Liouv}}$.

1.2. Statement of the theorems.

In this paper we obtain estimates for the second derivative of the functions $\text{Ent}_{\text{top}}$ and $\text{Ent}_{\text{Liouv}}$ at a metric of constant negative curvature
which allow us to establish that along a path through a metric of constant negative curvature $g_0$, locally, the only metric for which one has $\text{Ent}_{\text{top}}(g) = \text{Ent}_{\text{Liouv}}(g)$ is $g_0$. More exactly:

**Theorem A.** Let $g_0$ be a metric of constant negative curvature on a compact manifold $M$ of dimension $n$ and let $g_\varepsilon$ be a $C^2$ curve of $C^5$ metrics of constant volume. Then, if $g_\varepsilon$ is not tangent to the orbit of $g_0$ under the diffeomorphism group, the function

$$\varepsilon \mapsto \text{Ent}_{\text{top}}(g_\varepsilon) - \text{Ent}_{\text{Liouv}}(g_\varepsilon)$$

is strictly convex at $\varepsilon = 0$. In particular, it follows that along the path $g_\varepsilon$ and for small $\varepsilon$’s, the equality $\text{Ent}_{\text{top}}(g_\varepsilon) = \text{Ent}_{\text{Liouv}}(g_\varepsilon)$ occurs only at $g_0$.

In [Pol94], Pollicott proved that at a locally symmetric metric the topological entropy is convex for volume preserving conformal deformations. Along the way to the proof of Theorem A we find the following theorem.

**Theorem B.** Under the same hypothesis as Theorem A, the function

$$\varepsilon \mapsto \text{Ent}_{\text{top}}(g_\varepsilon),$$

is locally strictly convex at $g_0$.

For the measure theoretical entropy the volume normalization does not yield any useful convexity or concavity property. Surprisingly, we have:

**Theorem C.** There exists an hyperbolic 3-manifold $(M, g_0)$ for which the function $\text{Ent}_{\text{Liouv}} : \mathcal{M}^5(M) \to \mathbb{R}^+$ has second derivative at $g_0$ with mixed signature. In particular the entropy $\text{Ent}_{\text{Liouv}}$ of the Liouville measure does not have either a minimum nor a maximum at $g_0$, when restricted to $\mathcal{M}^5(M)$.

1.3. Outline of the proofs.

The scheme for the estimate can be divided into three parts. In the first part, since geodesic flows on a negatively curved manifold can be represented as a flow built under a function on a topologically mixing subshift of finite type, we investigate these flows.

We recall some basic definitions about cocycles. Let $T$ be a Borel $\mathbb{R}$-flow on a Borel space $X$. An ($\mathbb{R}$-valued) cocycle for the flow $T$ is a Borel function $c : X \times \mathbb{R} \to \mathbb{R}$ satisfying the relation

$$c(x, t + s) = c(x, t) + c(T^t x, s).$$
Cocycles for the flow $T$ form a group under addition and a cocycle $c$ for $T$ is called a coboundary if $c(x, t) = b(T^tx) - b(x)$ for some Borel function $b$. Coboundaries form a subgroup and the elements of the quotient group are the Borel cohomology classes of the flow. If a cocycle $c$ is "differentiable along the orbits", i.e. if $c(x, t) = \int_0^t A(T^sx)\, ds$, for some Borel function $A$ on $X$, we say that $A$ generates $c$. In a cohomology class we can always find a representative which is smooth along the orbits. If $X$ has a $C^\alpha$ structure, we say that a cohomology class is $C^\alpha$ if it has a representative generated by a $C^\alpha$ function on $X$.

If $(X, T)$ is a symbolic flow, i.e. a flow built over a topologically mixing subshift of finite type with a Hölder ceiling function $\lambda$, then to each $C^\alpha$ cohomology class $[c]$ is attached a $T$-invariant measure $m_{[c]}$ called the Gibbs state for $[c]$. If $A$ is a Hölder function generating a representative of $[c]$, it is customary to refer to $A$ as the potential for the Gibbs state $m_{[c]}$. The function identically equal to 1, generates a cocycle, called the length cocycle and its corresponding Gibbs state is the measure of maximal entropy for $T$.

We prove the following Proposition about derivatives of entropy of Gibbs states for symbolic flows in terms of the variation of the generating cocycles. The $\text{Cov}_u^T(v, w)$ below denotes the "covariance" of Hölder continuous functions $v, w$ for the flow $T$ and with respect to the Gibbs state for the potential $u$ (see [Rue78] and §2) and $\text{Var}_u^T(v) = \text{Cov}_u^T(v, v)$.

1.3.1. Proposition (A). Let $(\Sigma, \sigma)$ be a topologically mixing subshift of finite type and let $\lambda_\varepsilon$ be a $C^2$ curve positive $C^\alpha$ functions on $\Sigma$. Let $(X_\varepsilon, T_\varepsilon^t)$ be the special flow built over $(\Sigma, \sigma)$ with ceiling function $\lambda_\varepsilon$. Let $\delta_\varepsilon$ be a $C^\alpha$ function on $X_\varepsilon \approx \{(p, t) \mid p \in \Sigma, 0 \leq t < \lambda_\varepsilon(p)\}$ with the property that

$$\int_0^{\lambda_\varepsilon(p)} \delta_\varepsilon(p, t)\, dt = d/d\varepsilon \lambda_\varepsilon(p).$$

Then, denoting with primes differentiations with respect to $\varepsilon$, we have:

\begin{align*}
(A1) \quad & h_{t_0}^\prime(T_0) \equiv \frac{d}{d\varepsilon} h_{t_0}(T_\varepsilon|_{\varepsilon=0}) = -h_{t_0}(T_0)m_0(\delta l_0) \\
\end{align*}

where $m_0$ is the measure of maximal entropy for the flow $T_0$. If $h_{t_0}^\prime(T_0) = 0$ and $\delta^2 l_\varepsilon$ denotes a $C^\alpha$ function on $X_\varepsilon$ with the property that

$$\int_0^{\lambda_\varepsilon(p)} \delta^2 l_\varepsilon(p, t)\, dt = d^2/d\varepsilon^2 \lambda_\varepsilon(p),$$

then

\begin{align*}
(A2) \quad & h_{t_0}^\prime(T_0) = -h_{t_0}(T_0)m_0(\delta^2 l_0) + h_{t_0}(T_0)^2 \text{Var}_0^T(\delta l_0). \\
\end{align*}
(B). Let \( u_\varepsilon = u_\varepsilon(p,t) \), \( p \in \Sigma, \, t \in \mathbb{R} \), be a \( C^2 \) family of Hölder continuous functions of pressure zero for the flow \( T_\varepsilon \). Let \( m_{u_\varepsilon} \) denote the Gibbs state for the flow \( T_\varepsilon \) and potential \( u_\varepsilon \). We have:

\[
(B1) \quad h'(T_0, m_{u_0}) = \frac{d}{d\varepsilon} h(T_\varepsilon, m_{u_\varepsilon})_{\varepsilon=0} = \text{Cov}_{u_0}^{T_0}(u_0, \delta u_0) - h(T_0, m_{u_0}) m_{u_0}(\delta l_0),
\]

where \( \delta u_\varepsilon \) is the (generator of) first variation of the cocycle generated by \( u_\varepsilon \) and is defined by \( \int_0^{\lambda_\varepsilon} \delta u_\varepsilon(p,t) \, dt = d/d\varepsilon \int_0^{\lambda_\varepsilon} u_\varepsilon(p,t) \, dt \). If \( u_0 \) is cohomologous to a constant and \( m_{u_0}(\delta l_0) = 0 \), then \( h'(T, m_u) = 0 \) and

\[
(B2) \quad h''(T_0, m_{u_0}) = -\text{Var}_{u_0}^{T_0}(\delta u_0) - 2h(T_0, m_{u_0}) \text{Cov}_{u_0}^{T_0}(\delta l_0, \delta u_0)
\]

(C). Finally, if in addition to the hypothesis in (B), we have

(a) \( m_{u_\varepsilon}(\delta l_\varepsilon) = 0 \) for all \( \varepsilon \) and
(b) \( m_0 = m_{u_0} \), then

\[
(C1) \quad h'_{\text{top}}(T_0) = \text{Cov}_{u_0}^{T_0}(h_{\text{top}}(T_0) \delta l_0, \delta u_0 + h_{\text{top}}(T_0) \delta l_0).
\]
\[
(C2) \quad h''(T_0, m_{u_0}) = -\text{Cov}_{m_0}^{T_0}(\delta u_0, \delta u_0 + h_{\text{top}}(T_0) \delta l_0)
\]
\[
(C3) \quad h'_{\text{top}}(T_0) - h''(T_0, m_{u_0}) = \text{Var}_{m_0}^{T_0}(\delta u_0 + h_{\text{top}}(T_0) \delta l_0).
\]

The formula (A1) was essentially a step of [KKW91] and (A2) has also been proved independently by Pollicott [Pol94].

The above proposition yields formulas for the second derivative of \( \text{Ent}_{\text{top}} \) and \( \text{Ent}_{\text{Liouv}} \) along paths \( g_\varepsilon \) of metrics in \( \mathcal{M}(M) \) once one knows how to determine the variations of the cocycles generating the Margulis and the Liouville measures—i.e. the length cocycle and the Liapunov cocycle—in terms of the variation \( S = \frac{d}{d\varepsilon} g_\varepsilon \) of the Riemannian metric. In fact, since Gibbs states only depend on the cohomology class of the generating cocycles it is sufficient to determine the variation of the cohomology class of these cocycles. This is the second step of the proof.

1.3.2. Notation. For a symmetric covariant tensor field \( S \) of rank 2 on a Riemannian manifold \( (M,g_0) \), denote by \( S^V \) the quadratic form field \( S^V : v \mapsto S(v,v) \) defined on the sphere bundle \( S_{g_0}M \).

Then for the length cocycle we have \( \delta l_0 = \frac{1}{2} S^V |_{\varepsilon=0} \). For the Liapunov cocycle we have the following Proposition:

1.3.3. Proposition. Let \( g_\varepsilon \) be a \( C^1 \) curve of \( C^4 \) metrics on a manifold \( M \), and assume that \( g_0 \) has constant negative curvature. Let \( S = \frac{d}{d\varepsilon} g_\varepsilon \) at \( \varepsilon = 0 \).
Then the generator \( \delta u \) of the first variation of the Liapunov cocycle at \( \varepsilon = 0 \) is cohomologous to

\[
\delta u \approx T^\nabla + \frac{1-n}{2} S^\nabla
\]

where \( T \) is the symmetric tensor field defined by

\[
T = -\frac{1}{2} S + \frac{1}{4} \nabla^* \nabla S + \frac{1}{2} (\text{Tr}_g S) g - \frac{1}{2} \delta^* \delta S.
\]

In the Proposition above, \( \nabla^* \nabla \) is the rough Laplacian for the metric \( g_0 \), \( \delta^* \) denotes the symmetrization of the covariant derivative for \( g_0 \) and \( \delta \) denotes its formal adjoint, the divergence.

The final step consists in estimating the covariance \( \text{Cov}_{m_0}(\delta l_0, \delta u_0) = \text{Cov}_{m_0}(\frac{1}{2} S^\nabla, T^\nabla + \frac{1-n}{2} S^\nabla) \) which appears in the formulas (C1-3). We prove:

**1.3.4. Proposition.** Let \( g_0 \) be a metric on \( M \) of constant negative curvature \( -1. \) Let \( S \) be a \( C^3 \) a symmetric covariant tensor field of rank 2 on \( M \) and let

\[
T = -\frac{1}{2} S + \frac{1}{4} \nabla^* \nabla S + \frac{1}{2} (\text{Tr}_g S) g - \frac{1}{2} \delta^* \delta S.
\]

Then, we have:

\[
\text{Cov}_{\text{Liouv}(g_0)}^{\delta_0}(S^\nabla, T^\nabla) \geq \frac{n-2}{4} \text{Cov}_{\text{Liouv}(g_0)}^{\delta_0}(S^\nabla, S^\nabla)
\]

\[
\text{Cov}_{\text{Liouv}(g_0)}^{\delta_0}(T^\nabla, T^\nabla) \geq \left( \frac{n-2}{4} \right)^2 \text{Cov}_{\text{Liouv}(g_0)}^{\delta_0}(S^\nabla, S^\nabla)
\]

The above estimate is the heart of the proof. It follows from observing that since \( (M, g_0) \) is locally symmetric of constant negative curvature, the functions \( S^\nabla, T^\nabla \) on the unit tangent bundle \( S_{g_0}M \) lift to the orthonormal frame bundle \( FM \) of \( M \). The group \( G \approx SO_0(1, n) \) acts transitively on \( FM \), and the bilinear form \( \text{Cov}_{\text{Liouv}(g_0)}^{\delta_0} \) is diagonal with respect to this action of \( G \), i.e. it respects the decomposition of \( L^2(FM) \) into irreducible subspaces. But the linear map \( S \mapsto T \) also respects this decomposition and, on each irreducible subspace, \( T \) is a multiple of \( S \). In the end, the estimate reduces to an estimate of the smallest eigenvalue of the operator \( S \mapsto T \) on the space orthogonal to the orbit of \( g_0 \) under \( \text{Diff}(M) \). This is achieved via a suitable Weitzenböck formula.

