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SYNCHRONIZATION ANALYSIS OF KURAMOTO OSCILLATORS*

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Abstract. In this paper, we study the original Kuramoto oscillators and the generalized Kuramoto oscillators with directed coupling topology. For the original Kuramoto model with identical oscillators, we obtain that frequency synchronization can occur for all initial phase configurations distributed over the whole circle, which is proved by means of a new method based on the Łojasiewicz inequality for gradient systems of analytic functions. This improves the corresponding result in [S.-Y. Ha, T. Ha, and J.-H. Kim, Physica D 239, 1692–1700, 2010], where the authors only considered initial phase configurations distributed over the open half circle. For the generalized Kuramoto model with directed coupling topology, we show that when the phases of oscillators are distributed over the half circle and the coupling strength is sufficiently large, frequency synchronization is guaranteed. This improves and extends the previous results in [N. Chopra and M. W. Spong, IEEE Trans. Automat. Control 54, 353–357, 2009], [S.Y. Ha, T. Ha, and J.H. Kim, Physica D 239, 1692–1700, 2010], and [Y.P. Choi, S.Y. Ha, S. Jung, and Y. Kim, Physica D 241, 735–754, 2012], where the corresponding results hold in the original Kuramoto model for initial phase configurations whose diameters are smaller than $\frac{\pi}{2}$ or π. Finally, we extend the result to the case of switching topology.

Key words. Synchronization, Kuramoto model, Łojasiewicz inequality, switching topology.

AMS subject classifications. 92D25, 74A25, 76N10.

1. Introduction

In the past few decades, *synchronization* in complex networks has been a focus of interest for researchers from different disciplines. This phenomenon can be observed in many biological phenomena such as flashing of fireflies, chorusing of crickets, synchronous firing of a cardiac pacemaker, metabolic synchrony in yeast cell suspension, etc. (see [5, 22]). Among many models that have been proposed to address synchronization phenomena, we are here interested in the Kuramoto model [13, 14]. This model can be used to understand the emergence of synchronization in networks of oscillators. We refer the reader to the survey papers [23, 1] for the applications of the Kuramoto model in various biological synchronization phenomena.

The Kuramoto model consists of a population of N coupled nonlinear oscillators where the phase $\theta_i(t)$ of the ith oscillator evolves in time according to

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1,\ldots,N, \tag{1.1}$$

and subject to initial conditions

$$\theta_i(0) = \theta_i^0,$$

where Ω_i is the natural frequency of ith oscillator and $K > 0$ is the coupling strength. We now give the definition of phase-frequency synchronization.

Definition 1.1. Let $\{\theta_i(t)\}_{i=1}^N$ be the solution of the Kuramoto model (1.1).

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(1) We say the oscillators converge to phase synchronization if

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0, \text{ for } i \neq j.$$

(2) We say the oscillators converge to frequency synchronization if

$$\lim_{t\to\infty} |\omega_i(t) - \omega_j(t)| = 0, \text{ for } i \neq j,$$

where $\omega_i(t) = \dot{\theta}_i(t)$ is the frequency of ith oscillator.

There has been a large amount of literature on the Kuramoto model since the remarkable work of Kurmamoto [14]. For further discussion, we refer to the survey papers [3, 19, 20]. Various estimates of the critical coupling strength for the Kuramoto models were presented in several papers—Ermentrout [9], Hemmen and Wreszinski [24], Jadbabaie et al. [12], and Verwoerd and Mason [25]—using tools such as a Lyapunov functional, spectral graph theory, control theory, and fixed point theory. The linear stability of the phase-locked state and rigorous characterization of the spectrum for the Kuramoto model was treated by Mirollo and Strogatz [21] and Aeyels and Rogge [2]. Bonilla et al. [4] analyzed the Kuramoto model with randomly distributed frequencies and subject to independent external white noises in the thermodynamic limit. De Smet and Aeyels [8] investigated partial entrainment in the Kuramoto-Sakaguchi model.

