Curvature Estimates for Minimal Annuli and Non-Compact Douglas-Plateau Problem

YI FANG AND JENN-FANG HWANG

In this article we give curvature estimates for minimal annuli with convex boundary $L_1 \cup L_2$ in parallel planes and apply these estimates to solve some kinds of non-compact Douglas-Plateau problem. The estimates for minimal annuli also give various necessary conditions for the existence of minimal surfaces.

1. Introduction.

In this article we give curvature estimates for minimal annuli with convex boundary $L_1 \cup L_2$ in parallel planes and apply these estimates to solve some kinds of non-compact Douglas-Plateau problem. The estimates for minimal annuli also give various necessary conditions for the existence of minimal surfaces.

To state our results, let us fix some notations first.

Let $P_t = \{(x, y, z) \in \mathbb{R}^3; z = t\}$, $S(t_1, t_2) = \{(x, y, z) \in \mathbb{R}^3; t_1 \leq z \leq t_2\}$, and $S'(t_1, t_2) = \{(x, y, z) \in \mathbb{R}^3; t_1 \leq y \leq t_2\}$, where $t_1 < t_2$. Let $C_R$ be the solid cylinder $\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 < R^2\}$.

Let $A \subset S(t_1, t_2)$ be an embedded minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are convex Jordan curves. Denote $\mathcal{P}_\pi$ the perpendicular projection on a plane $\pi$ perpendicular to the $xy$-plane. Let $\Omega_\pi := \text{Int}(\mathcal{P}_\pi(A))$, the interior of $\mathcal{P}_\pi(A)$, and $\partial \Omega_\pi = \bigcup_{i=1}^4 \Gamma_i^\pi$, where $\Gamma_i^\pi = \mathcal{P}_\pi(L_i)$, $i = 1, 2$, and $\Gamma_3^\pi$ and $\Gamma_4^\pi$ are arcs connecting $\Gamma_1^\pi$ and $\Gamma_2^\pi$.

In Lemma 2.1 we estimate $|\Gamma_i|$, the length of $\Gamma_i$. We have

$$|\Gamma_1^\pi| + |\Gamma_2^\pi| \geq |\Gamma_3^\pi| + |\Gamma_4^\pi|,$$

and

$$|\Gamma_1^\pi||\Gamma_2^\pi| \geq (t_2 - t_1)^2, \quad |\Gamma_1^\pi| + |\Gamma_2^\pi| > 2(t_2 - t_1).$$

Although the proof of the former is a simple imitation of a technique in [fanghwang-js], and the proof of the latter comes from elementary Euclidean
Yi Fang and Jenn-Fang Hwang

geometry, the more accurate new estimate to $\Omega$ leads us to curvature estimates.

In Lemma 2.2 we prove that if $A \subset S(t_1, t_2) \cap S'(-R, R)$ and $r$ is the
yz-plane, then for any point $p \in \text{Int}(A)$,

$$\max\{\text{dist}(P_{\pi}(p), \Gamma_{t_1}), \text{dist}(P_{\pi}(p), \Gamma_{t_2})\} \geq \frac{(\sqrt{2} - 1)d^2}{R},$$

where $d = \text{dist}(p, \partial S(t_1, t_2))$. Note that $\partial S(t_1, t_2) = P_{t_1} \cup P_{t_2}$, so if $p = (x, y, z)$, then $d = \min\{z - t_1, t_2 - z\} = \min\{\text{dist}(p, P_{t_1}), \text{dist}(p, P_{t_2})\}$.

In Proposition 2.1, using Lemma 2.2 and by projecting on a suitable
direction we prove an interior curvature estimate for $A \subset S(t_1, t_2) \cap C_R$, that is, there is a constant $C_0 > 0$, such that for any $p \in \text{Int}(A)$, the
Gaussian curvature of $p$ satisfies

$$|K(p)| \leq \frac{C_0R^2}{d^4},$$

where $d = \text{dist}(p, \partial S(t_1, t_2))$.

This curvature estimate gives immediate generalization of the existence
result of Hoffman and Meeks, [fanghwang-hm9], to the continuous convex boundary case, see Theorem 2.2.

Proposition 2.2 gives another curvature estimate. It states that if $A \subset S(t_1, t_2) \cap S'(t'_1, t'_2)$ is a compact minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$ and $L_2 \subset P_{t_2}$ are $C^2$ convex Jordan curves, then there is a constant $C_1$ such that for any $p \in A$,

$$|K(p)| \leq C_1,$$

where $C_1$ only depends on the height $t_2 - t_1$, the width $t'_2 - t'_1$, and the boundary planar curvature bound of $\partial A$.

These curvature estimates then lead to compactness theorems, Theorem 2.1 and Theorem 2.3, which state that some sequence of $\{A_n\}$ as in Proposition 2.1 or 2.2 has a convergent subsequence.

As applications of Theorem 2.1 and 2.3, we solve various non-compact Douglas-Plateau problems in Section 3.

Recall that the Douglas-Plateau problem for two contours is as follows: Let $L_1$ and $L_2$ be two disjoint Jordan curves in $\mathbb{R}^3$, find a minimal annulus $A$ such that $\partial A = L_1 \cup L_2$.

Let $S_1$ and $S_2$ be area minimizing disks (when we say disks, we mean that they are homeomorphic to the unit disk in $\mathbb{C}$) such that $\partial S_1 = L_1$,
Curvature estimates annuli and non-compact Douglas-Plateau problem

\( \partial S_2 = L_2 \). Let \( S \) be the set of rectifiable annulus \( S \) such that \( \partial S = L_1 \cup L_2 \). Douglas [fanghwang-dogl] proved that if

\[
\inf_{S \in S} \{ \text{Area}(S) \} < \text{Area}(S_1) + \text{Area}(S_2),
\]

then there is an area minimizing (therefore minimal) annulus \( A \) such that \( \partial A = L_1 \cup L_2 \).

If \( L_1 \) and \( L_2 \) are coaxial unit circles in parallel planes, then it is well-known that there is a constant \( h > 0 \) such that when the distance between the centres is smaller than \( h \), there are exactly two catenoids bounded by \( L_1 \cup L_2 \); when the distance between the centres is equal to \( h \), there is only one catenoid bounded by \( L_1 \cup L_2 \); when the distance between the centres is larger than \( h \), there are no catenoids bounded by \( L_1 \cup L_2 \). Furthermore, by Shiffman's third theorem [fanghwang-shl], any minimal annuli bounded by \( L_1 \cup L_2 \) must be a rotation surface hence is a piece of a catenoid. Thus there are either two, one, or zero minimal annuli bounded by \( L_1 \cup L_2 \) depending on the distance between their centres.

Meeks and White [fanghwang-mwl] generalized the above observation to minimal annuli bounded by two smooth convex Jordan curves \( L_1 \cup L_2 \) in different parallel planes, i.e., there are either two, one, or zero minimal annuli bounded by \( L_1 \cup L_2 \). But unlike the coaxial circles case, there are no simple criteria to tell us when do we have two, one, or zero minimal annuli bounded by \( L_1 \cup L_2 \).

However, there are some partial conditions, either sufficient or necessary to the existence of a solution to some special Douglas-Plateau problems for two contours. For example, let us consider the Douglas-Plateau problem to the boundary consisting of two Jordan curves \( L_1 \cup L_2 \) in parallel and different planes, say \( L_1 \subset \partial P_0 \), \( L_2 \subset \partial P_d \), \( d > 0 \).

If \( L_1 \) and \( L_2 \) are smooth convex, then besides Douglas's sufficient condition, Hoffman and Meeks in [fanghwang-hm9] gave a sufficient condition to ensure that there are two solutions, i.e., if there is a connected compact non-planar minimal surface (could be branched) whose boundary is contained in open planar disks bounded by \( L_1 \) and \( L_2 \), then there are two minimal annuli bounded by \( L_1 \cup L_2 \). For the precise statement, please see Theorem 2.2 below.

The result of Hoffman and Meeks can be also treated as a necessary condition, i.e., let \( C_1 \) and \( C_2 \) be smooth convex Jordan curves such that \( L_1 \) and \( L_2 \) are contained in the open planar disks bounded by \( C_1 \) and \( C_2 \) respectively, then there is a connected compact minimal surface (maybe branched) bounded by \( L_1 \cup L_2 \) only if there are two minimal annuli bounded
There are other necessary conditions. For example, Theorem 2.2 combined with a J. C. C. Nitsche’s result [fanghwang-ni2], page 88, implies that the images of perpendicular projection of $L_1$ and $L_2$ on the $xy$-plane must intersect if $L_1$ and $L_2$ are convex.

Moreover, Osserman and Schiffer proved in [fanghwang-os1] that if $c$, $d$, $\delta_1$, and $\delta_2$ are positive constants and

$$L_1 \subset \left\{ (x, y, z); \left( x - \frac{c}{d} z \right)^2 + y^2 \leq \delta_1^2, z \leq 0 \right\},$$

$$L_2 \subset \left\{ (x, y, z); \left( x - \frac{c}{d} z \right)^2 + y^2 \leq \delta_2^2, z \geq d \right\},$$

are closed curves and they bound a minimal annulus, then

$$\delta_1 + \delta_2 > \sqrt{\frac{c^2}{2} + d^2}.$$  

Using the basic estimates in Lemma 2.1, in Corollary 2.2 we prove that if $\Sigma$ is a connected compact non-planar minimal surface (maybe branched) such that $\partial \Sigma = L_1 \cup L_2$ and

$$L_1 \subset \left\{ (x, y, z); -\delta_1 \leq x - \frac{c}{d} z \leq \delta_1, z \leq 0 \right\},$$

$$L_2 \subset \left\{ (x, y, z); -\delta_2 \leq x - \frac{c}{d} z \leq \delta_2, z \geq d \right\},$$

then

$$2 \max\{\delta_1, \delta_2\} > \sqrt{c^2 + d^2}.$$ 

Furthermore, if

$$L_1 \subset \{(x, y, 0); -\delta_1 \leq x \leq \delta_1\},$$

$$L_2 \subset \{(x, y, d); -\delta_2 \leq x - c \leq \delta_2\},$$

then

$$\delta_1 + \delta_2 > \sqrt{c^2 + d^2}.$$ 

We define Non-compact Douglas-Plateau problem of annular type for $n$ boundary curves as follows:

Let $L_i$, $i = 1, \cdots, n$, be disjoint, embedded proper complete curves, at least one of them is non-compact, find a minimal annulus $A$ such that $\partial A = \Gamma := \bigcup_{i=1}^{n} L_i$. 

Curvature estimates annuli and non-compact Douglas-Plateau problem

As we have seen, there are many necessary conditions restricting the solvability of even compact Douglas-Plateau problems for two contours, the solvability of the non-compact Douglas-Plateau problem seems should require more hypotheses than the compact case. We will see that in fact in our special cases discussed in Section 3, the same condition that ensures the existence of solutions for compact cases is also enough for non-compact cases.

It is known for more than one hundred years that for some non-compact boundaries we can find minimal annuli solving the corresponding "two contour" Douglas-Plateau problem. A classical example is a minimal annulus bounded by two parallel straight lines, a piece of one of Riemann's examples. Although a straight line is no longer a Jordan curve, it is a proper complete (convex) curve in $\mathbb{R}^3$.

In [fanghwang-f3], it was proved that if $L_1$ and $L_2$ are proper non-compact complete smooth planar convex curves in parallel planes with two symmetries, then there are two minimal annuli $A$ and $B$ such that $\partial A = \partial B = \Gamma$. Furthermore, $A$ and $B$ are foliated by strictly convex Jordan curves.

In Section 3, we prove the existence of various types of non-compact Douglas-Plateau problems. We will show that the symmetric conditions in [fanghwang-f3] is redundant, see Theorem 3.1. The proof of Theorem 3.1 is an application of Theorem 2.1, but we must first prove that there are barriers confining the approximate compact minimal annuli such that we can use Theorem 2.1, these barriers are established in Lemma 3.1.

In Theorem 3.2 we prove that there are at least two minimal annuli bounded by four straight lines $L_i$, $i = 1, 2, 3, 4$, such that $L_1 \subset P_{-1}$ and $L_2 \subset P_{-1}$ are parallel, $L_3 \subset P_1$ and $L_4 \subset P_1$ are parallel, but $L_1$ and $L_3$ are not parallel, if the distances between $L_1$ and $L_2$, and $L_3$ and $L_4$, are sufficiently large.

