

Aleksandrov reflection and geometric evolution of hypersurfaces

BENNETT CHOW AND ROBERT GULLIVER

Abstract. Consider a compact embedded hypersurface Γ_t in \mathbb{R}^{n+1} which moves with speed determined at each point by a function $F(\kappa_1, \dots, \kappa_n, t)$ of its principal curvatures, for $0 \leq t < T$. We assume the problem is degenerate parabolic, that is, that $F(\cdot, t)$ is nondecreasing in each of the principal curvatures $\kappa_1, \dots, \kappa_n$. We shall show that for $t > 0$ the hypersurface Γ_t satisfies local *a priori* Lipschitz bounds outside of a convex set determined by Γ_0 and lying inside its convex hull. Our method is the parabolic analogue of Aleksandrov's method of moving planes [A1], [A2], [A3], [A4], [AVo].

The flow of a smooth hypersurface may be generalized to the evolution of a closed set Γ_t described as the level set of a continuous function u_t which satisfies in the viscosity sense a degenerate parabolic PDE defined by F for $0 \leq t < \infty$, [ES], [CGG]. It has recently been noted that this level-set flow, even when starting from a smooth hypersurface Γ_0 , may develop a nonempty interior after the evolving hypersurface collides with itself or develops singularities [BP], [AIC], [AVe], [K]. We shall prove that the same local Lipschitz bounds as in the hypersurface case hold for the inner and outer boundaries of Γ_t .

As an application, we give some new results about $1/H$ flow for non-star-shaped hypersurfaces, which was recently investigated by Huisken and Ilmanen [HI]. We prove existence and asymptotic roundness, in the Lipschitz sense, for "extended" viscosity solutions in \mathbb{R}^{n+1} . In contrast, the evolving hypersurfaces given in [HI], which were used to prove a version of the Penrose conjecture, are solutions of a non-local variational problem, valid in general asymptotically flat Riemannian manifolds.

1. Main results.

Let Γ_t be a embedded connected C^2 hypersurface in \mathbb{R}^{n+1} evolving in the direction of its unit inward normal vector N with speed

$$(1.1) \quad V = F(\kappa_1, \dots, \kappa_n, t)$$

a function of the principal curvatures $\kappa_1 \leq \dots \leq \kappa_n$ and time $t \in [0, T)$ where $0 < T \leq \infty$. Here we assume that $F_t = F(\cdot, t)$ is uniformly Lipschitz continuous for each $t \geq 0$ and F is non-decreasing in each κ_i , which ensures the evolution equation is degenerate parabolic (weakly parabolic). Our sign convention is such that κ_i is positive on the standard sphere. We will not require that F_t be symmetric.

Choose a unit vector $\nu \in S^n$ and a number $\lambda \in \mathbb{R}$. Define the hyperplane

$$P = P^\lambda(\nu) = \{x \in \mathbb{R}^{n+1} : \langle x, \nu \rangle = \lambda\}$$

and the half-spaces

$$\begin{aligned} H_+ &= H_+^\lambda(\nu) = \{x : \langle x, \nu \rangle > \lambda\} \\ H_- &= H_-^\lambda(\nu) = \{x : \langle x, \nu \rangle < \lambda\}. \end{aligned}$$

We have the orthogonal reflection in $P^\lambda(\nu)$

$$\sigma^\lambda(\nu)(x) = x - 2(\langle x, \nu \rangle - \lambda)\nu.$$

For simplicity, when the choice of $\lambda \in \mathbb{R}$ and $\nu \in S^n$ are clear, we denote $x^* = \sigma^\lambda(\nu)(x)$.

Any embedded connected C^2 hypersurface Γ is the boundary of a connected bounded open set Ω , $\Gamma = \partial\Omega$.

Definition 1. We say that λ is *admissible* for Γ with respect to ν if

$$\sigma^\lambda(\nu)(\Gamma \cap H_-^\lambda(\nu)) \subset \bar{\Omega}.$$

We have the following Aleksandrov reflection-type result.

Theorem 2. *Let Γ_t be a family of embedded C^2 hypersurfaces evolving by equation (1.1).*

1. *If λ is admissible for Γ_0 with respect to ν , then λ is admissible for Γ_t with respect to ν for all $t \in [0, T)$.*

2. If Γ_0 is compact and $(-\infty, \lambda)$ is admissible for Γ_0 with respect to all ν in the neighborhood of ν_0 , then $\Gamma_t \cap H_-^\lambda(\nu_0)$ is a C^2 graph in the direction ν_0 with locally bounded gradient; this local bound is independent of t and F .

For an example where the first part of Theorem 2 is valid but not the second part, see the example of section 4.2 below.

Let $\lambda_{\max}(\nu)$ be the supremum of all λ such that $(-\infty, \lambda]$ is admissible for Γ_0 with respect to ν . For convenience, we shall write $H_{\pm}^{\max}(\nu)$ for the half-space $H_{\pm}^{\lambda_{\max}(\nu)}(\nu)$. According to Theorem 2, Part (2), it will also be important to consider the upper semicontinuous envelope $\bar{\lambda}$ of $\lambda_{\max} : S^n \rightarrow \mathbb{R}$.

Corollary 3. *Let K be the intersection of the closures of the halfspaces $H_+^{\bar{\lambda}(\nu)}(\nu)$, for $\nu \in S^n$. Then the part of the hypersurface Γ_t lying outside K satisfies local gradient bounds, which depend only on Γ_0 .*

Theorem 4. *Let $\Lambda = \Lambda(\Gamma_0) = \sup_{\nu \in S^n} [-\lambda_{\max}(\nu)]$. Then for all $t \in [0, T)$, the part of the hypersurface Γ_t lying outside the ball $\bar{B}_\Lambda(0)$ is a radial graph $r = \psi_t(\theta)$, $\theta \in S^n$, with gradient estimate*

$$|D\psi_t(\theta)| \leq \frac{r\Lambda}{\sqrt{r^2 - \Lambda^2}}.$$

For example, if Γ_0 is close in the Lipschitz norm to a round sphere, then the constant Λ of Theorem 4 is a small positive number, and for all $t > 0$, the part of Γ_t lying outside of $\bar{B}_\Lambda(0)$ will be Lipschitz close to a round sphere. In fact, $\log(\psi_t/\Lambda)$ satisfies Lipschitz estimates governed by its lower bound.

As another example, if Γ_t leaves every compact set, it must become round in the sense that after rescaling to get $|\Omega_t| = |B_1|$, Γ_t converges to the unit sphere in the Lipschitz norm. Note that in the literature, such convergence to a round sphere has been proven in the C^2 sense for certain strictly parabolic problems by first showing the curvatures all converge to 1. Our methods, in contrast, work for degenerate parabolic problems and do not require (nor imply) the pointwise convergence of curvatures.

