COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 13, Number 2, 439-459, 2005

On Stable Critical Points for a Singular Perturbation Problem

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We study a singular perturbation problem arising in the scalar two-phase field model. Assuming only the stability of the critical points for ε-problems, we show that the interface regions converge to a generalized stable minimal hypersurface as $\varepsilon \to 0$. The limit has an L^2 generalized second fundamental form and the stability condition is expressed in terms of the corresponding inequalities satisfied by stable minimal hypersurfaces. We show that the limit is a finite number of lines with no intersections when the dimension of the domain is 2.

1. Introduction.

In this paper, we consider the variational problem for the functional

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u : \Omega \to \mathbb{R}$ belongs to the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega) \,|\, \nabla u \in L^2(\Omega)\}$, $W : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is a double-well potential function and $\varepsilon > 0$ is a small parameter. This is a typical energy modeling the phase separation phenomena within the van der Waals - Cahn - Hilliard theory [3]. In this context, u represents the density of a two-phase fluid, where the zero points ± 1 of W correspond to stable fluid phases, and the free energy $E_\varepsilon(\cdot)$ depends both on the density potential and the density gradient. The sequence of minimizers is expected to converge in an appropriate sense to a function u_0 as $\varepsilon \to 0$, where u_0 takes values ± 1 and the interface $\partial\{u_0 = 1\} \cap \Omega$ is a hypersurface with the least possible area. The rigorous proof of this statement was given by Modica [10], Sternberg [15] with any given volume constraint $\int_\Omega u = m$ for the sequence of global minimizers via the Γ-convergence technique.

[1]Partly supported by Grant-in-Aid for Young Scientist, no. 14702001.

One of the natural questions concerning $E_\varepsilon(\cdot)$ that we address in this paper is the following: given a sequence of *stable* critical points with a uniform finite energy bound and $\varepsilon \to 0$, what can be said generally about the limit interface? Here we say u^ε is a *critical point* of E_ε if the Euler-Lagrange equation is satisfied:

$$\varepsilon \Delta u^\varepsilon = \frac{W'(u^\varepsilon)}{\varepsilon}. \tag{1.2}$$

We say that u^ε is *stable* if the second variation of E_ε is non-negative:

$$\frac{d^2}{dt^2}\bigg|_{t=0} E_\varepsilon(u^\varepsilon + t\phi) = \int_\Omega \varepsilon |\nabla \phi|^2 + \frac{W''(u^\varepsilon)}{\varepsilon}\phi^2 \geq 0 \tag{1.3}$$

for all $\phi \in C^1_c(\Omega)$. Since the functional $E_\varepsilon(\cdot)$ for small ε roughly approximates the area functional of the interface, one naturally expects that the limit interface should be a stable minimal hypersurface in a suitable sense. Recall that the stability for smooth minimal hypersurface $M \subset \Omega$ is equivalent to the following inequality:

$$\int_M |\mathbf{B}|^2 \phi^2 \leq \int_M |\nabla \phi|^2 \tag{1.4}$$

for any $\phi \in C^1_c(\Omega)$. Here, $\mathbf{B}$ is the second fundamental form of M [14]. Another important property satisfied by stable minimal hypersurfaces is the following inequality due to Schoen [12, 13]. Let ν be the unit normal vector field of M and let ν_0 be an arbitray constant unit vector. Then, there exists a constant c depending only on n such that

$$\int_M |\mathbf{B}|^2 \phi^2 \leq c \int_M (1 - (\nu \cdot \nu_0)^2) |\nabla \phi|^2 \tag{1.5}$$

for any $\phi \in C^1_c(\Omega)$. The right-hand side measures the oscillation of the unit normal in an integral form. The inequality (1.5) is one of the essential ingredients for the regularity theory for stable minimal hypersurfaces in [13]. In this paper, we show that there exists an L^2 second fundamental form defined on the limit interface and that it satisfies the same inequalities (1.4) and (1.5) in a generalized sense. Since we work in the setting of general critical points, the smoothness of the limit interfaces is not guaranteed in general. For this reason, we employ the notion of generalized second fundamental form for varifolds introduced by Hutchinson [7]. When $n = 2$, we show that the limit is a finite number of lines with no intersections or junctions. Note that intersecting lines are in fact stationary and stable for smooth variations of the length functional, so this shows that the limits of stable phase interfaces possess a better regularity property than stable critical points of the length functional in general.

There are numerous works related to the singular perturbation problem with double-well potential. Given a strict local minimizer for area functional with no constraint, Kohn and Sternberg [9] showed the existence of local minimizers for $E_\varepsilon(\cdot)$ which converge to the given limit. For such local minimizers, Córdoba and Caffarelli showed the local uniform convergence of the interface [4]. General stable critical points have been studied by Sternberg and Zumbrun, where the connectivity of interfaces for the ε-problem as well as the sharp interface problem on strictly convex domains was proved [16, 17]. Up to the boundary uniform convergence of the interface is also studied in this case [19]. For general critical points, Hutchinson and the author showed that the interfaces of any critical points with finite energy and with or without volume constraint converge to a locally constant mean curvature hypersurface with possible integer multiplicities [8].

The organization of the paper is as follows. In section 2, we recall the notions of measure-function pair and generalized second fundamental form ([7]). In section 3, we state the assumptions and main results, and in section 4 show the stability properties for the limit hypersurface for general dimensions as well as the complete regularity for $n = 2$.

2. Generalized second fundamental form.

In the next two subsections we collect definitions and theorems from [7] which we apply to the singular perturbation problems subsequently.

2.1. Measure-function pairs.

Let U be a subset of $\mathbb{R}^\alpha$. In the following application, U will be $G_{n-1}(\Omega) = \Omega \times G(n, n-1)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $G(n, n-1)$ is the set of $n-1$-dimensional unoriented subspaces in $\mathbb{R}^n$. Via the identification of $G(n, n-1)$ as a set of projections, we regard $G(n, n-1)$ to be a subset of $\mathbf{R}^{n^2}$.

Definition 1. *Suppose μ is a Radon measure on U and $f : U \to \mathbb{R}^\beta$ is a locally integrable function with respect to μ. (μ, f) is called a measure-function pair over U.*

Definition 2. *([7, 4.2.1]) Suppose $\{(\mu_k, f_k)\}_{k=1}^\infty$ and (μ, f) are measure-*

function pairs over U. Suppose $\mu_k \to \mu$ on U as $k \to \infty$. Then we say (μ_k, f_k) converges to (μ, f) in the weak sense if

$$\lim_{k\to\infty} \int_U < f_k, \phi > \, d\mu_k = \int_U < f, \phi > \, d\mu$$

for all $\phi \in C_c(U, \mathbb{R}^\beta)$. Here, $<,>$ is the standard inner product in $\mathbb{R}^\beta$.