The work most closely related to this paper is that in [7, 10, 6]. More precisely, for the Kuramoto model (1.1) with identical oscillators (i.e., $\Omega_i = \Omega_j$), the authors in [10] established phase and frequency synchronization results provided that the initial phase configurations are distributed over an open half circle, i.e.,

$$\left\{ \theta = (\theta_1, \ldots, \theta_N) \in \mathbb{T}^N : \ \mathcal{D}(\theta) := \max_{1\le i,j\le N} |\theta_i - \theta_j| < \pi \right\}.$$

On the other hand, for the Kuramoto model (1.1) with non-identical oscillators, the frequency synchronization results were obtained by [7, 10] if the coupling strength K is larger than a certain critical value and the initial phase configurations are such that

$$\left\{ \theta = (\theta_1, \ldots, \theta_N) \in \mathbb{T}^N : \ \mathcal{D}(\theta) = \max_{1\le i,j\le N} |\theta_i - \theta_j| < \frac{\pi}{2} \right\}.$$

Subsequently, the authors of [6] found that the results mentioned above still hold for the initial phase configurations distributed over an open half circle. However, all these results are not applied to the case when the diameter of the initial phase configurations is strictly larger than π. Even for the identical oscillators, the frequency synchronization result is still unknown when initial phase configurations are distributed over the whole circle. Another natural problem is whether the aforementioned results hold for the generalized Kuramoto model with directed coupling topology.

The contributions of this paper are threefold: first, we consider the Kuramoto model with identical oscillators and show that the frequency synchronization can occur for all possible initial phase configurations distributed over the whole circle. Our method is based on Łojasiewicz inequality for gradient systems of analytic functions. As far as we know, this method is not used to study the Kuramoto model in the literature. Our second contribution is to extend the results in [6] to the case of directed coupling topology. We provide an explicitly sufficient condition to ensure frequency synchronization for the generalized Kuramoto model with directed coupling topology when the initial phase configurations are distributed over a half circle. Specially, for the Kuramoto model (1.1), the obtained sufficient condition reduces to the one in [6]. As a final contribution, we extend our second work to the case of switching topology and present a similar sufficient condition to guarantee frequency synchronization for the generalized Kuramoto model with switching topology.

The paper is organized as follows. In Section 2, we consider the Kuramoto model for identical oscillators and establish the frequency synchronization for initial phase configurations whose diameters are smaller than 2π. In Section 3, we first give a sufficient condition on initial phase configurations and coupling strength a_{ij} to ensure frequency synchronization for the generalized Kuramoto model with directed coupling topology and then extend this result to the case of switching topology.

2. Synchronization for identical oscillators

In this section, we consider the Kuramoto model for identical oscillators:

$$\dot{\theta}_i = \Omega + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1,\ldots,N, \tag{2.1}$$

subject to initial conditions

$$\theta_i(0) = \theta_i^0,$$

where θ is the phase of the ith oscillator, Ω is its natural frequency, and K is the coupling strength. Due to the real analyticity of the right hand side of model (2.1), we know that its global solution exists and is also real analytic (see, for example, Hale's book [11]).

Now we define mean values and their associated fluctuations for phase θ_i and frequency $\dot{\theta}_i = \omega_i$:

$$\theta_c = \frac{1}{N}\sum_{i=1}^{N}\theta_i \quad \text{and} \quad \omega_c = \frac{1}{N}\sum_{i=1}^{N}\omega_i,$$

$$\hat{\theta}_i = \theta_i - \theta_c \quad \text{and} \quad \hat{\omega}_i = \omega_i - \omega_c.$$

Then we have

$$\dot{\theta}_c = \omega_c \equiv \Omega$$

and

$$\dot{\hat{\theta}}_i = \frac{K}{N}\sum_{j=1}^{N}\sin(\hat{\theta}_j - \hat{\theta}_i) \qquad i = 1,\ldots,N, \tag{2.2}$$

subject to initial condition

$$\hat{\theta}_i(0) = \hat{\theta}_i^0,$$

with the condition

$$\sum_{i=1}^{N}\hat{\theta}_i = 0.$$

We are now ready to state our main result for system (2.2).