**1.3.5. Corollary.** Under the hypothesis of Theorem A and setting \( S = \frac{d}{d\varepsilon} g_0 \)

at \( \varepsilon = 0 \), the second derivatives of the topological entropy and of the Liouville entropy satisfy:

\[
\text{Ent}_{\text{top}}''(g_0) \geq \frac{1}{4} \left( \frac{n-1}{2} \right) \text{Cov}_{\text{Liouv}(g_0)}^{\delta_0}(S^\nabla, S^\nabla),
\]

\[
\text{Ent}_{\text{top}}''(g_0) - \text{Ent}_{\text{Liouv}}''(g_0) \geq \frac{(n-2)^2}{4} \text{Cov}_{\text{Liouv}(g_0)}^{\delta_0}(S^\nabla, S^\nabla),
\]
Proof of Theorems A and B. The equality $\text{Cov}^g_{\text{Liouv}(g_0)}(S^V, S^V) = 0$, is equivalent to saying that $S^V$ is cohomologous to a constant, in our case to zero. Applying a theorem of Guillemin and Kazhdan [GK79], we obtain that this implies that the curve $g_e$ is tangent to the orbit of the diffeomorphism group at $g_0$, in contradiction to the hypothesis. Thus $\text{Cov}^g_{\text{Liouv}(g_0)}(S^V, S^V) > 0$ and by Corollary 1.3.5 we obtain $\text{Ent}_{\text{top}}''(g_0) > 0$ and $\text{Ent}_{\text{top}}''(g_0) - \text{Ent}_{\text{Liouv}}''(g_0) > 0$.

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§2. Derivatives of entropy for symbolic flows.

2.1.

For generalities on topologically mixing subshifts of finite type, pressure and Gibbs states we refer to [Bow75] and [Rue78].

Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing subshift of finite type. For any Hölder continuous potential $\phi$ denote by $\mu_{\phi}$ the Gibbs state for $\phi$, and by $P^\sigma(\phi)$ or $P(\sigma, \phi)$ the pressure of the potential $\phi$. We recall that pressure $P^\sigma(\phi)$ and the entropy $h(\sigma, \mu_{\phi})$ of $\sigma$ with respect to the Gibbs state $\mu_{\phi}$ are related by the variational principle:

$$P^\sigma(\phi) = h(\sigma, \mu_{\phi}) + \mu_{\phi}(\phi) = \sup_{\mu} (h(\sigma, \mu) + \mu(\phi)),$$

where the supremum is taken as $\mu$ ranges on the set of probability $\sigma$-invariant measures on $\Sigma$. It is also known that, for $\phi \in C^\alpha(\Sigma)$, the equality $P^\sigma(\phi) = h(\sigma, \mu_{\phi}) + \mu_{\phi}(\phi)$ completely characterizes the Gibbs state $\mu_{\phi}$ among the $\sigma$-invariant measures $\mu$ on $\Sigma$ (see [Bow75]).

We define the covariance of Hölder continuous functions $\psi$ and $\zeta$ with respect to the Gibbs state $\mu_{\phi}$ by

$$\text{Cov}_{\phi}^\sigma(\psi, \zeta) = \sum_{i=-\infty}^{\infty} \mu_{\phi} \left( \psi \cdot \zeta \circ \sigma^i - \mu_{\phi}(\psi) \mu_{\phi}(\zeta) \right).$$

We also set

$$\text{Var}_{\phi}^\sigma(\psi) = \text{Cov}_{\phi}^\sigma(\psi, \psi).$$
and we call the above limit is called the variance of $\psi$ w.r.t. the Gibbs state $\mu_\phi$. We have $\text{Cov}_\phi^\sigma(\psi, \zeta) = 0$ for all $\psi$ if and only if $\zeta$ is cohomologous to a constant, i.e. $\zeta = \chi \circ \sigma - \chi + \text{Const}$ for some Hölder continuous function $\chi : \Sigma \to \mathbb{R}$.

The convergence of the above series follows from the exponential rate of mixing of Hölder continuous functions.

2.2.

From [Con92] the map

$$P^\sigma : \phi \in C^\alpha(\Sigma) \mapsto P^\sigma(\phi) \in \mathbb{R}$$

is real analytic. Setting

$$D^n_P\phi_0(\phi_1, \ldots, \phi_n) = \frac{d}{dt_n} \cdots \frac{d}{dt_1} P^\sigma(\phi_0 + t_1 \phi_1 + \cdots + t_n \phi_n) \bigg|_{t_1 = t_2 = \cdots = t_n = 0},$$

we have, (cf. [Con92] and [Rue78, Ch.5, Exerc. 5])

(2.1)

$$D^1_P\phi_0(\phi_1) = \mu_{\phi_0}(\phi_1) \quad \text{and} \quad D^2_P\phi_0(\phi_1, \phi_2) = \text{Cov}_\phi^\sigma(\phi_1, \phi_2).$$

Thus the maps

$$\phi \in C^\alpha(\Sigma) \mapsto \mu_\phi \in C^\alpha(\Sigma)^*$$

and

$$\text{Cov}^\sigma : \phi \in C^\alpha(\Sigma) \mapsto \text{Cov}_\phi^\sigma \in L(C^\alpha(\Sigma), C^\alpha(\Sigma); \mathbb{R})$$

are real analytic.

2.2.1. Observation. If $\phi_\epsilon$ is a $C^2$ curve of potentials in $C^\alpha(\Sigma)$, writing $\phi_\epsilon = \phi_0 + \epsilon \phi_1 + \frac{\epsilon^2}{2} \phi_2 + o(\epsilon^2)$, we obtain

(2.2)

$$P^\sigma(\phi_\epsilon) = P^\sigma(\phi_0) + \epsilon \mu_{\phi_0}(\phi_1) + \frac{\epsilon^2}{2} \left( \text{Cov}_{\phi_0}^\sigma(\phi_1, \phi_1) + \mu_{\phi_0}(\phi_2) \right) + o(\epsilon^2).$$

2.3.

Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing subshift of finite type and $\lambda$ be a positive $C^\alpha$ function on $\Sigma$. 
We recall the definition of special flow $T$ built under $\lambda$. Let $\tilde{T}$ be the flow on $\Sigma \times \mathbb{R}$ defined by $\tilde{T}^t(p,s) = (p,s+t)$ and consider on $\Sigma \times \mathbb{R}$ the equivalence relation $\sim$ generated by $(p,t) \sim (\sigma p, t - \lambda(p))$. Then, setting $X = \Sigma \times \mathbb{R}/\sim$, the flow $\tilde{T}$ descends to a flow $T$ on $X$: the flow $(X,T)$ is called the special flow built under $\lambda$. Sometimes we shall simply write $(\Sigma, \sigma, \lambda, T)$ or $(\Sigma, \sigma, \lambda)$ to refer to $(X,T)$. Clearly a fundamental domain for $X$ is the set $\{(p,t) \in \Sigma \times \mathbb{R} : 0 \leq t < \lambda(p)\}$ and $X$ can be easily turned in a metric space (see [BW72]).

2.3.1. Notation. Let $A : X \rightarrow \mathbb{R}$ be a $C^\alpha$ continuous potential on $X$ and let $P^T(A) = P(T,A)$, $m_A$ and $h(T,m_A)$ denote respectively the pressure of $A$ for the flow $T$, the Gibbs state for the potential $A$ and the entropy of the flow $T$ with respect to the measure $m_A$.

In analogy to the case of shifts, the pressure $P^T(A)$ and the entropy $h(T,m_A)$ of $T$ with respect to the Gibbs state $m_A$ are related by the variational principle:

$$P^T(A) = h(T,m_A) + m_A(A) = \sup_m (h(T,m) + m(A)),$$

where the supremum is taken as $m$ ranges on the set of probability $T$-invariant measures on $X$. Also, $m_A$ is the unique $T$-invariant measure on $X$ for which the equality $P^T(A) = h(T,m_A) + m_A(A)$ is achieved.

2.3.2. Definition. Let $c$ be the cocycle for the special flow $(X,T) = (\Sigma, \sigma, \lambda)$ generated by $A : X \rightarrow \mathbb{R}$. Then the induced cocycle on $\Sigma$ is the cocycle for $\sigma$ generated by $I[A] : \Sigma \rightarrow \mathbb{R}$ where $I[A]$ is defined by

$$I[A](p) = \int_0^{\lambda(p)} A(T^t p) \, dt.$$

2.3.3. Notation. Given a function $A : X \rightarrow \mathbb{R}$ let $\Phi[A] : \Sigma \rightarrow \mathbb{R}$ be the function defined by

$$\Phi[A] = I[A] - P^T(A) \lambda = I[A - P^T(A)]$$

If $A : X \rightarrow \mathbb{R}$ is a Hölder continuous on $X$, then $I[A]$ and $\Phi[A]$ are also Hölder continuous on $\Sigma$.

We can reduce the study of flows to the case of shifts by the following theorem that clarifies the relation between Gibbs states for $(\Sigma, \sigma)$ and $(X,T)$.
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2.3.4. Theorem (Bowen, Franco-Sanchez). The Gibbs measure $m_A$ for the flow $T$ and the potential $A$ is given by

\[
m_A = \frac{\mu_{\Phi[A]} \otimes dt}{(\mu_{\Phi[A]} \otimes dt)(X)} = \frac{\mu_{\Phi[A]} \otimes dt}{\mu_{\Phi[A]}(\lambda)}
\]

where $\mu_{\Phi[A]}$ denotes the Gibbs state of the potential $\Phi[A]$ on $\Sigma$. Furthermore the pressure $P^\sigma(\Phi[A])$ of $\Phi[A]$ is zero.

2.3.5. Remark. From the above Theorem, Contreras' result mentioned in 2.2, the formula 2.1 for the first derivative for $P^T(A)$ and the implicit function theorem, it follows that for $\lambda \in C^\beta(\Sigma)$ and $\alpha \leq \beta$, the map

\[
P^T : A \in C^\alpha(X) \mapsto P^T(A) \in \mathbb{R}
\]

is real analytic. The maps $m : A \in C^\alpha(X) \mapsto m_A \in C^\alpha(X)^*$ and $A \in C^\alpha(X) \mapsto h(T, m_A) \in \mathbb{R}$ are real analytic as well.

2.3.6. Remark. The variational characterization of Gibbs states implies that the measure of maximal entropy for the flow $T$ is the Gibbs state for the potential zero (or, more generally, for a potential cohomologous to a constant).

2.3.7. Remark. Again by the variational principle, the potential $A$ has pressure zero if and only if

\[h(T, m_A) = -m_A(A)\]

2.4.

We recall now the definition of covariance for special flows. Since an exponential mixing rate for Hölder functions is not guaranteed, the definition is not as immediate as in the case of shifts.

2.4.1. Notation. Retaining the previous notations let $B : X \to \mathbb{R}$ be another Hölder continuous function on $X$. Define $\Psi_A[B] : \Sigma \to \mathbb{R}$ by setting

\[
\Psi_A[B] = I[B] - m_A(B) \lambda = I[B - m_A(B)].
\]

The definition of covariance was given by Marina Ratner who proved in [Rat73a] the following theorem:
2.4.2. Theorem (Ratner). For a Hölder continuous function $B : X \rightarrow \mathbb{R}$ and any Gibbs measure $m_A$ on $X$ the limit

$$\text{Var}_A^T(B) = \lim_{T \to \infty} \frac{1}{T} \int_X \left( \int_0^T (B \circ T^t - m_A(B)) \, dt \right)^2 \, dm_A$$

exists, equals $\frac{1}{\mu_{\Phi[A]}(\lambda)} \text{Var}_{\Phi[A]}^T(\Psi_A[B])$ and it is called the variance of $B$ with respect to the Gibbs state $m_A$.

By polarization, given any three Hölder continuous functions $A, B, C : X \rightarrow \mathbb{R}$, the limit

$$\text{Cov}_A^T(B, C) = \lim_{T \to \infty} \frac{1}{T} \int_X \left( \int_0^T (B \circ T^t - m_A(B)) \, dt \right) \cdot 
\cdot \left( \int_0^T (C \circ T^{t'} - m_A(C)) \, dt' \right) \, dm_A$$

exists and we have

$$\text{(2.4)} \quad \text{Cov}_A^T(B, C) = \frac{1}{\mu_{\Phi[A]}(\lambda)} \text{Cov}_{\Phi[A]}^T(\Psi_A[B], \Psi_A[C]).$$

We call $\text{Cov}_A^T(B, C)$ the covariance of $B$ and $C$ with respect to the Gibbs state $m_A$. We have $\text{Cov}_A^T(B, C) = 0$ for all $B$ if and only if $C$ is cohomologous to a constant, i.e. if there exists a Hölder function $D$ differentiable along the flow $T^t$ such that $C = \frac{dD \circ T^t}{dt} \bigg|_{t=0} + \text{Const}$ (see [Rat73a]).

2.4.3. Remark. Notice that if the integral $\int_{-\infty}^{\infty} |m_A(B \cdot C \circ T^t) - m_A(B)m_A(C)| \, dt$ exists, then

$$\text{Cov}_A^T(B, C) = \int_{-\infty}^{\infty} (m_A(B \cdot C \circ T^t) - m_A(B)m_A(C)) \, dt.$$

2.5.

Let $\lambda_\varepsilon$ be a $C^r$ curve of positive $C^\alpha$ functions on $\Sigma$ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under $\lambda_\varepsilon$.

Let $u_\varepsilon$ be the a $C^r$ family of $C^\alpha$ potentials on $X_\varepsilon$, i.e. a curve admitting a $C^r$ lift to $C^\alpha(\Sigma \times \mathbb{R})$. 
2.5.1. **Definition.** For $\lambda_\varepsilon$ and $u_\varepsilon$ as above we define the $i$-th variation of the cocycle (generated by) $u_\varepsilon$ as (the cocycle generated by) any $C^\alpha$ function $\delta^i u_\varepsilon$ be a on $X_\varepsilon$ with the property that

$$I_\varepsilon[\delta^i u_\varepsilon](p) = \int_0^{\lambda_\varepsilon(p)} \delta^i u_\varepsilon(p, t) \, dt = \frac{d^i}{d\varepsilon^i} \int_0^{\lambda_\varepsilon(p)} u_\varepsilon(p, t) \, dt.$$  

(for simplicity, $\delta^1 = \delta$ and $\delta^0 u_\varepsilon = u_\varepsilon$). In other words the $i$-th variation $\delta^i u_\varepsilon$ induces on $\Sigma$ the $i$-th derivative of the induced cocycle $I_\varepsilon[u_\varepsilon]$. Notice that the $i$-th variation $\delta^i u_\varepsilon$ of the cocycle $u_\varepsilon$ is only defined up to a coboundary for the flow $T_\varepsilon$.