Using the notion of *graph measures*, the following theorem was proved [18, 7]. The theorem holds for more general setting, but we only state it in the form necessary for our use:

Theorem 1. *([7, 4.4.2(i,ii)]) If $\{(\mu_k, f_k)\}_{k=1}^\infty$ is a sequence of measure- function pairs over U with $\liminf \mu_k(U) < \infty$ and $\liminf \int_U |f_k|^2 \, d\mu_k < \infty$, then some subsequence of $\{(\mu_k, f_k)\}_{k=1}^\infty$ converges in the weak sense to a measure-function pair (μ, f) for some f. Moreover,*

$$\int_U |f|^2 \, d\mu \le \liminf_{k\to\infty} \int_U |f_k|^2 \, d\mu_k.$$

2.2. Generalized second fundamental form.

Let $M \subset \mathbb{R}^n$ be a smooth hypersurface. For $x \in M$, let $S_x = [(S_x)_{ij}]_{1 \le i,j \le n}$ be the $n \times n$ orthogonal projection matrix corresponding to the projection onto T_xM, where T_xM is the tangent space to M at x. The *second fundamental form* of M at x is defined by

$$\mathbf{B} : T_xM \times T_xM \to (T_xM)^\perp, \qquad \mathbf{B}(v, w) = (D_v w)^\perp,$$

where $(T_xM)^\perp$ is the normal space and $D_v w$ is the covariant differentiation in $\mathbb{R}^n$. It is bilinear and it depends only on v and w at x. We also extend the domain of $\mathbf{B}$ to $\mathbb{R}^n \times \mathbb{R}^n$ by defining

$$\mathbf{B}(v, w) = \mathbf{B}(S_x v, S_x w).$$

For the standard orthonormal basis $\{\mathbf{e}_i\}_{i=1}^n$, define the component of $\mathbf{B}$ by

$$B_{ij}^k = < \mathbf{B}(\mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_k > .$$

In the following, we use the usual summation convention whenever no ambiguity arises. The mean curvature vector is given by $B_{jj}^k \mathbf{e}_k$. The component of $\mathbf{B}$ is expressed in terms of the projection matrix S (omitting x) by

$$B_{ij}^k = S_{lj} \delta_i S_{kl},$$

where $\delta_i = S_{ij}\frac{\partial}{\partial x_j}$. This is because $B_{ij}^k =< D_{S\mathbf{e}_i}S\mathbf{e}_j, \mathbf{e}_k^\perp >=< \delta_i S_{lj}\mathbf{e}_l, \mathbf{e}_k - S_{sk}\mathbf{e}_s >= \delta_i S_{kj} - (\delta_i S_{lj})S_{lk} = S_{lj}\delta_i S_{kl}$. We used $S_{lk} = S_{kl}$ and $S_{kj} = S_{kl}S_{lj}$. We also use the fact that

$$\delta_i S_{jk} = B_{ij}^k + B_{ik}^j.$$

This follows from $\delta_i S_{jk} = \delta_i(S_{jl}S_{lk}) = S_{jl}\delta_i S_{lk} + S_{lk}\delta_i S_{jl} = B_{ij}^k + B_{ik}^j$. If M is a level set $\{u = \text{const}\}$, $S = I - \nu \otimes \nu$, $\nu = \frac{\nabla u}{|\nabla u|} = (\nu_1, \cdots, \nu_n)$, one computes

$$B_{ij}^k = -\nu_k \delta_i \nu_j$$

$$(2.1) \qquad = -\frac{u_{x_k}}{|\nabla u|^2}(u_{x_i x_j} - \nu_j \nu_l u_{x_l x_i} - \nu_i \nu_l u_{x_l x_j} + \nu_i \nu_s \nu_j \nu_l u_{x_l x_s}).$$

If we choose a coordinate system such that $\nu(x) = (0, \cdots, 0, 1)$, we have

$$B_{ij}^n = -\frac{u_{x_i x_j}}{u_{x_n}} \qquad \text{for } i, j \in \{1, \cdots, n-1\}$$

and $B_{ij}^k = 0$ otherwise. In particular, $2\sum_{ijk}(B_{ij}^k)^2 = \sum_{ijk}(\delta_i S_{jk})^2$ in this case.

Now let $\phi \in C^1(\Omega \times \mathbb{R}^{n^2})$ be a "test function" to be used to define the generalized second fundamental form. Denote the differentiations with respect to x_i and S_{ij} by $D_i\phi$ and $D_{ij}^*\phi$, respectively. Apply the standard divergence theorem on submanifold M with $X = \phi(x, S_x)\mathbf{e}_i$ and $S_x = T_x M$ to obtain

$$0 = \int_M \operatorname{div}_M(X^\top) = \int_M \delta_r(S_{ir}\phi)$$
$$= \int_M S_{ij}D_j\phi + (\delta_i S_{jk})D_{jk}^*\phi + (\delta_j S_{ij})\phi.$$

Motivated by this we define a generalized second fundamental form for varifold. Here we briefly state the definitions and notations for varifolds. For more comprehensive account of rectifiable set and integral varifold, see [1, 14]. Radon measures on $G_{n-1}(\Omega)$ are called $(n-1)$-*varifolds*. For $\phi \in C_c(\Omega)$, we define

$$||V||(\phi) = \int_{\Omega \times G(n,n-1)} \phi(x) dV(x, S).$$

We denote the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. V is called the *integral varifold* if there exist an $(n-1)$-rectifiable set $M \subset \Omega$ and $\mathcal{H}^{n-1}$

measurable non-negative integer-valued function $\theta(x)$ on M such that, for $\phi \in C_c(G_{n-1}(\Omega))$,

$$V(\phi) = \int_M \phi(x, T_x M)\theta(x)\, d\mathcal{H}^{n-1}(x).$$

Here, $T_x M$ is the approximate tangent plane which exists uniquely $\mathcal{H}^{n-1}$ a.e. on M. We say that V is *stationary* if, for any $g \in C_c^1(\Omega : \mathbf{R}^n)$,

$$\int_{G_{n-1}(\Omega)} Dg(x) \cdot S\, dV(x, S) = 0.$$

Here, $Dg(x)$ is the $n \times n$ first derivative matrix and $A \cdot B = \mathrm{tr}(AB)$.

Definition 3. *([7, 5.2.1]) A varifold V is said to have a generalized second fundamental form if there exist functions A_{ijk}, $1 \le i, j, k \le n$, defined V a.e. on $G_{n-1}(\Omega)$ such that*

1. *$(V, \{A_{ijk}\})$ is a measure-function pair,*
2. *$0 = \int_{G_{n-1}(\Omega)} (S_{ij} D_j \phi + A_{ijk} D^*_{jk} \phi + A_{jij} \phi)\, dV(x, S)$, $i = 1, \cdots, n$, for all $\phi \in C^1(\Omega \times \mathbb{R}^{n^2})$ with a compact support in the x variables.*

One proceeds to define:

Definition 4. *([7, 5.2.5]) The generalized second fundamental form $\mathbf{B}$: $G_{n-1}(\Omega) \to \mathbb{R}^{n^3}$ is $\mathbf{B} = \{B^k_{ij}\}$ defined V a.e. by*

$$B^k_{ij}(x, S) = S_{lj} A_{ikl}(x, S), \qquad 1 \le i, j, k \le n.$$

We write $|\mathbf{B}|^2 = \sum_{i,j,k=1}^n (B^k_{ij})^2$.

The generalized second fundamental form is uniquely determined V a.e. on $G_{n-1}(\Omega)$ if V is an integral varifold ([7, 5.2.2]).