THEOREM 2.1. *Let* $\{\hat{\theta}_i(t)\}_{i=1}^N$ *be the solution of system* (2.2) *with all initial phase differences satisfying* $|\hat{\theta}_i^0 - \hat{\theta}_j^0| < 2\pi$ *for* $1 \le i, j \le N$. *Then* $\hat{\omega}_i(t) \to 0$ *as* $t \to \infty$. *Moreover, there exists* θ_{ij} *such that* $\hat{\theta}_i(t) - \hat{\theta}_j(t) \to \theta_{ij}$ *as* $t \to \infty$.

Note that the condition "$\hat{\omega}_i(t) \to 0$ as $t \to \infty$" is equivalent to the condition "$\omega_i(t) \to \omega_c = \Omega$ as $t \to \infty$" and $\hat{\theta}_i(t) - \hat{\theta}_j(t) = \theta_i(t) - \theta_j(t)$, from which the following result for (2.1) immediately follows.

COROLLARY 2.2. *Let* $\{\theta_i(t)\}_{i=1}^N$ *be the solution of system* (2.1) *satisfying the initial phase condition* $|\theta_i^0 - \theta_j^0| < 2\pi$ *for* $1 \le i, j \le N$. *Then* $\omega_i(t) \to \Omega$ *as* $t \to \infty$. *Moreover, there exists* θ_{ij} *such that* $\theta_i(t) - \theta_j(t) \to \theta_{ij}$ *as* $t \to \infty$.

REMARK 2.3. We here use a new method based on the Łojasiewicz inequality for gradient systems of analytic functions to study system (2.1). To the best of our knowledge, this inequality is first used to investigate the Kuramoto model. Compared with the existing work, the result of Corollary 2.2 holds for all initial phase configurations distributed over the whole circle, instead of an arc of length smaller than π. The results for identical oscillators in [10] can be easily derived from Corollary 2.2 when initial phase configurations are located on an arc whose length is smaller than π.

For Kuramoto model (2.1), the following example shows that we can not obtain phase synchronization, i.e., $\theta_{ij} = 0$ in Corollary 2.2 when the initial phase differences are lying in $[0, 2\pi)$.

EXAMPLE 2.4. Consider the Kuramoto model for three identical oscillators:

$$\dot{\theta}_1 = \Omega + \frac{K}{3}(\sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1)),$$
$$\dot{\theta}_2 = \Omega + \frac{K}{3}(\sin(\theta_1 - \theta_2) + \sin(\theta_3 - \theta_2)),$$
$$\dot{\theta}_3 = \Omega + \frac{K}{3}(\sin(\theta_1 - \theta_3) + \sin(\theta_2 - \theta_3)),$$

with initial condition $(\theta_1(0), \theta_2(0), \theta_3(0)) = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$. Then it is easy to see that the corresponding solution is $\theta_1(t) = \Omega t$, $\theta_2(t) = \frac{2\pi}{3} + \Omega t$, and $\theta_3(t) = \frac{4\pi}{3} + \Omega t$. Hence there is no phase synchronization.

To show Theorem 2.1, we need the following lemma.

LEMMA 2.5. *Let* $\{\hat{\theta}_i(t)\}_{i=1}^N$ *be the solution of system* (2.2) *with all initial phase differences satisfying* $|\hat{\theta}_i^0 - \hat{\theta}_j^0| < 2\pi$ *for* $1 \le i, j \le N$. *Then we have* $|\hat{\theta}_i(t)| < 2\pi$ *for* $t \ge 0$, $i = 1, \ldots, N$.

Proof. Assume that $\{\hat{\theta}_i(t)\}_{i=1}^N$ is the solution of system (2.2) with initial conditions $|\hat{\theta}_i^0 - \hat{\theta}_j^0| < 2\pi$ for $1 \le i, j \le N$. For $i, j \in \{1, \ldots, N\}$, there are only two cases for initial states: (i) $\hat{\theta}_i^0 = \hat{\theta}_j^0$, (ii) $\hat{\theta}_i^0 \ne \hat{\theta}_j^0$.