We denote by $\delta^i l_\varepsilon$, the $i$-th variation of the length cocycle:

$$I_\varepsilon[\delta^i l_\varepsilon](p) = \int_0^{\lambda_\varepsilon(p)} \delta^i l_\varepsilon(p, t) \, dt = \frac{d^i}{d\varepsilon^i} \lambda_\varepsilon(p).$$

Having stated the set up the proof of Proposition 1.3.1 is rather elementary. To simplify notation, we denote with primes differentiation with respect to $\varepsilon$ and suppress the dependence on $\varepsilon$, e.g. $P'(T, u) = d/d\varepsilon P(T_\varepsilon, u_\varepsilon)$ and $P'(T_0, u_0) = d/d\varepsilon P(T_\varepsilon, u_\varepsilon)\vert_\varepsilon = 0$.

2.5.2. **Proposition (Derivatives of the Pressure and Entropy).** Let $\lambda_\varepsilon$ be a $C^2$ curve of positive $C^\alpha$ functions on $\Sigma$ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under $\lambda_\varepsilon$. Let $u_\varepsilon$ be the a $C^2$ family of $C^\alpha$ potentials on $X_\varepsilon$. Then, retaining the previous notation for the first and second variation for length cocycle and the cocycle generated by $u_\varepsilon$ and setting

$$v_\varepsilon = \delta u_\varepsilon - P(T_\varepsilon, u_\varepsilon) \delta l_\varepsilon \quad \text{and} \quad w_\varepsilon = \delta^2 u_\varepsilon - P(T_\varepsilon, u_\varepsilon) \delta^2 l_\varepsilon$$

we have:

$$P'(T, u) = m_u(v)$$
$$P''(T, u) = \text{Var}_u(v) + m_u(w) - 2m_u(v) m_u(\delta l)$$

and

$$h'(T, m_u) = -\text{Cov}_u(v, u) + h(T, m_u) m_u(\delta l)$$
$$h''(T, m_u) = -D^3 P_u(T, u, v, v) - 2h(T, m_u) \text{Cov}_u(\delta l, v)$$
$$+ 2 \left( \text{Cov}_u(T, v, u) + h(T, m_u) m_u(\delta l) \right) m_u(\delta l)$$
$$- \text{Cov}_u(T, u, w) + 2m_u(v) \text{Cov}_u(T, u, \delta l)$$
$$- h(T, m_u) m_u(\delta^2 l) - \text{Cov}_u(T, v, v).$$
We shall not give the proof of the proposition above. However, for com-
p| pleteness, we shall give independent proofs of the following corollaries which
we use in this paper.

2.6.1. Corollary. Let \( \lambda_\varepsilon \) be a \( C^2 \) curve of positive \( C^\alpha \) functions on \( \Sigma \) and
let \((X_\varepsilon, T_\varepsilon)\) be the family of special flows built under \( \lambda_\varepsilon \). We retain here
the previous notation and denote by \( h_{\text{top}}(T_\varepsilon) \) the topological entropy of the
flow \( T_\varepsilon \) and by \( M_\varepsilon \) the measure of maximal entropy for the flow \( T_\varepsilon \). Also
denote by \( \text{Var}_{T_\varepsilon}^T \) the variance of the flow \( T_\varepsilon \) with respect to the measure of
maximal entropy \( m_\varepsilon \). Then, suppressing the dependence on \( \varepsilon \), we have
\[
2.7 \quad h'_{\text{top}}(T) = -h_{\text{top}}(T) M(\delta l).
\]
If \( h'_{\text{top}}(T_0) = 0 \) we have
\[
2.8 \quad h''_{\text{top}}(T_0) = -h_{\text{top}}(T_0) M_0(\delta^2 l_0) + h_{\text{top}}(T_0)^2 \text{Var}_{T_0}^T(\delta l_0).
\]

Proof. Observe that since \( \varepsilon \mapsto \lambda_\varepsilon \in C^\alpha(\Sigma) \) is \( C^2 \), the first and second vari-
ation \( \delta l_\varepsilon \) and \( \delta^2 l_\varepsilon \) of the length cocycle exist (as Hölder functions). By the
Remark 2.3.6 the topological entropy equals the pressure for the potential
0 and from Theorem 2.3.4 and 2.3 we obtain that
\[
m_\varepsilon = \frac{\mu_{\phi(\varepsilon)} \otimes dt}{\mu_{\phi(\varepsilon)}(\lambda_\varepsilon)},
\]
where \( \phi(\varepsilon) = -h_{\text{top}}(T_\varepsilon) \lambda_\varepsilon \). Again by Theorem 2.3.4, we have that
\( P^\sigma(\phi(\varepsilon)) = 0 \) for all \( \varepsilon \). By Remark 2.3.5, the curve \( \phi(\varepsilon) \) is a \( C^2 \) curve
of \( C^\alpha \) potentials on \( \Sigma \). Thus from (2.2) we obtain that
\[
2.9 \quad \mu_{\phi}(\phi') = 0 \quad \text{and} \quad \text{Var}_{\phi}^T(\phi') + \mu_{\phi}(\phi'') = 0.
\]
Since \( \phi' = -h'_{\text{top}}(T)\lambda - h_{\text{top}}(T)\lambda' \) from the first of (2.9) we conclude that
\[
h'_{\text{top}}(T) = -\frac{h_{\text{top}}(T) \mu_{\phi}(\lambda')}{\mu_{\phi}(\lambda)} = -\frac{h_{\text{top}}(T) \mu_{\phi} \otimes dt(\delta l)}{\mu_{\phi}(\lambda)} = -h_{\text{top}}(T) m_0(\delta l)
\]
proving (2.7).

The hypothesis \( h'_{\text{top}}(T_0) = 0 \) implies that \( \phi'(0) = -h'_{\text{top}}(T_0) \lambda_0 \) and
\( \phi''(0) = -h''_{\text{top}}(T_0) \lambda_0 - h_{\text{top}}(T_0) \lambda''_0 \). Thus, using the second of the formu-
las (2.9), we obtain
\[
\text{Var}_{\phi(0)}^T(-h_{\text{top}}(T_0) \lambda'_0) + \mu_{\phi(0)}(-h''_{\text{top}}(T_0) \lambda_0 - h_{\text{top}}(T_0) \lambda''_0) = 0.
\]
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We conclude that

\[
 h''_{\text{top}}(T_0) = -h_{\text{top}}(T_0) \frac{\mu \phi(0)(\lambda_0')}{\mu \phi(0)(\lambda_0)} + h_{\text{top}}(T_0)^2 \frac{\text{Var}_{\phi}(\lambda_0)}{\mu \phi(0)(\lambda_0)} 
\]

\[
 = -h_{\text{top}}(T_0) M_0(\delta l_0) + h_{\text{top}}(T_0)^2 \frac{\text{Var}_{\phi}(I[\delta \lambda_0])}{\mu \phi(0)(\lambda_0)} 
\]

From \( h'_{\text{top}}(T_0) = 0 \) we have \( M_0(\delta l_0) = 0 \) and therefore \( I[\delta \lambda_0] = \Psi_0[l_0] \). Using (2.4) we conclude

\[
 \frac{\text{Var}_{\phi}(I[\delta \lambda_0])}{\mu \phi(0)(\lambda_0)} = \text{Var}_{M_0}(\delta l_0) 
\]

proving (2.8) and Part A of Proposition 1.3.1. \( \square \)

2.6.2. Corollary. Let \( \lambda_\varepsilon \) be a \( C^2 \) curve of positive \( C^\alpha \) functions on \( \Sigma \) and let \( (X_\varepsilon, T_\varepsilon) \) be the family of special flows built under \( \lambda_\varepsilon \). Let \( u_\varepsilon \) be the a \( C^2 \) family of \( C^\alpha \) potentials on \( X_\varepsilon \). In the hypothesis that \( P(T_\varepsilon^t, u_\varepsilon) = 0 \) for all \( \varepsilon \) we have:

(2.10) \[
 h'(T, m_u) = -\text{Cov}_{T_\varepsilon}(u, \delta u) - h(T, m_u) m_u(\delta l). 
\]

If \( u_0 \) is cohomologous to a constant and \( m_{u_0}(\delta l_0) = 0 \), then \( h'(T, m_u) = 0 \)

and

(2.11) \[
 h''(T_0, m_{u_0}) = -\text{Var}_{u_0}(\delta u_0) - 2 h(T_0, m_{u_0}) \text{Cov}_{u_0}(\delta l_0, \delta u_0) - h(T_0, m_{u_0}) m_{u_0}(\delta^2 l_0), 
\]

Proof. Since by hypothesis \( P(T_\varepsilon^t, u_\varepsilon) = 0 \) for all \( \varepsilon \), we have that \( \Phi[u_\varepsilon](p) = \int_0^{\lambda_\varepsilon(p)} u_\varepsilon(T_\varepsilon^t p) dt \). For simplicity set \( \psi_\varepsilon = \Phi[u_\varepsilon] \). The Gibbs state \( m_{u_\varepsilon} \) is thus given by

\[
 m_{u_\varepsilon} = \frac{\mu_{\psi_\varepsilon} \otimes dt}{\mu_{\psi_\varepsilon}(\lambda_\varepsilon)}, 
\]

where \( \mu_{\psi_\varepsilon} \) denotes as before the Gibbs measure for the shift \( \sigma \) and the potential \( \psi_\varepsilon \). From equation 2.2 and the fact that \( P(\sigma, \psi_\varepsilon) = 0 \) for all \( \varepsilon \) (see 2.2), we obtain

(2.12) \[
 \mu_{\psi}(\psi') = 0. 
\]

By the variational characterization of Gibbs states and \( P(\sigma, \psi_\varepsilon) = 0 \) we have \( h(\sigma, \mu_{\psi}) = -\mu_{\psi}(\psi) \) and, using (2.12), we obtain

(2.13) \[
 h'(\sigma, \mu_{\psi}) = -\text{Cov}_{\psi}(\psi, \psi') - \mu_{\psi}(\psi') = -\text{Cov}_{\psi}(\psi, \psi'). 
\]
Differentiating w.r.t. $\varepsilon$ Abramov’s formula

$$h(\sigma, \mu_{\psi_e}) = \mu_{\psi_e}(\lambda_e) h(T_e, m_{u_e}),$$

and using (2.13), we have

$$-\text{Cov}_{\psi}^\sigma(\psi, \psi') = h(T, m_u) \left( \text{Cov}_{\psi}^\sigma(\lambda, \psi') + \mu_{\psi_e}(\lambda') \right) + \mu_{\psi_e}(\lambda) h'(T, m_u)$$

or

(2.14)

$$h'(\sigma, m_u) = -\frac{\text{Cov}_{\psi}^\sigma(\psi + h(T, m_u) \lambda, \psi')}{\mu_{\psi_e}(\lambda)} - h(T, m_u) \frac{\mu_{\psi_e}(\lambda')}{\mu_{\psi_e}(\lambda)}.$$

Observe that we have $\Psi_u[u] = \psi - m_u(u) \lambda = \psi + h(T, m_u) \lambda$ and $\Psi_u[\delta u] = \psi' - m_u(\delta u) \lambda = \psi' - \mu(\psi') \lambda = \psi'$. Thus, using (2.4) we have $\text{Cov}_{\psi}^\sigma(\psi + h(T, m_u) \lambda, \psi') = \text{Cov}_{\psi}^\sigma(u, \delta u) \mu_{\psi_e}(\lambda)$, which proves (2.10).

Assume now that the $u_0$ is cohomologous to a constant. Then $P(T, u) = 0$ implies that the function $u_0 + h(T, m_{u_0})$ is cohomologous to zero and therefore also $\psi_0 + h(m_{u_0}) \lambda_0$ is cohomologous to a zero. The further assumption $m_{u_0}(\delta l_0) = 0$ entails $\mu_{\psi_0}(\lambda_0) = 0$ and therefore $h'(T_0, m_{u_0}) = 0$. In this case one further differentiation of (2.14) yields

$$h''(m_{u_0}) = -\frac{\text{Cov}_{\psi_0}^\sigma(\psi'_0 + h(m_{u_0}) \lambda'_0, \psi_0')}{\mu_{\psi_0}(\lambda_0)} - h(m_{u_0}) \left[ \frac{\text{Cov}_{\psi_0}^\sigma(\psi_0', \lambda'_0)}{\mu_{\psi_0}(\lambda_0)} + \frac{\mu_{\psi_0}(\lambda'_0)}{\mu_{\psi_0}(\lambda_0)} \right]$$

$$= -\frac{\text{Cov}_{\psi_0}^\sigma(\psi_0', \psi_0')}{\mu_{\psi_0}(\lambda_0)} - h(m_{u_0}) \left[ 2 \frac{\text{Cov}_{\psi_0}^\sigma(\psi_0', \lambda_0)}{\mu_{\psi_0}(\lambda_0)} + \frac{\mu_{\psi_0}(\lambda_0)}{\mu_{\psi_0}(\lambda_0)} \right].$$

and (2.11) is proved. Part B of Proposition 1.3.1 is now proved.

\begin{proof}
\end{proof}

2.6.3. Corollary. Let $\lambda_e$ be a $C^2$ curve of positive $C^\alpha$ functions on $\Sigma$ and let $(X_e, T_e)$ be the family of special flows built under $\lambda_e$. Let $u_e$ be the a $C^2$ family of $C^\alpha$ potentials on $X_e$. Assume that

1. $P(T'_e, u_e) = 0$ for all $e$;
2. $m_{u_e}(\delta l_e) = 0$ for all $e$;
3. $u_0$ is cohomologous to a constant.
Then \( m_0 = m_{u_0} \), then we obtain \( m_0(\delta^2 l_0) = -\text{Cov}^T_m(\delta l_0, \delta u_0) \). Therefore we obtain \( h'(T_0, m_{u_0}) = 0 \) and

\[
\begin{align*}
\text{Cov}^T_{m_0}(h^T_{\text{top}}(T_0) \delta l_0, u'_0) &= -\text{Cov}^T_{m_0}(u'_0, u'_0 + h^T_{\text{top}}(T_0) \delta l_0) \\
\text{Cov}^T_{m_0}(h^T_{\text{top}}(T_0) - h^T_{\text{top}}(T_0, m_{u_0}), u'_0 + h^T_{\text{top}}(T_0) \delta l_0)
\end{align*}
\]

Proof. The claim \( m_0 = m_{u_0} \) is equivalent to \( u_0 \) being cohomologous to a constant. Retain the notation of the proof of the previous Corollary. Since \( m_{u_\varepsilon}(\delta l_\varepsilon) = 0 \) implies \( \mu\psi_\varepsilon(\lambda'_{\varepsilon}) = 0 \) for all \( \varepsilon \), differentiating we obtain

\[
\text{Cov}^g_{\psi_\varepsilon}(\lambda'_{\varepsilon}, \psi'_{\varepsilon}) + \mu\psi_\varepsilon(\lambda''_{\varepsilon}) = 0.
\]

From \( m_{u_\varepsilon}(\lambda') = 0 \) we obtain \( \Psi_{u_\varepsilon}[\delta l] = \lambda' \) and since \( \Psi_{u_\varepsilon}[\delta u] = \psi' \), using 2.4, we obtain that

\[
m_u(\delta^2 \lambda) = -\text{Cov}^T_u(\delta l, \delta u).
\]

The rest is now mere rephrasing. This also concludes the proof of Proposition 1.3.1.

\[\square\]

§ 3. The first variation of the length and Liapunov cocycles.