3. Assumptions and main results.

First we state the assumptions on W and u^ε and recall some definitions and known results.

We assume that

(1) $W \in C^3$, $W(\pm 1) = 0$, $W''(\pm 1) > 0$, $W \ge 0$ has only three critical points.

(2) A sequence of $C^3(\Omega)$ functions $\{u^{\varepsilon_i}\}$, where $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ satisfy (1.2) and (1.3) with ε there replaced by ε_i for all i.

(3) There exist C, $E_0 < \infty$ such that $E_{\varepsilon_i}(u^{\varepsilon_i}) \le E_0$ and $\sup_\Omega |u^{\varepsilon_i}| \le C$ for all i.

To characterize the limit interface, we next define a sequence of $(n-1)$-varifolds V^{ε_i} from u^{ε_i}. For $\phi \in C_c(G_{n-1}(\Omega))$, define

$$V^{\varepsilon_i}(\phi) = \frac{1}{\sigma}\int_{\Omega\cap\{|\nabla u^{\varepsilon_i}|>0\}} \phi(x, I - \nu^{\varepsilon_i}(x)\otimes\nu^{\varepsilon_i}(x))\frac{\varepsilon_i}{2}|\nabla u^{\varepsilon_i}|^2,$$

where $\nu^{\varepsilon_i} = \frac{\nabla u^{\varepsilon_i}}{|\nabla u^{\varepsilon_i}|}$ and $\sigma = \int_{-1}^1 \sqrt{W(s)/2}\, ds$. Due to the uniform energy bound (3), there always exists some converging subsequence in the sense of Radon measures. The following result applies to any such subsequence and the limit.

First, even without the stability condition (1.3), we have

Theorem 2. *([8, Theorem 1]) Suppose* $V = \lim_{i\to\infty} V^{\varepsilon_i}$ *as Radon measures on* $G_{n-1}(\Omega)$*. Then,*

(1) V *is a stationary integral varifold.*

(2) $\lim_{i\to\infty}\int_{\tilde\Omega}\left|\varepsilon_i\frac{|\nabla u^{\varepsilon_i}|^2}{2} - \frac{W(u^{\varepsilon_i})}{\varepsilon_i}\right| = 0$ *for all* $\tilde\Omega \subset\subset \Omega$.

(3) For any $1 > s > 0$ *and* $\tilde\Omega \subset\subset \Omega$, $\{|u^{\varepsilon_i}| < 1 - s\}\cap\tilde\Omega$ *converge to* $\mathrm{spt}\|V\|\cap\tilde\Omega$ *in the Hausdorff distance sense.*

With the stability condition (1.3), we prove

Theorem 3. *The limit varifold* V *has a generalized second fundamental form* $\mathbf{B}$ *with*

$$\int_{G_{n-1}(\Omega)} |\mathbf{B}|^2\phi^2\, dV \le \int_\Omega |\nabla\phi|^2\, d\|V\| \tag{3.1}$$

for all $\phi \in C^1_c(\Omega)$*. Moreover, we have*

$$\sum_{j=1}^n B^k_{jj} = 0 \text{ and } B^k_{ij} = B^k_{ji} \qquad V \text{ a.e. } (x,S)\in G_{n-1}(\Omega). \tag{3.2}$$

The first and second conditions of (3.2) correspond to the zero mean curvature and the symmetry of the second fundamental form, respectively. Due to the fact that V is integral, by defining $\mathbf{B}(x) = \mathbf{B}(x, T_x M)$ with $M = \mathrm{spt}||V||$, $\mathbf{B}$ may be regarded as a function defined on Ω instead of $G_{n-1}(\Omega)$. Hence with this implicitly assumed, we may write (3.1) as

$$\int_\Omega |\mathbf{B}|^2 \phi^2 \, d||V|| \leq \int_\Omega |\nabla \phi|^2 \, d||V||$$

as well. In addition to (3.1), we also prove

Theorem 4. *There exists a constant c depending only on n such that, for any $T \in G(n, n-1)$,*

$$\int_\Omega |\mathbf{B}|^2 \phi^2 \, d||V|| \leq c \int_{G_{n-1}(\Omega)} |\nabla \phi|^2 (I - S) \cdot T \, dV(x, S) \tag{3.3}$$

for all $\phi \in C_c^1(\Omega)$. Here, $(I - S) \cdot T = \mathrm{tr}((I - S)T)$.

Note that, if $\mathrm{spt}||V|| = M$ and M is a C^1 hypersurface, ν is the unit normal vector field of M and $T = I - \nu_0 \otimes \nu_0 \in G(n, n-1)$, then $(I - S) \cdot T \, dV(x, S) = (\nu \otimes \nu) \cdot (I - \nu_0 \otimes \nu_0) \, d\mathcal{H}^{n-1}(x) = (1 - (\nu \cdot \nu_0)^2) \, d\mathcal{H}^{n-1}(x)$. Thus, this is the direct analogue to (1.5) in the setting of varifolds. For $n = 2$, we prove

Theorem 5. *For any open ball $B \subset\subset \Omega$, $\mathrm{spt}||V|| \cap B$ consists of finite line segments $\cup_{j=1}^N (\mathbf{a}_j, \mathbf{b}_j)$, with $\mathbf{a}_j, \mathbf{b}_j \in \partial B$ (boundary of B) and $[\mathbf{a}_j, \mathbf{b}_j] \cap [\mathbf{a}_k, \mathbf{b}_k] = \emptyset$ for $j \neq k$.*

For $n \geq 3$, there is no general regularity theory for stationary integral varifolds with L^2 second fundamental form. By Allard's regularity theory [1] for stationary integral varifolds, $\mathrm{spt}||V||$ is real analytic on a dense open set $O \subset \mathrm{spt}||V||$. If we assume that $\mathrm{spt}||V||$ is a regular hypersurface outside of a closed singular subset $X \subset \mathrm{spt}||V||$ and that $\mathcal{H}^{n-3}(X) = 0$, then Schoen and Simon's result on stable minimal hypersurfaces [13, Theorem 3] shows that $\mathrm{spt}||V||$ is regular (i.e. X is empty) for $n \leq 7$ and $\mathcal{H}^{n-8+\delta}(X) = 0$ for all $\delta > 0$ for general dimensions.

4. Proof of the theorems.

In the following proposition, we see a phase field analogue of (1.4). We omit i from ε_i in the following computations for simplicity.