(i) For $\hat{\theta}_i^0 = \hat{\theta}_j^0$, consider the following equation:

$$\frac{d(\hat{\theta}_i - \hat{\theta}_j)}{dt} = \frac{K}{N} \sum_{l=1}^{N} \sin(\hat{\theta}_l - \hat{\theta}_i) - \sin(\hat{\theta}_l - \hat{\theta}_i + \hat{\theta}_i - \hat{\theta}_j),$$

subject to the initial condition $\hat{\theta}_i^0 - \hat{\theta}_j^0 = 0$. By the existence and uniqueness of solutions for the above equation, it follows that the unique solution is $\hat{\theta}_i(t) - \hat{\theta}_j(t) \equiv 0$. That is, if $\hat{\theta}_i^0 = \hat{\theta}_j^0$, then $\hat{\theta}_i(t) = \hat{\theta}_j(t)$.

(ii) For $\hat{\theta}_i^0 \neq \hat{\theta}_j^0$, we now make the following claim: for all $t \geq 0$,

$$\hat{\theta}_i(t) \neq \hat{\theta}_j(t) \quad \text{and} \quad |\hat{\theta}_i(t) - \hat{\theta}_j(t)| < 2\pi.$$

We first prove $\hat{\theta}_i(t) \neq \hat{\theta}_j(t)$. If not, let t_0 be the first collision time, i.e.,

$$\hat{\theta}_i(t) \neq \hat{\theta}_j(t),\ t \in [0, t_0) \quad \text{and} \quad \hat{\theta}_i(t_0) = \hat{\theta}_j(t_0).$$

Then from case (i) we obtain that $\hat{\theta}_i(t) = \hat{\theta}_j(t)$ for $t \geq t_0$. However, by Corollary 1.2.6 in [15], if two analytic functions $\hat{\theta}_i(t)$ and $\hat{\theta}_j(t)$ are equal on some open set, then the two functions must be equal on the whole existence domain. Hence we get a contradiction which implies that $\hat{\theta}_i(t) \neq \hat{\theta}_j(t)$, for all $t \geq 0$.

We now show that

$$|\hat{\theta}_i(t) - \hat{\theta}_j(t)| < 2\pi \quad \text{for } t \geq 0.$$

Assume the contrary. Then there exist i and j such that

$$\hat{\theta}_i(t_0) = \hat{\theta}_j(t_0) + 2\pi \ \text{ and } \ \hat{\theta}_i(t) < \hat{\theta}_j(t) + 2\pi, \ \text{ for } t \in [0, t_0). \tag{2.3}$$

It follows that

$$\begin{aligned}
\frac{d\hat{\theta}_i}{dt}(t_0) &= \frac{K}{N}\sum_{l=1}^{N} \sin(\hat{\theta}_l(t_0) - \hat{\theta}_i(t_0)) \\
&= \frac{K}{N}\sum_{l=1}^{N} \sin(\hat{\theta}_l(t_0) - \hat{\theta}_j(t_0) - 2\pi) \\
&= \frac{K}{N}\sum_{l=1}^{N} \sin(\hat{\theta}_l(t_0) - \hat{\theta}_j(t_0)) \\
&= \frac{d\hat{\theta}_j}{dt}(t_0).
\end{aligned}$$

That is, $\hat{\omega}_i(t_0) = \hat{\omega}_j(t_0)$. By differentiating system (2.2), we have

$$\begin{aligned}
\frac{d\hat{\omega}_i}{dt}(t_0) &= \frac{K}{N}\sum_{l=1}^{N} \cos(\hat{\theta}_l(t_0) - \hat{\theta}_i(t_0))(\hat{\omega}_l(t_0) - \hat{\omega}_i(t_0)) \\
&= \frac{K}{N}\sum_{l=1}^{N} \cos(\hat{\theta}_l(t_0) - \hat{\theta}_j(t_0) - 2\pi)(\hat{\omega}_l(t_0) - \hat{\omega}_i(t_0)) \\
&= \frac{K}{N}\sum_{l=1}^{N} \cos(\hat{\theta}_l(t_0) - \hat{\theta}_j(t_0))(\hat{\omega}_l(t_0) - \hat{\omega}_j(t_0)) \\
&= \frac{d\hat{\omega}_j}{dt}(t_0).
\end{aligned}$$

Similarly, we can obtain

$$\frac{d^n\hat{\omega}_i}{dt^n}(t_0) = \frac{d^n\hat{\omega}_j}{dt^n}(t_0), \text{ for } n \geq 2.$$