The results above extend to generalized solutions (**viscosity solutions**). In particular, let Γ_0 be an embedded compact (but not necessarily connected) C^0 hypersurface bounding a bounded open (not necessarily con-

nected) set Ω_0 . Let $K > 0$. Define $u_0 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ by

$$u_0(x) = \begin{cases} \text{dist}(x, \Gamma_0) \wedge K & \text{if } x \in \mathbb{R}^{n+1} \setminus \Omega_0 \\ -\text{dist}(x, \Gamma_0) & \text{if } x \in \Omega_0 \end{cases},$$

which is the signed distance function to Γ_0 (cut off by the constant K .) Then $\Gamma_0 = \{x : u_0(x) = 0\}$. Consider the level set flow (see [ES] and [CGG]). That is, for $(x, t) \in \mathbb{R}^{n+1} \times [0, \infty)$ let $u_t(x)$ be the unique continuous solution of the degenerate parabolic equation

$$\frac{\partial u_t}{\partial t} = -|Du_t| F(\kappa_1, \dots, \kappa_n, t)$$

where $\kappa_1 \leq \dots \leq \kappa_n$ are the eigenvalues of $D(|Du_t|^{-1} Du_t)$, with the given initial condition $u_0(x)$. The evolving zero-set (which is not necessarily a hypersurface) is given by

$$\Gamma_t = \{x : u_t(x) = 0\}.$$

Γ_t is called the *generalized solution* to the evolution problem (1.1), and is unique. Recall that for $t > 0$, Γ_t is uniquely determined by Γ_0 independent of the choice of u_0 . Also, since Γ_0 is compact, Γ_t is a compact set for each $t \in [0, \infty)$. Define further

$$\begin{aligned} \Omega_t &= \{x : u_t(x) < 0\} \\ E_t &= \{x : u_t(x) > 0\}. \end{aligned}$$

We then have $\mathbb{R}^{n+1} = \Gamma_t \cup \Omega_t \cup E_t$ as a disjoint union. In fact (contrary to a conjecture stated for the case $F = \kappa_1 + \dots + \kappa_n$ by E. DeGiorgi, L.C. Evans and J. Spruck) a smooth hypersurface Γ_0 may evolve into a set Γ_t with nonempty interior after a finite time. See [BP], [K], [AVe] and section 4 below.

Definition 5. Let Γ, Ω, E be disjoint sets such that $\mathbb{R}^{n+1} = \Gamma \cup \Omega \cup E$. Given $\nu \in S^n$ and $\lambda \in \mathbb{R}$, we say that λ is *admissible for the triple* (Γ, Ω, E) with respect to ν if both

$$\begin{aligned} \sigma^\lambda(\nu) \left(\Omega \cap H_-^\lambda(\nu) \right) &\subset \Omega \cap H_+^\lambda(\nu) \\ \sigma^\lambda(\nu) \left(E \cap H_-^\lambda(\nu) \right) &\supset E \cap H_+^\lambda(\nu). \end{aligned}$$

Since $\mathbb{R}^{n+1} = \Gamma \cup \Omega \cup E$ is a disjoint union, these containments are respectively equivalent to

$$\begin{aligned} \sigma^\lambda(\nu) \left((E \cup \Gamma) \cap H_-^\lambda(\nu) \right) &\supset (E \cup \Gamma) \cap H_+^\lambda(\nu) \\ \sigma^\lambda(\nu) \left((\Omega \cup \Gamma) \cap H_-^\lambda(\nu) \right) &\subset (\Omega \cup \Gamma) \cap H_+^\lambda(\nu). \end{aligned}$$

When Γ is an embedded compact connected C^2 hypersurface, Ω is the bounded open set with boundary Γ and E is their complement, this definition agrees with our previous definition of admissibility.

Theorem 6. *Let Γ_t be the zero-set of a solution u_t to a degenerate parabolic geometric level-set flow.*

1. *If λ is admissible for the triple $(\Gamma_0, \Omega_0, E_0)$ with respect to ν , then λ is admissible for the triple $(\Gamma_t, \Omega_t, E_t)$ with respect to ν for all $t \geq 0$.*
2. *If Γ_0 is compact and $(-\infty, \lambda_0)$ is admissible for the triple $(\Gamma_0, \Omega_0, E_0)$ with respect to all ν in a neighborhood of ν_0 , then $\Gamma_t \cap H_-^{\lambda_0}(\nu_0)$ is the closed set between two locally Lipschitz graphs in the direction ν_0 , with Lipschitz bounds independent of F and of t .*

Remark 1. It will be apparent to the reader that in the context of level-set solutions, the conclusions of Corollary 3 and of Theorem 4 hold for the inner boundary and for the outer boundary of the level-set solution Γ_t . Similarly, the following corollary is stated for level-set solutions, but applies to Γ_t itself in the smooth case.

Corollary 7. *Let $\Lambda := \sup_{\nu \in S^n} [-\lambda_{\max}(\nu)]$ be determined by Γ_0 . For $t \geq 0$, let $\partial_0(\Gamma_t)$ be a connected component of either the inner or the outer boundary of the level-set solution Γ_t . There is a constant $C_0 = 4.603$ such that if $\partial_0(\Gamma_t)$ contains a point x_0 with $|x_0| > C_0\Lambda$, then $\partial_0(\Gamma_t)$ is a radial graph in an annulus of width $(C_0 - 1)\Lambda$. Moreover, any other connected components of the inner or outer boundary (respectively) must lie inside $\overline{B_\Lambda(0)}$.*

Remark 2. It will be seen from the proof that $C_0 = \sqrt{\sigma^2 + 1}$, where $\sigma = \arctan \sigma = \pi$, and that $\pi < C_0 - 1 < 3\pi/2$.

This paper is closely connected with the authors' 1996 paper [CG1]. In the special case that Γ_t is a strictly convex hypersurface, it may be represented by its support function $v_t : S^n \rightarrow \mathbb{R}$. Then $v = v_t$ satisfies the

evolution PDE

$$\frac{\partial v}{\partial t} = G(\nabla \nabla v + vg, t),$$

where, writing $G(\cdot, t)$ in terms of the eigenvalues of its matrix argument, $G(\lambda_1, \dots, \lambda_n, t) = F((\lambda_1)^{-1}, \dots, (\lambda_n)^{-1}, t)$ and where g is the standard Riemannian metric on S^n . For this special case, results such as Corollary 3 of the present paper follow from Theorem 3.1, part (iv) of [CG1].

2. Proofs.

We first define admissibility for functions on \mathbb{R}^{n+1} .

Definition 8. Let $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a continuous function. Given $\nu \in S^n$ and $\lambda \in \mathbb{R}$, we say that λ is *admissible for u* with respect to ν if

$$u(x) \geq u(x^*) \quad \text{for all } x \in H_-^\lambda(\nu),$$

where $x^* = \sigma^\lambda(\nu)(x) = x - 2(\langle x, \nu \rangle - \lambda)\nu$.