Proposition 1. *([11, Proposition 2.6]) For $\phi \in C_c^1(\Omega)$ and u^ε satisfying (1.2) and (1.3),*

$$\int_{\Omega\cap\{|\nabla u^\varepsilon|>0\}} \varepsilon \left\{ \sum_{i,j=1}^n (u^\varepsilon_{x_i x_j})^2 - \frac{1}{|\nabla u^\varepsilon|^2} \sum_{j=1}^n (\sum_{i=1}^n u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2 \right\} \phi^2$$

$$\leq \varepsilon \int_\Omega |\nabla \phi|^2 |\nabla u^\varepsilon|^2. \tag{4.1}$$

Proof. We include the computation for the reader's convenience. In (1.3), replace ϕ by $\phi|\nabla u^\varepsilon|$. The computation shows that

$$\int_{\Omega\cap\{|\nabla u^\varepsilon|>0\}} \varepsilon \left(\phi^2 \sum_j \frac{(\sum_i u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2}{|\nabla u^\varepsilon|^2} + 2\phi u^\varepsilon_{x_i} \phi_{x_j} u^\varepsilon_{x_i x_j} + |\nabla u^\varepsilon|^2 |\nabla \phi|^2 \right)$$

$$+ \frac{W''(u^\varepsilon)}{\varepsilon} |\nabla u^\varepsilon|^2 \phi^2 \geq 0. \tag{4.2}$$

Differentiate the equation (1.2) with respect to x_j and multiply $u^\varepsilon_{x_j}\phi^2$ to obtain

$$\varepsilon(\Delta u^\varepsilon_{x_j}) u^\varepsilon_{x_j} \phi^2 = \frac{W''(u^\varepsilon)}{\varepsilon} (u^\varepsilon_{x_j})^2 \phi^2.$$

After summing over j and integrating by parts, we have

$$\int_\Omega \frac{W''(u^\varepsilon)}{\varepsilon} |\nabla u^\varepsilon|^2 \phi^2 = -\varepsilon \int_\Omega \sum_{i,j} \{ (u^\varepsilon_{x_i x_j})^2 \phi^2 + 2 u^\varepsilon_{x_i x_j} u^\varepsilon_{x_j} \phi \phi_{x_i} \}.$$

By substituting this into (4.2), we obtain (4.1). □

Remark 1. Note that the integrand on the left-hand side is invariant under orthogonal rotation, and if we choose the coordinate system at x with $\frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|} = (0, \cdots, 0, 1)$, it is

$$\varepsilon \{ \sum_{i,j=1}^{n-1} (u^\varepsilon_{ij})^2 + \sum_{i=1}^{n-1} (u^\varepsilon_{ni})^2 \} \phi^2$$

at x. The first term divided by $\varepsilon|\nabla u^\varepsilon|^2$ corresponds to the length square of the second fundamental form of the level set $\{u^\varepsilon = \text{const}\}$. □

For $u^\varepsilon, (x,S) \in \Omega \times G(n,n-1)$, with $|\nabla u^\varepsilon(x)| \neq 0$ we define

$$\nu^\varepsilon(x) = \frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|},$$

$$A^\varepsilon_{ijk}(x,S) = \delta_i(-\nu^\varepsilon_j \nu^\varepsilon_k) = -S_{il}(\nu^\varepsilon_j \nu^\varepsilon_k)_{x_l},$$

$$(B^k_{ij})^\varepsilon(x,S) = S_{lj} A^\varepsilon_{ikl}(x,S),$$

$$H^\varepsilon_i(x,S) = H^\varepsilon_i(x) = \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u)}{\varepsilon}\right)_{x_i} \frac{1}{\varepsilon |\nabla u^\varepsilon|^2},$$

$i = 1, \cdots, n$. $(B^k_{ij})^\varepsilon(x,S)$ with $S = I - \nu^\varepsilon \otimes \nu^\varepsilon$ corresponds to the second fundamental form of $\{u^\varepsilon = \text{const}\}$.

Proposition 2. *For $\phi \in C^1(\Omega \times \mathbf{R}^{n^2})$ with a compact support in the first set of variables, we have*

$$\int_{G_{n-1}(\Omega)} (S_{ij} D_j \phi + A^\varepsilon_{ijk} D^*_{jk}\phi + H^\varepsilon_i \phi) dV^\varepsilon(x,S) = 0,$$
$$i = 1, \cdots, n.$$

Proof. Fix $i, \phi \in C^1_c(\Omega)$ and multiply the equation (1.2) by $\phi u^\varepsilon_{x_i}$. After two integrations by parts, one obtains

$$\int_\Omega \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 \phi_{x_i} - \varepsilon u^\varepsilon_{x_i} u^\varepsilon_{x_j} \phi_{x_j} + \frac{W(u^\varepsilon)}{\varepsilon}\phi_{x_i} = 0$$

and consequently,

$$\int_\Omega (\phi_{x_i} - \nu^\varepsilon_i \nu^\varepsilon_j \phi_{x_j})\varepsilon|\nabla u^\varepsilon|^2 + \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u^\varepsilon)}{\varepsilon}\right)_{x_i} \phi = 0. \tag{4.3}$$

Now, for $\phi \in C^1_c(\Omega \times \mathbf{R}^{n^2})$ and $s > 0$, define $\phi^s(x) = \phi\left(x, I - \frac{\nabla u^\varepsilon \otimes \nabla u^\varepsilon}{s^2 + |\nabla u^\varepsilon|^2}\right)$. Then $\phi^s \in C^1_c(\Omega)$, and substitution in (4.3) with $s \to 0$ gives

$$\int_\Omega (I - \nu^\varepsilon \otimes \nu^\varepsilon)_{ij}(D_j\phi - (\nu^\varepsilon_l \nu^\varepsilon_k)_{x_j} D^*_{lk}\phi)\varepsilon|\nabla u^\varepsilon|^2$$
$$+ \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{W(u^\varepsilon)}{\varepsilon}\right)_{x_i} \phi = 0.$$

Since $\psi(x,S) dV^\varepsilon(x,S) = \frac{1}{2\sigma}\psi(x, I - \nu^\varepsilon \otimes \nu^\varepsilon)\varepsilon|\nabla u^\varepsilon|^2 dx$, and by the definition of $A^\varepsilon_{ijk}, H^\varepsilon_i$, we obtain the stated identity. □

Proposition 3. *For $\phi \in C_c^1(\Omega)$, we have*

$$\int \sum_{i,j,k=1}^{n} |A_{ijk}^{\varepsilon}|^2 \phi^2 dV^{\varepsilon} \le 2 \int_{\Omega} |\nabla \phi|^2 d\|V^{\varepsilon}\|, \tag{4.4}$$

$$\int \sum_{i,j,k=1}^{n} |(B_{jk}^{i})^{\varepsilon}|^2 \phi^2 dV^{\varepsilon} \le \int_{\Omega} |\nabla \phi|^2 d\|V^{\varepsilon}\|, \tag{4.5}$$

$$\int_{\Omega} \sum_{i=1}^{n} |H_i^{\varepsilon}|^2 \phi^2 d\|V^{\varepsilon}\| \le C(n) \int_{\Omega} |\nabla \phi|^2 d\|V^{\varepsilon}\|. \tag{4.6}$$