We will also prove that there are minimal annuli bounded by four parallel straight lines in two different parallel planes, if the boundary satisfies some kind of Hoffman-Meeks condition. See Theorem 3.3.

Acknowledgement. We sincerely thank our colleagues Chun-Chung Hsieh, Fei-Tsen Liang, and Derchyi Wu for helpful discussions. The first author also thanks the financial support of Australian Research Council and the hospitality of the Institute of Mathematics, Academia Sinica, Taipei, this research was started and finished there during visitings.
2. Curvature Estimates for Minimal Annuli.

**Lemma 2.1.** Let $A \subset S(t_1, t_2)$ be a minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P_{t_1}$, $L_2 \subset P_{t_2}$ are convex Jordan curves. Let $\pi$ be a plane perpendicular to the $xy$-plane, $P_\pi$ the perpendicular projection on $\pi$. Then $\Omega_\pi := \text{Int}(P_\pi(A)) = P_\pi(A) - (\bigcup_{i=1}^4 \Gamma_i^\pi)$ is a domain in $\pi$ bounded by $\Gamma_1^\pi = P_\pi(L_i)$, $i = 1, 2$, and $\Gamma_3^\pi$ and $\Gamma_4^\pi$, two curves connecting $\Gamma_1^\pi$ and $\Gamma_2^\pi$.

Let $|\Gamma_i^\pi|$ be the arc length of $\Gamma_i^\pi$, $i = 1, 2, 3, 4$. Then

\begin{equation}
|\Gamma_1^\pi| + |\Gamma_2^\pi| \geq |\Gamma_3^\pi| + |\Gamma_4^\pi|,
\end{equation}

\begin{equation}
|\Gamma_1^\pi| |\Gamma_2^\pi| > (t_2 - t_1)^2, \quad |\Gamma_1^\pi| + |\Gamma_2^\pi| > 2(t_2 - t_1).
\end{equation}

In particular, if $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$, take $\pi$ to be the $yz$-plane, then from $|\Gamma_i^\pi| \leq t_2 - t_1', i = 1, 2$, we have

\begin{equation}
\min\{ |\Gamma_1^\pi|, |\Gamma_2^\pi| \} > \frac{(t_2 - t_1)^2}{t_2' - t_1'},
\end{equation}

and

\begin{equation}
\min\{ |\Gamma_1^\pi|, |\Gamma_2^\pi| \} > \frac{(t_2 - t_1)^2}{t_2' - t_1'}.
\end{equation}

Furthermore, if $A \subset S(t_1, t_2) \cap C_R$, then

\begin{equation}
\min\{ |\Gamma_1^\pi|, |\Gamma_2^\pi| \} > \frac{(t_2 - t_1)^2}{4R^2} \max\{ |\Gamma_1^\pi|, |\Gamma_2^\pi| \},
\end{equation}

and

\begin{equation}
\min\{ |\Gamma_1^\pi|, |\Gamma_2^\pi| \} > \frac{(t_2 - t_1)^2}{4R}.
\end{equation}

**Proof.** Select a coordinate system such that $\pi$ is the $yz$-plane and $P_\pi(x, y, z) = (y, z)$.

For simplicity, write $\Omega_\pi$ as $\Omega$, $P_\pi$ as $P$, etc.

By Shiffman's first theorem in [fanghwang-sh1], every level curve $A \cap P_t$, $t_1 < t < t_2$, is a strictly convex Jordan curve. Thus $P(A \cap P_t)$ is a line.
Curvature estimates annuli and non-compact Douglas-Plateau problem

segment and $A \cap P_t$ consists of two graphs on $\mathcal{P}(A \cap P_t)$, $u^+(y, z) \geq u^-(y, z)$ such that $u^+(y, z) = u^-(y, z)$ if and only if $(y, z)$ is one of the two ends of $\mathcal{P}(A \cap P_z)$.

We orient the $yz$-plane such that $((0,1,0), (0,0,1))$ has positive orientation. Then $\Omega$ is bounded by $\Gamma_1 = \mathcal{P}(A \cap P_{t_1})$, $\Gamma_2 = \mathcal{P}(A \cap P_{t_2})$, and $\Gamma_3$ and $\Gamma_4$ consisting of the set of end points of $\mathcal{P}(A \cap P_t)$ such that for any $t$, if $(y_1, t) \in \Gamma_3$, $(y_2, t) \in \Gamma_4$, then $y_1 < y_2$, thus we may say that $\Gamma_3$ is the left side boundary, $\Gamma_4$ is the right side boundary.

Recall that $\pi$ is the $yz$-plane. Let $S^2$ be the unit sphere in $\mathbb{R}^3$ and $S^1_1 := S^2 \cap \pi$. Let $N : \text{Int}(A) \to S^2$ be the Gauss map and $p$ be any interior point of $A$. Since $\mathcal{P}(p) \in \Gamma_3 \cup \Gamma_4$ if and only if the tangent vector of $A \cap P_t$ at $p$ is in the direction $\pm(1,0,0)$ and since each $A \cap P_t$ is strictly convex, we see that $\mathcal{P}(p) \in \Gamma_3 \cup \Gamma_4$ if and only if $p \in N^{-1}(S^1_1)$. Since it is proved in [fanghwang-mwl] that $N$ is one-to-one and harmonic, we know that $N^{-1}(S^1_1)$ is smooth and its tangent directions are not pointed at $\pm(1,0,0)$, therefore $\Gamma_3 \cup \Gamma_4 = \mathcal{P}(N^{-1}(S^1_1))$ is smooth in its interior.

Note that for an interior point $\mathcal{P}(p)$ of $\Gamma_3$, the tangent line of $\mathcal{T}(p)$ at $\mathcal{P}(p)$ is where $T_p A$ is the tangent plane of $A$ at $p$. Since $A$ is minimal, there are points of $A$ at both sides of $T_p A$ in any neighbourhood of $p$ in $\mathbb{R}^3$. Thus since $\Gamma_3$ is the left side boundary of $\Omega$, there are points of $\overline{\Omega}$ on the left side of $\mathcal{P}(T_p A)$ in any neighbourhood of $\mathcal{P}(p)$, hence $\Gamma_3$ is locally on the left side of its tangent line at $\mathcal{P}(p)$. If $\Gamma_3$ is not convex, then there is another point $\mathcal{P}(p_1) \in \Gamma_3 \cap \mathcal{P}(T_p A)$. Thus there would be another point $\mathcal{P}(q)$ in the interior of $\Gamma_3$ and is located between $\mathcal{P}(p)$ and $\mathcal{P}(p_1)$ such that $\mathcal{P}(q)$ is on the left side of $\mathcal{P}(T_p A)$ and the distance from $\mathcal{P}(q)$ to $\mathcal{P}(T_p A)$ is a local maximum, thus $T_q A$ is parallel to $T_p A$. Therefore in a small neighbourhood of $q$ in $\mathbb{R}^3$ there are no points of $A$ on the left side of $T_q(A)$, a contradiction. Similarly we can prove that $\Gamma_4$ is convex.

We now prove that $\mathcal{P}(A) - \bigcup_{i=1}^{4} \Gamma_i$ is a domain and $\Omega = \mathcal{P}(A) - \bigcup_{i=1}^{4} \Gamma_i$. In fact if $\mathcal{P}(p) \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, then $p \notin N^{-1}(S^1_1)$, there is a neighbourhood $U$ of $p$ in $\mathbb{R}^3$ such that $U \cap N^{-1}(S^1_1) = \emptyset$ and $\mathcal{P}(U \cap A)$ is open in $\pi$ and $\mathcal{P}(U \cap A) \cap (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4) = \emptyset$, i.e., $\mathcal{P}(p) \in \text{Int}(\mathcal{P}(A))$, hence $\Omega = \text{Int}(\mathcal{P}(A)) = \mathcal{P}(A) - (\bigcup_{i=1}^{4} \Gamma_i)$ and $\partial \Omega = \bigcup_{i=1}^{4} \Gamma_i$. Finally, $\Omega$ is connected since $\mathcal{P}(A)$ is connected and $\bigcup_{i=1}^{4} \Gamma_i$ does not separate $\mathcal{P}(A)$. See Figure 1 below.
Since $\Gamma_1$ and $\Gamma_2$ are straight line segments, $\Omega$ is a domain with piecewise smooth boundary and $\partial \Omega$ has only four corner points $[\Gamma_1 \cap (\Gamma_3 \cup \Gamma_4)] \cup [\Gamma_2 \cap (\Gamma_3 \cup \Gamma_4)]$. Let $\nu$ be the outward unit normal vector of $\partial \Omega$.

Let us consider the graph defined by $u = u^+$. Then $u$ satisfies the minimal surface equation $\text{div} \, Tu = 0$, where $Tu = Du/\sqrt{1 + |Du|^2}$. The Gauss map is given by

$$N(x, y, z) = \frac{1}{\sqrt{1 + |Du|^2}} (1, -Du)(y, z), \quad (y, z) \in \Omega, \quad x = u(y, z).$$

If $\mathcal{P}(p)$ is an interior point of $\Gamma_3 \cup \Gamma_4$, then since $N(p) = (0, b, c)$, $\mathcal{P}(N(p)) = N(p)$. Since $N(p)$ is perpendicular to the tangent vector $v(p)$ along $N^{-1}(S_1)$ at $p$, $N(p)$ is also perpendicular to the tangent vector $\tilde{v}((\mathcal{P}(p)) = \mathcal{P}(v(p))$ along $\Gamma_3$ or $\Gamma_4$ at $\mathcal{P}(p)$, thus $\nu(\mathcal{P}(p)) = \mathcal{P}(N(p))$. We then have that

$$\nu(\mathcal{P}(p)) = \mathcal{P}(N(p)) = N(p) = (0, b, c).$$

Therefore, $\nu \cdot N = 1$ along the interior of $\Gamma_3$ and $\Gamma_4$. Thus by the expression of $N$ we see that $u$ satisfies the boundary condition

$$\nu \cdot Tu = -\nu \cdot N = -1, \quad \text{on} \quad \Gamma_3 \cup \Gamma_4.$$

First assume that $L_1$ and $L_2$ are smooth and strictly convex, then $Du$ exists on $\Gamma_1 \cup \Gamma_2$. We have

$$\int_{\partial \Omega} Tu \cdot \nu = \int_{\Omega} \text{div} \, Tu = 0.$$
Curvature estimates annuli and non-compact Douglas-Plateau problem

Now since $|Tu \cdot \nu| \leq 1$,

$$|\Gamma_3| + |\Gamma_4| = - \int_{\Gamma_3 \cup \Gamma_4} Tu \cdot \nu = \int_{\Gamma_1 \cup \Gamma_2} Tu \cdot \nu \leq |\Gamma_1| + |\Gamma_2|.$$ 

If $L_1$ or $L_2$ is only continuously convex, then by Shiffman's theorem $A \cap P_t$ is smooth and strictly convex for any $t \in (t_1, t_2)$. Consider $A \cap S(t_1 + \epsilon, t_2 - \epsilon)$, $0 < \epsilon < (t_2 - t_1)/2$, then (2.1) is true to the corresponding $\Gamma_i$'s of $A \cap S(t_1 - \epsilon, t_2 + \epsilon)$. Since $A$ is continuous up to boundary, letting $\epsilon \to 0$, we have proved (2.1).

To prove (2.2) we replace $\Gamma_3$ by the line segment $\Gamma_3'$ connecting the two end points of $\Gamma_3$, replace $\Gamma_4$ by the line segment $\Gamma_4'$ connecting the two end points of $\Gamma_4$. Then by comparison principle for minimal surfaces, $|\Gamma_3'| < |\Gamma_3|$, $|\Gamma_4'| < |\Gamma_4|$. Furthermore we replace the four-gon $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ by a trapezoid $\Gamma_1'' \cup \Gamma_2'' \cup \Gamma_3'' \cup \Gamma_4''$ such that $|\Gamma_3''| = |\Gamma_4''|$, and $|\Gamma_1''| = |\Gamma_1|$, $|\Gamma_2''| = |\Gamma_2|$ and $\Gamma_1''$ and $\Gamma_2''$ are parallel.