Lemma 9. Let Γ be the zero-set of a continuous function u . If λ is admissible for u with respect to ν , then λ is admissible for the triple (Γ, Ω, E) with respect to ν , where $\Omega = \{x : u(x) < 0\}$ and $E = \{x : u(x) > 0\}$. If u is the signed distance function, then the converse is true.

Proof. Write $H_\pm = H_\pm^\lambda(\nu)$. If λ is admissible for u with respect to ν , then for all $x \in \Omega \cap H_-$ we have $u(x^*) \leq u(x) < 0$, which implies $x^* \in \Omega \cap H_+$. Similarly, if $x \in E \cap H_+$, then $u(x^*) \geq u(x) > 0$, which implies $x^* \in E \cap H_-$. That is, λ is admissible for the triple (Γ, Ω, E) with respect to ν .

Conversely, suppose λ is admissible for the triple (Γ, Ω, E) with respect to ν . We want to show that the signed distance function $u(x) \geq u(x^*)$ for all $x \in H_-$.

First, suppose $x \in (E \cup \Gamma) \cap H_-$. If the closest point $y \in \Gamma$ to x is in H_+ , then $|x^* - y| \leq |x - y|$, so that $u(x^*) \leq u(x)$. Otherwise, y is in H_- , in which case $y^* \in (\Omega \cup \Gamma) \cap H_+$. At the same time, $\sigma^\lambda(\nu)[(\Omega \cup \Gamma) \cap H_-] \subset (\Omega \cup \Gamma) \cap H_+$, hence

$$\begin{aligned} u(x^*) &= \text{dist}(x^*, (\Omega \cup \Gamma)) \leq \text{dist}(x^*, y^*) \\ &= \text{dist}(x, (\Omega \cup \Gamma) \cap H_-) = u(x). \end{aligned}$$

Second, suppose $x \in (\Omega \cup \Gamma) \cap H_-$. Now $\sigma^\lambda(\nu)((E \cup \Gamma) \cap H_-) \supset (E \cup \Gamma) \cap H_+$ implies

$$\begin{aligned} \text{dist}(x, (E \cup \Gamma) \cap H_+) &\geq \text{dist}(x^*, (E \cup \Gamma) \cap H_+) \\ &\geq \text{dist}(x^*, \sigma^\lambda(\nu)[(E \cup \Gamma) \cap H_-]) \\ &= \text{dist}(x, (E \cup \Gamma) \cap H_-), \end{aligned}$$

and hence

$$\begin{aligned} u(x) &= -\text{dist}(x, (E \cup \Gamma) \cap H_-) \\ &\geq -\text{dist}(x^*, (E \cup \Gamma) \cap H_+) = u(x^*). \quad \square \end{aligned}$$

In particular, the lemma implies that if u is the signed distance function of a C^2 hypersurface Γ , then λ is admissible for u with respect to ν if and only if λ is admissible for Γ with respect to ν .

Hence Theorem 2 follows from Theorem 6 and we only give the

Proof of Theorem 6. Let u_0 be the signed distance function to Γ_0 . Since λ is admissible for the triple $(\Gamma_0, \Omega_0, E_0)$ with respect to ν , the lemma implies λ is admissible for u_0 with respect to ν . That is, $u_0(x) \geq u_0(x^*)$ for all $x \in H_-^\lambda(\nu)$. For $x \in P^\lambda(\nu)$, we have $x = x^*$. Thus, by the maximum principle for viscosity solutions, the solution u_t to the geometric level set flow satisfies $u_t(x) \geq u_t(x^*)$ for all $x \in H_-^\lambda(\nu)$ and $t \geq 0$. That is, λ is admissible for u_t with respect to ν for all $t \geq 0$. Applying the lemma again implies λ is admissible for the triple $(\Gamma_t, \Omega_t, E_t)$ with respect to ν for all $t \geq 0$.

Before considering part (2) of Theorem 6, we shall prove the

Proposition 10. *If $(-\infty, \lambda_0)$ is admissible for the function u with respect to ν , then u is nonincreasing in the ν direction on $H_-^{\lambda_0}(\nu)$.*

Proof. Let $y \in P^{\lambda_0}(\nu)$, and consider any two points $x = y + s\nu$ and $x' = y + s'\nu$ with $s < s' \leq 0$. Their bisecting hyperplane is $P^\lambda(\nu)$, where $\lambda = \lambda_0 + \frac{1}{2}(s + s') < \lambda_0$, and we have $x \in H_-^\lambda(\nu)$, while

$$\begin{aligned} x^* &:= x - 2(\langle x, \nu \rangle - \lambda)\nu = y + s\nu - 2(\lambda_0 + s - \lambda)\nu \\ &= y + (2\lambda - 2\lambda_0 - s)\nu = y + s'\nu = x'. \end{aligned}$$

So the first part of Theorem 6 shows $u_t(x) \geq u_t(x^*)$, that is, u_t is nonincreasing in the ν -direction when restricted to the half-space $H_-^{\lambda_0}(\nu)$. \square

Proof of Theorem 6, Part (2). Let ε be the radius in S^n of a ball so that $(-\infty, \lambda_0)$ is admissible for $(\Gamma_0, \Omega_0, E_0)$ with respect to each ν in the ball. As we have shown above, $(-\infty, \lambda_0)$ is also admissible for u_t with respect to each ν in the ball. It now follows from Proposition 10 that u_t is nonincreasing in the ν direction on $H_-^{\lambda_0}(\nu)$ for all ν with $\text{dist}(\nu, \nu_0) < \varepsilon$.

Consider two points $\hat{x}, \tilde{x} \in H_-^{\lambda_0}(\nu_0)$ which are in the outer boundary $\partial_{out}(\Gamma_t) := \Gamma_t \cap \overline{E_t}$. Then there are $\hat{x}_k \rightarrow \hat{x}$ with $u_t(\hat{x}_k) > 0$ for all k . Since $H_-^{\lambda_0}(\nu_0)$ is an open halfspace, we have $\hat{x}, \tilde{x} \in H_-^{\lambda_0}(\nu)$ for all $\nu \in S^n$ with $\text{dist}(\nu, \nu_0) < \hat{\varepsilon}$ for some $\hat{\varepsilon} \in (0, \varepsilon]$ which depends on \hat{x} and \tilde{x} . Write $\hat{x} = \hat{y} + \hat{s}\nu_0$ and $\tilde{x} = \tilde{y} + \tilde{s}\nu_0$, where $\tilde{s}, \hat{s} \in (-\infty, 0)$ and $\tilde{y}, \hat{y} \in P^{\lambda_0}(\nu_0)$. We shall show that

$$(2.1) \quad |\tilde{s} - \hat{s}| \leq \cot \hat{\varepsilon} |\tilde{y} - \hat{y}|.$$

This inequality shows that inside $H_-^{\lambda_0}(\nu_0)$, the outer boundary $\partial_{out}(\Gamma_t)$ is the graph of a function: $P^{\lambda_0} \rightarrow (-\infty, 0)$ with local Lipschitz constant $\cot \hat{\varepsilon}$.