In this section we collect some well known facts that we allow us to connect the results of §2 to the study of our original problem. Some of the theorems stated are valid in a more general setting of Anosov or Axiom-A flows. We state them for the case of geodesic flows for metrics of negative curvature.

3.1. Symbolic Coding.

The geodesic flow \( T_g \) of a metric of negative curvature \( g \) on the unit tangent bundle \( S_gM \) of a manifold \( M \) is isomorphic to a symbolic flow [Rat69], [Rat73b], [Bow72b]. More exactly, there is symbolic flow \((X, T)\) built over a topologically mixing subshift of finite type \((\Sigma, \sigma)\) with a positive Hölder continuous ceiling function \( \lambda \) and a finite-to-one Hölder continuous surjection \( \pi : X \to S_gM \) such that \( \pi \) intertwines the flow \( T \) on \( X \) with the flow \( T_g \) on \( S_gM \). Furthermore, denoting by \( m_u \) and \( n_{u,\pi} \) respectively the Gibbs state for \( u \in C^\alpha(S_gM) \) with respect to the flow \( T_g \) and Gibbs state for \( u \circ \pi \in C^\beta(X) \) with respect to the flow \( T \), we have that the surjection
\( \pi \) is a measure theoretical isomorphism of \((X, T, n_{\omega(T)}\)) onto \((S_g, T_g, m_u)\). In particular \( u \in C^\alpha(S_gM) \) is cohomologous to zero if and only if \( u \circ \pi \) is cohomologous to zero.

The Margulis measure and the Liouville measure on \( S_gM \) are the Gibbs states respectively for the length cocycle and for the Liapunov cocycle. We recall the definition of Liapunov cocycle.

**3.1.1. The Liapunov cocycle.** For \( v \in S_gM \) let \( W^{ss}(v) \) denote the strong stable manifold passing through \( v \), i.e. the set of points \( w \in S_gM \) with \( d(\phi^tv, \phi^tw) \to 0 \) as \( t \to +\infty \). The Liapunov cocycle is given by the growth of the volume of strong stable manifold: more exactly, if \( \mu^{ss} \) is the volume form induced on strong stable manifolds by some Riemannian metric on \( S_gM \), the Liapunov cocycle is defined by \( \mathcal{L}(v, t) = \frac{d(\phi^t)}{d\mu^{ss}(v)} \). Changing the Riemannian metric on \( S_gM \) does not affect the cohomology class of the Liapunov cocycle.

The Liapunov cocycle has an interpretation in terms of Jacobi fields. The natural projection \( p : S_gM \to M \) maps bijectively the strong stable manifold \( W^{ss}_g(v) \) onto the stable horosphere \( H_g(v) \) of \( v \). The second fundamental form \( U_g(v) \) of the stable horosphere \( H(v) \) at \( p(v) \) considered as a tensor of type \((1,1)\) satisfies \(^2\) along the orbit \( T^t_gv \) the Riccati equation

\[
\frac{d}{dt} U_g(T^t_gv) + U^{ss}_g(T^t_gv) = R(p_*T^t_gv, \cdot)p_*T^t_gv.
\]

The Liapunov cocycle can be expressed as

\[
\mathcal{L}_g(v, t) = \int_0^t \text{Tr} U_g(T^\tau_gv) d\tau.
\]

It is not difficult to see that \( \text{Tr} U_g \) is a Hölder continuous function on \( S_gM \).

**3.1.2.** The construction yielding \((\Sigma, \sigma, \lambda)\) and the semi-conjugacy \( \pi \), proceeds by choosing in \( S_gM \) finitely many disjoint smooth disks \( D_i \) transversal to the flow \( T_g \). Inside each \( D_i \), we choose closed connected sets \( R_i \) so that each \( R_i \) equals the closure of its interior \( R_i^\circ \) and the union \( R \) of all \( R_i \)'s forms a cross-section to the flow \( T_g \). Let \( P \) be the Poincaré map of \( R \). Then \( \Sigma \) is obtained as the collection of sequences \( \omega = (\omega_i)_{i \in \mathbb{Z}} \) such that

\(^2\)Technically the map \( v \mapsto U_g(v) \) is a section of the pull-back to \( S_gM \) of the bundle \( T^{(1,1)}M \to M \) via the natural projection \( p : SM \to M \). The Riccati equation can then be interpreted in these terms, \( \nabla \) being the pull-back to \( S_gM \) of the Levi-Civita connection on \( M \).
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\[ \cap_{i \in \mathbb{Z}} P^{-i} R_{\omega_i} \neq \emptyset. \] By the expansiveness of the flow \( T_g \) if \( \omega \in \Sigma \) then the set \( \cap_{i \in \mathbb{Z}} P^{-i} R_{\omega_i} \) contains exactly one point \( x = x(\omega) \) of \( R \), thus defining a surjection \( \pi_\Sigma : \omega \mapsto x(\omega) \). For each \( p \in R \) the first return time \( \bar{\lambda}(p) \) to \( P \) under the Poincaré map of \( P \) is well defined and positive. Let \( \lambda = \bar{\lambda} \circ \pi_\Sigma \) and let \( \sigma \) be the shift on \( \Sigma \) and define \( (X, T) \) as the symbolic flow \( (X, T) \) built over \( (\Sigma, \sigma) \) with ceiling function \( \lambda \). It is plain that \( p_\Sigma \) extents to a a surjection \( \pi : X \to S_g M \) intertwining the flow \( T \) on \( X \) with the flow \( T_g \) on \( S_g M \).

### 3.2. Structural Stability.

With \( M \) a compact manifold, denote by \( SM \) the bundle of oriented directions on \( M \), i.e. \( SM = \{ v \in TM \mid v \neq 0 \} / \sim \), where \( v \sim v' \) if and only if \( \exists c > 0, v = cv' \). For each Riemannian metric \( g \) there is a natural identification between the bundle \( SM \) and the unit tangent bundle \( S_g M \). By means of this identification, we regard the geodesic flow \( T_g \) for the metric \( g \) as a flow on \( SM \), which we also denote by \( T_g \) with abuse of notation.

Denote by \( \mathcal{U} \) the space of \( C^2 \)-metrics of negative curvature on \( M \), and more generally by \( \mathcal{U}^s \) the space of \( C^s \)-metrics of negative curvature on \( M \).

It is well known, (cf. [Mor24], [Ano67] and [Gro]), that the various flows \( T_g \) for \( g \in \mathcal{U} \) are all orbit-equivalent. This means that if \( g_0 \) and \( g \in \mathcal{U} \), there is a homeomorphism \( H_g : SM \to SM \) sending the orbits of the flow \( T_{g_0} \) to the orbits of the flow \( T_g \). In fact the homeomorphism \( H_g \) is homotopic to the identity and can always be chosen smooth along the orbits of \( T_{g_0} \) and Hölder continuous on \( SM \). We call such an homeomorphism a \((g_0, g)\)-Morse correspondence. The Morse correspondence is not unique: if \( H_1 \) and \( H_2 \) are two \((g_0, g)\)-Morse correspondences then there exists a real valued function \( t(v) \) on \( SM \) such that \( \forall v \in SM, H_1^{-1} \circ H_2(v) = T_{g_0}^{t(v)}(v) \). Thus we have a sort of transversal uniqueness:

#### 3.2.1. Definition. For \( g \) in a neighbourhood \( V \subseteq \mathcal{U} \), we will call the family of Morse correspondences normal at \( g_0 \), the family \( g \mapsto H_g \) of \((g_0, g)\)-Morse correspondences \( H_g \) which is uniquely defined by the property

"for each \( v \in SM \) the footpoint of \( H_g(v) \) belongs to the hypersurface obtained by exponentiating in the metric \( g_0 \) a small ball in the subspace \( g_0\)-perpendicular to \( v \)"

We have described some of the generalities of the construction of the symbolic flow in order to understand what happens if we perturb the flow
Let $\pi_{g_0}$ be a semi-conjugacy between a symbolic flow $(\Sigma, \sigma, \lambda_{g_0})$ and $(SM, T_{g_0})$ mentioned in Section 3.1.

For $g$ in some subset of $\mathcal{U}$, denoting by $H_g$ the family of Morse correspondences normal at $g_0$, from the fact that $H_g$ sends the orbits of the flow $T_{g_0}$ to the orbits of the flow $T_g$, we obtain that the set $H_g \circ \pi_{g_0}(\Sigma)$ provides a global cross-section for the flow $T_g$ with a Hölder continuous return function $\lambda_g : \Sigma \to \mathbb{R}$.

We obtain a finite-to-one Hölder continuous surjection $\pi_g : X_g \to SM$ intertwining the symbolic flow on $X_g = \Sigma \times \mathbb{R} / [(p, \lambda_g(p) + t) \sim (\sigma p, t)]$ and the flows $T_g$ on $SM$. [Equivalently we can look at the composition $H_g \circ \pi$ as giving an orbit equivalence between $(\Sigma, \sigma, \lambda_{g_0})$ and $(SM, T_g)$.] Thus we have the following diagram

\[
\begin{array}{ccc}
(\Sigma, \sigma, \lambda_{g_0}) & \longrightarrow & (SM, T_{g_0}) \\
\uparrow H_g & & \uparrow H_g \\
(\Sigma, \sigma, \lambda_g) & \longrightarrow & (SM, T_g)
\end{array}
\]

with dashed arrows representing orbit equivalences. Observe that, identifying the space $X_g$ supporting the flow $(\Sigma, \sigma, \lambda_g)$ with $X_g = \{(p, t) \in \Sigma \times \mathbb{R} | 0 \leq t < \lambda_g\}$ and $\Sigma$ with $\Sigma \times \{0\}$, the orbit equivalences $H_g$ are the identity on $\Sigma$.

### 3.3. Variation of cocycles.

Now we are in the situation in which we can apply the results of §2. To this purpose we need to pull-back the length and the Liapunov cocycles to the flow $(\Sigma, \sigma, \lambda_g)$ or $(\Sigma, \sigma, \lambda_{g_0})$, and determine in geometric terms their first variations $\delta l, \delta u$ which entered in Proposition 1.3.1. Finally we need to resolve the smoothness issues related to this construction.

Assume that for each $g \in \mathcal{U}$ we are given a cocycle $c_g$ for the flow $T_g$ and orbit equivalences $H_g : SM \rightarrow SM$ sending the orbits of the flow $T_{g_0}$ to the orbits of the flow $T_g$.

**3.3.1. Definition.** The pull-back of the family of cocycles $c_g$ along the orbit equivalences $H_g$ is the family of cocycles $H_g^* c_g$ for the flow $T_{g_0}$ given by

$$H_g^* c_g(v, t) = c_g(H_g v, t'),$$

where $t'$ is defined by $T_g t' H_g v = H_g T_{g_0} v$. 

3.3.2. Example. The length cocycles \( \ell_g(v,t) = t \) for the flow \( T_g \) is generated by \( A_g = 1 \). If the orbit equivalences are differentiable along the flow \( T_{g_0} \), then, denoting by \( Y_g \) the generator of the flow \( T_g \), we have

\[
(H_g)_*X_{g_0} := \frac{d}{dt} H_g \circ T_{g_0}^t = l_{H_g} Y_g
\]

or equivalently

\[
H_g T_{g_0}^t v = T_g^t L_{H_g}(T_{g_0}^s v) ds H_g v.
\]

Therefore the pull-back of the length cocycle \( \ell_g \) along \( H_\phi \) is given by

\[
H_\phi^* \ell_g(v,t) = \int_0^t l_{H_g}(T_{g_0}^s v) ds
\]

and it has as generator the function \( l_{H_g} \) defined in 3.3. Denoting by \( p \) the natural projection \( SM \to M \), and by \( \| \cdot \|_g \) the norm induced by \( g \) on \( TM \), from \( \| p_* Y_g \|_g = 1 \) we have

\[
l_{H_g}(v) = \| p_*(H_g)_*X_{g_0}(v) \|_g.
\]

The following Proposition is elementary.

3.3.3. Proposition. If the cocycles \( c_g \) have generators \( A_g \), then their pull-back along a family \( H_g \) of \( T_{g_0} \)-differentiable orbit equivalences is differentiable along the flow \( T_{g_0} \) and it is generated by \( (A_g \circ H_g)_l H_g \).