Proof. Since $\sum_{i,j,k} |A_{ijk}^{\varepsilon}|^2 dV^{\varepsilon} = \frac{1}{2\sigma} \varepsilon 2 \sum_{i,j=1}^{n-1} (u_{x_i x_j}^{\varepsilon})^2$ with a suitable coordinate system, Proposition 1 shows immediately (4.4). The inequality (4.5) is similar. For (4.6), using the equation (1.2),

$$\begin{aligned} \varepsilon |\nabla u^{\varepsilon}|^2 (H_i^{\varepsilon})^2 &= (\varepsilon |\nabla u^{\varepsilon}|^2)^{-1} (\varepsilon u_{x_j}^{\varepsilon} u_{x_i x_j}^{\varepsilon} - \frac{W'}{\varepsilon} u_{x_i})^2 \\ &= \varepsilon |\nabla u^{\varepsilon}|^{-2} (u_{x_j}^{\varepsilon} u_{x_i x_j}^{\varepsilon} - \Delta u^{\varepsilon} u_{x_i}^{\varepsilon})^2 \\ &= \begin{cases} \varepsilon (\sum_{j=1}^{n-1} u_{x_j x_j}^{\varepsilon})^2 & i = n, \\ \varepsilon (u_{x_i x_n}^{\varepsilon})^2 & i \ne n, \end{cases} \end{aligned}$$

in the coordinate system with $\nabla u^{\varepsilon}(x) = (0, \cdots, 0, u_{x_n}^{\varepsilon})$. With a suitable choice of $C(n)$ with Proposition 1 and Remark 1, (4.6) follows. □

Since the right-hand side of the above estimates are bounded by the energy bound E_0 and ϕ, Theorem 1 gives weak limit functions $A_{ijk}, B_{ij}^k, H_i, 1 \le i, j, k \le n$ defined V a.e. with $(A_{ijk}^{\varepsilon_l}, V^{\varepsilon_l})$, $((B_{ij}^k)^{\varepsilon_l}, V^{\varepsilon_l})$, $(H_i^{\varepsilon_l}, V^{\varepsilon_l})$ converging in the weak sense to (A_{ijk}, V), (B_{ij}^k, V) and (H_i, V), respectively. Moreover, for any $\phi \in C_c(\Omega)$

$$\int_{G_{n-1}(\Omega)} \phi^2 \sum_{i,j,k} |B_{ij}^k|^2 dV \le \liminf_{l \to \infty} \int_{G_{n-1}(\Omega)} \phi^2 \sum_{i,j,k} |(B_{jk}^i)^{\varepsilon_l}|^2 dV^{\varepsilon_l}$$

$$\le \liminf_{l \to \infty} \int_{\Omega} |\nabla \phi|^2 d\|V^{\varepsilon_l}\| = \int_{\Omega} |\nabla \phi|^2 d\|V\|$$

by (4.5). Also by the definition of weak convergence, we have for $\phi \in C_c^1(\Omega \times \mathbf{R}^{n^2})$,

$$\int_{G_{n-1}(\Omega)} (S_{ij} D_j \phi + A_{ijk} D_{jk}^* \phi + H_i \phi) dV(x, S) = 0, \ i = 1, \cdots, n.$$

On the other hand, by (2) of Theorem 2, for $\phi \in C_c^1(\Omega)$,

$$\int_{G_{n-1}(\Omega)} H_i^{\varepsilon_l}(x,S)\phi(x)dV^{\varepsilon_l}(x,S) = \int_\Omega \left(\frac{\varepsilon_l}{2}|\nabla u^{\varepsilon_l}|^2 - \frac{W}{\varepsilon_l}\right)_{x_i}\phi$$
$$= -\int_\Omega \phi_{x_i}\left(\tfrac{\varepsilon_l}{2}|\nabla u^{\varepsilon_l}|^2 - \tfrac{W}{\varepsilon_l}\right) \to 0 \quad \text{as} \quad l\to\infty.$$

Hence,

$$\int_{G_{n-1}(\Omega)} H_i(x,S)\phi(x)dV(x,S) = 0 = \int_\Omega H_i(x,T_xM)\phi(x)d\|V\|,$$

where M is an $(n-1)$- rectifiable set (identified with spt $\|V\|$) and T_xM is the approximate tangent plane at x. This shows that $H_i(x,T_xM)=0$ $\|V\|$ a.e. on Ω. Thus we obtain

$$\int_{G_{n-1}(\Omega)} (S_{ij}D_j\phi + A_{ijk}D^*_{jk}\phi)dV(x,S) = 0,\ i=1,\cdots,n.$$

By the definition of the weak convergence, we also have

$$B^i_{jk} = S_{lj}A_{ikl} \quad V \ a.e. \quad \text{on} \quad G_{n-1}(\Omega).$$

This ends the proof of the existence of the generalized second fundamental form which satisfies (3.1). To prove (3.2), with $\phi(x)S_{si}$ in place of ϕ above, we obtain

$$0 = \int_{G_{n-1}(\Omega)} (S_{ij}D_j\phi S_{si} + \phi A_{isi})dV = \int_{G_{n-1}(\Omega)} (S_{sj}D_j\phi + A_{isi}\phi)dV.$$

Since $\int_{G_{n-1}(\Omega)} S_{ij}D_j\phi\, dV = 0$ by the stationarity of V, we have $A_{isi}=0$ V a.e. on $G_{n-1}(\Omega)$, $s=1,\cdots,n$, which is just another way of saying that V is stationary. Using $S_{pm}S_{mi}\phi(x)$ as a test function and using $A_{isi}=0$, one can also prove that $B^k_{jj}=0$ for all k, V a.e. on $G_{n-1}(\Omega)$. $B^k_{ij}=B^k_{ji}$ follows from the fact that $(B^k_{ij})^\varepsilon = (B^k_{ji})^\varepsilon$ (see (2.1)). Thus, we complete the proof of Theorem 3. □

Next we proceed to prove Theorem 4. Without loss of generality, we assume the given $T\in G(n,n-1)$ is the projection onto the first $n-1$ coordinate hyperplane, i.e., $T(x_1,\cdots,x_n)=(x_1,\cdots,x_{n-1},0)$. We first derive an inequality which is analogous to (4.1).

Proposition 4. *For $\phi\in C_c^1(\Omega)$ and u^ε satisfying (1.2) and (1.3),*

$$\int_\Omega \varepsilon\left\{\sum_{j=1}^n\sum_{i=1}^{n-1}(u^\varepsilon_{x_ix_j})^2 - \sum_{j=1}^n \frac{\left(\sum_{i=1}^{n-1} u^\varepsilon_{x_i}u^\varepsilon_{x_ix_j}\right)^2}{\sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2}\right\}\phi^2$$

$$\leq \int_\Omega |\nabla\phi|^2 \varepsilon \sum_{i=1}^{n-1} (u^\varepsilon_{x_i})^2. \tag{4.7}$$

Here, on $\left\{\sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2 = 0\right\}$, *the second term of the left-hand side is understood to be 0.*

Proof. In (1.3), replace ϕ by $\phi\sqrt{\delta + \sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2}$, $\delta > 0$. The computation shows that

$$\int \varepsilon \left\{ \frac{\sum_{j=1}^n \left(\sum_{i=1}^{n-1} u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j}\right)^2}{\delta + \sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2}\phi^2 + 2\sum_{j=1}^{n}\sum_{i=1}^{n-1} \phi\phi_{x_j} u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} \right.$$

$$\left. + |\nabla\phi|^2(\delta + \sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2) \right\} + \frac{W''}{\varepsilon}(\delta + \sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2)\phi^2 \geq 0\,, \tag{4.8}$$

and we let $\delta \to 0$. Differentiate the equation (1.2) with respect to $x_1, \cdots, x_{n-1}$ variables and multiply $u^\varepsilon_{x_i}\phi^2$. After summation over $i = 1, \cdots, n-1$ and integration, we obtain

$$\int_\Omega \sum_{i=1}^{n-1} \varepsilon(\Delta u^\varepsilon_{x_i}) u^\varepsilon_{x_i}\phi^2 = \int_\Omega \frac{W''}{\varepsilon}\sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2\phi^2.$$