Suppose inequality (2.1) is false; without loss of generality we may assume $\tilde{s} < \hat{s} < 0$. Write the unit vector

$$\nu_k := \frac{\hat{x}_k - \tilde{x}}{|\hat{x}_k - \tilde{x}|} \rightarrow \nu := \frac{\hat{x} - \tilde{x}}{|\hat{x} - \tilde{x}|}.$$

We may compute

$$\langle \nu, \nu_0 \rangle = \frac{\langle \hat{y} - \tilde{y}, \nu_0 \rangle + (\hat{s} - \tilde{s})}{\sqrt{|\hat{y} - \tilde{y}|^2 + (\hat{s} - \tilde{s})^2}} > \frac{1}{\sqrt{\tan^2 \hat{\varepsilon} + 1}} = \cos \hat{\varepsilon},$$

which implies $\langle \nu_k, \nu_0 \rangle > \cos \hat{\varepsilon}$, and hence $\text{dist}(\nu_k, \nu_0) < \hat{\varepsilon}$, for k sufficiently large. Note that $\langle \tilde{x}, \nu_k \rangle < \langle \hat{x}_k, \nu_k \rangle$. Proposition 10 shows that u_t is nonincreasing in $H_-^{\lambda_0}(\nu_k)$, which implies $0 = u_t(\tilde{x}) \geq u_t(\hat{x}_k) > 0$, a contradiction. This proves inequality (2.1) for any $\tilde{x}, \hat{x} \in \partial_{out}(\Gamma_t)$ in the open half-space $H_-^{\lambda_0}(\nu_0)$.

A similar argument shows that, inside $H_-^{\lambda_0}(\nu_0)$, the inner boundary $\partial_{in}(\Gamma_t) := \Gamma_t \cap \overline{\Omega_t}$ is the graph of a second function: $P^{\lambda_0}(\nu_0) \rightarrow (-\infty, 0)$ satisfying the same local Lipschitz condition.

Applying Proposition 10 one more time, one sees that inside $H_-^{\lambda_0}(\nu_0)$, $\partial_{in}(\Gamma_t)$ lies above $\partial_{out}(\Gamma_t)$ in terms of the ν_0 direction. Since the Lipschitz bound $\cot \hat{\varepsilon}$ depends only on \tilde{x}, \hat{x} and Γ_0 , it is independent of t and F . \square

Proof of Theorem 4. Note first that necessarily $\Lambda \geq 0$, since $\lambda_{max}(\nu) + \lambda_{max}(-\nu) \geq 0$.

Let x_0 be a point of Γ_t , $|x_0| =: r_0 > \Lambda$. Then for all $\nu \in S^n$, since $\lambda_{max}(\nu) \geq -\Lambda$, we have $x_0 \in H_-^{max}(\nu)$ for all ν such that $\langle \nu, x_0 \rangle < -\Lambda$. Write $x_0 = r_0 \theta_0$, where $\theta_0 \in S^n$, and choose a unit tangent vector e to S^n at θ_0 . The inward radial unit vector $\nu_0 = -\partial/\partial r$ satisfies $\langle \nu_0, x_0 \rangle = -r_0 < -\Lambda$, implying via Theorem 2, Part (2) that Γ_t is a Lipschitz graph $r = \psi_t(\theta)$ in the radial direction in some neighborhood of x_0 . The vector $\tau := -re - D\psi_t(e)\nu_0$ is tangent to Γ_t ; according to Theorem 2, Part (2), τ is transverse to ν for all $\nu \in S^n$ satisfying $\langle \nu, x_0 \rangle < -\Lambda$. Therefore

$$-\frac{rD\psi_t(e)}{\sqrt{r^2 + (D\psi_t(e))^2}} = \left\langle \frac{\tau}{|\tau|}, x_0 \right\rangle \geq -\Lambda.$$

Equivalently, after a little algebra

$$D\psi_t(e) \leq \frac{r\Lambda}{\sqrt{r^2 - \Lambda^2}}, \quad \square$$

Proof of Corollary 7. According to Theorem 4, applied now in the level-set context (cf. Remark 1), near any point outside of $\overline{B_\Lambda}(0)$, $\partial_0(\Gamma_t)$ is locally a radial graph $r = \psi_t(\theta)$ where the Lipschitz function $\psi_t : S^n \rightarrow \mathbb{R}$ satisfies almost everywhere

$$|D\psi_t| \leq \frac{r\Lambda}{\sqrt{r^2 - \Lambda^2}}.$$

We define the constant C_0 by

$$\pi = \int_1^{C_0} \frac{\sqrt{r^2 - 1}}{r} dr.$$

Write $x_0 = r_0 \theta_0$, where $r_0 = |x_0| > C_0\Lambda$, and consider any other point $x_1 = r_1 \theta_1 \in \partial_0(\Gamma_t)$. Write θ_\perp for a unit vector orthogonal to θ_0 such that $\theta_1 = \theta_0 \cos s_1 + \theta_\perp \sin s_1$ for some $0 < s_1 \leq \pi$. Consider the unit speed great-circle arc on S^n from θ_0 to θ_1 :

$$s \mapsto \theta(s) := \theta_0 \cos s + \theta_\perp \sin s, \quad 0 \leq s \leq s_1.$$

Since, outside $\overline{B_\Lambda}(0)$, $\partial_0(\Gamma_t)$ is a locally Lipschitz graph in the radial direction, there is a unique continuous lift from $\theta(s)$ to $r(s)\theta(s) \in \partial_0(\Gamma_t)$ on $0 \leq s < s^*$, where s^* is the first value in $[0, s_1]$, if any, where $r(s^*) = \Lambda$, and $s^* = s_1$ otherwise.

According to Theorem 4, we have

$$\frac{dr}{ds} \geq -\frac{r(s)\Lambda}{\sqrt{(r(s))^2 - \Lambda^2}}$$

on $[0, s^*]$, with the initial condition $r(0) = r_0$. That is, $r(s)$ is a supersolution to an ordinary differential equation (evolute equation) whose exact solution $\tilde{r}(s)$ is given by $\tilde{r}(s) = \Lambda\sqrt{(\sigma(s))^2 + 1}$, where

$$s =: \arctan \sigma(s) - \sigma(s) - \arctan \sigma_0 + \sigma_0$$

and $\sigma_0 := \sqrt{r_0^2 \Lambda^{-2} - 1}$. The hypothesis $r_0 > C_0 \Lambda$ implies $\sigma_0 - \arctan \sigma_0 > \pi$. On the other hand, $s \leq s^* \leq s_1 \leq \pi$. This forces $\sigma(s) \geq 0$ and $r(s) \geq \Lambda$ on $[0, s^*]$. Hence $s^* = s_1$ and $r(s_1) \geq \Lambda$.