3.3.4. Definition. For a curve of metrics \( g_\varepsilon \) we define the \( i \)-th variation of the cocycle \( c_{g_\varepsilon} \) as the cocycle for the flow \( T_{g_0} \) defined by \( d^i / d\varepsilon^i H_{g_\varepsilon}^* c_{g_\varepsilon} \) at \( \varepsilon = 0 \).

3.3.5. Remark. Let \( u_g \) be the generator of a cocycle \( c_g \) for the flow \( (\Sigma, \sigma, \lambda_g) \). Let \( H_g \) be orbit equivalences sending the orbits of the flow \( (\Sigma, \sigma, \lambda_{g_0}) \) to the orbits of the flow \( (\Sigma, \sigma, \lambda_g) \). Assume further that \( H_g \) is the identity on \( \Sigma \) (identifying the space \( X_g \) supporting the flow \( (\Sigma, \sigma, \lambda_g) \) with \( X_g = \{(p,t) \in \Sigma \times \mathbb{R} | 0 \leq t < \lambda_g \} \) and \( \Sigma \) with \( \Sigma \times \{0\} \)). Then the pull-back of the length cocycle for \( (\Sigma, \sigma, \lambda_g) \) along the orbit equivalences \( H_g \) has a generator \( l_g \) satisfying

\[
\int_0^{\lambda_{g_0}} l_g(p,t) dt = \lambda_g(p)
\]
and the pull-back $H_g^* c_g$ is the cocycle generated by a function $\delta^0 u_g$ such that
\[
\int_0^{\lambda_{g_0}} \delta^0 u_g(p, t) \, dt = \int_0^{\lambda_g} u_g(p, t) \, dt.
\]
We conclude that the two definitions 2.5.1 and 3.3.4 of the $i$-th variation of a cocycle agree.

Now we are ready to tackle the question of the smoothness of the dependence on $g$.

Improving on previous results of [dILMM86], Katok et al. proved:

3.3.6. Theorem ([KKPW89]). For sufficiently small $\beta > 0$, in a neighbourhood $V^s$ of $g_0$ in $U^s$ the family of Morse correspondences normal at $g_0$, $g \mapsto H_g$, exists and has the following properties:

1. The homeomorphisms $H_g$ are $T_{g_0}$-differentiable, that is they are differentiable along the orbits of the flow $T_{g_0}$.

2. The map $g \mapsto H_g$ is of class $C^{s-2}$ as a map from the Banach manifold $V^s$ to the Banach manifold of $T_{g_0}$-differentiable $C^\beta$-maps of $SM$ into itself.  

3. The map $g \mapsto l_{H_g}$ is of class $C^{s-2}$ as a map with values in $C^\beta(SM, \mathbb{R}^+)$.  

4. The topological entropy $g \mapsto \text{Ent}_{\text{top}}(g)$ is of class $C^{s-1}$.

Furthermore Contreras showed:

3.3.7. Theorem ([Con92]). Under the same hypotheses, denoting by $E_g^s$ the stable bundle for the flow $T_g$ and by $u_g$ the generator of the Liapunov cocycle for $T_g$ we have:

5. the maps $g \in V^s \to u_g \circ H_g$ and $g \in V^s \to E_g^s \circ H_g$ is of class $C^{s-3}$ as a map with values respectively in $C^\beta(SM)$ and in the space of $C^\beta$ distributions on $SM$.

\[^3\]The theorem proved in [KKPW89] was stated for Anosov flows and it is valid more generally for hyperbolic attractors: see also [Con92]. We quote its application to geodesic flows. The loss of one further derivative is due to the fact that the vector field defining the geodesic flow depends on the first derivatives of the metric.

\[^4\]More exactly the target space is the space $C^\beta_g(SM, SM)$ of $C^\beta$ continuous maps $f : SM \to SM$ whose derivatives $X_{g_0} f$ along the orbits of the flow $T_{g_0}$ are $C^\beta$ continuous maps $SM \to T(SM)$ endowed with the norm $\|f\|_\beta + \|X_{g_0} f\|_\beta$. 
6. The entropy of Liouville measure \( g \mapsto \text{Ent}_{\text{Liouv}}(g) \) as a function of \( g \in \mathcal{V}^s \) is of class \( C^{s-3} \).

3.3.8. In particular, from (2) above, if \( s \geq 3 \) the derivative of the \((g_0, g)\)-Morse correspondence \( H_g \) normal at \( g_0 \), is a linear map from the \( C^3 \) sections \( S \) of the symmetric tensor bundle \( S^2 M \) to H"older continuous vector fields \( \Xi_g(S) \) along \( H_g \).

3.3.9. Definition. We call the vector field \( \Xi_{g_0}(S) \) the infinitesimal Morse correspondence at \( g_0 \) in the direction of \( S \).

3.3.10. The infinitesimal Morse correspondence \( \Xi_{g_0}(S) \) is differentiable along the \( g_0 \)-geodesics and it satisfies a differential equation given in [FF93]. Furthermore, by the definition of normal Morse correspondence, the vector field \( \Xi_{g_0}(S) \) is everywhere perpendicular to \( X_{g_0} \) in the natural metric on \( S_{g_0}M \), i.e. the projection of \( \Xi_{g_0}(S) \) at \( v \in S_{g_0}M \) on \( M \) is \( g_0 \)-perpendicular to \( v \):

\[
(3.5) \quad g_0(v, p_\star \Xi_{g_0}(S)(v)) = 0.
\]

3.3.11. Lemma. Let \( g_\varepsilon \) is a \( C^1 \)-curve of \( C^5 \) Riemannian negatively curved metrics on a compact manifold \( M \), and let \( S_\varepsilon = \frac{d}{d\varepsilon} g_\varepsilon \big|_{\varepsilon = 0} \).

Then the generator of the first variation of the length cocycle at \( g_\varepsilon \) at \( \varepsilon = 0 \) is is cohomologous to the function:

\[
v \in SM \mapsto \delta l_0(v) = \frac{1}{2} S_0'(v) = \frac{1}{2} S_0(v, v),
\]

and the generator of the first variation of the Liapunov cocycle \( u_\varepsilon \) at \( \varepsilon = 0 \) is cohomologous to the H"older function:

\[
v \in SM \mapsto \delta u_0(v) = \frac{d}{d\varepsilon} u_\varepsilon \circ H_{g_\varepsilon}(v) \big|_{\varepsilon = 0} + \frac{1}{2} S_0(v, v) u_0(v).
\]

Proof. By Proposition 3.3.3, the pull-back of the \( T_{g_\varepsilon} \)-cocycle generated by \( u_\varepsilon \) has generator \( u_\varepsilon(H_{g_\varepsilon}) l_\varepsilon \) and by (1)-(6) above the derivatives \( d u_\varepsilon \circ H_{g_\varepsilon}/d\varepsilon \) and \( d l_\varepsilon/d\varepsilon \) both exist as limits in \( C^3(SM) \). Thus for each \( g_0 \)-geodesic closed orbit \( \gamma_0 \), denoting by \( \gamma_\varepsilon \) the geodesic \( g_\varepsilon \)-geodesic closed orbit homotopic to
\(\gamma_0\) we have

\[
\frac{d}{d\varepsilon} \int_{\gamma_\varepsilon} u_\varepsilon(T^s_{g_\varepsilon} v) \, ds = \frac{d}{d\varepsilon} \int_{\gamma_0} u_\varepsilon(H_{g_\varepsilon}T^s_{g_0} v) l_{H_{g_\varepsilon}}(T^s_{g_\varepsilon} v) \, ds
\]

\[
= \int_{\gamma_0} \left[ \frac{d}{d\varepsilon} \left( u_\varepsilon(H_{g_\varepsilon}T^s_{g_0} v) \right) l_{H_{g_\varepsilon}}(T^s_{g_0} v) + u_\varepsilon(H_{g_\varepsilon}T^s_{g_0} v) \frac{d}{d\varepsilon} l_{H_{g_\varepsilon}}(T^s_{g_0} v) \right] \, ds
\]

At \(\varepsilon = 0\), we have \(l_{H_{g_\varepsilon}}(T^s_{g_0} v) = 1\) and, using 3.4 and 3.5, we see from the equation of the geodesics (cf. [FF93]) that

\[
\frac{d}{d\varepsilon} l_{H_{g_\varepsilon}}(w) \bigg|_{\varepsilon = 0} = \frac{1}{2} S(w, w).
\]

Since, by a well known theorem of Livšic [Liv71], the collection of integrals along periodic orbits determine the cohomology class of Hölder cocycles, our claim is proved. \(\square\)

\section*{§ 4. Relation of the Liapunov cocycle to the variation of metric.}

In this section we find a formula for the first variation of the Liapunov cocycle in terms of the first order variation of the Riemannian metric.

\subsection*{4.1.}

Throughout this section, \(M\) denotes a compact connected manifold with no boundary. We denote by \(SM\) the bundle of oriented directions on \(M\). As usual given a Riemannian metric \(g\) of class \(C^s\) on \(M\) we identify \(SM\) with the unit tangent bundle \(S_gM = \{v \in TM \mid g(v, v) = 1\}\), via the obvious \(C^s\)-diffeomorphism. The symbol \(g^t\) denotes both the geodesic flow on \(S_gM\) determined by \(g\) can and the flow induced on \(SM\), via the above identification.

We shall consider a \(C^1\) path \(\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto g_\varepsilon\) of \(C^s\)-metrics, with \(s \geq 4\) and set \(S_\varepsilon = \frac{dg_\varepsilon}{d\varepsilon}\). We denote by \(H_\varepsilon : SM \to SM\) the \((g_0, g_\varepsilon)\)-Morse correspondence normal at \(g_0\) and denote by \(\Xi_{g_0}(S_{g_0})\) the infinitesimal Morse correspondence.
As usual let $p : SM \to M$ be the canonical projection. Let $\gamma_0(t)$ be a unit-speed geodesic of initial velocity $\gamma_0'(0) = v \in SM$. Clearly we have $\gamma_0(t) = p(g_0^t v)$. Define

$$\overline{\gamma} : D = \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \mapsto \overline{\gamma}(t, \varepsilon) = p(h_\varepsilon (g_0^t v)) \in M.$$

We set, for all $t$, $\xi = \frac{\partial \overline{\gamma}}{\partial \varepsilon} |_{\varepsilon = 0}$, $Y = \frac{\partial \gamma}{\partial t} |_{\varepsilon = 0} = 0$. Of course $Y(t) = g_0^t v$ and $\xi(t) = p_*(\Xi_{g_0} (S_{g_0})(g_0^t v))$.

The map $\overline{\gamma}$ satisfies the following conditions:

1. The map $\overline{\gamma}$ is differentiable, by Theorem 3.3.6 and our smoothness hypotheses.

2. The curves $\gamma_\varepsilon : t \mapsto \overline{\gamma}(t, \varepsilon)$ are geodesics for the metric $g_\varepsilon$ on $M$, by definition of Morse correspondence.

3. We have $g_0(\xi, Y) = 0$, since we are considering the Morse correspondence normal at $g_0$.

4. We have also $g_0(Y, Y) = 1$, since $\gamma_0(t)$ is a unit-speed geodesic.

Let $E$ be the pullback to $D$ of the tangent bundle of $M$ via the map $\overline{\gamma} : D \to M$. Then the bundle $E$ is endowed with Riemannian metrics $g_\varepsilon$ on $E$, Levi-Civita connections $\nabla_\varepsilon$ for the metrics $g_\varepsilon$, and we let $R_\varepsilon$ denote curvature of $\nabla_\varepsilon$.

Let $U$ be the section of $E^* \otimes E$, i.e. the $\binom{1}{1}$-tensor field on $M$ along $\overline{\gamma}$, defined as $U(t, \varepsilon) = U_{g_\varepsilon}(h_\varepsilon (g_0^t v))$, where, as usual $U_g(v)$ is the second fundamental form of the stable horosphere of the metric $g$ at $p(v)$ considered as tensor of type $\binom{1}{1}$. By Theorem 3.3.7, the section $U$ is $C^1$.

Finally to lighten the notation we write $g_\varepsilon, g_0^t, S_\varepsilon, S_0, \gamma_\varepsilon, \nabla_\varepsilon, R_\varepsilon, \nabla_0, R_0, \xi, U_\xi$, etc. for $g_0, g_0^t, S_0, \gamma_0, \nabla, R, \nabla_0, R_0$, etc.

4.2.1. Lemma. Denoting by $u_\varepsilon$ the generator of the Liapunov cocycle for that flow $g_\varepsilon^t$, along the geodesic $\gamma(t) = \gamma_0(t)$, we have

$$\frac{d}{d\varepsilon} u_\varepsilon \circ H_{g_\varepsilon}(g_\varepsilon^t v) |_{\varepsilon = 0} = L_\xi \text{Tr } U(\gamma(t)) = \text{Tr } \nabla_\xi U(\gamma(t))$$

Proof. The first equality follows from (3.2) and the definition of $\xi$. The second follows from the fact that taking traces of endomorphisms commutes with covariant differentiation. □
4.2.2. **Notation.** Set for simplicity

\[
B(v) = \frac{d}{d\varepsilon} u_\varepsilon \circ H_{g_\varepsilon}(g^t v) |_{\varepsilon = 0}.
\]

Recall that the first variation of the Liapunov cocycle along the path \( g_\varepsilon \) at \( \varepsilon = 0 \) is given by

\[
\delta u(v) = B(v) + \frac{1}{2} S(v, v) \text{Tr} \, U.
\]

4.2.3. **Lemma.** Along the geodesic \( \gamma(t) \), the field of endomorphisms \( \nabla_\xi U(\gamma(t)) \) satisfies the following differential equation:

\[
\nabla_Y (\nabla_\xi U) + U \circ \nabla_\xi U + \nabla_\xi U \circ U = -\frac{1}{2} S(Y, Y) \nabla Y U - [\Gamma(Y), U] - [R(\xi, Y), U] - S(Y, Y) U^2 + \frac{\partial R^\varepsilon_{\gamma(t)}}{\partial \varepsilon} |_{\varepsilon = 0} (Y, \cdot) Y + \nabla_\xi R(Y, \cdot) Y + R(\nabla Y, \xi) Y + R(Y, \cdot) \nabla Y \xi.
\]

where \([\cdot, \cdot]\) denotes the ordinary commutator of endomorphisms and \( \Gamma \) denotes the \((\frac{1}{2})\)-tensor field along \( \gamma \) given by \( \Gamma = \frac{d\nabla^\varepsilon}{d\varepsilon} |_{\varepsilon = 0} \). For simplicity we have written \( \Gamma(Y) \) for the contraction \( \frac{d\nabla^\varepsilon_Y}{d\varepsilon} |_{\varepsilon = 0} \).