After integration by parts and substituting this into (4.8), we obtain (4.7). $\square$

Remark 2. If we rotate the coordinate system at x while fixing $\{x_n = 0\}$ so that $\frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|} = (0, \cdots, 0, \nu_{n-1}, \nu_n)$, the integrand on the left-hand side is

$$\varepsilon\left\{\sum_{j=1}^{n}\sum_{i=1}^{n-2}(u^\varepsilon_{x_i x_j})^2\right\}\phi^2.$$

The second derivatives which do not appear here are $u^\varepsilon_{x_{n-1}x_{n-1}}$, $u^\varepsilon_{x_{n-1}x_n}$ and $u^\varepsilon_{x_n x_n}$. The idea of the proof of Theorem 4 is the following. Since we know that the second fundamental form has at most rank $n-1$ and that the trace (the mean curvature) is 0, the control of $n \times (n-2)$ elements of the second fundamental form is enough to bound all the element. The idea is similar to Schoen's [12, 13] but the setting is quite different.

For a projection matrix $S \in G(n, n-1)$, we define a linear map L_S from $\mathbb{R}^n$ to $\mathbb{R}^n$ by

$$L_S = \begin{cases} T - \frac{T\nu \otimes T\nu}{|T\nu|^2} & \text{if } T \neq S, \\ 0 & \text{if } T = S. \end{cases}$$

Here, ν is a unit vector such that $S\nu = 0$. When $S \neq T$, L_S is the orthogonal projection map onto the $(n-2)$-dimensional subspace $S \cap T$. We write $L_S = [(L_S)_{ij}]_{1 \leq i,j \leq n}$.

Now, for u^ε and $(x, S) \in \Omega \times G(n, n-1)$ with $|\nabla u^\varepsilon(x)| \neq 0$, we define

$$(\tilde{B}^k_{ij})^\varepsilon(x, S) = (L_S)_{li}(B^k_{lj})^\varepsilon(x, S).$$

Note that this means

$$(\tilde{B}^k_{ij})^\varepsilon(x, I - \nu \otimes \nu) = < \mathbf{B}(L_{I-\nu\otimes\nu}\mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_k >,$$

where $\mathbf{B}$ is the second fundamental form of the level set $\{u^\varepsilon = \text{const}\}$ and $\nu = \frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|}$.

Proposition 5. *There exists a constant $c = c(n)$ such that*

$$(4.9) \qquad \int_{G_{n-1}(\Omega)} \sum_{i,j,k=1}^n |(\tilde{B}^k_{ij})^\varepsilon|^2 \phi^2 \, dV^\varepsilon \leq c \int_{G_{n-1}(\Omega)} (I - S) \cdot T |\nabla \phi|^2 \, dV^\varepsilon$$

holds for $\phi \in C^1_c(\Omega)$.

Proof. The quantity $\sum_{i,j,k} |(\tilde{B}^k_{ij})^\varepsilon|^2$ is invariant under any orthogonal rotation fixing T. For a fixed $x \in \Omega$ with $|\nabla u^\varepsilon(x)| \neq 0$, we choose a coordinate system such that $\nu = \frac{\nabla u^\varepsilon(x)}{|\nabla u^\varepsilon(x)|} = (0, \cdots, 0, \nu_{n-1}, \nu_n)$. Since $L_S \mathbf{e}_i = \mathbf{e}_i$ for $i = 1, \cdots, n-2$ and $= 0$ for $i = n-1, n$, we have $(\tilde{B}^k_{ij})^\varepsilon(x, I - \nu \otimes \nu) = (B^k_{ij})^\varepsilon(x, I - \nu \otimes \nu)$ for $i \neq n-1$, n. Direct computation using the formula (2.1) shows that (with the evaluation at $(x, I - \nu \otimes \nu)$)

$$(\tilde{B}^k_{ij})^\varepsilon = \begin{cases} 0 & \text{for } k \in \{1, \cdots, n-2\} \text{ or } i \in \{n-1, n\}, \\ -\frac{u^\varepsilon_{x_i x_j} \nu_k}{|\nabla u^\varepsilon|} & \text{for } k \in \{n-1, n\} \text{ and } i, j \in \{1, \cdots, n-2\}, \\ -\frac{\nu_k}{|\nabla u^\varepsilon|}\{u^\varepsilon_{x_i x_j} - \nu_j(\nu_{n-1} u^\varepsilon_{x_i x_{n-1}} + \nu_n u^\varepsilon_{x_i x_n})\} & \\ \qquad \text{for } j, k \in \{n-1, n\} \text{ and } i \in \{1, \cdots, n-2\}. & \end{cases}$$

The point here is that there is no $u^\varepsilon_{x_n x_n}$, $u^\varepsilon_{x_{n-1} x_n}$, $u^\varepsilon_{x_{n-1} x_{n-1}}$ appearing in the expressions. Thus, with a suitable choice of c depending only on n, we have

$$|\nabla u^\varepsilon|^2 \sum_{i,j,k=1}^n |(\tilde{B}^k_{ij})^\varepsilon|^2 \leq c \sum_{j=1}^n \sum_{i=1}^{n-2} (u^\varepsilon_{x_i x_j})^2.$$

Then, this with (4.7) and Remark 2 shows the desired inequality. Note that $\varepsilon\sum_{i=1}^{n-1}(u^\varepsilon_{x_i})^2dx = \varepsilon|\nabla u^\varepsilon|^2(1-\nu_n^2)dx = (I-S)\cdot T\,dV^\varepsilon(x,S)$. □

Since the right-hand side of (4.9) is bounded uniformly in terms of E_0 and ϕ, Theorem 1 gives weak limit functions $\tilde{B}^k_{ij}$, $1\le i,j,k\le n$, defined V a.e., with $((\tilde{B}^k_{ij})^{\varepsilon_l}, V^{\varepsilon_l})$ converging in the weak sense to $(\tilde{B}^k_{ij}, V)$. Furthermore, we have

$$\int_{G_{n-1}(\Omega)}\sum_{i,j,k=1}^{n}|\tilde{B}^k_{ij}|^2\phi^2\,dV \le c\int_{G_{n-1}(\Omega)}(I-S)\cdot T|\nabla\phi|^2\,dV \tag{4.10}$$

for $\phi\in C^1_c(\Omega)$. We next prove

Lemma 1.

$$\tilde{B}^k_{ij}(x,S) = (L_S)_{li}B^k_{ij}(x,S), \qquad 1\le i,j,k\le n$$

for V a.e. on $\Omega\times(G(n,n-1)\setminus\{T\})$.