By a similar argument, we may show that

$$\frac{r_1 - r_0}{\Lambda} \geq C_0 - 1.$$

In fact, comparing any two intervals of the form $[r_0, r_1] \subset [0, \infty)$ on which $\arctan \sigma_1 - \sigma_1 - \arctan \sigma_0 + \sigma_0 = \pi$, where $r_k^2 =: \Lambda^2(\sigma_k^2 + 1)$, $k = 0, 1$, we see that the shortest such interval starts at $r_0 = 0$. This shows that $\partial_0(\Gamma_t)$ is a radial graph $r = \psi_t(\theta)$, where ψ_t maps S^n to an interval of length at most $\Lambda(C_0 - 1)$.

Now consider any point $x_2 = r_2 \theta_2 \in \partial_{out}(\Gamma_t)$ (or in $\partial_{in}(\Gamma_t)$, respectively) with $r_2 = |x_2| > \Lambda$. We may apply Theorem 6, part (2), with $\nu = -\theta_2$ and $\lambda_0 = \Lambda \geq -\lambda_{max}(\nu)$. Both $x_2 = r_2 \theta_2$ and $x_3 := \psi_t(\theta_2) \theta_2$ lie in $H_-^{max}(-\theta_2)$, and therefore lie on the graph in the θ_2 -direction of the same function; hence $x_2 = x_3 \in \partial_0(\Gamma_t)$. This shows that outside $\overline{B_\Lambda(0)}$, $\partial_{out}(\Gamma_t)$ (or $\partial_{in}(\Gamma_t)$, respectively) agrees with $\partial_0(\Gamma_t)$. \square

3. Application: Flow by $1/H^+$.

Several papers have been published recently which investigate properties of a hypersurface which expands with velocity equal to the reciprocal of mean curvature [G], [U1], [U2], [HI]. As a simple application of the results of the previous sections, we shall construct an “extended” solution Γ_t for $1/H^+$ flow, and will show that as $t \rightarrow \infty$, and after suitable normalization, Γ_t converges in Lipschitz norm to a round sphere or around annulus. Our approach is in a certain sense opposite to that of [G], [U2] and [HI], since we first show asymptotic roundness in the Lipschitz sense and only later,

if at all, are in a position to prove that the curvatures are asymptotically constant. As in the recent result of Huisken and Ilmanen on the Penrose conjecture [HI], we do not need to assume the initial hypersurface is star-shaped. In contrast to the solutions constructed in [HI], which result from a globally defined variational problem, our hypersurfaces are limits of solutions to local problems (see Remark 5 below.)

Consider an embedded, compact initial hypersurface Γ_0 . In the next few paragraphs, we shall construct the solution Γ_t to the extended evolution problem $V = 1/H^+$. Γ_t will be the level set of the limit of viscosity solutions to a one-parameter family of geometric PDE's. We refer to this as an *extended* evolution problem, since the values of the right-hand side $1/H^+$ are extended real numbers in $(0, \infty]$. Similarly, the corresponding PDE, the level sets of whose solutions are extended solutions to the $1/H^+$ -flow, may be called an *extended PDE* :

$$(3.1) \quad \frac{\partial u}{\partial t} = \begin{cases} -\frac{|Du|}{\left(\operatorname{div} \frac{Du}{|Du|}\right)^+} & \text{if } Du \neq 0 \\ 0 & \text{if } Du = 0 \end{cases} .$$

As mentioned above, the solution u will not necessarily be a viscosity solution, but instead will be the monotone limit of viscosity solutions of a family of regularized PDEs.

The first step is to regularize the problem. Given $\varepsilon > 0$, we approximate $1/H^+$ with a smooth, strictly decreasing function $\Phi^\varepsilon : \mathbb{R} \rightarrow (0, \infty)$ such that $\Phi^\varepsilon(H) = 1/H$ if $H \geq \varepsilon$ and $1/\varepsilon < \Phi^\varepsilon(H) < 1 + 1/\varepsilon$ for $H \in (-\infty, \varepsilon)$. In addition, we require that $\Phi^\varepsilon(H)$ be a nonincreasing function of ε .

Choose $K > 0$, and define initial values as in Section 1 above:

$$(3.2) \quad u_0(x) = \begin{cases} \operatorname{dist}(x, \Gamma_0) \wedge K & \text{if } x \in E_0 \\ -\operatorname{dist}(x, \Gamma_0) & \text{if } x \in \Omega_0 \end{cases} ,$$

where Ω_0 is the bounded open set with $\partial\Omega_0 = \Gamma_0$ and E_0 is their open complement. For each $\varepsilon > 0$, there is a unique, continuous viscosity solution u^ε to the equation

$$(3.3) \quad \frac{\partial u^\varepsilon}{\partial t} = -|Du^\varepsilon| \Phi^\varepsilon \left(\operatorname{div} \frac{Du^\varepsilon}{|Du^\varepsilon|} \right)$$

for $(x, t) \in \mathbb{R}^{n+1} \times (0, \infty)$ satisfying the initial condition $u^\varepsilon(x, 0) = u_0(x)$ [CGG]. Observe that, since $\Phi^\varepsilon(H)$ is a nonincreasing function of ε , $u^\varepsilon(x)$ is a

nondecreasing function of ε . Namely, if $0 < \varepsilon_0 < \varepsilon$, then u^{ε_0} is a subsolution of the equation (3.3). Also, u^ε is bounded above by the constant K and below by the infimum of u_0 on Ω_0 . Therefore, u^ε converges pointwise and monotonically to a function u as $\varepsilon \rightarrow 0$. We define the zero level set Γ_t of $u(\cdot, t)$ to be an *extended solution* of $1/H^+$ flow.

The solution u will in general have discontinuities, and we will not claim it is a viscosity solution of the extended PDE (3.1). However, it will have the approximate-symmetry properties we have shown in Section 2 above for viscosity solutions of geometric PDE's. To see this, suppose $(-\infty, \lambda_0)$ is admissible for the initial hypersurface Γ_0 with respect to a unit vector $\nu \in S^n$. Then we may apply Proposition 10 to the viscosity solutions u^ε of the regularized PDE (3.3). The conclusion of Proposition 10 is that u^ε is nonincreasing in the ν direction on $H_-^{\lambda_0}(\nu)$. Since $u^\varepsilon \rightarrow u$ pointwise, it follows that u itself is nonincreasing in the ν direction on $H_-^{\lambda_0}(\nu)$. We conclude that the level set Γ_t of $u(\cdot, t)$ satisfies the conclusions of Theorem 6, and all the hypotheses of Corollary 7 except the existence of $x_0 \in \Gamma_t$ with $|x_0| > C_0\Lambda$.