Note that $|\Gamma_3''| = |\Gamma_4''|$ and $2|\Gamma_3''| \leq |\Gamma_3| + |\Gamma_4| < |\Gamma_3| + |\Gamma_4| \leq |\Gamma_1| + |\Gamma_2|$. Now let $h = t_2 - t_1$ and without loss of generality suppose that $|\Gamma_1| \leq |\Gamma_2|$, thus $|\Gamma_2| = c|\Gamma_1|$, $0 < c \leq 1$, and $2|\Gamma_3''| < (1 + c)|\Gamma_1|$. Since

$$|\Gamma_3''| = \sqrt{\left(\frac{|\Gamma_1| - |\Gamma_2|}{4}\right)^2 + h^2} = \sqrt{\left(\frac{(1 - c)^2|\Gamma_1|^2}{4}\right) + h^2},$$

we obtain that

$$(1 - c)^2|\Gamma_1|^2 + 4h^2 < (1 + c)^2|\Gamma_1|^2.$$ 

Thus

$$(2.7) \quad h^2 < c|\Gamma_1|^2 = |\Gamma_1||\Gamma_2|.$$ 

By comparison principle for minimal surfaces,

$$2h \leq |\Gamma_3' + |\Gamma_4'| < |\Gamma_3| + |\Gamma_4| \leq |\Gamma_1| + |\Gamma_2|,$$

(2.2) is proved.

If $A \subset S(t_1, t_2) \cap S'(t_1', t_2')$, then $|\Gamma_1| \leq t_2' - t_1'$, so by (2.7)

$$c > \frac{h^2}{(t_2' - t_1')^2},$$

(2.3) is true. From (2.2),

$$|\Gamma_2| > \frac{h^2}{|\Gamma_1|} \geq \frac{h^2}{t_2' - t_1'},$$
(2.4) is true.

If $A \subset S(t_1, t_2) \cap C_R$, then $t'_2 - t'_1 \leq 2R$. By a rotation, we see that (2.5) and (2.6) are true.

The proof is now complete. \qed

Similar arguments as in Lemma 2.1 give us further information of the domain $\text{Int}(P(A))$.

**Lemma 2.2.** Suppose $A \subset S(t_1, t_2) \cap S'(-R, R)$ is a minimal annulus such that $\partial A = L_1 \cup L_2$, where $L_1 \subset P(t_1)$ and $L_2 \subset P(t_2)$ are convex Jordan curves. Let $\pi$ be the $yz$-plane and $P_\pi$ the perpendicular projection on $\pi$. Let $\Omega^\pi = \text{Int}(P_\pi(A))$ be the domain in $\pi$ and $\partial \Omega^\pi = \Gamma_1^\pi \cup \Gamma_2^\pi \cup \Gamma_3^\pi \cup \Gamma_4^\pi$ be as in Lemma 2.1. Let $p$ be any interior point of $A$ and $d = \text{dist}(p, \partial S(t_1, t_2))$. Then

$$d' := \max\{\text{dist}(P_\pi(p), \Gamma_3^\pi), \text{dist}(P_\pi(p), \Gamma_4^\pi)\} > \frac{(\sqrt{2} - 1)d^2}{R}.$$ 

**Proof.** Let us write $\Omega^\pi = \Omega$ etc.. Since $\Gamma_3$ and $\Gamma_4$ are compact, there are $q_1 \in \Gamma_3$ and $q_4 \in \Gamma_4$ such that $|P(p) - q_3| = \text{dist}(P(p), \Gamma_3), |P(p) - q_4| = \text{dist}(P(p), \Gamma_4)$. Connecting $P(p)$ and $q_3, P(p)$ and $q_4$ by line segments $l_3$ and $l_4$ to form two subdomains $\Omega_1$ and $\Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2 \cup l_3 \cup l_4$ and $\partial \Omega_1 = \Gamma_1 \cup l_3 \cup l_4 \cup \Gamma_3 \cup \Gamma_4, \partial \Omega_2 = \Gamma_2 \cup l_3 \cup l_4 \cup \Gamma_3' \cup \Gamma_4'$, where $\Gamma_i' \subset \Gamma_i, \Gamma_i'' \subset \Gamma_i$, for $i = 3, 4$. Note that $l_3$ and $l_4$ are contained in $\Omega$ except at end points on $\Gamma_3$ and $\Gamma_4$ so that $\Gamma_i' \cup \Gamma_i'' = \Gamma_i$, and $|\Gamma_i'| + |\Gamma_i''| = |\Gamma_i|$, for $i = 3, 4$. See Figure 2 below.

Let $p = (x, y, z), t_1 < z < t_2$. Let $T_1$ and $T_2$ be the line segments connecting $P(p)$ and the two ends of $\Gamma_1$, then

$$|T_1| + |T_2| < 2d' + |\Gamma_3'| + |\Gamma_4'|.$$ 

**Figure 2**
Curvature estimates annuli and non-compact Douglas-Plateau problem

Elementary geometry tells us that

$$|T_1| + |T_2| \geq 2\sqrt{\frac{|\Gamma_1|^2}{4} + (z - t_1)^2},$$

thus

$$2d' + |\Gamma_3'| + |\Gamma_4'| > 2\sqrt{\frac{|\Gamma_1|^2}{4} + (z - t_1)^2} = |\Gamma_1|\sqrt{1 + \frac{4(z - t_1)^2}{|\Gamma_1|^2}}.$$  

Similarly, we have

$$2d' + |\Gamma_3''| + |\Gamma_4''| > 2\sqrt{\frac{|\Gamma_2|^2}{4} + (t_2 - t_1)^2} = |\Gamma_2|\sqrt{1 + \frac{4(t_2 - t_1)^2}{|\Gamma_2|^2}}.$$  

Now since $d = \min\{(z - t_1), (t_2 - t_1)\} \leq (t_2 - t_1)/2$ and $\max\{|\Gamma_1|, |\Gamma_2|\} \leq 2R$, by (2.2) and (2.6),

$$\min\{|\Gamma_1|R, |\Gamma_2|R\} \geq \frac{|\Gamma_1||\Gamma_2|}{2} \geq \frac{(t_2 - t_1)^2}{2} \geq 2d^2,$$

thus

$$\max\left\{\frac{2d^2}{|\Gamma_1|R'}, \frac{2d^2}{|\Gamma_2|R}\right\} \leq 1.$$  

Note that $\sqrt{1 + x} \geq 1 + (\sqrt{2} - 1)x$ for $0 \leq x \leq 1$, we obtain

$$|\Gamma_1|\sqrt{1 + \frac{4(z - t_1)^2}{|\Gamma_1|^2}} \geq |\Gamma_1|\sqrt{1 + \frac{2d^2}{|\Gamma_1|R}} \geq |\Gamma_1|\left(1 + \frac{2(\sqrt{2} - 1)d^2}{|\Gamma_1|R}\right)$$

$$= |\Gamma_1| + \frac{2(\sqrt{2} - 1)d^2}{R},$$

$$|\Gamma_2|\sqrt{1 + \frac{4(t_2 - z)^2}{|\Gamma_2|^2}} \geq |\Gamma_2| + \frac{2(\sqrt{2} - 1)d^2}{R}.$$  

Therefore,

$$4d' + |\Gamma_3| + |\Gamma_4| > |\Gamma_1|\sqrt{1 + \frac{4(z - t_1)^2}{|\Gamma_1|^2}} + |\Gamma_2|\sqrt{1 + \frac{4(t_2 - z)^2}{|\Gamma_2|^2}}$$

$$\geq |\Gamma_1| + |\Gamma_2| + \frac{4(\sqrt{2} - 1)d^2}{R}.$$
But by (2.1)
\[|\Gamma_1| + |\Gamma_2| \geq |\Gamma_3| + |\Gamma_4|,\]
we obtain that
\[d' > \frac{(\sqrt{2} - 1)d^2}{R}.\]

From Lemma 2.2 we obtain an interior estimate of curvature.

**Proposition 2.1 (Interior Curvature Estimate).** If \( A \subset S(t_1, t_2) \cap C_R \) is a minimal annulus such that \( \partial A = L_1 \cup L_2 \), where \( L_1 \subset P_{t_1} \) and \( L_2 \subset P_{t_2} \) are convex Jordan curves, then there is an absolute constant \( C_0 > 0 \), such that

\[
|K(p)| \leq \frac{C_0 R^2}{d^4},
\]

where \( d = \text{dist}(p, \partial S(t_1, t_2)) \).

**Proof.** Let \( \mathcal{P}_\theta \) be the projection on a plane \( \pi_\theta \) with normal
\[v_\theta = (\cos \theta, \sin \theta, 0).\]
Then by Lemma 2.1 \( \text{Int}(\mathcal{P}_\theta(A)) \) is a domain bounded by \( \Gamma_i^\theta, i = 1, 2, 3, 4 \). We give an orthonormal basis in \( \pi_\theta, (e_1, e_2) \), such that \( (v_\theta, e_1, e_2) \) is an positive basis of \( \mathbb{R}^3 \). The convention is that under this coordinate system, \( \Gamma_3^\theta \) has a smaller first coordinate than that for \( \Gamma_4^\theta \). Define
\[f(\theta) = \text{dist}(\mathcal{P}_\theta(p), \Gamma_3^\theta) - \text{dist}(\mathcal{P}_\theta(p), \Gamma_4^\theta).\]
Then \( f \) is continuous and \( f(\theta + \pi) = -f(\theta) \). Thus there exists at least one \( \theta = \theta_0 \), such that \( f(\theta_0) = 0 \), i.e., \( \text{dist}(\mathcal{P}_{\theta_0}(p), \Gamma_3^{\theta_0}) = \text{dist}(\mathcal{P}_{\theta_0}(p), \Gamma_4^{\theta_0}) = d' \).

By Lemma 2.2, \( d' > D := (\sqrt{2} - 1)d^2/R \), where \( d = \text{dist}(p, \partial S(t_1, t_2)) \). Thus there is a round disk centred at \( \mathcal{P}(p) \) of radius greater than or equal to \( r = \min\{D, d\} \) contained in \( \Omega \) and \( p = (u(\mathcal{P}(p)), \mathcal{P}(p)) \) is on a minimal graph generated by \( u = u^+ \) or \( u^- \). A theorem of Osserman [fanghwang-os1], page 107, says that there is an absolute constant \( C > 0 \) such that
\[|K(p)| \leq \frac{C}{r^2}.\]
Curvature estimates annuli and non-compact Douglas-Plateau problem

Finally by (2.2) and the definition of $d$, $$4R \geq |\Gamma_{1}^{t_{0}}| + |\Gamma_{2}^{t_{0}}| > 2(t_{2} - t_{1}) \geq 4d > 4(\sqrt{2} - 1)d,$$
we know that $\min\{D, d\} = D$. Taking $C_{0} = \frac{C}{(\sqrt{2} - 1)^{2}}$, we complete the proof. \hfill \Box

Recall that the Gaussian curvature bound ensures compactness, as stated in the following summarization appeared in [fanghwang-ands],

**Lemma 2.3 (Compactness Lemma).** Let $\Omega$ be a bounded domain in a complete Riemannian 3-manifold $N^{3}$ and let $\{M_{i}\}$ be a sequence of minimally immersed surfaces in $\Omega$. Suppose there is a constant $C$ such that the Gauss curvature $K_{M_{i}}(x)$ satisfies $|K_{M_{i}}(x)| < C$ for all $i$. Then a subsequence of $\{M_{i}\}$ converges smoothly (in the $C^{k}$-topology, $k \geq 2$) to an immersed minimal surface $M_{\infty}$ (with multiplicity) in $\Omega$ and $|K_{M_{\infty}}(x)| \leq C$.

If each $M_{i}$ is embedded, then $M_{\infty}$ is also embedded.

**Theorem 2.1.** Let $\{A_{n}\}$ be a sequence of embedded minimal annuli contained in $S(t_{1}, t_{2}) \cap C_{R}$ such that $\partial A_{n} = L_{1}^{n} \cup L_{2}^{n}$, where $L_{1}^{n} \subset P_{t_{1}}, L_{2}^{n} \subset P_{t_{2}}$ are convex Jordan curves. Then there is a subsequence of $\{A_{n}\}$ converging to an embedded minimal annulus $A \subset S(t_{1}, t_{2}) \cap C_{R}$ such that $\partial A = L_{1} \cup L_{2}$, where $L_{1} \subset P_{t_{1}}, L_{2} \subset P_{t_{2}}$ are convex Jordan curves.