Proof. Note that L_S is a smooth function on $\{T\ne S\}$. Let $\psi\in C(G(n,n-1))$ be a function which vanishes in a neighborhood of $T=\{x_n=0\}$. Then, for $\phi\in C_c(G_{n-1}(\Omega))$,

$$\int_{G_{n-1}(\Omega)}\tilde{B}^k_{ij}(x,S)\psi(S)\phi(x,S)\,dV(x,S) = \lim_{m\to\infty}\int_{G_{n-1}(\Omega)}(\tilde{B}^k_{ij})^{\varepsilon_m}\psi\phi\,dV^{\varepsilon_m}$$

$$= \lim_{m\to\infty}\int_{G_{n-1}(\Omega)}(B^k_{lj})^{\varepsilon_m}(L_S)_{li}\psi\phi\,dV^{\varepsilon_m}$$

by the definition of $(\tilde{B}^k_{ij})^\varepsilon$. Since $(L_S)_{li}\psi\phi\in C_c(G_{n-1}(\Omega))$, by the definition of the weak convergence, we have

$$= \int_{G_{n-1}(\Omega)}B^k_{ij}(L_S)_{li}\psi\phi\,dV.$$

Since this holds for any ϕ, we have $\psi\tilde{B}^k_{ij} = \psi B^k_{ij}(L_S)_{li}$ for V a.e. on $G_{n-1}(\Omega)$. Since ψ may take arbitrary value on $G(n,n-1)\setminus\{T\}$, we then have $\tilde{B}^k_{ij} = B^k_{ij}(L_S)_{li}$ for V a.e. on $\Omega\times(G(n,n-1)\setminus\{T\})$. □

Proof of Theorem 4. Let $x\in\operatorname{spt}||V||$ be a point such that the unique weak tangent plane S_x exists (which holds for $||V||$ a.e.) and that $S_x\ne T$. Choose a coordinate system such that the unit normal vector ν orthogonal to S_x has a form $\nu = (0, \cdots, 0, \nu_{n-1}, \nu_n)$. By Lemma 1 and $L_S \mathbf{e}_i = 0$ for $i = n-1, n$ and $= \mathbf{e}_i$ for $i = 1, \cdots, n-2$, we have

$$\tilde{B}^k_{ij}(x, S_x) = \begin{cases} B^k_{ij}(x, S_x) & \text{for } i = 1, \cdots, n-2 \text{ and } 1 \le j, k \le n, \\ 0 & \text{for } i = n-1, n \text{ and } 1 \le j, k \le n. \end{cases}$$

By (3.2), we also have

$$\tilde{B}^k_{ij}(x, S_x) = B^k_{ji}(x, S_x) \qquad \text{for } i = 1, \cdots, n-2 \text{ and } 1 \le j, k \le n.$$

By the definition of B^k_{ij} and since $S_x = I - \nu \otimes \nu$, at (x, S_x) for each k,

$$[B^k_{ij}]_{n-1 \le i,j \le n} = [A_{ikl}(S_x)_{lj}]_{n-1 \le i,j \le n}$$

$$= \begin{bmatrix} A_{n-1,k,n-1} & A_{n-1,k,n} \\ A_{n,k,n-1} & A_{n,k,n} \end{bmatrix} \begin{bmatrix} 1 - \nu^2_{n-1} & -\nu_{n-1}\nu_n \\ -\nu_{n-1}\nu_n & 1 - \nu^2_n \end{bmatrix}.$$

Since $1 = \nu^2_{n-1} + \nu^2_n$,

$$\begin{bmatrix} 1 - \nu^2_{n-1} & -\nu_{n-1}\nu_n \\ -\nu_{n-1}\nu_n & 1 - \nu^2_n \end{bmatrix} = \begin{bmatrix} \nu_n \\ -\nu_{n-1} \end{bmatrix} \begin{bmatrix} \nu_n & \nu_{n-1} \end{bmatrix}.$$

If we set $(A_{n-1,k,n-1}\nu_n - A_{n-1,k,n}\nu_{n-1})/\nu_n = c_k$, $B^k_{n,n-1} = B^k_{n-1,n}$ implies with a little computation that

$$[B^k_{ij}]_{n-1 \le i,j \le n} = c_k \begin{bmatrix} \nu^2_n & -\nu_{n-1}\nu_n \\ -\nu_{n-1}\nu_n & \nu^2_{n-1} \end{bmatrix}.$$

By (3.2), we have $\sum_{j=1}^n B^k_{jj}(x, S_x) = 0$, so in particular,

$$c_k = -\sum_{j=1}^{n-2} B^k_{jj}(x, S_x) = -\sum_{j=1}^{n-2} \tilde{B}^k_{jj}(x, S_x).$$

Thus, with a suitable choice of $c = c(n)$, we showed that

$$|\mathbf{B}|^2 = \sum_{i,j,k=1}^n |B^k_{ij}(x, S_x)|^2 \le c \sum_{i,j,k=1}^n |\tilde{B}^k_{ij}(x, S_x)|^2$$

whenever $S_x \neq T$. This shows with (4.10) that

$$\int_{\Omega \times (G(n,n-1) \setminus \{T\})} |\mathbf{B}|^2 \phi^2 \, dV \le c \int_{G_{n-1}(\Omega)} |\nabla \phi|^2 (I - S) \cdot T \, dV.$$

To obtain the full inequality, let $\{T_i\} \subset G(n, n-1)$ be a sequence converging to T and $T \neq T_i$. For each T_i, we have the above inequality with T replaced by T_i. Then,

$$\begin{aligned} &\int_{G_{n-1}(\Omega)} |\mathbf{B}|^2 \phi^2 \, dV \\ &\leq \int_{\Omega \times (G(n,n-1)\setminus\{T\})} |\mathbf{B}|^2 \phi^2 \, dV + \int_{\Omega \times (G(n,n-1)\setminus\{T_i\})} |\mathbf{B}|^2 \phi^2 \, dV \\ &\leq c \int_{G_{n-1}(\Omega)} |\nabla \phi|^2 \{(I - S) \cdot T + (I - S) \cdot T_i\} \, dV. \end{aligned}$$

Since $\lim_{i\to\infty} \int_{G_{n-1}(\Omega)} (I - S) \cdot T_i |\nabla \phi|^2 \, dV = \int_{G_{n-1}(\Omega)} (I - S) \cdot T |\nabla \phi|^2 \, dV$, we show the inequality (3.3) with a suitable choice of c. □

For the proof of Theorem 5, we first show

Lemma 2. *Given $0 < s < 1$ and $\tilde{\Omega} \subset\subset \Omega$, there exists a positive constant c depending only on W and s such that, for all sufficiently small $\varepsilon > 0$,*

$$\frac{c}{\varepsilon} \leq |\nabla u^\varepsilon(x)| \tag{4.11}$$

for all $x \in \{|u^\varepsilon| < 1 - s\} \cap \tilde{\Omega}$.