We shall show next that the zero level set Γ_t of u_t expands outward toward infinity as $t \rightarrow \infty$. Without loss of generality, suppose that the origin O of \mathbb{R}^{n+1} lies inside Ω_0 , and choose a radius ρ_0 such that $B_{\rho_0}(O) \subset \Omega_0$. Define $v_0 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ to be the signed distance function from $\partial B_{\rho_0}(O)$:

$$v_0(x) := (|x| - \rho_0) \wedge K.$$

Note that $v_0(x) \geq u_0(x)$. Starting from the initial condition $v^\varepsilon(x, 0) = v_0(x)$, there is a unique solution $v^\varepsilon(x, t)$ to the regularized PDE (3.3) for $(x, t) \in \mathbb{R}^{n+1} \times [0, \infty)$. Since v^ε and u^ε are viscosity solutions of the same PDE (3.3), we have $v^\varepsilon(x, t) \geq u^\varepsilon(x, t)$ for all $(x, t) \in \mathbb{R}^{n+1} \times [0, \infty)$ (see e.g. Theorem 4.1 of [CGG]). It follows that $u(x, t) \leq u^\varepsilon(x, t) \leq v^\varepsilon(x, t)$ for all $t \geq 0$. But the zero level set of $v^\varepsilon(\cdot, t)$ is $\partial B_{\rho(t)}(O)$, where $\rho(t) = \rho(t, \varepsilon)$ satisfies the ODE

$$\frac{d\rho}{dt} = \Phi^\varepsilon\left(\frac{n}{\rho(t)}\right) = \frac{\rho(t)}{n} \quad \text{for } \rho(t) \leq \frac{n}{\varepsilon},$$

with initial condition $\rho(0, \varepsilon) = \rho_0$. We have $v^\varepsilon(x, t) < 0$ whenever $|x| < \rho(t, \varepsilon)$. For $\varepsilon \leq (n/\rho_0) \exp(-t/n)$, we find $\rho(t, \varepsilon) = \rho_0 \exp(t/n)$ independent of ε , by means of a straightforward computation. This implies that $u_t = u(\cdot, t) < 0$ on a ball $B_{\rho(t)}(O)$ of exponentially growing radius. Therefore Γ_t lies entirely outside an arbitrarily large ball for large t , and the remaining hypothesis of Corollary 7 is satisfied for sufficiently large t . We conclude that

$\partial_{out}(\Gamma_t)$, after rescaling to enclose volume equal to $\text{Vol}(B_1(O))$, converges at an exponential rate to a round sphere in the Lipschitz norm. The same conclusion holds for $\partial_{in}(\Gamma_t)$.

Remark 3. In order to obtain convergence of the curvatures of $\partial_{in}(\Gamma_t)$ and of $\partial_{out}(\Gamma_t)$ to constants, it would be necessary to derive curvature estimates for level sets of u^ε , for example. Such estimates are a topic for another paper.

Remark 4. We would like to show that these extended solutions satisfy a *weak barrier principle*. Suppose that Γ_0 and $\tilde{\Gamma}_0$ are compact hypersurfaces in \mathbb{R}^{n+1} and let Ω_0 resp. $\tilde{\Omega}_0$ be a bounded union of components of the complement. Choose $u_0 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ and $\tilde{u}_0 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ to be the signed distance functions from Γ_0 resp. $\tilde{\Gamma}_0$. Let u_t resp. \tilde{u}_t be the solutions of the extended PDE (3.1) and let $\Gamma_t, \tilde{\Gamma}_t$ be their zero sets at time t . If $\tilde{\Gamma}_0$ lies inside Ω_0 , or more generally if $\tilde{\Omega}_0 \subset \Omega_0$, then for all $t > 0$, $\tilde{\Omega}_t \subset \Omega_t$. That is, one extended solution Γ_t of $1/H^+$ flow acts as a barrier to any other extended solution $\tilde{\Gamma}_t$.

The proof is similar to the proof above of approximate-symmetry properties. As in the proof of Lemma 9, the inclusion $\tilde{\Omega}_0 \subset \Omega_0$ implies that $\tilde{u}_0 \geq u_0$. The weak maximum principle then allows us to conclude that for all $t > 0$, $\tilde{u}_t \geq u_t$, which implies that $\tilde{\Omega}_t \subset \Omega_t$, as required.

The reader might well ask about the consistency of our definition of extended solutions. The following result is clear.

Proposition 11. *If M^n is a compact n -manifold and $X : M^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ is a classical, embedded C^2 solution to the inverse mean curvature flow*

$$\frac{\partial X}{\partial t} = \frac{1}{H}N$$

with $H > 0$ (N is the unit outward normal), then our extended solution u with the same initial data $\{x : u_0(x) = 0\} = X(0)(M^n)$ is equal to the classical solution:

$$(3.4) \quad \{x : u_t(x) = 0\} = X(t)(M^n)$$

for all $t \in [0, T)$.

Proof. Given any $\tau < T$, let $\varepsilon_\tau = \min_{M^n \times [0, \tau]} H$. Then equation (3.3) defines a viscosity solution $u = u^\varepsilon$ (independent of ε) to (5) on $\mathbb{R}^{n+1} \times [0, \tau]$ with

$u(0) = u_0$ for any $\varepsilon \in (0, \varepsilon_\tau]$ and the proposition follows from the uniqueness of the solution (see [CGG]).

Remark 5. By means of a fairly simple example, we would like to compare our notion of “extended” solutions to $1/H^+$ -flow with the variational solutions recently introduced by Huisken and Ilmanen [HI]. Let Γ_0 be the union of two disjoint spheres lying outside each other. Then for t in a certain time interval $[0, T]$, an embedded solution Γ_t of $1/H$ flow will continue to exist. In fact, Γ_t will be the union of two spheres corresponding to the two components of Γ_0 . If one of the original components of Γ_0 has radius r_0 , then the corresponding sphere in Γ_t has the same center and radius $r_0 e^t$. This solution ceases to be embedded at the first time T when the two spheres of Γ_t intersect. According to Proposition 11, Γ_t will be the extended solution of $1/H^+$ -flow for all $0 \leq t < T$. In contrast, Example 1.5 of [HI] indicates how the Huisken-Ilmanen solution of $1/H^+$ -flow agrees with Γ_t only for $t < T'$, where T' is strictly less than T . Specifically, if the two spheres which comprise Γ_t are sufficiently close together, then there will be a unique catenoid tangent to both of them, at small circles C_1 and C_2 , one circle in each sphere of Γ_t . Then T' is the time at which the area of the segment of the catenoid between C_1 and C_2 equals the area of the union of the two spherical caps of Γ_t having the C_1 and C_2 as boundary.