**Proof.** First observe that as uniformly bounded convex Jordan curves $\{L_{1}^{n}\}$ and $\{L_{2}^{n}\}$ are equicontinuous.

In fact, by Lemma 2.1, the arc lengths of $\{L_{1}^{n}\}$ and $\{L_{2}^{n}\}$ are at least $(t_{2} - t_{1})^{2}/R$. Since $\{L_{1}^{n}\}$ and $\{L_{2}^{n}\}$ are contained in $C_{R}$ and are convex, their arc lengths have an upper bound too.

Thus a subsequence of $\{L_{1}^{n}\}$, still denote by $\{L_{1}^{n}\}$, has a convergent arc length, i.e., $l_{n} := |L_{1}^{n}| \to l \geq (t_{2} - t_{1})^{2}/R$.

Since $L_{1}^{n}$ is convex, it has tangent almost everywhere. Let $\alpha_{n} : [0, l_{n}] \to P_{t_{1}}$ be the embedding of $L_{1}^{n}$ such that $|\alpha_{n}| = 1$ almost everywhere. Define $s : [0, l] \to [0, l_{n}]$ by $s(t) = l_{n}t/l$ and $\beta_{n} : [0, l] \to P_{t_{1}}$ by $\beta_{n}(t) = \alpha_{n}(s(t))$, then $|\beta_{n}| = l_{n}/l$ almost everywhere. Thus $\{\beta_{n}\}$ is equicontinuous and uniformly bounded.

By Ascoli-Arzela theorem, a subsequence of $\beta_{n}$, still denote by $\beta_{n}$, uniformly converges to a continuous mapping $\beta : [0, l]$. By (2.6) the image of $\beta$ cannot be a line segment or a point. Since each $\beta_{n}$ is one-to-one except at the two ends, $\beta$ defines a Jordan curve $L_{1}$. Let $D_{1}$ be the domain enclosed
by $L_1$ and $D_1^n$ the domain enclosed by $L_1^n$, then $D_1^n \to D_1$. Since each $D_1^n$ is convex, $D_1$ is convex. Hence $L_1$ is a convex Jordan curve. Similarly we can treat $\{L_2^n\}$ and we may assume that a subsequence of $\{\partial A_n\}$, still denote by $\{\partial A_n\}$, has a limit $L_1 \cup L_2$, and $L_1 \subset P_t$ and $L_2 \subset P_t$ are convex Jordan curves.

Let $\epsilon_m \to 0$ as $m \to \infty$ and $A_m^n = A_n \cap S(t_1 + \epsilon_m, t_2 - \epsilon_m) \subset C_R$. Then by Proposition 2.1 for fixed $m$, $\{A_m^n\}$ has a uniform curvature bound, thus by Lemma 2.3, there is a subsequence $\{A_m^n\}$ converging in $S(t_1 + \epsilon_m, t_2 - \epsilon_m) \cap C_R$. Thus the subsequence $\{A_m^n\}$ converges in the interior of $S(t_1, t_2)$ to an open minimal surface $A$.

Since each $A_m^n \cap P_t$ is a strictly convex Jordan curve in $C_R \cap P_t$ and $\{A_m^n\}$ converges in $C^k$ topology, $k \geq 2$, $A \cap P_t$ must be a convex Jordan curve, thus $A$ is a minimal annulus.

Now $\lim_{m \to \infty} \partial A_m^n = L_1 \cup L_2$. We only need prove that $\partial A = L_1 \cup L_2$.

In fact, let $P_\pi$ be the perpendicular projection on a plane $\pi$ perpendicular to the $xy$-plane and $\Omega_m^\pi = \text{Int}(P_\pi(A_m^n))$, then by Lemma 2.1, $A_m^n$ consists of two simply connected graphs $G_{m}^+$ and $G_{m}^-$ with continuous boundary. Similarly, $A$ consists of two simply connected graphs $G_{+}$ and $G_{-}$ on a domain in the plane $\pi$.

Let $X^+_m : D \to \mathbb{R}^3$ and $X^-_m : D \to \mathbb{R}^3$ be conformal embeddings from the closed unit disk $D$ for $G_{+}^m$ and $G_{-}^m$ respectively, normalized such that $X^+_m(p_i) = q^+_m$, for three fixed points $p_i \in \partial D$ and $\lim_{m \to \infty} q^+_m = q_i \in L_1$. Similarly we require the three points condition for $X^-_m$.

Since $\partial A_m^n$ consists of two convex Jordan curves in $\partial S(t_1, t_2) \cap C_R$, the arc lengths of $\partial A_m^n$ is uniformly bounded, hence by isoperimetric inequality, the areas of $A_m^n$, hence of $G_{+}^m$, $G_{-}^m$, are uniformly bounded. Now since $X^+_m$ are conformal, $\int_D |DX^+_m|^2$ is uniformly bounded. Therefore, by Courant-Lebesgue Lemma (see Theorem 3 on page 238 of [fanghwang-dhkw]), $X^+_m$ is uniformly continuous and converges on $D$.

Similarly we can prove that $X^-_m$ converges on $D$. Since $G_{+}^m \to G_{+}$ and $G_{-}^m \to G_{-}$, and $\partial(G_{+}^m \cup G_{-}^m) \cap \partial S(t_1, t_2) = \partial A_m^n$ converges to $L_1 \cup L_2$, and the plane $\pi$ was arbitrary, we see that $\partial A = L_1 \cup L_2$ and $\overline{A}$ is continuous up to boundary and $\partial A = L_1 \cup L_2$.

Remark 2.1. The argument used in the proof that $\overline{A}$ is continuous actually gives an alternative proof that a subsequence of $\{A_n\}$ converges to a minimal annulus without curvature estimates and Lemma 2.3, i.e., via the uniformly boundedness of $\int_D |DX_m|^2$ and the Courant-Lebesgue Lemma, with the help of Lemma 2.1 and 2.2.
With Theorem 2.1, we can give a generalization of a theory dealing with smooth convex boundary developed by Hoffman and Meeks in [fanghwang-hm9], and Meeks and White in [fanghwang-mw1], to the continuous convex boundary case.

**Theorem 2.2.** Suppose $D_1$ and $D_2$ are two open disks lying on parallel planes, and suppose their boundaries $L_1$ and $L_2$ are continuous convex Jordan curves.

1. If $A'$ is a connected non-planar compact (maybe branched) minimal surface such that $\partial A' \subset D_1 \cup D_2$, then there exist at least two embedded compact minimal annuli $A$ and $B$, $\partial A = \partial B = L_1 \cup L_2$.

2. If $A$ is stable and has the property that for any disks $D' \subset D_1$ and $D'' \subset D_2$ with continuous boundaries, if there is a connected compact (maybe branched) minimal surface $N$ such that $\partial N = \partial D' \cup \partial D''$, then $N$ is contained in the solid $V$ bounded by $A \cup D_1 \cup D_2$. In particular, if $A \neq N$, then $\text{Int}(A) \cap \text{Int}(N) = \emptyset$. On the other hand, $B$ is unstable and $\text{Int}(B) \cap \text{Int}(N) \neq \emptyset$.

3. If merely $\partial A' \subset \overline{D_1} \cup \overline{D_2}$, then there exists at least one embedded minimal annulus $C$ such that $\partial C = L_1 \cup L_2$. Such a $C$ is almost stable in the sense that the first eigenvalue of the second variation of $C$ is larger than or equal to zero. Let $N$ be a connected compact (maybe branched) minimal surface such that $\partial N = \partial D' \cup \partial D''$, then $N$ is contained in the solid $V$ bounded by $C \cup D_1 \cup D_2$. In particular, if $C \neq N$, then $\text{Int}(C) \cap \text{Int}(N) = \emptyset$.

4. Furthermore, the symmetry groups of $A$ and $B$, or $C$, are the same as the symmetry group of $L_1 \cup L_2$.

**Proof.** If $L_1$ and $L_2$ are smooth, or in the cases of existence of $A$ and $C$ in conclusions 1 or 3 for merely continuous $L_1$ and $L_2$, the theorem is a combination of Theorem 1.1, 1.2 of [fanghwang-hm9], and Lemma 2.1 of [fanghwang-mw1], with "exact" replacing "at least" in 1.

In general for the case of $L_1$ and $L_2$ are merely continuous, let $\text{Symm}(L_1 \cup L_2)$ be the symmetry group of $L_1 \cup L_2$. We can construct smooth convex Jordan curves $L_1^n$ and $L_2^n$ such that $\lim_{n \to \infty} L_1^n = L_1$, $\lim_{n \to \infty} L_2^n = L_2$, $\text{Symm}(L_1^n \cup L_2^n) = \text{Symm}(L_1 \cup L_2)$, and $L_1$ and $L_2$ are enclosed in the disks bounded by $L_1^n$ and $L_2^n$ respectively. Therefore there are corresponding minimal annuli $A_n$ and $B_n$ etc., satisfying all the properties stated in
the theorem, and being contained in some $C_R$. Using Theorem 2.1 we get our limiting minimal annuli $A$ and $B$ bounded by $L_1 \cup L_2$, they satisfy all properties stated in the theorem, especially, $A \neq B$. 

An immediate corollary of Lemma 2.1 and Theorem 2.2 is

**Corollary 2.1.** If $\Sigma \subset S(t_1, t_2)$ is a connected compact non-planar minimal surface (maybe branched) such that $\partial \Sigma \subset P_t \cup P_t$. Let $L(\theta, t)$ be the length of $\Sigma \cap P_t$ projected on the plane with normal $(\cos \theta, \sin \theta, 0)$, for $(\theta, t) \in [0, \pi] \times [t_1, t_2]$. Then for any $t_1 \leq s < t \leq t_2$ and $\theta \in [0, \pi]$, 

$$L(\theta, s) + L(\theta, t) > 2(t - s), \quad L(\theta, s) L(\theta, t) > (t - s)^2.$$

Thus

$$L(\theta, t) > \max \left\{ \frac{(t - t_1)^2}{L(\theta, t_1)}, \frac{(t_2 - t)^2}{L(\theta, t_2)} \right\}, \quad t \in (t_1, t_2).$$

In particular, if $\partial \Sigma \subset S(t_1, t_2) \cap S'(t_1', t_2')$, then $t_2' - t_1' > t_2 - t_1$. Furthermore, if $\Sigma \subset S(t_1, t_2) \cap C_R$, then

$$L(\theta, t) > \max \left\{ \frac{(t - t_1)^2}{2R}, \frac{(t_2 - t)^2}{2R} \right\}, \quad t \in (t_1, t_2).$$

**Proof.** First observe that by comparison principle for minimal surfaces $\Sigma \cap P_t$ is a variety without isolated points, hence we can apply Theorem 2.2. Thus we can construct a minimal annulus $A(\theta) \subset S(s, t)$ with convex Jordan curves boundary in $P_s$ and $P_t$ respectively, which enclose $\Sigma \cap P_s$ and $\Sigma \cap P_t$. Then by Lemma 2.1

$$|\Gamma_1| + |\Gamma_2| > 2(t - s), \quad |\Gamma_1||\Gamma_2| > (t - s)^2,$$

where $\Gamma_1$ and $\Gamma_2$ are the projection of $\partial A(\theta)$. Note that we can make $\partial A(\theta)$ such that

$$L(\theta, s) = |\Gamma_1|, \quad L(\theta, t) = |\Gamma_2|.$$

Then all the conclusions are trivial by Lemma 2.1. 