**Proof.** The field of endomorphisms \( U \) satisfies, along each geodesic \( \gamma_\varepsilon \), the Riccati equation 3.1 suitably corrected to take into account the fact that \( t \mapsto \gamma_\varepsilon(t) \) is not a \( g_\varepsilon \)-unit speed geodesic. Thus:

\[
(4.1) \quad \left\| \frac{\partial \gamma}{\partial t} \right\|_{\varepsilon} \left\| \nabla^\varepsilon U \right\|_{\varepsilon} + \left\| \frac{\partial \gamma}{\partial t} \right\|_{\varepsilon}^2 U^2 = R^\varepsilon \left( \frac{\partial \gamma}{\partial t}, \cdot \right) \frac{\partial \gamma}{\partial t}
\]

where \( R^\varepsilon \) denotes the Riemann curvature tensor of the metric \( g^\varepsilon \).

Observe that we have:

\[
(4.2) \quad \frac{\partial}{\partial \varepsilon} \left( \left\| \frac{\partial \gamma}{\partial t} \right\|_{\varepsilon}^2 \right) \bigg|_{\varepsilon = 0} = \frac{d}{d\varepsilon} \left( g^\varepsilon_{\gamma(t, \varepsilon)} \left( \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right) \right) \bigg|_{\varepsilon = 0} = S(Y, Y) + 2g(\nabla_\xi Y, Y) = S(Y, Y).
\]
where in the last equality we have used the fact that $\nabla_Y \xi$ and $Y$ are $g$-perpendicular.

Similarly one has

\begin{equation}
L \frac{\partial}{\partial \varepsilon} \left( \frac{\partial \gamma}{\partial t} \right) \bigg|_{\varepsilon=0} = \frac{1}{2} S(Y, Y)
\end{equation}

since $\|Y\|_0 = 1$.

Furthermore:

\begin{equation}
\nabla_{\frac{\partial}{\partial \varepsilon}} \left( \nabla_{\frac{\partial}{\partial \varepsilon}} U_{\gamma(t, \varepsilon)} \right) \bigg|_{\varepsilon=0} = \Gamma(Y) U_{\gamma(t, 0)} - U_{\gamma(t, 0)} \Gamma(Y) + \nabla_{\frac{\partial}{\partial \varepsilon}} \left( \nabla_{\frac{\partial}{\partial \varepsilon}} U_{\gamma(t, \varepsilon)} \right) \bigg|_{\varepsilon=0} = \Gamma(Y) U_{\gamma(t, 0)} - U_{\gamma(t, 0)} \Gamma(Y) + \nabla_Y \left( \nabla_\xi U_{\gamma(t, 0)} \right) + R(\xi, Y) U_{\gamma(t, 0)} - U_{\gamma(t, 0)} R(\xi, Y)
\end{equation}

where, in the last equality, we have used the fact that for commuting vector fields $X, Z$ and $\gamma$ tensor fields $W$ one has:

$$
$$

Finally observe that

\begin{equation}
\nabla_{\frac{\partial}{\partial \varepsilon}} \left( R^e \left( \frac{\partial \gamma}{\partial t}, \cdot \right) \frac{\partial \gamma}{\partial t} \right) \bigg|_{\varepsilon=0} = \frac{\partial R^e_{\gamma(t, 0)}}{\partial \varepsilon} \bigg|_{\varepsilon=0} (Y, \cdot) Y + \nabla_{\frac{\partial}{\partial \varepsilon}} \left( R \left( \frac{\partial \gamma}{\partial t}, \cdot \right) \frac{\partial \gamma}{\partial t} \right) \bigg|_{\varepsilon=0} = \frac{\partial R^e_{\gamma(t, 0)}}{\partial \varepsilon} \bigg|_{\varepsilon=0} (Y, \cdot) Y + \nabla_\xi R(Y, \cdot) Y + R(\nabla_\xi Y, \cdot) Y + R(Y, \cdot) \nabla_\xi Y = \frac{\partial R^e_{\gamma(t, 0)}}{\partial \varepsilon} \bigg|_{\varepsilon=0} (Y, \cdot) Y + \nabla_\xi R(Y, \cdot) Y + R(\nabla_\xi Y, \cdot) Y + R(Y, \cdot) \nabla_\gamma \xi.
\end{equation}

Taking the covariant derivative of equation (4.1) along $\xi$ and using the above observations (4.2)–(4.5) we obtain that $\nabla_\xi U$ satisfies the given differential equation along the geodesic $\gamma$. \hfill \Box

4.3.

Assume that $(M, g)$ is a rank-1 locally symmetric space of dimension $n$. Then, we can find parallel orthonormal vector fields $Y_1 = Y, Y_2, \ldots, Y_n$
along the geodesic $\gamma$ satisfying $R(Y, Y_i)Y = \lambda_i^2 Y_i$, with $\lambda_1 = 0$, $\lambda_j = 2$ for $1 < j \leq r$, and $\lambda_j = 1$ for $r < j \leq n$. Here $r = 1, 2, 4$ or 8 depending on the type of symmetric space.

Furthermore, from the definition of $U$, along the geodesic $\gamma$, we have $U(\cdot) = -\sqrt{R(\cdot, \cdot)} Y$, i.e. $UY_j = -\lambda_j Y_j$.

4.3.1. Lemma. Retaining the above notation, define $B_j = \langle (\nabla \xi U) Y_j, Y_j \rangle$, $j = 1, \ldots, n$, where $\langle \cdot, \cdot \rangle$ denotes the $g$-inner-product. Then, along the geodesic $\gamma$, we have that the $B_j$ satisfy the differential equation

$$L_Y B_j - 2\lambda_j B_j = -S(Y, Y)\lambda_j^2 + \left. \frac{\partial R^e}{\partial e} \right|_{e=0} \langle Y, Y_j \rangle Y, Y_j$$

Proof. Since $U$ is symmetric, we have:

(4.6) $\langle (U \circ \nabla \xi U + \nabla \xi U \circ U) Y_j, Y_j \rangle = \langle (\nabla \xi U) Y_j, U Y_j \rangle + \langle (\nabla \xi U) U Y_j, Y_j \rangle = -2\lambda_j B_j.$

Also:

(4.7) $\langle [\Gamma(Y), U] Y_j, Y_j \rangle = \langle \Gamma(Y) U Y_j, Y_j \rangle - \langle U \Gamma(Y) Y_j, Y_j \rangle = \langle \Gamma(Y) U Y_j, Y_j \rangle - \langle \Gamma(Y) Y_j, U Y_j \rangle = 0.$

Similarly,

(4.8) $\langle [R(\xi, Y), U] Y_j, Y_j \rangle = 0.$

From $\langle \nabla Y \xi, Y \rangle = 0$, and since we can also assume $R(Y_j, Y) Y_j = \lambda Y$, we have:

(4.9) $\langle R(\nabla Y \xi, Y_j) Y, Y_j \rangle = \langle R(Y_j) (\nabla Y \xi), Y_j \rangle = \langle \nabla Y \xi, R(Y_j, Y) Y_j \rangle = \lambda \langle \nabla Y \xi, Y \rangle = 0.$

Similarly

(4.10) $\langle R(Y, Y_j) \nabla Y \xi, Y_j \rangle = 0.$

Evaluating on $Y_j$ the right hand-side of the equation of Lemma 4.2.3 and taking the inner product of with $Y_j$, after using 4.6–4.10 and noticing that $\nabla Y U = 0$, $\nabla R = 0$, and $\langle U^2 Y_j, Y_j \rangle = \lambda_j^2$,

we obtain the given equation for $B_j$. \qed
4.3.2. Corollary. If $(M, g)$ has constant negative curvature $-1$ and dimension $n$, $\nabla_\xi \text{Tr} U$, along $\gamma$, satisfies the following equation:

\[ L_Y(\nabla_\xi \text{Tr} U) - 2(\nabla_\xi \text{Tr} U) = (1 - n)S(Y, Y) - \left. \frac{\partial \text{Ric}^\xi}{\partial \xi} \right|_{\xi=0} (Y, Y). \]

Proof.
Clearly $\sum_j B_j = \sum_j \langle (\nabla_\xi U)Y_j, Y_j \rangle = \nabla_\xi \text{Tr} U$. From \n
\[ \text{Ric}(X, Z) = \text{Tr}(W \mapsto R(X, W)Z), \]

we have $\sum_j \left. \frac{\partial \text{Ric}^\xi}{\partial \xi} \right|_{\xi=0} (Y, Y_j) = \left. \frac{\partial \text{Ric}^\xi}{\partial \xi} \right|_{\xi=0}$. Summing over $j$ the equation of Lemma 4.3.1, since $\lambda_j = 1$ we obtain our claim. \qed

4.4. Let $\nabla^* \nabla$ denote the rough Laplacian for the metric $g = g_0$ on $M$. Recall that the Lichnerowicz Laplacian for a metric $g$ is defined as [Bes87, 1.180b]:

\[ \Delta_L S = \nabla^* \nabla S + \text{Ric} \circ S + S \circ \text{Ric} - 2R^0(S), \]

where $\circ$ denotes the contraction symmetric tensors of rank 2 identified with $\{1\}$ tensor via $g$ and $R^0(S)$ is defined by

\[ R^0(S)(Y, Z) = \text{Tr}_g S(R(\cdot, X)Y, \cdot). \]

Then [Bes87, 1.174]:

\[ \left. \frac{\partial \text{Ric}^\xi}{\partial \xi} \right|_{\xi=0} = \frac{1}{2} \Delta_L S - \delta^* \delta S - \nabla d(\text{Tr} S) \]

where $\delta^*$ is the symmetrization of covariant derivative and $\delta$ is its formal adjoint, the divergence.

Proof of Proposition 1.3.3. Form Corollary 4.3.2 we obtain that along the geodesic $\gamma$ we have

\[ B(g^t u)\nabla_\xi = \text{Tr} U(\gamma(t)) \approx \frac{n-1}{2} S(Y(t), Y(t)) + \frac{1}{2} \left. \frac{\partial \text{Ric}^\xi}{\partial \xi} \right|_{\xi=0} (Y(t), Y(t)), \]

\[^5\text{Our definition agrees with [Bes87, 1.131], in spite of the opposite convention on the sign of the Riemann curvature $R$.}\]
with \( \approx \) meaning that the difference is a coboundary. In fact, the difference at \( \gamma(t) = p(g^t v) \) is \( \frac{1}{2} L_Y(\nabla_\xi \text{Tr} U)(\gamma(t)) = \frac{1}{2} \frac{\partial}{\partial t} B(g^t v) \)

We claim that since \( g \) has constant curvature \(-1\) we have:

\[
\Delta L S(Y, Y) = \nabla^* \nabla S(Y, Y) - 2n S(Y, Y) + 2 \text{Tr}_g S.
\]

In fact, \( \text{Ric} = -(n - 1)g \), and therefore

\[
\text{Ric} \circ S + S \circ \text{Ric} = -2(n - 1)S.
\]

Furthermore, since \( R(Y, X)Z = -(X, Z)Y + (Y, Z)X \),

\[
(4.14) \quad R^o(S)(Y, Y) = \sum_i S(R(Y_i, Y)Y_i) = \\
= \sum_i S(-Y_i + \langle Y, Y_i \rangle Y_i) = -\text{Tr}_g(S) + S(Y, Y).
\]

proving our claim. We obtain that

\[
\nabla_\xi \text{Tr} U \approx \frac{n-1}{2} S(Y, Y) + \frac{1}{4} \nabla^* \nabla S(Y, Y) - \frac{n}{2} S(Y, Y) + \\
+ \frac{1}{2} \text{Tr}_g S - \frac{1}{2} \delta^* \delta S(Y, Y) - \frac{1}{2} \nabla d(\text{Tr} S)(Y, Y) \\
\approx -\frac{1}{2} S(Y, Y) + \frac{1}{4} \nabla^* \nabla S(Y, Y) + \frac{1}{2} \text{Tr}_g S - \\
- \frac{1}{2} \delta^* \delta S(Y, Y) - \frac{1}{2} \nabla d(\text{Tr} S)(Y, Y).
\]

and setting

\[
T = -\frac{1}{2} S + \frac{1}{4} \nabla^* \nabla S + \frac{1}{2} (\text{Tr}_g S)g - \frac{1}{2} \delta^* \delta S
\]

and noticing that the term \( \nabla d(\text{Tr} S)(Y, Y) = Y^2(\text{Tr} S) \) is cohomologous to zero, we have concluded that

\[
B(g^t v) = \nabla_\xi \text{Tr} U(\gamma(t)) \approx T^v(g^t v)
\]

or \( B \approx T^v \). Thus the first variation of the Liapunov cocycle along the path \( g_\varepsilon \) at \( \varepsilon = 0 \) is

\[
\delta u = B + \frac{1}{2} S^v \text{Tr} U \approx T^v - \frac{n-1}{2} S^v.
\]

\[\square\]
§ 5. Proof of Proposition 1.3.4 and Theorem C.

5.1.