4. Examples.

4.1. Two circles.

Belletini and Paolini have proposed a simple and intriguing initial condition consisting of $\Gamma_0 =$ two circles in \mathbb{R}^2 bounding disjoint disks. They allowed Γ_0 to flow as a level set, that is, $\Gamma_t = \{x : u(x, t) = 0\}$, where u satisfies equation (1.1) with F in the form $F(\kappa, t) = \kappa + g(t)$, where κ is the curvature of Γ_t . For certain choices of the forcing term $g(t)$, the initial radii r_{10} and r_{20} , and the distance between the centers of the two circles of Γ_0 , they show that Γ_t has positive Lebesgue measure in \mathbb{R}^2 after a time $t^* > 0$ [BP]¹. After rescaling, we may assume that the two circles comprising Γ_0 are $\partial B_{r_{10}}((-1, 0))$ and $\partial B_{r_{20}}((+1, 0))$. Let $r_1(t)$ and $r_2(t)$ be the solutions of the ODE

$$r_i'(t) = -(r_i(t))^{-1} + g(t), \quad 0 \leq t < T$$

¹That is, “ballooning” or “fattening” of the solution occurs at time t^* .

with initial conditions

$$r_i(0) = r_{i0} \quad (i = 1, 2),$$

and where $T \in (0, \infty]$ is the smaller of the blow up times for the two ODE-initial value problems. Belletini and Paolini show that ballooning occurs at time $t^* < T$ provided that $g(t) \geq 0$ for all t and that the sum of the solutions $r_1(t) + r_2(t)$ reaches the maximum value 2 at $t = t^*$.

We shall apply Theorem 6 to show that for each $t > 0$, Γ_t is the closed set between two locally Lipschitz graphs:

$$\Gamma_t = \{(x_1, x_2) : \varphi_{\text{in}}^t(x_1) \leq |x_2| \leq \varphi_{\text{out}}^t(x_1)\}$$

where $\varphi_{\text{in}}^t, \varphi_{\text{out}}^t : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and uniformly Lipschitz away from the x_1 -axis. The Lipschitz bounds are independent of t , and in fact take the simple form

$$(4.1) \quad -1 \leq \varphi^t(x_1) \frac{d}{dx_1} \varphi^t(x_1) + x_1 \leq 1,$$

for both φ_{in}^t and φ_{out}^t . The local Lipschitz estimate (4.1) holds whether or not ballooning occurs; if Γ_t is a smooth curve, then $\varphi_{\text{in}}^t \equiv \varphi_{\text{out}}^t$.

Note in particular that inequality (4.1) forces Γ_t , or the inner or outer boundary of a ballooned Γ_t , to have a vertical tangent line whenever it crosses the x_1 -axis outside the closed interval $-1 \leq x_1 \leq 1$, but that the Lipschitz bounds lose all force near $(-1, 1)$ on the x_1 -axis.

To prove inequality (4.1), we first compute $\lambda_{\text{max}}(\nu)$ for an arbitrary $\nu = (\cos \theta, \sin \theta) \in S^1$. Given $\lambda \in \mathbb{R}$, we observe that the portion of a given circle in $\Gamma_0 \cap H_-^\lambda(\nu)$ reflects to the interior of the circle if and only if the center of the circle lies in $H_+^\lambda(\nu)$. That is, λ is admissible for Γ_0 with respect to $\nu = (\cos \theta, \sin \theta)$ if and only if both centers $(\pm 1, 0)$ lie in $\overline{H_+^\lambda}$, that is, $\lambda \leq -|\cos \theta|$. Thus $\lambda_{\text{max}}(\nu) = -|\cos \theta|$. According to Theorem 6, for any $t \in (0, T)$, λ will remain admissible for Γ_t with respect to ν whenever $\lambda \leq -|\cos \theta|$.

Now suppose $\nu_0 = (\cos \theta_0, \sin \theta_0)$, and consider a point $x = (x_1, x_2) \in H_-^{\text{max}}(\nu_0)$, which is equivalent to

$$(4.2) \quad \langle x, \nu_0 \rangle < -|\cos \theta_0|.$$

Then inequality (4.2) remains true for all $\nu \in S^1$ sufficiently close to ν_0 , that is,

$$(4.3) \quad \langle x, \nu \rangle < -|\cos \theta| = \lambda_{\text{max}}(\nu),$$

for all θ near θ_0 , writing $\nu = (\cos \theta, \sin \theta)$. Thus, according to Part 2 of Theorem 6, for each ν satisfying (4.3) and for all $t \geq 0$, $\Gamma_t \cap H_-^{\max}(\nu)$ is the closed set between two locally Lipschitz graphs in the direction ν .

In particular, with $\theta_0 = -\pi/2$, we have $\lambda_{\max}(\nu_0) = 0$, and inequality (4.2) holds whenever $x_2 > 0$. That is, $H_-^{\max}(0, -1)$ is the open upper half-plane. This allows us to conclude, for any given $t \in (0, T)$, that the intersection of Γ_t with the upper half-plane is the relatively closed set

$$\{x : x_2 > 0, \varphi_{\text{in}}^t(x_1) \leq x_2 \leq \varphi_{\text{out}}^t(x_1)\}$$

for some functions $\varphi_{\text{in}}^t, \varphi_{\text{out}}^t : \mathbb{R} \rightarrow [0, \infty)$ which are locally Lipschitz away from zeroes.

We next show the sharp Lipschitz bound (4.1). With either $\varphi = \varphi_{\text{in}}^t$ or $\varphi = \varphi_{\text{out}}^t$, consider any point $x = (x_1, x_2)$ in the upper half-plane on the inner or outer boundary, respectively, of $\Gamma_t : x_2 = \varphi(x_1) > 0$. For small $\varepsilon > 0$, write $x^+ = (x_1, x_2 + \varepsilon)$ and $x^- = (x_1, x_2 - \varepsilon)$. By Theorem 6, for any $\nu = (\cos \theta, \sin \theta)$ satisfying inequality (4.3), $x \in H_-^{\max}(\nu)$ and $\Gamma_t \cap H_-^{\max}(\nu)$ is the relatively closed set between two Lipschitz graphs in the ν -direction (as well as in the ν_0 -direction). We restrict θ to the interval $(-\pi, 0)$ and note that inequality (4.3) holds for θ in an interval on both sides of $\theta_0 = -\pi/2$. Choose ε small enough that $x^\pm \in H_-^{\max}(\nu)$. Then the sign (0 or +1, resp.) of u_t is constant along the (upward) ray $x^+ - s\nu$, $0 \leq s < \infty$. Similarly, the sign (-1 or 0, resp.) of u_t is constant along the segment $x^- + s\nu$, $0 \leq s < -\langle x, \nu \rangle - |\cos \theta| + \varepsilon \sin \theta$. In terms of the function φ , this means that a line segment starting from $x^+ = (x_1, \varphi(x_1) + \varepsilon)$ and having slope $\tan \theta$ lies above the graph of φ , and that a line segment starting from x^- with the same slope lies below the graph of φ . Note, using inequality (4.3), that the lengths of these segments may be chosen independent of ε . Letting $\varepsilon \rightarrow 0$ gives bounds for $\varphi'(x_1)$, assuming for convenience that this derivative exists (otherwise, we find the same bounds on difference quotients with sufficiently small denominators). The bound states that $\varphi'(x_1) \leq \tan \theta$ when $\tan \theta \geq 0$, i.e., when $\theta \leq -\pi/2$. In this case, inequality (4.3) holds whenever $x_2 \tan \theta > 1 - x_1$, which shows $\varphi(x_1) \varphi'(x_1) \leq 1 - x_1$, which is the right-hand side of inequality (4.1). When $\tan \theta \leq 0$, i.e. $-\pi/2 \leq \theta$, this is a lower bound $\varphi'(x_1) \geq \tan \theta$, for all θ such that $x_2 \tan \theta < 1 - x_1$, and the other side of inequality (4.1) follows. This proves the Lipschitz bound (4.1). Moreover, the estimate (4.1) is sharp. In fact, equality holds on the right-hand side of (4.1) for a circle centered at $(1, 0)$, and on the left-hand side for a circle centered at $(-1, 0)$. In particular, this is the case for Γ_t , $0 \leq t < t^*$.