**Corollary 2.2.** Let $c, d, \delta_1,$ and $\delta_2$ be positive constant numbers. If $\Sigma$ is a connected compact non-planar minimal surface (maybe branched) such that $\partial \Sigma = L_1 \cup L_2$ and

$$L_1 \subset D'_1 := \left\{ (x, y, z); -\delta_1 \leq x - \frac{c}{d} z \leq \delta_1, \ z \leq 0 \right\},$$
Curvature estimates annuli and non-compact Douglas-Plateau problem

\[ L_2 \subset D' := \{(x, y, z); -\delta_2 \leq x - \frac{c}{d}z \leq \delta_2, z \geq d\}, \]

then

\[ 2 \max\{\delta_1, \delta_2\} > \sqrt{c^2 + d^2}. \tag{2.9} \]

Furthermore, if

\[ L_1 \subset D_1 := \{(x, y, 0); -\delta_1 \leq x \leq \delta_1\}, \]

\[ L_2 \subset D_2 := \{(x, y, d); -\delta_2 \leq x - c \leq \delta_2\} \]

then

\[ \delta_1 + \delta_2 > \sqrt{c^2 + d^2}. \tag{2.10} \]

Proof. Let \( X : M \to \mathbb{R}^3 \) be a conformal parametrization of \( \Sigma \) and consider the function

\[ \phi = X_1 - \frac{c}{d}X_3, \]

then \( \phi \) is harmonic. Thus let \( R := \max\{\delta_1, \delta_2\} \), by maximum principle for harmonic functions,

\[ \Sigma \subset \{(x, y, z); -R \leq x - \frac{c}{d}z \leq R\}. \]

Since \( \partial\Sigma \) is contained in \( \{z \leq 0\} \cup \{z \geq d\} \), \( \Sigma \cap S(0, d) \) is a minimal surface whose boundary \( L_1' \cup L_2' \) satisfies that

\[ L_1' \subset \{(x, y, 0); -R \leq x \leq R\}, \quad L_2' \subset \{(x, y, d); -R \leq x - c \leq R\}. \]

Let \( C_1 \subset P_0, C_2 \subset P_d \) be convex Jordan curves such that they enclose \( L_1' \) and \( L_2' \) respectively and

\[ C_1 \subset \{(x, y, 0); -R \leq x \leq R\}, \quad C_2 \subset \{(x, y, d); -R \leq x - c \leq R\}. \]

By Theorem 2.2, there is a minimal annulus \( A \) bounded by \( C_1 \) and \( C_2 \). Let \( \pi \) be the \( xz \)-plane and \( \Gamma_i, i = 1, 2, 3, 4, \) be as defined in Lemma 2.1. By comparison principle for minimal surfaces, \( \Gamma_3 \) and \( \Gamma_4 \) are not line segments. Then by Lemma 2.1,

\[ 4R \geq |\Gamma_1| + |\Gamma_2| \geq |\Gamma_3| + |\Gamma_4| > |l_3| + |l_4|, \]
where \( l_3 \) and \( l_4 \) are line segments such that \( \Gamma_1 \cup \Gamma_2 \cup l_3 \cup l_4 \) consists of the boundary of a convex four-gon in the \( xz \)-plane. Elementary calculation shows that
\[
|l_3| + |l_4| = 2\sqrt{c^2 + d^2}.
\]
Similarly we can prove (2.10) by requiring that
\[
C_1 \subset \{(x,y,0); -\delta_1 \leq x \leq \delta_1\}, \quad C_2 \subset \{(x,y,d); -\delta_2 \leq x - c \leq \delta_2\}.
\]
Then we have
\[
2(\delta_1 + \delta_2) \geq |\Gamma_1| + |\Gamma_2| \geq |\Gamma_3| + |\Gamma_4| > |l_3| + |l_4|,
\]
and
\[
|l_3| + |l_4| = \sqrt{(\delta_2 - \delta_1 + c)^2 + d^2} + \sqrt{(\delta_1 - \delta_2 + c)^2 + d^2} \geq 2\sqrt{c^2 + d^2}.
\]

To establish the existence of solutions to non-compact Douglas-Plateau problem with four parallel straight lines as boundary in Section 3, we need another curvature estimate for minimal annuli.

**Proposition 2.2.** Let \( A \subset S(t_1, t_2) \cap S'(t'_1, t'_2) \) be a compact embedded minimal annulus such that \( \partial A = L_1 \cup L_2 \), where \( L_1 \subset \mathcal{P}_t \), \( L_2 \subset \mathcal{P}_t \) are \( C^2 \) convex Jordan curves. Let \( E > 0 \) such that \( |\kappa(p)| \leq E \) for any \( p \in \partial A \), where \( \kappa \) is the planar curvature. Then there is a constant \( C_1 > 0 \) only depending on \( t_2 - t_1, t'_2 - t'_1 \) and \( E \), such that
\[
|K(p)| \leq C_1,
\]
where \( K(p) \) is the Gaussian curvature of \( A \).

*Proof.* The proof is a generalization of the proof in [fanghwang-mrl] of a special case of this Proposition, the estimates in Lemma 2.1 enables us to make this generalization.

By a homothety, we can assume that \( t_1 = -1, t_2 = 1 \). By a translation we can assume that \( t'_1 = -R, t'_2 = R \) for some \( R > 0 \).

If the Proposition is not true, then there are minimal annuli \( B_n \subset S(-1,1) \cap S'(-R,R) \) such that \( \partial B_n \) consists of two convex Jordan curves.
Curvature estimates annuli and non-compact Douglas-Plateau problem

in $P_{-1}$ and $P_1$, whose planar curvatures are bounded by $E$, and $\exists p_n \in B_n$ such that

$$-C_n := K_{B_n}(p_n) \leq K_{B_n}(p), \quad \forall p \in B_n; \quad \lim_{n \to \infty} K_{B_n}(p_n) = -\infty.$$ 

Let $\tilde{B}_n = B_n - p_n := \{ p \in \mathbb{R}^3, p + p_n \in B_n \}$. Note that $p_n = (x_n, y_n, z_n)$, $-R \leq y_n \leq R$, $-1 \leq z_n \leq 1$. By a rotation if necessary, we may assume that $-1 \leq z_n \leq 0$. So that $\tilde{B}_n = B_n - p_n \subset S(-1,2) \cap S'(-2R,2R)$ contains the origin, and $\partial\tilde{B}_n \subset P_{-1-z_n} \cup P_{1-z_n}$.

Let $\tilde{B}_n = \sqrt{C_n}B_n$ be the homothety of $\tilde{B}_n$ and $K_{\tilde{B}_n}$ be the Gaussian curvature of $\tilde{B}_n$, then $|K_{\tilde{B}_n}| \leq 1$. Let $D_m$ be the ball centred at origin with radius $m$. Then by Lemma 2.3 a subsequence of $\{\tilde{B}_n\}$ converges in $D_m$. By a diagonal argument, a subsequence of $\{\tilde{B}_n\}$, still denote by $\{\tilde{B}_n\}$, converges to an embedded minimal surface $\tilde{M}$ in $\mathbb{R}^3$. $\tilde{M}$ is not a plane, since it has a point (the origin) with Gaussian curvature $-1$.

Since the Gauss map $N : \tilde{B}_n \to S^2$ is one-to-one and $N \neq \pm(0,0,1)$, [fanhwang-mw1], we have $\int_{\tilde{B}_n} K_{\tilde{B}_n} dA > -4\pi$, see [fanhwang-os1]. It forces that $\tilde{M}$ must have total curvature at least $-4\pi$.

Since the boundaries of $\tilde{B}_n$ are on $P\sqrt{C_n(-1-z_n)}$ and $P\sqrt{C_n(1-z_n)}$ and $C_n \to \infty$ as $n \to \infty$, if $\tilde{M}$ has a boundary, it must be that $\lim_{n \to \infty} \sqrt{C_n}(-1-z_n)$ exists in $\mathbb{R}$ or $\lim_{n \to \infty} \sqrt{C_n}(1-z_n)$ exists in $\mathbb{R}$. Since $-1 \leq z_n \leq 0$, it must be $\lim_{n \to \infty} \sqrt{C_n}(-1-z_n) = t_0$ exists and $\partial\tilde{M} \subset P_{t_0}$. Since $B_n \cap P_t$ are convex and have uniform planar curvature bound, $\partial\tilde{M} = \lim_{n \to \infty} \partial\tilde{B}_n \cap \sqrt{C_n(1-z_n)}$ exists.

Since $\partial B_n$ has uniform planar curvature bound $E$, it turns out the planar curvature of $\partial\tilde{M}$ is bounded by $E/\sqrt{C_n} \to 0$, hence $\partial\tilde{M}$ must be a straight line $l$ if $\partial\tilde{M} \neq \emptyset$.

If $\partial\tilde{M} \neq \emptyset$, then rotating $\tilde{M}$ around $l = \partial\tilde{M}$ by $\pi$ degree, we get a complete minimal surface without boundary, its total curvature is at least $-8\pi$, and it contains a straight line. But such a surface does not exist by classification, see for example [fanhwang-lo]. Thus $\partial\tilde{M} = \emptyset$.

It forces that $\tilde{M}$ must be a catenoid since that it is non-flat completely embedded without boundary, and its total curvature is at least $-4\pi$.

Thus $\tilde{M} \cap P_0$ is a circle and since $\tilde{B}_n \to \tilde{M}$, the length of $\tilde{B}_n \cap P_0$ should be bounded, i.e., there is an $F > 0$, such that

$$|\tilde{B}_n \cap P_0| \leq F.$$ 

But since $B_n \cap P_t$ is a convex Jordan curve for each $t \in (-1,1)$, and $B_n \cap S(-1,1) \cap S'(-R,R)$, recalling that $-1 \leq z_n \leq 0$ and applying (2.4) to
 Remark 2.2. Since we assume that $\partial A$ is $C^2$, the estimate is a global one, not just interior. But we can only prove by contradiction that the curvature of $A \subset S(t_1, t_2) \cap S'(t'_1, t'_2)$ has an a priori bound. The ideal proof is to give an a priori estimate of curvature bound for $A \subset S(t_1, t_2) \cap S'(t'_1, t'_2)$ explicitly involving the boundary planar curvature of $\partial A$, the height $t_2 - t_1$, and the width $t'_2 - t'_1$. Such a concrete estimate will be useful in many other cases.

Then we have another compactness theorem.

**Theorem 2.3.** If $\{A_n\} \subset S(t_1, t_2) \cap S'(t'_1, t'_2)$ is a sequence of minimal annuli such that $\partial A_n = L^n_1 \cup L^n_2$, where $L^n_1 \subset P_1$, $L^n_2 \subset P_2$ are $C^2$ convex Jordan curves with uniform planar curvature bound $E$ and $\lim_{n \to \infty} L^n_1 = L_1$, $\lim_{n \to \infty} L^n_2 = L_2$, then there is a subsequence of $\{A_n\}$ which converges to an embedded minimal surface $A$ such that $\partial A = L_1 \cup L_2$.

**Proof.** Let $D_r$ be the ball centred at origin with radius $r$, then $\mathbb{R}^3 = \bigcup_{m=1}^{\infty} D_m$. Since $\lim_{n \to \infty} L^n_1 = L_1$, $\lim_{n \to \infty} L^n_2 = L_2$, for $m$ and $n$ large enough, $A_n \cap D_m \neq \emptyset$. By Proposition 2.2 $\{A_n\}$ has a uniform curvature bound, so Lemma 2.3 applied in $D_m$ gives a convergent subsequence $\{A_{n_m} \cap D_m\}$, then $\{A_{n_m}\}$ is a subsequence of $\{A_n\}$ which converges to an embedded minimal surface $A$ in any compact set. Since $\lim_{n \to \infty} L^n_1 = L_1$, $\lim_{n \to \infty} L^n_2 = L_2$, we have $\partial A = L_1 \cup L_2$. □

**Remark 2.3.** Note that the limit minimal surface $A$ is not necessarily an annulus. In fact, it may be even not connected.

### 3. Applications to Non-Compact Douglas-Plateau Problem

First let us define various boundaries for which we want to solve the related Douglas-Plateau problem.

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ be a properly embedded complete convex curve, and let $L = \alpha(\mathbb{R})$. Suppose that $L$ is not a straight line, then $\mathbb{R}^2 - L$ has two components, only one of them is convex.
Definition 3.1 (Standard Boundary). We call $\Gamma := L_1 \cup L_2$ a standard boundary if:

- $L_1 \subset P_1$ and $L_2 \subset P_1$ are two continuously embedded, proper, complete, non-compact, non-flat convex curves.

- Let $Y_1 \subset P_{-1}$ and $Y_2 \subset P_1$ be the two convex domains bounded by $L_1$ and $L_2$ respectively. Let $\tilde{Y}_i \subset P_0$ and $\tilde{L}_i \subset P_0$ be the perpendicular projections of $Y_i$ and $L_i$, for $i = 1, 2$. Then $\tilde{Y}_1 \cap \tilde{Y}_2$ is a bounded convex domain.