Let \( g_\varepsilon \in \mathcal{U} \) be a curve of metrics of constant volume on \( M \). Then if we set \( S_\varepsilon = \frac{\partial g_\varepsilon}{\partial \varepsilon} \) and denoting by \( s_n \) the volume of the \( n \)-sphere, we have,

\[
\frac{1}{2} \int_{Sg_\varepsilon M} S^\varepsilon(v) \, d\text{Liouv}(g_\varepsilon)(v) = \frac{s_{n-1}}{2n} \int_M \text{Tr}_{g_\varepsilon} S_\varepsilon \, d\text{Vol}_{g_\varepsilon} = -\frac{s_{n-1}}{n} \frac{\partial \text{Vol}_{g_\varepsilon}(M)}{\partial \varepsilon} = 0.
\]

Since \( \frac{1}{2}S^\varepsilon \) is the first variation \( \delta l_\varepsilon \) of the length cocycle \( l_\varepsilon \), and \( \text{Liouv}(g_\varepsilon) \) the Gibbs state for the Liapunov cocycle \( u_\varepsilon \) of the flow \( T_{g_\varepsilon} \), we obtain that \( m_{u_\varepsilon}(\delta l_\varepsilon) = 0 \) for all \( \varepsilon \). If \( g_0 \) is locally symmetric, the measure of maximal entropy \( M_0 \) and the Liouville measure \( \text{Liouv}(g_0) \) coincide and thus all the conditions of Proposition 1.3.1 (C) are verified. Denoting for simplicity \( g_0 \) by \( g \), the maximal measure entropy \( M_0 \) by \( m \) and the covariance of the flow \( T_g \) with respect to the measure of \( m \) by \( \text{Cov} \), we can summarize the results so far achieved (cf. Propositions 1.3.1, 1.3.3) in the following Proposition:

5.1.1. Proposition. Let \( g_\varepsilon \) be a \( C^2 \)-curve of \( C^5 \)-metrics on a manifold \( M \) of constant volume, and assume that that \( g = g_0 \) has constant negative curvature \(-1\). Denoting with \( \delta l \) and \( \delta u \) the first variation of the length cocycle and of Liapunov cocycle at \( \varepsilon = 0 \) we have:

\[
\text{Ent}''_{\text{top}}(g) = \text{Cov} ((n - 1)\delta l + \delta u, (n - 1)\delta l)
\]

and

\[
\text{Ent}''_{\text{Liouv}}(g) = -\text{Cov} ((n - 1)\delta l + \delta u, \delta u).
\]

Setting \( T = T(S) := -\frac{1}{2}S + \frac{1}{4} \nabla^* \nabla S + \frac{1}{2}(\text{Tr}_g S)g - \frac{1}{2} \delta^* \delta S \), since \( \delta l \approx \frac{1}{2}S^\varepsilon \) and \( \delta u \approx T^\varepsilon - \frac{n-1}{2}S^\varepsilon \), we have

\[
\text{Ent}''_{\text{top}}(g) = \text{Cov} (T^\varepsilon, \frac{n-1}{2}S^\varepsilon),
\]

\[
\text{Ent}''_{\text{Liouv}}(g) = -\text{Cov} (T^\varepsilon, T^\varepsilon - \frac{n-1}{2}S^\varepsilon)
\]

and

\[
\text{Ent}''_{\text{top}}(g) - \text{Ent}''_{\text{Liouv}}(g) = \text{Cov} (T^\varepsilon, T^\varepsilon).
\]
If $g$ has constant sectional curvature, then the group $G$ of isometries of $\tilde{M}$, the universal cover of $M$, is isomorphic to $SO_0(1,n)$, where $n = \dim(M)$. We have an identification of $M$ with the space $\Gamma\backslash G/K$, where $\Gamma \approx \pi_1(M)$ is a discrete group of isometries acting without fixed points on $\tilde{M}$, and $K \approx SO(n)$ is the stabilizer of a point $p_0 \in \tilde{M}$. Furthermore, the unit tangent unit bundle $S_gM$ and the orthonormal frame bundle $FM$ of $M$ are identified with $\Gamma\backslash G/K_1$ and $\Gamma\backslash G$, where $K_1 \approx SO(n - 1)$ is the stabilizer of a vector $v_0 \in T_{p_0}\tilde{M}$. The parallel transport of an orthonormal frame $(v_1, v_2, \ldots, v_n) \in FM$ along the geodesic determined by $v_1$ is identified with the action on $\Gamma\backslash G$ given by multiplication on the right by the split-Cartan $A_1 \approx \mathbb{R}$ commuting with $K_1$. It is plain that this parallel transport projects via the natural projection $(v_1, v_2, \ldots, v_n) \in FM \mapsto v_1 \in S_gM$ to the geodesic flow on $S_gM$.

5.2.1. Notation. Denote by $C^r(S^2M)$ the space of $C^r$-sections of the bundle $S^2M$ of symmetric covariant tensors of rank 2. Similarly $L^2(S^2M)$ denotes the $L^2$ sections of this bundle. Clearly the map $\nabla : S \to S^\prime$ defined, for $v \in S_gM$, by $S^\prime(v) = S(v, v)$ maps $C^r(S^2M)$ to $C^r(S_gM)$ and $L^2(S^2M)$ to $L^2(S_gM)$.

Observe also that we have an injection $L^2(S_gM) \hookrightarrow L^2(FM)$, regarding $L^2(S_gM)$ as the subset of $L^2(FM)$ of $K_1$-invariant vectors. Similarly, $C^r(S_gM) \hookrightarrow C^r(FM)$. In the sequel these identifications will be implicit.

5.2.2. Left invariant differential operators on $G$ commuting with $K_1$ act on $C^\infty(S_gM)$. This is in particular true of the the Casimir operator $\text{Cas}_{SO(1,n)}$ of $G$, which we normalize so that on $C^\infty(M)$ it coincides with the rough Laplacian $\nabla^*\nabla$.

Observe that the differential operators $\nabla^*\nabla, \delta, \delta^*$ etc. acting on symmetric tensor field can be viewed by means of the above maps as left invariant differential operators on $C^\infty(FM)$ commuting with $K_1$. In fact, denote by $P_i$ the parallel transport of an orthonormal frame $(v_1, v_2, \ldots, v_n) \in FM$ along the geodesic determined by $v_i$ and let $Y_i$ be vector field generating the flow $P_i$. For $i < j$, set $R_{ij} = [Y_i, Y_j]$. Then it is plain that the vector fields $Y_i, R_{ij}, (i < j)$, form a basis for the Lie algebra $\mathfrak{g} \approx so(1,n)$ of $G$ and that $R_{ij}$ exponentiate to the flow $\exp(\theta R_{ij})$ which rotates an orthonormal frame $(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \in FM$ to $(v_1, \ldots, \cos(\theta)v_i - \sin(\theta)v_j, \ldots, \sin(\theta)v_i + \cos(\theta)v_j, \ldots, v_n)$. In particular given $S \in C^\infty(S^2M)$,
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setting, for \((v_1, \ldots, v_n) \in FM,\)

\[ S_{ij}(v_1, \ldots, v_n) = S(v_i, v_j) \]

and identifying \(S_{11}\) with the lift to \(FM\) of \(S^\vee\) we have

\[ S_{1i} = \frac{1}{2} L_{R_{1i}} S^\vee. \]

Thus, if \((v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \in FM\) and \(S \in C^2(S^2M),\) we have:

\[
(\delta S)^\vee (v_1) = (\delta S)(v_1) = -\sum_i \nabla_{v_i} S(v_i, v_1) = -\sum_i \left. \frac{\partial}{\partial t} S(P^t_i v_i, P^t_i v_1) \right|_{t=0} \\
= -\sum_i S_{1i}(P^t_i v_1, \ldots, P^t_i v_n) = -\sum_i L_{Y_i} S_{1i}(v_1, \ldots, v_n) \\
= \frac{1}{2} \sum_i L_{Y_i} L_{R_{1i}} S^\vee(v_1, \ldots, v_n)
\]

Similarly, we have

\[
\nabla^* \nabla = -\sum_i L_{Y_i}^2 = (-\sum_i L_{Y_i}^2 + \sum_{i<j} R_{ij}^2) - \sum_{i<j} R_{ij}^2 = \text{Cas}_{SO(1,n)} - \text{Cas}_{SO(n)},
\]

where \(\text{Cas}_G\) denotes the Casimir operator of the group \(G.\) Since, for \(S \in C^\infty(S^2M),\) we have \(\text{Cas}_{SO(n)} S^\vee = 2 \text{Tr} S - 2n S^\vee,\) we can express the operator \(\mathcal{T}(S)\) defined in 5.1.1 as

\[
(5.1) \\
\mathcal{T}(S) = -\frac{1}{2} S + \frac{1}{4} \nabla^* \nabla S + \frac{1}{2} (\text{Tr}_g S) \otimes g - \frac{1}{2} \delta^* \delta S \\
= -\frac{1}{2} S + \frac{1}{4} (\text{Cas}_{SO(1,n)} - \text{Cas}_{SO(n)}) S + \frac{1}{2} (\text{Tr}_g S) \otimes g - \frac{1}{2} \delta^* \delta S \\
= \frac{n-1}{2} S + \frac{1}{4} \text{Cas}_{SO(1,n)} S - \frac{1}{2} \delta^* \delta S.
\]

5.3.

By a well known theorem on the unitary representation of \(SO_0(1,n),\) the Hilbert space \(L^2(FM) \approx \Gamma\backslash G\) splits as a direct sum \(L^2(FM) = \sum H_i\) of topologically irreducible components \(H_i\) of the right action of \(G.\)
5.3.1. Lemma. Each irreducible representation space \( H_i \) splits as a direct sum of irreducible representation \( H_j^\tau \approx \tau \) for the action of the maximal compact subgroup \( K \approx SO(n) \) of \( G \). Let \( \tau \) be an irreducible representation of \( SO(n) \). Then the multiplicity of \( \tau \) in \( H_i \) is at most one.

Proof. The first statement is trivial. Denote by \( P = MAN \) a minimal parabolic of \( G = SO_0(1,n) \). By a Theorem of Harish-Chandra [War72, 5.5.1.7], there is a irreducible representation \( \sigma \) of \( M \approx SO(n-1) \), such that the multiplicity of \( \tau \) in \( H_i \) is at most the multiplicity of \( \sigma \) in the restriction of \( \tau \) to \( M \). This concludes the proof, since the multiplicity of any irreducible representations of \( SO(n-1) \) in a given irreducible representation of \( SO(n) \) is at most one (cf. [Zel73, §129] and [GC50]).

\[ \square \]

5.3.2. Notation. For each equivalence class of irreducible representations \( \tau \) of \( K \), let \( H_\tau \) be the subspace of \( H \) of \( K \)-type \( \tau \) i.e. the set of vectors transforming under \( K \) according to the irreducible representation \( \tau \) of \( K \).

The following lemma is an easy consequence of the ellipticity of the rough Laplacian \( \nabla^* \nabla \) acting on symmetric tensors. For \( S \in C^\infty(S^p M) \), we set \( S^\tau(v) = S(v,v,\ldots,v) \).

5.3.3. Lemma. Let \( S \in C^\infty(S^p M) \) be a symmetric tensor field of rank \( p \) and \( K \)-type \( \tau \). Let \( S^\tau = \sum s_i, s_i \in L^2(FM) \), be its decomposition in irreducible components. Then we have

1. For each \( s_i \) there exists a symmetric tensor field \( S_i \in C^\infty(S^p M) \) of rank \( p \) and \( K \)-type \( \tau \) with \( s_i = S_i^\tau \).
2. Each \( S_i \) is an eigenfunction of the rough Laplacian \( \nabla^* \nabla \) and, in particular, \( S_i \in C^\infty \).
3. The series \( S = \sum S_i \), or equivalently \( S^\tau = \sum s_i \), converges in the \( C^\infty \) topology.

We will only be interested with symmetric tensor field of rank 2 and therefore with only two \( K \)-types: the trivial \( K \)-type and the \( K \)-type of traceless symmetric tensor of rank 2.

5.3.4. Notation. Set

\[ \text{Conf}^\tau(g) = \{ S : S = f \otimes g, f \in C^\tau(M) \} \]
and

$$\text{Teich}^r(g) := \{ S \in C^r(S^2M) : \delta S = 0, \text{Tr}_g S = 0 \}.$$ 

Then Conf$^r(g)$ is the space tangent to $C^r$-conformal deformation of $g$ and it is the space of vectors of the trivial $K$-type in $C^r(FM)$. The space Teich$^r(g)$ is the formal $L^2$-orthogonal space in $C^r(S^2M)$ to the conformal fibers and to the local orbit of $g$ under the diffeomorphism group. Furthermore, each $S \in \text{Teich}^r(g)$ has the $K$-type of traceless symmetric tensor of rank 2.

5.3.5. **Notation.** Let $S \in L^2(S^2M)$. Let $H(S)$ be the cyclic subspace of $L^2(FM)$ generated by $G$ acting on $S'$.

5.3.6. **Corollary.** Let $R \in \text{Conf}^\infty(g)$ and write $R^\gamma = \sum_i r_i$, with $r_i \in H_i$.

Then, for all $i$, we have $r_i = R_i^\gamma$ with $R_i \in \text{Conf}^\infty(g)$.

Similarly, let $S \in \text{Teich}^\infty(g)$ and write $S^\gamma = \sum_i s_i$, with $s_i \in H_i$. Then, for all $i$, we have $s_i = S_i^\gamma$ with $S_i \in \text{Teich}^\infty(g)$.

**Proof.** The first claim is a mere restatement of Lemma 5.3.3. If $S \in \text{Teich}^\infty(g)$ and $S^\gamma = \sum_i s_i$, by Lemma 5.3.3 we have $s_i = S_i^\gamma$ with $S_i$ smooth traceless symmetric tensor field of rank 2. The discussion in 5.2.2 implies that the divergence operator on smooth traceless symmetric tensor field of rank 2 coincides with an element of the enveloping algebra of $g \approx so(1,n)$.

Thus we have $\delta s_i \in H_i$ for all $i$. We conclude that $\delta S = 0$ implies $\delta S_i = 0$ for all $i$, showing that $S_i \in \text{Teich}^\infty(g)$ for all $i$.