Proof. Suppose for a contradiction that there exists $\bar{x} \in \{|u^\varepsilon| < 1 - s\} \cap \tilde{\Omega}$ with $|\nabla u^\varepsilon(\bar{x})| < \frac{c}{\varepsilon}$, where we set $c^2 = \frac{1}{4} \min_{|t|<1-s/2} W(t) > 0$. Then there exists $r = r(W)$ such that $|\nabla u^\varepsilon(x)| < \frac{2c}{\varepsilon}$ and $|u^\varepsilon(x)| \leq 1 - s/2$ for $x \in B_{\varepsilon r}(\bar{x})$, since we may obtain C^1 estimate for the rescaled equation $\Delta u = W'(u)$ after the change of variables $\tilde{x} = \frac{x - \bar{x}}{\varepsilon}$. Hence we have

$$\xi^\varepsilon = \left(\frac{W(u)}{\varepsilon} - \frac{\varepsilon}{2} |\nabla u|^2 \right) \geq \frac{W(u)}{\varepsilon} - \frac{2c^2}{\varepsilon}$$

$$\geq \frac{W(u)}{2\varepsilon} \geq \frac{2c^2}{\varepsilon} \text{ on } B_{\varepsilon r}(\bar{x}). \tag{4.12}$$

On the other hand, by the Poincaré inequality and for $B_R(\bar{x})$ and $R < \frac{1}{2}\mathrm{dist}(\tilde{\Omega}, \partial\Omega)$,

$$\begin{aligned} &(\int_{B_R(\bar{x})} |\xi - \bar{\xi}|^2)^{\frac{1}{2}} \leq c_0 \int_{B_R(\bar{x})} |\nabla \xi| = c_0 \int_{B_R(\bar{x})} |\varepsilon u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} - \frac{W'}{\varepsilon} u^\varepsilon_{x_j}| \\ &\leq c_0 \int_{B_R(\bar{x})} \varepsilon |u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j} - \Delta u^\varepsilon u^\varepsilon_{x_j}| \quad \text{by (1.2).} \end{aligned}$$

By the same argument as before for H_i^ε, using (4.1) as well as the Hölder inequality,

(4.13)

$$\begin{aligned} &\leq c_0 \left(\int_{B_R(\bar{x})} \varepsilon |\nabla u^\varepsilon|^2 \right)^{1/2} \left(\int_{B_R(\bar{x})} \varepsilon \{ (u^\varepsilon_{x_i x_j})^2 - \frac{1}{|\nabla u^\varepsilon|^2} (u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2 \} \right)^{1/2} \\ &\leq c_0 \left(\int_{B_R(\bar{x})} \varepsilon |\nabla u^\varepsilon|^2 \right)^{1/2} C(\Omega, \tilde{\Omega}) E_0^{1/2}. \end{aligned}$$

By the monotonicity formula for the scaled energy on concentric balls [8, Proposition 1], we have

$$\begin{aligned} \frac{1}{R} \int_{B_R(\bar{x})} \varepsilon |\nabla u^\varepsilon|^2 &\leq \frac{1}{R_1} \int_{B_{R_1}(\bar{x})} \left(\varepsilon |\nabla u|^2 + \frac{2W}{\varepsilon} \right) + c_1 R_1 \\ &\leq \frac{2E_0}{R_1} + c_1 R_1 \end{aligned}$$

for $R < R_1 = \text{dist}(\tilde{\Omega}, \Omega)/2$. Then choose R small so that

$$c_0 (R (\frac{2E_0}{R_1} + c_1 R_1))^{1/2} c(\Omega, \tilde{\Omega}) E_0^{1/2} \leq c^2 r \sqrt{\pi}$$

is satisfied.

By (2) of Theorem 2, note that $\displaystyle\frac{1}{\pi R^2} \int_{B_R(\bar{x})} \xi = \bar{\xi} \to 0$ as $\varepsilon \to 0$,

and by (4.12),

$$\left(\int_{B_R(\bar{x})} |\xi|^2 \right)^{1/2} \geq \left(\int_{B_{\varepsilon r}(\bar{x})} |\xi|^2 \right)^{1/2} \geq \sqrt{\pi} (\varepsilon r) \frac{2c^2}{\varepsilon} = 2c^2 r \sqrt{\pi}.$$

This contradicts (4.13) for small ε. □

Proposition 6. *For $0 < s < 1$ and all small ε,*

$$\int_{-1+s}^{1-s} \left(\int_{\{u^\varepsilon = t\} \cap \tilde{\Omega}} (\kappa^\varepsilon)^2 d\mathcal{H}^1 \right) dt \leq C(\Omega, \tilde{\Omega}, W, E_0).$$

Here, κ^ε is the geodesic curvature of the level curve of u^ε.

Proof. Since $|\mathbf{B}^\varepsilon|^2 = (\kappa^\varepsilon)^2$ for $n = 2$, (4.5), (4.11) and the coarea formula [5] yield the stated inequality. □

Proof of Theorem 5. By Fatou's Lemma, we have

$$\int_{-1+s}^{1-s} \liminf_{i\to\infty} \left(\int_{\{u^{\varepsilon_i}=t\}} (\kappa^{\varepsilon_i})^2 d\mathcal{H}^1 \right) dt \ < \ \infty,$$

so that we may choose $t \in [-1+s, 1-s]$ such that

$$\liminf_{i\to\infty} \int_{\{u^{\varepsilon_i}=t\}} (\kappa^{\varepsilon_i})^2 d\mathcal{H}^1 < \infty.$$

By (4.11), each curve $\tilde{\Omega}\cap\{u^{\varepsilon_i} = t\}$ is a finite number of curves with a uniform $C^{1,1/2}$ bound. Also by Theorem 2, $\{u^{\varepsilon_i} = t\} \cap \tilde{\Omega}$ converge to spt $\|V\|$ in the Hausdorff distance sense. Thus, locally, spt $\|V\|$ is expressed as $\cup_{j=1}^m$ graph g_j, $g_j \in C^{1,1/2}$ and $g_1 \le \cdots \le g_m$ over a suitable line segment. On the other hand, the support of one dimensional stationary integral varifold is locally either a line segment or a junction point ([2]). Since $g_1 \le \cdots \le g_m$ and $g_j \in C^{1,1/2}$, there cannot be any junction point in Ω. □

5. Remarks.

1. Though we do not know how to utilize it, we point out that the following identity holds for u satisfying (1.2):

$$\sum_{i=1}^n (H_i^\varepsilon)_{x_i} = -\sum_{i=1}^n |H_i^\varepsilon|^2$$
$$+ \left\{ \sum_{i,j=1}^n (u^\varepsilon_{x_i x_j})^2 - \frac{1}{|\nabla u^\varepsilon|^2} \sum_{j=1}^n (\sum_{i=1}^n u^\varepsilon_{x_i} u^\varepsilon_{x_i x_j})^2 \right\} / |\nabla u^\varepsilon|^2.$$

Here, $(H_1^\varepsilon, \cdots, H_n^\varepsilon)$ may be considered as an approximate mean curvature vector for the ε-problem (see Proposition 2). The identity can be checked by direct computation and the equation (1.2). For $n = 2$, the right-hand side is equal to 0. For stable critical points with a uniform energy bound, the integral with respect to $||V^\varepsilon||$ of the right-hand side is bounded locally uniformly due to (4.1) and (4.6), so there is a uniform estimate for the divergence of the approximate mean curvature vector in this sense.

2. One may speculate that the supports of limit varifolds of stable critical points are always smooth for $n \le 7$. For $n = 2$, we have a complete

regularity result in this paper. For $n = 3$, with some extra work, one can show that the tangent cones of the limit varifolds are always 2-planes with integer multiplicities. This does not give a complete regularity for $n = 3$, since around a point of multiplicities greater than 1, no regularity theory is available even if the tangent cones are 2-planes at every point.

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Received October 11, 2003.