- There is a connected compact non-planar (maybe branched) minimal surface $\Sigma$ such that $\partial \Sigma \subset Y_1 \cup Y_2$.

Remark 3.1. By Corollary 2.2, the last condition of Definition 3.1 implies that $\tilde{Y}_1 \cap \tilde{Y}_2 \neq \emptyset$. We will call this condition **H-M condition** to $Y_1 \cup Y_2$, it first appeared in [fanghwang-hm9] by Hoffman and Meeks.

Let $D_r \subset P_0$ be the disk centred at $(0,0)$ with radius $r$. It is well-known that if $r$ is large enough and $D_r \subset \tilde{Y}_1 \cap \tilde{Y}_2$, then there is a piece of catenoid $C$ such that $\partial C \subset Y_1 \cup Y_2$. Hence the H-M condition is satisfied.

Our first existence theorem is:

**Theorem 3.1.** Let $\Gamma$ be a standard boundary. Then there exist two embedded minimal annuli $A$ and $B$ such that $\partial A = \partial B = \Gamma$. The minimal annuli $A$ and $B$ have the following properties:

1. For each $t \in (-1, 1)$, $P_t \cap A$ and $P_t \cap B$ are strictly convex Jordan curves.

2. $\operatorname{Int}(A) \cap \operatorname{Int}(B) = \emptyset$.

3. Let $N$ be a connected compact non-planar (maybe branched) minimal surface such that $\partial N \subset \tilde{Y}_1 \cup \tilde{Y}_2$, then

$$\operatorname{Int}(A) \cap \operatorname{Int}(N) = \emptyset, \quad B \cap N \neq \emptyset.$$ 

4. $A$ and $B$ have the same symmetry groups as that of $\Gamma$. 

Remark 3.2. If we change the last condition in the definition of standard boundary so that $\partial \Sigma \subset \overline{Y}_1 \cup \overline{Y}_2$, then there is at least one embedded minimal annulus $C$ such that $\partial C = \Gamma$. Furthermore, $C$ behaves just like $A$ in the sense that they satisfy the same properties in 1, 3 and 4 of Theorem 3.1.

A limit case of standard boundary is that of *straight line boundary*.

**Definition 3.2 (Straight Line Boundary).** A straight line boundary is as follows:

- $\Gamma = \bigcup_{i=1}^{4} L_i$, where $L_i$ are straight lines such that $L_1$ and $L_2$ are contained in $P_{-1}$ and parallel, while $L_3$ and $L_4$ are contained in $P_1$ and parallel. But $L_1$ and $L_3$ are not parallel.

- Let $Y_1 \subset P_{-1}$ be the open strip bounded by $L_1$ and $L_2$, $Y_2 \subset P_1$ be the open strip bounded by $L_3$ and $L_4$. Then $Y_1 \cup Y_2$ satisfies the H-M condition.

**Remark 3.3.** Note that since $L_1$ and $L_3$ are not parallel, the perpendicular projections $\tilde{Y}_1$ and $\tilde{Y}_2$ of $Y_1$ and $Y_2$ have bounded intersection.

**Theorem 3.2.** Let $\Gamma$ be a straight line boundary. Then there exist two embedded minimal annuli $A$ and $B$ such that $\partial A = \partial B = \Gamma$. The minimal annuli $A$ and $B$ have all the properties stated in Theorem 3.1.

**Remark 3.4.** The Remark 3.2 also applies to the straight line boundary.

Now let us consider parallel straight lines. It is also an limit case of a standard boundary. Indeed if we consider a standard boundary such that there is a straight line $L \subset P_0$ which intersects $\tilde{L}_1$ and $\tilde{L}_2$ in exactly one point respectively. We may change $L_1$ and $L_2$ such that the single intersection points of $L$ with $\tilde{L}_1$ and $\tilde{L}_2$ go to infinity in opposite directions, and $\tilde{L}_1$ and $\tilde{L}_2$ both break into two straight lines parallel to $L$. Thus we give the following definition of a *parallel line boundary*:

**Definition 3.3 (Parallel Boundary).** A parallel boundary is as follows:

- $\Gamma = \bigcup_{i=1}^{4} L_i$, where $L_i$ are parallel straight lines such that $L_1$ and $L_2$ are contained in $P_{-1}$, while $L_3$ and $L_4$ are contained in $P_1$. The strip bounded by $L_1$ and $L_3$ and the strip bounded by $L_2$ and $L_4$ are disjoint.
Curvature estimates annuli and non-compact Douglas-Plateau problem

- Let $Y_1 \subset \text{annulus bounded by } L_1 \text{ and } L_2$, $Y_2 \subset P_1 \text{ be the open strip bounded by } L_3 \text{ and } L_4 \text{ in } P_1$. Then $Y_1 \cup Y_2$ satisfies the H-M condition.

**Remark 3.5.** By Corollary 2.1, a necessary condition for $\Gamma = \bigcup_{i=1}^{4} L_i$ being a parallel boundary is that the product of the widths of the strips $Y_1$ and $Y_2$ is larger than 4.

Given a parallel boundary $\Gamma$ as above, we always assume that the straight lines $L_i$ are parallel to the $x$-axis.

A special case of parallel boundary is a lattice boundary (it defines a lattice in the $yz$-plane):

**Definition 3.4 (Lattice Boundary).** Let $\Gamma$ be a parallel boundary. Let $p_i$ be the intersection points of the $L_i$ with the $yz$-plane. If the $p_i$'s are the vertices of a parallelogram, then we call the parallel boundary a lattice boundary.

Let $F$ be the parallelogram with $p_i$ as vertices. Then we select the bisectrice point (the intersection of the two diagonals) of $F$ as the origin of $\mathbb{R}^3$.

Our third existence theorem is the following:

**Theorem 3.3.** Let $\Gamma$ be a parallel boundary, then there exists an embedded minimal annulus $\mathcal{D}$ such that $\partial \mathcal{D} = \Gamma$. Also $\mathcal{D}$ satisfies:

1. For $-1 < t < 1$, $P_t \cap \mathcal{D}$ are strictly convex Jordan curves.
2. Let $N$ be a connected compact non-planar (maybe branched) minimal surface such that $\partial N \subset Y_1 \cup Y_2$, then $\mathcal{D} \cap N \neq \emptyset$.
3. $\mathcal{D}$ is invariant under the reflection $(x, y, z) \rightarrow (-x, y, z)$.
4. If $\Gamma$ is a lattice boundary, then $\mathcal{D}$ is invariant under the rotation of angle $\pi$ around the $x$-axis.

**Remark 3.6.** In [fanghwang-mrl], Meeks and Rosenberg gave a proof of the existence of $\mathcal{D}$ with a lattice boundary in order to construct doubly periodic minimal surfaces.
3.1. Proof of Theorem 3.1.

The idea for the proof of Theorem 3.1 is to construct sequences of compact minimal annuli \( \{A_n\} \) and \( \{B_n\} \) whose convex boundaries approaching the given \( \Gamma \).

Now we approach \( L_1 \) and \( L_2 \) by convex Jordan curves \( L_1^1 \subset P_1 \) and \( L_2^2 \subset P_1 \) such that for any \( R > 0 \), there is an \( N_R > 0 \) such that whenever \( n > N_R \),

\[
(L_1^1 \cup L_2^2) \cap C_R = (L_1 \cup L_2) \cap C_R.
\]

Let \( D_1^1 \) and \( D_2^2 \) be the disks bounded by \( L_1^1 \) and \( L_2^2 \). We can make \( L_1^1 \) and \( L_2^2 \) such that

\[
\partial \Sigma \subset D_1^1 \cup D_2^2, \quad D_1^1 \subset D_1^{n+1}, \quad D_2^2 \subset D_2^{n+1},
\]

where \( \Sigma \) is the surface in the H-M condition.

By Theorem 2.2 there are two minimal annuli \( A_n \) and \( B_n \), such that \( \partial A_n = \partial B_n = L_1^1 \cup L_2^2 \).

To prove that there are convergent subsequences of \( \{A_n\} \) and \( \{B_n\} \), we need the following lemma. The proof of this technical lemma is quite involved. In order not to interrupt the main argument, at this moment let us assume the lemma is true.

First let us fix some more notations. Let \( Q_a := \{(x,y,z) \in \mathbb{R}^3; x = a\} \), \( H^+_a := \{(x,y,z) \in \mathbb{R}^3; x \geq a\} \), \( H^-_a := \{(x,y,z) \in \mathbb{R}^3; x \leq a\} \). Let \( W_a = \{(x,y,z); -a \leq x \leq a\} \). Denote the \( xz \)-plane by \( P_0 \).

Lemma 3.1. Let \( Y_1 \) and \( Y_2 \) be as defined in Definition 3.1. Let \( A \subset S(-1,1) \) be a compact minimal surface. Suppose that \( \partial A = C_1 \cup C_2 \) such that \( C_1 \subset Y_1 \), \( C_2 \subset Y_2 \). Then we can choose coordinates \( (x,y,z) \) such that for any \( a > 0 \), there is an \( S(a) > 0 \) such that \( A \cap W_a \subset S'(-S(a),S(a)) \). We can choose that if \( a > b > 0 \), then \( S(a) \geq S(b) \). And for any \( t \in (-1,1) \), there is an \( R(t) > 0 \) such that

\[
A \cap P_t \subset C_{R(t)}.
\]

Moreover, we can make that \( R(t) = R(-t) \) and \( R(s) \leq R(t) \) whenever \( |s| < |t| \).

As proved in the proof of Lemma 3.1, we can choose coordinates \( (x,y,z) \) of \( \mathbb{R}^3 \), such that if \( a > 0 \) is large enough, we have \( Q_a \cap L_2 = \emptyset \) and \( Q_{-a} \cap L_1 = \emptyset \).
Curvature estimates annuli and non-compact Douglas-Plateau problem

\( \emptyset \) or, \( Q_a \cap L_1 = \emptyset \) and \( Q_{-a} \cap L_2 = \emptyset \), but not both. Without loss of generality, we assume that it is the former and keep this convention in this paper.

By Lemma 3.1, \( A_n \cap S(-t, t) \subset C_{R(t)} \) and \( B_n \cap S(-t, t) \subset C_{R(t)} \), hence for each \( t \in (0, 1) \), they are uniformly bounded. By Shiffman’s first theorem [fanghwa-shl], \( A_n \cap P_t \) and \( B_n \cap P_t \) are strictly convex Jordan curves.

Now we can use Theorem 2.1 to prove subsequences of \( \{A_n\} \) and \( \{B_n\} \) converge to \( A \) and \( B \) in the interior of \( S(-1, 1) \). In fact, there are subsequences of \( \{A_n\} \) and \( \{B_n\} \) which are convergent to embedded compact minimal annuli \( A_{tm} \subset S(-t_m, t_m) \) and \( B_{tm} \subset S(-t_m, t_m) \) in \( S(-t_m, t_m) \) for any \( t_m \), where \( t_m \to 1 \) as \( m \to \infty \). By a diagonal argument we see that subsequences of \( \{A_n\} \) and \( \{B_n\} \) converge to embedded minimal surfaces \( A \) and \( B \). Since for each \( s \in (-t_m, t_m) \), \( A_n \cap P_s \) and \( B_n \cap P_s \subset C_{R(tm)} \) is uniformly bounded convex Jordan curves and the convergence is smooth, \( A \cap P_s \) and \( B \cap P_s \) are convex Jordan curves. Since for \( t \in (-1, 1) \), \( A \cap P_t \) and \( B \cap P_t \) are convex, by Shiffman’s theorem again, we know that \( A \cap P_t \) and \( B \cap P_t \) are strictly convex, hence \( A \) and \( B \) are minimal annuli.

Still denote these subsequences by \( \{A_n\} \) and \( \{B_n\} \), we only need prove that \( A \) and \( B \) are continuous up to boundary and \( \partial A = \partial B = \Gamma \).

Now by Lemma 2.1 \( A_n \) consists of two simply connected minimal graphs over a domain \( \Omega_n \subset P_0' \), say \( G_n^+, G_n^- \).