This fact, together with analyticity of $\hat{\omega}_i$ and $\hat{\omega}_j$, and Corollary 1.2.5 in [15], shows that

$$\hat{\omega}_i(t) = \hat{\omega}_j(t),\ t \in (0,T) \quad \text{for } T > t_0,$$

which is in contradiction with the fact (2.3). This shows that

$$|\hat{\theta}_i(t) - \hat{\theta}_j(t)| < 2\pi \quad \text{for } t \geq 0.$$

Based on the above analysis for cases (i) and (ii), it follows that for $t \geq 0$, $i = 1,\dots,N$,

$$\begin{aligned}|\hat{\theta}_i(t)| = |\theta_i(t) - \theta_c(t)| &\leq \frac{1}{N}\sum_{j=1}^{N}|\theta_i(t) - \theta_j(t)| \\ &= \frac{1}{N}\sum_{j=1}^{N}|\hat{\theta}_i(t) - \hat{\theta}_j(t)| < 2\pi,\end{aligned}$$

which completes the proof of Lemma 2.5. □

Proof of Theorem 2.1.

Proof. We will apply Theorem A.2 to show Theorem 2.1. To this end, we write (2.2) in the form of gradient system (A.1). Let $\hat{\theta} = (\hat{\theta}_1,\dots,\hat{\theta}_N)$ and

$$f(\hat{\theta}) = 1 - \frac{K}{2N}\sum_{1 \leq i,j \leq N}\cos(\hat{\theta}_i - \hat{\theta}_j).$$

Then (2.2) can be rewritten as

$$\dot{\hat{\theta}} = -\nabla f(\hat{\theta}). \tag{2.4}$$

From Lemma 2.5, it follows that the solution $\hat{\theta}(t)$ is bounded, which implies that there exist a vector $\hat{\theta}_\infty = (\hat{\theta}_{1\infty},\dots,\hat{\theta}_{N\infty})$ and some sequence $t_n \to \infty$ such that $\hat{\theta}(t_n) \to \hat{\theta}_\infty$ as $n \to \infty$. We now use Theorem A.2 to conclude that $\hat{\theta}(t) \to \hat{\theta}_\infty$ as $t \to \infty$, which implies that there exists θ_{ij} such that $\hat{\theta}_i(t) - \hat{\theta}_j(t) \to \theta_{ij}$ as $t \to \infty$. In addition, vector $\hat{\theta}_\infty$ satisfies

$$\sum_{1=j}^{N}\sin(\hat{\theta}_{j\infty} - \hat{\theta}_{i\infty}) = 0, \quad i = 1,\dots,N,$$

and $\hat{\omega}_i(t) = \dot{\hat{\theta}}_i(t) = \frac{K}{N}\sum_{j=1}^{N}\sin(\hat{\theta}_j - \hat{\theta}_i) \to 0$ when $t \to \infty$. This completes the proof of Theorem 2.1.□

3. Synchronization for non-identical oscillators

In this section, we study the generalized Kuramoto model for non-identical oscillators. We first consider the generalized Kuramoto model with fixed topology and derive a sufficient condition to ensure exponential frequency synchronization, and then we extend this result to the case of switching topology.

3.1. Fixed topology. We consider the generalized Kuramoto model for non-identical oscillators:

$$\dot{\theta}_i = \Omega_i + \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \tag{3.1}$$

subject to initial conditions

$$\theta_i(0) = \theta_i^0$$

where Ω_i is the natural frequency of the ith oscillator and $a_{ij} \geq 0$ is the coupling strength.

For the solution $\{\theta_i(t)\}_{i=1}^N$ of (3.1), define extremal phases $\theta_M(t)$, $\theta_m(t)$ and phase diameter $D(\theta(t))$:

$$\theta_M(t) := \max_{1\leq i\leq N} \theta_i(t),\ \theta_m(t) := \min_{1\leq i\leq N} \theta_i(t),\ \text{and } \mathcal{D}(\theta(t)) := \theta_M(t) - \theta_m(t).$$

Similarly, define

$$\omega_M(t) := \max_{1\leq i\leq N} \omega_i(t),\ \omega_m(t) := \min_{1\leq i\leq N} \omega_i(t),\ \text{and } \mathcal{D}(\omega(t)) := \omega_M(t) - \omega_m(t).$$

The natural frequency diameter is defined by

$$\mathcal{D}(\Omega) := \max_{1\leq i,j\leq N} |\Omega_i - \Omega_j|.$$

We now state the main result for system (3.1).