5.3.7. **Lemma.** Let $R \in \text{Conf}^\infty(g)$ and $S \in \text{Teich}^\infty(g)$ then $H(R) \perp H(S)$.

**Proof.** By Corollary 5.3.6, is sufficient to consider the case in which $R$ and $S$ belong to the same irreducible subspace $H_i$. Arguing by contradiction, we may assume that $H_i = H(R) = H(S)$. Then $R = r \otimes g$ with $r \in C^\infty(M)$. By the ergodicity of the geodesic flow, we have that $\delta^* \delta^* r = \nabla dr$ is not zero. In fact, for $v \in S_g M$ we have $(\delta^* \delta^* r)^{\gamma'}(v) = \nabla^2_{u,v} r = d^2/dt^2 r(g^t v)|_{t=0}$, thus $\delta^* \delta^* r = 0$ would imply that $r$ is constant, in which case there is nothing to prove.

It is not difficult to see from the compactness of $M$ that the traceless part $(\nabla dr)_0 := \nabla dr - \frac{\Delta r}{n} \otimes g$ of $\nabla dr$ is not zero. Since $S$ is traceless, the $K$ orbits of $(\nabla dr)^\gamma$ and $S^\gamma$ generate $K$-representations both isomorphic to the representation of $SO(n)$ on the space of harmonic homogeneous polynomial
of degree two on \(\mathbb{R}^n\). By Lemma 5.3.1 the multiplicity of this representation in \(H_1\) is 1, and therefore \((\nabla dr)_0^\vee\) and \(S^\vee\) have the same \(K\)-orbit. However, since \((\nabla dr)_0^\vee\) and \(S^\vee\) are both \(K_1\)-invariant we obtain that, for some constant \(C \neq 0\), we have \(S^\vee = C(\nabla dr)_0^\vee\) and equivalently \(S = C(\nabla dr)_0\). From \(\langle S, (\nabla dr)_0 \rangle = \langle S, (\delta^* \delta^* r)_0 \rangle = \langle S, \delta^* \delta^* r \rangle\), we obtain \(\langle S, \delta^* \delta^* r \rangle \neq 0\). This is impossible because we have \(\langle S, \delta^* \delta^* r \rangle = \langle \delta S, \delta^* r \rangle\), and the latter term is zero since \(S \in \text{Teich}(g)\) and therefore \(S\) is divergence free.

\[\square\]

### 5.3.8. Remark. A simple modification of the proof above shows that if \(S \in \text{Teich}^\infty(g)\), then \(H(S)\) is orthogonal to all one forms. This means that if \(S \in \text{Teich}^\infty(g)\), then the minimal \(K\)-type occurring in \(H(S)\) is the \(K\)-type of \(S\).

### 5.3.9. Lemma. Let \(S_1, S_2 \in C^\infty(S^0M)\) and assume that \(H(S_1)\) and \(H(S_2)\) are orthogonal. Then

\[
\text{Cov}(S_1^\vee, S_2^\vee) = \text{Cov}(\mathcal{T}(S_1)^\vee, S_2^\vee) = \text{Cov}(\mathcal{T}(S_1)^\vee, \mathcal{T}(S_1)^\vee) = 0.
\]

**Proof.** By formula 5.1 the operator \(\mathcal{T}\) belongs to the enveloping algebra of \(G\) and therefore \(\mathcal{T}(S_i) \in H(S_i)\).

Since the flow \(g^t\) (lifted to \(FM\)) is given by right translation of by some split-Cartan in \(G\), by the exponential decay of matrix coefficients, we have

\[
\text{Cov}(F, G) = \int_{-\infty}^{\infty} \langle F, G \circ g^t \rangle \, dt
\]

for any pair of smooth functions \(F\) and \(G\) on \(FM\), provided that either \(F\) or \(G\) integrates to zero on \(FM\). Thus if \(F\) and \(G\) belong to orthogonal \(G\)-invariant subspaces we have that at least one of them has average zero on \(FM\) and \(\text{Cov}(F, G) = 0\). The lemma follows.

\[\square\]

**Proof of Proposition 1.3.4.** Recall that for simplicity we write \(g\) instead of \(g_0\). It is sufficient to prove the claim for a dense set of \(S \in C^\delta(S^2M)\), e.g. for \(S \in C^\infty(S^2M)\), since the bilinear form \(\text{Cov}\) is continuous in the \(C^\alpha\) topology. For \(S\) in \(C^\infty(S^2M)\) we can write

\[
S = S_T + S_C + S_0,
\]
with $S_T \in \text{Teich}^\infty(g)$, $S_C \in \text{Conf}^\infty(g)$ and with $S_0$ belonging to the image under $\delta^*$ of the space $\Lambda^1(M)$ of $C^\infty$ one-forms. This decomposition of $S$ is unique, since the sum

$$C^\infty(S^2M) = \text{Teich}^\infty(g) + \text{Conf}^\infty(g) + \delta^*(\Lambda^1(M))$$

is direct (Cf. [Bes87, Lemma 4.57]). By the linearity of the operator $T$, we can decompose $T$ as $T = T(S_T) + T(S_C) + T(S_0)$. Now, we observe that, in computing $\text{Cov}(T^\vee, S^\vee)$ and $\text{Cov}(T^\vee, T^\vee)$, the terms $S_0$ and $T(S_0)$ are irrelevant: in fact the terms $S_0^\vee$ and $T(S_0)^\vee$ are cohomologous to zero, since we have $S_0 = L_X g$ for some $C^\infty$ vector field $X$ on $M$. Thus the proof of Proposition 1.3.4 will be complete if we prove it simply for $S \in \text{Teich}^\infty(g) + \text{Conf}^\infty(g)$.

By Lemma 5.3.9, we have

$$\text{Cov}(T(S_C)^\vee, S^\vee_T) = \text{Cov}(T(S_T)^\vee, S^\vee_C) = \text{Cov}(T(S_C)^\vee, T(S_T)^\vee) = 0$$

and thus it suffices to prove the claim of Proposition 1.3.4 separately for $S \in \text{Teich}^\infty(g)$ and for $S \in \text{Conf}^\infty(g)$.

Let $S \in \text{Teich}^\infty(g)$. From $T(S) = -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\text{Tr}_g S)g - \frac{1}{2}\delta^*\delta S$ and the definition of $\text{Teich}^\infty(g)$ we obtain

$$T(S) = -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S,$$

(5.2)

Regard $S$ as a one-form with values in $T^*M$. Let $d^\vee$ be the differential induced on $\Lambda^r(M) \otimes T^*(M)$ by the Levi-Civita connection and let $d^\vee^*$ be its formal adjoint. We have the following Weitzenbock formula (cf. [Bes87, p.335])

$$(d^\vee^*d^\vee + d^\vee d^\vee^*)S = \nabla^*\nabla S - R^\circ(S) + S \circ \text{Ric},$$

where $R^\circ(S)$ was defined in (4.12). Using the formula (4.14) and the fact that $\text{Ric} = -(n-1)g$, we obtain

$$(d^\vee^*d^\vee + d^\vee d^\vee^*)S = \nabla^*\nabla S - S - (n-1)S = \nabla^*\nabla S - nS.$$  

Since the left hand side above is positive, we conclude that the spectrum of the restriction of the $\nabla^*\nabla$ to $\text{Teich}^\infty(g)$ lies in $[n, \infty)$ and (5.2) implies that the spectrum of the restriction of $T(S)$ to $\text{Teich}^\infty(g)$ lies in the interval $[(n-2)/4, \infty)$.

By Lemma 5.3.3, we can decompose $S$ as a $C^\infty$ convergent series $S = \sum S_i$ with $S_i^\vee$ belonging to the irreducible component $H_i$ and $S_i$ eigenfunction of $\nabla^*\nabla$ of eigenvalue $\lambda_i \geq (n-2)/4$. Then for $i \neq j$ the spaces $H(S_i)$ and $H(S_j)$ are orthogonal.
Lemma 5.3.9 implies that
\[
\text{Cov} (S_i^\vee, S_j^\vee) = \text{Cov} (\mathcal{T}(S_i)^\vee, S_j^\vee) = \text{Cov} (\mathcal{T}(S_i)^\vee, \mathcal{T}(S_j)^\vee) = 0.
\]
We conclude that
\[
(5.3) \quad \text{Cov} (\mathcal{T}(S)^\vee, S^\vee) = \sum_i \text{Cov} (\mathcal{T}(S_i)^\vee, S_i^\vee) = \sum_i \lambda_i \text{Cov} (S_i^\vee, S_i^\vee) 
\geq \sum_i \frac{n-2}{4} \text{Cov} (S_i^\vee, S_i^\vee) = \frac{n-2}{4} \text{Cov} (S^\vee, S^\vee)
\]
and, similarly,
\[
(5.4) \quad \text{Cov} (\mathcal{T}(S)^\vee, \mathcal{T}(S)^\vee) = \sum_i \text{Cov} (\mathcal{T}(S_i)^\vee, \mathcal{T}(S_i)^\vee) = \sum_i \lambda_i^2 \text{Cov} (S_i^\vee, S_i^\vee) 
\geq \sum_i \left(\frac{n-2}{4}\right)^2 \text{Cov} (S_i^\vee, S_i^\vee) = \left(\frac{n-2}{4}\right)^2 \text{Cov} (S^\vee, S^\vee),
\]
proving the desired estimate for the case \(S \in \text{Teich}^\infty(g)\).

Consider now the case \(S \in \text{Conf}^\infty(g)\). We have \(\text{Tr}_g S = nS\) and therefore \(\mathcal{T}(S) = \frac{n-1}{2} S + \frac{1}{4} \nabla^* \nabla S - \frac{1}{2} \delta^* \delta S\). Writing \(S = F \otimes g\), with \(F \in C^\infty(M)\), we see that the term \(\delta^* \delta S\) appearing in \(\mathcal{T}(S)\) can be rewritten as \(\delta dF\) and it gives rise in \(\mathcal{T}(S)^\vee\) to the term \((\delta^* \delta S)^\vee = (\delta dF)^\vee\), which is cohomologous to zero. Thus, in computing \(\text{Cov} (\mathcal{T}(S)^\vee, S^\vee)\) and \(\text{Cov} (\mathcal{T}(S)^\vee, T^\vee)\) we can replace \(\mathcal{T}(S)\) by
\[
\mathcal{T}_0(S) := \frac{n-1}{2} S + \frac{1}{4} \nabla^* \nabla S.
\]
The positivity of the rough Laplacian \(\nabla^* \nabla\) implies that the elliptic operator \(\mathcal{T}_0\) has spectrum in \([ (n-1)/2, \infty)\). Reasoning as before we obtain , for \(S \in \text{Conf}^\infty(g)\), estimates similar to (5.3) and (5.4), but with the constant \((n-1)/2\) replacing the smaller constant \((n-2)/4\). Thus the estimates (5.3) and (5.4) are also valid for \(S \in \text{Conf}^\infty(g)\). This concludes our proof \(\square\)

**Proof of Theorem C.** Let \(n = \dim M = 3\). By Proposition 5.1.1 and formulas (5.1) and (5.2), we see that, for \(S \in \text{Teich}^\infty(g)\),
\[
\text{Ent}_{\text{Liouv}}''(g) = - \text{Cov} (T^\vee, T^\vee - S^\vee) = 
- \text{Cov} \left( S^\vee + \frac{1}{4} \text{Cas}_{SO(1,3)} S^\vee, \frac{1}{4} \text{Cas}_{SO(1,3)} S^\vee \right).
\]
Thus assuming that $\text{Cas}_{SO(1,3)} S^\gamma = \lambda S^\gamma$, we obtain $\text{Ent}''_{\text{Liouv}}(g) = -\frac{1}{4} \lambda (1 + \frac{1}{4} \lambda) \text{Cov}(S^\gamma, S^\gamma)$. Thus $\text{Ent}''_{\text{Liouv}}(g) < 0$ is equivalent to $\lambda > 0$ or $\lambda < -4$. The latter eventuality is impossible, since we have shown that, in dimension $3$, the spectrum of the Laplacian on $\text{Teich}(g)$ is contained in the interval $[3, \infty)$ and $\text{Cas}_{SO(1,3)} = \nabla^* \nabla + \text{Cas}_{SO(3)} = \nabla^* \nabla - 6$. In any case, since the spectrum of $\nabla^* \nabla$ is discrete, we find that on an infinite dimensional subspace of $\text{Teich}(g)$ the derivative $\text{Ent}''_{\text{Liouv}}(g)$ in a curve of metrics in a direction in this subspace is negative.

However if the operator $\text{Cas}_{SO(1,3)}$ on $\text{Teich}^\infty(g)$ has an eigenvector $S$ of eigenvalue $\lambda < 0$, it follows that the Liouville entropy has positive second derivative in the direction of $S$. Now recall that such an $S$ would generate an irreducible representation of $SO(1, 3)$ with minimal $K$-type given by the $K$-orbit of $S$, i.e. with minimal $K$-type $\tau_2$ (here we have denoted by $\tau_2$ the representation of $K \approx SO(3)$ on the space of traceless symmetric tensors of rank $2$). The unitary irreducible representations of $SO(1, 3)$ with minimal $K$-type $\tau_2$ belong to the unitary principal series of $SO(1, 3)$ and determined up to unitary equivalence by the value of the Casimir operator on them. From [Tay86] we obtain that the values of the eigenvalues of the Casimir operator, with our normalization, are given by $\mu^2 - 3$, $\mu \in \mathbb{R}$. Thus, since the Plancherel measure has support on all the interval $[-3, \infty)$, by Theorem 5.4 of [DW79], we find a cocompact lattice $\Gamma$ in $SO(1, 3)$, with spectrum of the Casimir on $\text{Teich}^\infty(g)$ in $(-3, 0)$. Then $\Gamma \backslash SO(1, 3)/SO(3)$ provides us with the desired counterexample. 

\[\square\]

References.


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