Since \( A_n = G_n^+ \cup G_n^- \), \( A_n \cap Q_a \) and \( A_n \cap Q_{-a} \) are the unions of two graphs respectively, hence they are simple curves. Similarly, \( A \cap Q_a \) and \( A \cap Q_{-a} \) are simple curves. Thus \( G_n^+ \cap W_a, G_n^- \cap W_a, G^+ \cap W_a, \) and \( G^- \cap W_a \) are all simply connected.

Let \( \Omega_a = \Omega \cap W_a \), then \( \Omega_a \) is bounded and has piecewise smooth boundary as proved in Lemma 2.1, hence \( \partial \Omega_a \) has finite length. Also by Lemma 3.1, \( A \cap W_a \) is also bounded, thus we know that \( G^+ \cap W_a, G^- \cap W_a \) have finite area.

Let \( D \) be the closed unit disk in \( \mathbb{C} \) and \( X_n : D \to \mathbb{R}^3 \) be a conformal embedding of \( G_n^+ \cap W_a \) such that for three fixed points \( p_i \in \partial D \), \( X_n(p_i) = q_i \), where \( q_i \in \partial G_n^+ \cap (L_1 \cup L_2) \cap W_a, \) \( i = 1, 2, 3 \). Since \( \partial A_n \to L_1 \cup L_2 \), this is always possible. Since \( G_n^+ \to G^+ \), the areas of \( G_n^+ \cap W_a \) are uniformly bounded and by the conformality of \( X_n \), \( \int_D |DX_n|^2 \) are uniformly bounded.

By Courant-Lebesgue Lemma, \( G_n^+ \cap W_a \cap X_n(D) \) converges to \( (G^+ \cup (L_1 \cup L_2)) \cap W_a \) and is continuous up to boundary. Similar argument for \( G^- \) also holds. Thus we see that \( \partial(A \cap W_a) \cap (P_{-1} \cup P_1) = (L_1 \cup L_2) \cap W_a \) for all \( a > 0 \) large enough. Moreover, it is clear that \( \partial A \subset P_{-1} \cup P_1 \). Therefore \( \partial A = L_1 \cup L_2 \) and similarly \( \partial B = L_1 \cup L_2 \) and they are continuous up to boundary.

Let \( N \) be a connected non-planar compact (maybe branched) minimal
surface such that \( \partial N \subset \overline{Y}_1 \cup \overline{Y}_2 \). Let \( V_n \) be the solid bounded by \( A_n \cup D_n^1 \cup D_n^2 \), and \( V \) be the solid bounded by \( A \cup \overline{Y}_1 \cup \overline{Y}_2 \). We know that \( N \subset V_n \) and \( \text{Int}(A_n) \cap \text{Int}(N) = \emptyset \). Since \( V_n \to V, N \subset V \). By the comparison principle for minimal surfaces, either \( A = N \) or \( \text{Int}(A) \cap \text{Int}(N) = \emptyset \). Since \( N \) is compact and \( A \) is not compact, \( \text{Int}(N) \cap \text{Int}(A) = \emptyset \). In particular, \( \text{Int}(\Sigma) \cap \text{Int}(A) = \emptyset \).

Since \( B_n \cap N \neq \emptyset \), \( \lim_{n \to \infty} B_n = B \), and \( N \) is compact, we know that \( B \cap N \neq \emptyset \).

In particular, \( \text{Int}(\Sigma) \cap \text{Int}(B) \neq \emptyset \). Thus \( A \neq B \).

Let \( V_n' \) be the solid bounded by \( B_n \cup D_n^1 \cup D_n^2 \), and \( V' \) be the solid bounded by \( B \cup \overline{Y}_1 \cup \overline{Y}_2 \). Then since \( V_n' \subset V_n, \lim_{n \to \infty} V_n = V \), and \( \lim_{n \to \infty} V_n = V', \)

\( V' \subset V \). By the comparison principle for minimal surfaces, we have that \( \text{Int}(A) \cap \text{Int}(B) = \emptyset \).

By Theorem 2.2, we can construct the approaching sequences \( \{A_n\} \) and \( \{B_n\} \) such that they have the same symmetry groups as that of \( \Gamma \), thus the limits, \( A \) and \( B \), have the same symmetry groups as that of \( \Gamma \).

The proof of Theorem 3.1 is complete except that we still need prove Lemma 3.1.

The idea for the proof of Lemma 3.1 is to construct various barriers and use the comparison principle for minimal surfaces. To establish these barriers, let us quote a Lemma in [fanghwang-chm].

**Lemma 3.2 (Lemma 4 in [fanghwang-chm]).** Let \( L_0 \) consist of two non-collinear rays emanating from the origin in the plane \( P_0 \), and let \( L_1 \) be their vertical translation into the plane \( P_1 \). Then \( L_0 \cup L_1 \) is the boundary of a unique properly embedded minimal surface contained in the convex hull of \( L_0 \cup L_1 \). This minimal surfaces is a graph over an infinite strip and hence is simply connected.

**Remark 3.7.** As pointed out in the proof of Lemma 3.2 in [fanghwang-chm], \( A \) is asymptotic to a flat strip as it diverges to infinity.

**Proof of Lemma 3.1.** First we claim that \( \tilde{Y}_1 \cup \tilde{Y}_2 \) is contained in an unbounded domain \( \Omega \) with four rays as boundary. And if we adjust the angles between the boundary rays of \( \Omega \) we can assume that there is a straight line \( l \) contained in \( \Omega \). Note that this implies that \( \mathbb{R}^3 - \overline{\Omega} \) consists of two unbounded convex domains.

In fact, since \( \tilde{Y}_1 \cap \tilde{Y}_2 \) is convex and bounded, there are exactly two unbounded components on \( \tilde{L}_1 - \tilde{L}_2 \) and \( \tilde{L}_2 - \tilde{L}_1 \). Say \( \alpha_i \subset \tilde{L}_1 - \tilde{L}_2, \beta_i \subset \)
\( \bar{L}_1 - \bar{L}_2, i = 1, 2, \) are the four unbounded components. Take a point on each of these unbounded components, say \( p \in \alpha_1, p' \in \alpha_2, q \in \beta_1, q' \in \beta_2. \) Since \( \bar{L}_i \) are convex, there are straight lines passing through these four points such that \( \bar{L}_i \) are on the same side of these lines. Denote these lines by \( l_p, l_{p'}, l_q, \) and \( l_{q'}. \)

If \( l_p \) and \( l_{p'} \) intereset, then \( \bar{Y}_1 \) is contained in a wedge \( \Omega_1 \) (a convex domain bounded by two rays issuing from one point) bounded by rays in \( l_p \) and \( l_{p'}. \) If \( l_p \) and \( l_{p'} \) are parallel, then since \( \bar{L}_1 \) is non-compact convex, \( \bar{Y}_1 \) is contained in the strip \( S \) bounded by \( l_p \) and \( l_{p'}. \) Since \( \bar{L}_1 \) is non-flat, we can find a wedge \( \Omega_1 \) such that \( \bar{Y}_1 \subset \Omega_1. \) Similarly, there is a wedge \( \Omega_2 \supset \bar{Y}_2. \)

Since \( \bar{Y}_1 \cap \bar{Y}_2 \) is compact, by parallel translations or, if necessary, vary the angles of the wedges, we can assume that \( \partial \Omega_1 \cap \partial \Omega_2 = \{P, Q\}. \) Take \( \Omega \) to be the domain bounded by rays in \( \partial \Omega_1 \cup \partial \Omega_2 \) issuing from \( P \) and \( Q, \) then clearly \( \bar{Y}_1 \cup \bar{Y}_2 \subset \Omega. \) By enlarging \( \Omega \) if necessary, we can assume that the straight line \( l \) which is the bisector of the line segment \( PQ \) is contained in \( \Omega. \) Take \( l \) as the \( x \)-axis, then the coordinate system of \( (x, y, z) \) satisfies that if \( a > 0 \) large enough, we have \( Q_a \cap L_2 = \emptyset \) and \( Q_{-a} \cap L_1 = \emptyset \) or, \( Q_a \cap L_1 = \emptyset \) and \( Q_{-a} \cap L_2 = \emptyset, \) but not both.

Denote the two components of \( \partial \Omega \) by \( l^1 \) and \( l^2. \)

Now let \( A \subset S(-1,1) \) be a compact minimal surface such that \( \partial A = C_1 \cup C_2, C_1 \subset Y_1, C_2 \subset Y_2. \) Then \( C_1 \cap L_2 = \emptyset \) and \( C_2 \cap L_1 = \emptyset. \) See Figure 3 below.

![Figure 3](image-url)
We first prove that for any \( a > 0 \), there is an \( S(a) > 0 \) such that \( A \cap W_a \subset S'(-S(a), S(a)) \).

We use the barrier in Lemma 3.2.

\[
(3.3) \quad l^1 \cap (\bar{Y}_1 \cup \bar{Y}_2) = \emptyset.
\]

Now let us parallel translate \( l^1 \) along the \( z \)-axis into \( P_{-1} \) and \( P_1 \) and call them \( l_{1,-1}^1 \) and \( l_1^1 \) respectively. By Lemma 3.2, there is a minimal graph \( M \) bounded by \( l_{1,-1}^1 \) and \( l_1^1 \). By (3.3), \( \partial A \cap (l_{1,-1}^1 \cup l_1^1) = \emptyset \). By the comparison principle for minimal surfaces, \( A \cap M = \emptyset \). Thus for any \( a \neq 0 \), there is an \( R_a^2 > 0 \) such that \( A \cap W_a \) is contained in the half-space \( \{y \leq -R_a^2\} \). Since \( M \) is continuous, we can select \( R_a^2 \) such that it is nondecreasingly continuous respect to \( a \).

Similarly, using \( l^2 \) to make minimal graph \( M' \), we can find an \( R_a^2 > 0 \) such that \( \mathcal{P}(A) \cap W_a \) is contained in the half-space \( \{y \geq -R_a^2\} \) and \( R_a^2 \) is nondecreasingly continuous respect to \( a \). Take \( S(a) = \max\{R_a^1, R_a^2\} \), we have proved that \( A \cap W_a \subset S'(-S(a), S(a)) \) and \( S(a) \) is nondecreasingly continuous respect to \( a \).

Next, the minimal graph \( M \) bounded by \( l_{1,-1}^1 \cup l_1^1 \) is contained in the convex hull of \( l_{1,-1} \cup l_1 \). Let \( \mathcal{P} \) be the perpendicular projection on the \( xy \)-plane. Then \( \mathcal{P}(M) \) is contained in the convex domain bounded by \( l^1 \). Similarly, \( \mathcal{P}(M') \) is contained in the convex domain bounded by \( l^2 \). Therefore, \( \mathcal{P}(M) \cap (\bar{Y}_1 \cup \bar{Y}_2) = \emptyset \) and \( \mathcal{P}(M') \cap (\bar{Y}_1 \cup \bar{Y}_2) = \emptyset \). Parallel moving \( l^1 \) and \( l^2 \) to \( L^1 \) and \( L^2 \) along the \( \pm y \)-direction and denote the non-convex domain bounded by \( L^1 \cup L^2 \) by \( \Omega' \), then by the Remark 3.7 we can make

\[
(3.4) \quad (\mathcal{P}(M) \cup \mathcal{P}(M')) \cap \Omega' = \emptyset, \quad \mathcal{P}(A) \subset \Omega'.
\]

Still denote \( L^1 \) by \( l^1 \), \( L^2 \) by \( l^2 \).

Let \( l_u = \{(x, y, z); x = u, z = 0\} \). We have proved and made the convention that \( \bar{L}_1 \cap l_{-d} = \emptyset, \bar{L}_2 \cap l_d = \emptyset \) for \( d > 0 \) large enough.

Let \( \Omega_1 \) be the unbounded convex domain bounded by part of \( l^1 \cup l^2 \cup l_d \).

Then \( \partial \Omega_1 = \Gamma_1 \cup \Gamma_2 \), where \( \Gamma_1 \subset l^1 \cup l^2 \) consists of two rays issued from the intersections of \( l_d \) with \( l^1 \cup l^2 \), and \( \Gamma_2 \) is the line segment in \( l_d \) between the two intersection points.