THEOREM 3.1. *Let $\{\theta_i(t)\}_{i=1}^N$ be the solution of system* (3.1) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi,\ \mathcal{C} > \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)},$$

where

$$\mathcal{C} = \min_{i\neq j} \left\{ a_{ij} + a_{ji} + \sum_{\substack{k=1\\k\neq i,j}}^{N} \min\{a_{ik}, a_{jk}\} \right\}.$$

Then we have, for some time $T > 0$,

$$\mathcal{D}(\omega(t)) \leq \mathcal{D}(\omega(T)) e^{-\mathcal{C}(\cos \mathcal{D}^\infty)(t-T)}, \quad \text{for } t \geq T,$$

where $\mathcal{D}^\infty$ satisfies

$$\sin \mathcal{D}(\theta^0) = \sin \mathcal{D}^\infty,\ \mathcal{D}^\infty \in \left(0, \frac{\pi}{2}\right).$$

In the case of identical coupling strength, i.e., $a_{ij} = \frac{K}{N}$, Theorem 3.1 reduces to the following known result.

COROLLARY 3.2. *Let $\{\theta_i(t)\}_{i=1}^N$ be the solution of system* (3.1) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi,\ K > \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)}.$$

Then we have, for some time $T>0$,

$$\mathcal{D}(\omega(t))\leq\mathcal{D}(\omega(T))e^{-K(\cos\mathcal{D}^{\infty})(t-T)},\quad \textit{for } t\geq T,$$

where $\mathcal{D}^{\infty}$ *satisfies*

$$\sin\mathcal{D}(\theta^0)=\sin\mathcal{D}^{\infty},\ \mathcal{D}^{\infty}\in\left(0,\frac{\pi}{2}\right).$$

REMARK 3.3. The result of Corollary 3.2 has been obtained in [6] (see Theorem 3.1 of [6]), which improves the main results in [7, 10]. Note that in Theorem 3.1, the coupling matrix $A=(a_{ij})$ can be asymmetric.

To give the proof of Theorem 3.1, we need the following lemma.

LEMMA 3.4. *Let the conditions of Theorem 3.1 be satisfied. Then we have, for some time* $T>0$,

$$\mathcal{D}(\theta(t))\leq\mathcal{D}^{\infty},\ t\geq T. \tag{3.2}$$

Proof. Let $I_{\max}(t)$ be the set of indices of phases $\theta_1(t),\dots,\theta_N(t)$ that are equal to the counterclockwise maximum, and define $I_{\min}(t)$ similarly. Note that the upper Dini derivative of $D(\theta(t))$ along the system (3.1) is defined by [16, Lemma 2.2]

$$D^+\mathcal{D}(\theta(t))=\limsup_{h\downarrow 0}\frac{\mathcal{D}(\theta(t+h))-\mathcal{D}(\theta(t))}{h}=\dot{\theta}_{\bar{M}}(t)-\dot{\theta}_{\bar{m}}(t),$$

where $\bar{M}$ and $\bar{m}$ are indices which have the properties that

$$\dot{\theta}_{\bar{M}}(t)=\max\{\dot{\theta}_M(t):\ M\in I_{\max}(t)\}\text{ and }\dot{\theta}_{\bar{m}}(t)=\min\{\dot{\theta}_m(t):\ m\in I_{\min}(t)\}.$$