We next consider reflections $\sigma^\lambda(\nu_0)$ with $\nu_0 = (0, 1)$; as we have seen, $H_-^{\max}(0, 1)$ is the lower half-plane. We shall show that Γ_t is symmetric under reflection $\sigma^{\max}(\nu_0)$ in the x_1 -axis. From the proof of Part 1 of Theorem 6, $(-\infty, 0]$ is admissible for u_t with respect to $\nu_0 = (0, 1)$; recall $\Gamma_t = \{x : u_t(x) = 0\}$. In particular, u_t is a nonincreasing function of $x_2 \in (-\infty, 0]$ for each fixed $x_1 \in \mathbb{R}$. Further, since $\lambda = 0$ is admissible for u_t with respect to $\nu_0 = (0, 1)$, we have $u_t(x_1, x_2) \leq u_t(x_1, -x_2)$ for all $x_2 \geq 0$, by Definition 6. On the other hand, $\lambda = 0$ is also admissible for u_t with respect to $-\nu_0 = (0, -1)$, so that $u_t(x_1, x_2) \geq u_t(x_1, -x_2)$ for all $x_2 \geq 0$. Therefore $u_t(x_1, \cdot)$ is an even function for all $x_1 \in \mathbb{R}$, and Γ_t is invariant under reflection in the x_1 -axis.

Higher Dimensions. Results analogous to [BP] have been recently published for hypersurfaces in \mathbb{R}^{n+1} . Koo [K] has shown that if an immersed surface Σ_t evolves by $V = -H + g(t)$ for some nonnegative function $g(t)$, and touches itself from the outside at time t^* without crossing, then the corresponding generalized solution Γ_t has positive Lebesgue measure for times t in an interval $(t^*, t^* + \delta)$.

In particular, fattening occurs for the initially smooth example of a pair of evolving spheres whose centers lie at an appropriately chosen distance. An application of Part 1 of Theorem 6, completely analogous to the case $n = 1$ just discussed, shows that for all $t > t^*$, Γ_t is the closed set between the hypersurfaces of revolution generated by two graphs $r = \varphi_{\text{in}}^t(x_1)$ and $r = \varphi_{\text{out}}^t(x_1)$, in cylindrical coordinates about the axis containing the centers of the two initial spheres. The functions φ_{in}^t and $\varphi_{\text{out}}^t : \mathbb{R} \rightarrow [0, \infty)$ satisfy the estimate (4.1) and hence are locally Lipschitz away from zeroes.

4.2. Non-Lipschitz Example.

The purpose of this example is to highlight the distinction between the first and second parts of Theorems 2 and 6. Namely, if a point x is in $H_-^\lambda(\nu_0)$ but $\langle x, \nu \rangle > \lambda_{\max}(\nu)$ for all ν in a punctured neighborhood of ν_0 , then Part 1 of the theorem applies near x , but not Part 2. In this example, the conclusion of Part 2 fails: for a certain $t_0 > 0$, Γ_t is not a Lipschitz submanifold at x .

We begin with the case $n = 1$ of an evolving curve in the plane. Let (x_0, x_1) be Euclidean coordinates for \mathbb{R}^2 . Choose four line segments parallel to the x_0 -axis, all ending at the line $x_0 = -1$, the outer two starting at $x_0 = +1$, and the inner two at the x_1 -axis. Now add four semicircles to complete a connected, embedded C^1 curve Γ_0 . Write $\nu_0 = (1, 0)$. Then for $\nu \neq \pm\nu_0$, $\lambda = \lambda_{\max}(\nu)$ is the largest value such that the centers of all four

semicircles lie in $\overline{H}_+(\nu)$.

In particular, as $\nu \rightarrow \nu_0$, $\lambda_{\max}(\nu) \rightarrow -1$. However, $\lambda_{\max}(\nu_0) = 0$. Thus, even though $(-\infty, 0)$ is admissible for ν_0 , it cannot be concluded from Theorem 6, Part (2) that Γ_t , or its inner and outer boundaries, are Lipschitz hypersurfaces inside the half-plane $H_-^0(\nu_0) = H_-^{\max}(\nu_0)$.

In fact, this conclusion fails, at least for the fully degenerate case of the Hamilton-Jacobi equations. If the curve moves outward with prescribed constant velocity $-F(\kappa_1, \dots, \kappa_n, t) = 1$, then the inner two line segments will collide at a certain time $t^* > 0$, where $2t^*$ is the distance between the inner segments of Γ_0 . The level set Γ_{t^*} is singular, since it includes a triple point where two semicircles are externally tangent and are continued by a segment of their common tangent. Moreover, this line segment has the bounded open set Ω_{t^*} on both sides.

This example shows that Lipschitz or even topological regularity of Γ_t , or its inner and outer boundaries, inside the open half-space $H_-^{\max}(\nu)$ is not guaranteed. Nevertheless, regardless of the choice of Γ_0 and F , a limited regularity follows from Proposition 10: $\Gamma_t \cap H_-^{\max}(\nu)$ is the closed set between the graphs of two semicontinuous functions.

Higher Dimensions. If the original curve Γ_0 is chosen to be symmetric about the x_0 -axis, then rotation about the x_0 -axis forms a hypersurface of \mathbb{R}^{n+1} such that the solution of the outward Hamilton-Jacobi flow $F = -1$ starting from this hypersurface has analogous properties to those described above for $n = 1$. For example, after a certain time $t^* > 0$, Γ_{t^*} contains the segment $0 \leq x_0 < 1$ of the x_0 -axis, which is also in the interior of $\overline{\Omega_{t^*}}$.

With the inward Hamilton-Jacobi flow $F = +1$, and with the same rotationally symmetric initial hypersurface Γ_0 , the evolving level set becomes singular at some positive time t^{**} : $\Gamma_{t^{**}}$ includes a cylindrical segment which bounds $E_{t^{**}}$ on both sides.

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UNIVERSITY OF MINNESOTA

E-mail addresses: benchow@math.ucsd.edu gulliver@math.umn.edu

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