Similarly, let \( \Omega_2 \) be the unbounded convex domain bounded by part of \( l^1 \cup l^2 \cup l_{-d} \). Then \( \partial \Omega_2 = \Gamma_3 \cup \Gamma_4 \), where \( \Gamma_3 \subset l^1 \cup l^2 \) consists of two rays issued from the intersections of \( l_{-d} \) with \( l^1 \cup l^2 \), and \( \Gamma_4 \subset l_{-d} \) is a line segment. See Figure 4 below.

We construct two minimal graphs \( G_1 \) and \( G_2 \) by solving Dirichlet problems for minimal surface equation on the domains \( \Omega_1 \) and \( \Omega_2 \), with the
Curvature estimates annuli and non-compact Douglas-Plateau problem

boundary data:

\[
\begin{cases}
  u_1 = -1 & \text{on } \Gamma_1 \\
  u_1 = \infty & \text{on } \Gamma_2; \\
  u_2 = 1 & \text{on } \Gamma_3 \\
  u_2 = -\infty & \text{on } \Gamma_4.
\end{cases}
\]

Figure 4

Such minimal graphs exist and are unique. For example, let \( \Omega_a \) be the domain \( \{(x,y) \in \Omega_1; x < a\} \), then \( \partial \Omega_a = \Gamma_2 \cup \Gamma_a \cup \Gamma_1^\alpha \), where \( \Gamma_a \) is the segment in \( l_a \) between the intersection points of \( l_a \) with \( \Gamma_1 \); \( \Gamma_1^\alpha \) is \( \Gamma_1 \cap \{x < a\} \).

Then clearly when \( a \) large enough, we have

\[ |\Gamma_1^\alpha| > |\Gamma_2| + |\Gamma_a|. \]

By a theorem of Jenkins and Serrin [fanghwang-js], there is a unique solution \( u_a \) for the Dirichlet problem of minimal surface equation with the boundary value:

\[
\begin{align*}
  u_a &= -1 \text{ on } \Gamma_1^\alpha, \\
  u_a &= \infty \text{ on } \Gamma_2 \cup \Gamma_a.
\end{align*}
\]
Now if $c > a$, then by comparison principle for minimal surfaces, $u_c < u_a$ in $\Omega_a$. For any $p \in \Omega_1$, if $a$ is large enough then $p \in \Omega_a$. Let $c_n \to \infty$ then $\{u_{c_n}\}$ is a decreasing sequence, by the monotone convergence theorem [fanghwang-j], there is a $u = \lim_{n \to \infty} u_{c_n}$ which solves our Dirichlet problem. By the maximum principle in [fanghwang-nil], page 256, $\sup_{x \geq d+1} |u_1(x, y)|$ and $\sup_{x \leq -d-1} |u_2(x, y)|$ are finite. Hence by Theorem 3 of [fanghwang-hwa1], the solution is unique.

An important property for $G_i$ is that $\lim_{(x,y) \to \infty} u_i(x, y) = \mp 1$, $i = 1, 2$, see [fanghwang-mr2], Theorem 3.1. Thus by (3.4), for any $t \in (-1, 1)$, there is an $R_1(t) > 0$ such that

$$G_1 \cap P_t \subset H_{R_1(t)}^-, \quad G_2 \cap P_t \subset H_{R_1(t)}^+.$$  

By the construction of $G_i$, $\partial A \cap \partial G_i = \emptyset$ and $\mathcal{P}(A \cap H_d^+) \subset \mathcal{P}(G_1) = \Omega_1$, $\mathcal{P}(A \cap H_{-d}) \subset \mathcal{P}(G_2) = \Omega_2$. By the comparison principle for minimal surfaces, we conclude that $A \cap G_i = \emptyset$, for $i = 1, 2$. Since $\mathcal{P}(A) \cap (\bar{Y}_1 \cap \bar{Y}_2) \neq \emptyset$ and $\mathcal{P}(G_i) \cap (\bar{Y}_1 \cap \bar{Y}_2) = \emptyset$, $i = 1, 2$, by the maximum principle $A \cap P_t \subset H_{R_1(t)}^- \cap H_{-R_1(t)}^+$, $R_1(t) \geq d$, for any $t \in (-1, 1)$. Then we have $S(R_1(t)) > 0$, such that $A \cap P_t \subset W_{R_1(t)} \cap S'(-S(R_1(t)), S(R_1(t)))$.

Take $R(t) = \max\{\sqrt{2}R_1(t), \sqrt{2}S(R_1(t))\}$, then $A \cap P_t \subset C_{R(t)}$. Since $G_1$ and $G_2$ are continuous, $R_1(t)$ is nondecreasingly continuous respect to $t$. Thus $R(t)$ is nondecreasingly continuous respect to $t$.

The proof of Lemma 3.1 is complete. 

**3.2. Proof of Theorem 3.2.**

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1. The key point is that we can confine approaching minimal annulus sequences $\{A_n\}$ and $\{B_n\}$ by four minimal barriers, i.e., the minimal graphs as in Lemma 3.2, using the eight rays issuing form the four intersection points of $\bar{L}_i$, $i = 1, 2, 3, 4$. Thus Lemma 3.1 is true for a straight line case. The other arguments are either exactly the same as the arguments in the proof of Theorem 3.1 or are slightly variations of them.

**3.3. Proof of Theorem 3.3.**

We start with $L_n^1$ being the convex curve consisting of the two line segments $(L_1 \cup L_2) \cap C_n$, and two round arcs smoothly connecting the two pairs of end points, note that we use the same arcs up to a translation or reflection. Similarly define $L_n^2$. By this construction, $L_n^1$ and $L_n^2$ are invariant under the
Curvature estimates annuli and non-compact Douglas-Plateau problem

reflection about the $yz$-plane and have uniformly bounded boundary planar curvature.

If $\Gamma$ is a lattice boundary, then we can make $L_n^2$ to be the image of $L_n^1$ under the rotation of angle $\pi$ around the $x$-axis.

By Theorem 2.2, there are embedded minimal annuli $B_n$ bounded by $L_n^1 \cup L_n^2$ such that $\text{Int}(B_n) \cap \text{Int}(\Sigma) \neq \emptyset$, where $\Sigma$ is the minimal surface in the H-M condition of Definition 3.3. Moreover, $B_n$ has the same symmetry group as that of $L_n^1 \cup L_n^2$. Note that there is an $R > 0$ such that $B_n \subset S'(-R, R)$ and by the construction, we see that $L_n^1 \cup L_n^2$ have uniform boundary planar curvature bound and $\lim_{n \to \infty} L_n^1 \cup L_n^2 = \Gamma$. Hence we can apply Theorem 2.3 to conclude that there is a subsequence of $\{B_n\}$ which converges to an embedded minimal surface $\mathcal{D}$ such that $\partial \mathcal{D} = \Gamma$.

Since $B_n$'s satisfy the symmetry conditions in Theorem 3.3, $\mathcal{D}$ also satisfies the symmetry conditions.

It remains to prove that $\mathcal{D}$ is an annulus and satisfies the other properties claimed in Theorem 3.3.

To establish that $\mathcal{D}$ is an annulus, it is sufficient to prove that $\mathcal{D} \cap P_t$ is a strictly convex Jordan curve, for any $-1 < t < 1$.

Since $B_n \cap \Sigma \neq \emptyset$ and $\Sigma$ is compact, we see that $\mathcal{D} \cap \Sigma \neq \emptyset$.

We observe that $P_t \cap \mathcal{D}$ is the smooth limit of a sequence of strictly convex Jordan curves. If $P_t \cap \mathcal{D}$ is compact, then it must be a convex Jordan curve. Thus the only thing left to be proved is that $\mathcal{D} \cap P_t$ must be compact.

Note that $P_t \cap \mathcal{D}$ is invariant under the reflection about the $yz$-plane. If $P_t \cap \mathcal{D}$ is not compact, then clearly $P_t \cap \mathcal{D} \cap \{x = \pm s\} \neq \emptyset$, for any $s > 0$, otherwise $P_t \cap \mathcal{D}$ is bounded. This forces that $P_t \cap \mathcal{D}$ consists of two graphs generated by uniformly bounded functions $y_1(x, t)$ and $y_2(x, t)$, $y_1 \geq y_2$, $-\infty < x < \infty$, or in the limit case, $y_1 \equiv y_2$. Let $y_1^n$ and $y_2^n$ be the functions defined by $P_t \cap B_n$, then $y_1^n$ is concave and $y_2^n$ is convex. Since $y_1(x) = \lim_{n \to \infty} y_1^n$, $y_2(x) = \lim_{n \to \infty} y_2^n$, in $C^k$ topology, $\forall k > 0$, $y_1^n(x) = y_2^n(-x)$, $i = 1, 2$, $y_1$ is concave with maximum $y_1(0, t)$ and $y_2$ is convex with minimum $y_2(0, t)$. By Lemma 2.1

$$|y_1(0, t) - y_2(0, t)| > \max \left\{ \frac{(1-t)^2}{4R^2}, \frac{(t+1)^2}{4R^2} \right\} \geq \frac{1}{4R^2},$$

for $-1 < t < 1$. If $P_t \cap \mathcal{D}$ is not compact, then $y_1$ and $y_2$ are both defined on $(-\infty, \infty)$, and $y_1$ is concave, $y_2$ is convex. Thus $P_t \cap \mathcal{D}$ is the union of two parallel straight lines which are parallel to the $x$-axis.

Consider the arc length functions

$L(t) = \text{arc length of } P_t \cap \mathcal{D}, \quad \text{and} \quad L_n(t) = \text{arc length of } P_t \cap B_n.$
Then $L_n(t) \to L(t)$ when $n \to \infty$. By [fanghwang-osh], $L_n$ is convex with respect to $t$. We see that if for some $-1 < t_0 < 1$, $L(t_0) = \infty$, then $L_n(t_0) \to \infty$ and there is a closed interval containing $t_0$ in $(-1,1)$ such that on which $L_n(t) \to \infty$. Thus we can assume that the set

$$\{ t \in (-1,1) \mid P_t \cap \mathcal{D} \text{ is not compact} \}$$

contains an open interval in $(-1,1)$. Hence there are $-1 < t_1 < t_2 < 1$ such that $P_t \cap \mathcal{D}$ consists of two straight lines parallel to the $x$-axis for $t_1 < t < t_2$. Thus $\mathcal{D}$ is contained in a ruled minimal surface. Since the only non-planar ruled minimal surface is the Helicoid and its generating straight lines are not parallel, $\mathcal{D}$ is contained in two planes $P_1$ and $P_2$ such that $L_1 \cup L_2 \cup L_3 \cup L_4 \subset P_1 \cup P_2$. Now since $\mathcal{D}$ is embedded, we have that $P_1$ is the plane containing $L_1 \cup L_3$, $P_2$ is the plane containing $L_2 \cup L_4$. But by the comparison principle for minimal surfaces, $\Sigma \cap (P_1 \cup P_2) = \emptyset$, and $\Sigma \cap \mathcal{D} = \emptyset$, a contradiction. This contradiction proves that $P_t \cap \mathcal{D}$ is compact.

As before, once we know that $P_t \cap \mathcal{D}$ is convex for $-1 < t < 1$, then it is strictly convex by quoting Shiffman’s first theorem.

The remaining properties claimed in Theorem 3.3 can be proved in the same way as in the proof of Theorem 3.1.

The proof of Theorem 3.3 is complete now. \qed

**Remark 3.8.** There are other cases of boundaries such that similar lemma as Lemma 3.1 is true, thus with H-M condition, there are two solutions. For example, $L_1$ becomes two parallel straight lines and $\tilde{Y}_1 \cap \tilde{Y}_2$ is bounded, etc.

Is the solution in Theorem 3.3 unique? Similar questions can be asked. For example, are there other solutions besides the two given in Theorem 3.1 and 3.2? is there a theory about non-compact smooth convex boundary as that established by Meeks and White in [fanghwang-mwl]? Furthermore, can the theory of Meeks and White, together with its generalization to non-compact cases (if it is generalizable), be generalized to the continuous case? We would like to know the answers.

**References.**


Curvature estimates annuli and non-compact Douglas-Plateau problem


Yi Fang and Jenn-Fang Hwang

Received September 2, 1998.

Australian National University
Canberra, ACT 0200, Australia

and

Academia Sinica
Nankang, Taipei, Taiwan 11529
Republic of China
E-mail addresses: yi@maths.anu.edu.au
                MAJFH@ccvax.sinica.edu.tw