By the assumption $\mathcal{C}>\frac{\mathcal{D}(\Omega)}{\sin\mathcal{D}(\theta^0)}$, we can obtain

$$\begin{aligned}D^+\mathcal{D}(\theta^0)&=\Omega_{\bar{M}}+\sum_{k=1}^{N}a_{\bar{M}k}\sin(\theta_k^0-\theta_{\bar{M}}^0)-\Omega_{\bar{m}}-\sum_{k=1}^{N}a_{\bar{m}k}\sin(\theta_k^0-\theta_{\bar{m}}^0)\\&\leq\Omega_{\bar{M}}-\Omega_{\bar{m}}-(a_{\bar{M}\bar{m}}+a_{\bar{m}\bar{M}})\sin(\theta_{\bar{M}}^0-\theta_{\bar{m}}^0)\\&\qquad-\sum_{\substack{k=1\\k\neq\bar{M},\bar{m}}}^{N}\min\{a_{\bar{M}k},a_{\bar{m}k}\}[\sin(\theta_{\bar{M}}^0-\theta_k^0)+\sin(\theta_k^0-\theta_{\bar{m}}^0)]\\&\leq\mathcal{D}(\Omega)-\min_{i\neq j}\left\{a_{ij}+a_{ji}+\sum_{\substack{k=1\\k\neq i,j}}^{N}\min\{a_{ik},a_{jk}\}\right\}\sin(\theta_{\bar{M}}^0-\theta_{\bar{m}}^0)\\&=\mathcal{D}(\Omega)-\mathcal{C}\sin\mathcal{D}(\theta^0)<0,\end{aligned}\tag{3.3}$$

where the second inequality follows from the fact that

$$\begin{aligned}&\sin(\theta_{\bar{M}}^0-\theta_k^0)+\sin(\theta_k^0-\theta_{\bar{m}}^0)\\&=2\sin\left(\frac{\theta_{\bar{M}}^0-\theta_{\bar{m}}^0}{2}\right)\cos\left(\frac{\theta_{\bar{M}}^0+\theta_{\bar{m}}^0}{2}-\theta_k^0\right)\\&\geq 2\sin\frac{\theta_{\bar{M}}^0-\theta_{\bar{m}}^0}{2}\cos\frac{\theta_{\bar{M}}^0-\theta_{\bar{m}}^0}{2}\\&=\sin(\theta_{\bar{M}}^0-\theta_{\bar{m}}^0).\end{aligned}$$

This implies that $\mathcal{D}(\theta(t))$ is strictly decreasing for $t\in[0,\varepsilon)$ when ε is sufficiently small. We first claim that

$$\mathcal{D}(\theta(t))<\mathcal{D}(\theta^0) \quad \text{for } t>0. \tag{3.4}$$

Assume the contrary, so that there exists a first time $t_0>0$ such that

$$\mathcal{D}(\theta(t))<\mathcal{D}(\theta^0) \quad \text{for } t\in[0,t_0) \quad \text{and } \mathcal{D}(\theta(t_0))=\mathcal{D}(\theta^0).$$

On the other hand, we note that

$$\begin{aligned} D^-\mathcal{D}(\theta(t_0)) &\le \mathcal{D}(\Omega)-\mathcal{C}\sin\mathcal{D}(\theta(t_0)) \\ &= \mathcal{D}(\Omega)-\mathcal{C}\sin\mathcal{D}(\theta^0)<0. \end{aligned}$$

This is in contradiction with the fact that $\mathcal{D}(\theta(t))<\mathcal{D}(\theta(t_0))$ for $t\in[0,t_0)$. Hence we prove our claim (3.4). We next prove (3.2).

Case (i) If $\mathcal{D}(\theta^0)\in\left(0,\frac{\pi}{2}\right)$, then one has $\mathcal{D}(\theta^0)=\mathcal{D}^\infty$. The desired result (3.2) then follows from (3.4).

Case (ii) For $\mathcal{D}(\theta^0)\in\left(\frac{\pi}{2},\pi\right)$, we have $\mathcal{D}^\infty<\mathcal{D}(\theta^0)$. When $\mathcal{D}^\infty\le\mathcal{D}(\theta(t))\le\mathcal{D}(\theta^0)$, it follows that $D^+\mathcal{D}(\theta(t))$ takes the form

$$\begin{aligned} D^+\mathcal{D}(\theta(t)) &\le \mathcal{D}(\Omega)-\mathcal{C}\sin\mathcal{D}(\theta(t)) \\ &\le \mathcal{D}(\Omega)-\mathcal{C}\sin\mathcal{D}(\theta^0)<0, \end{aligned}$$

where the second inequality $\sin\mathcal{D}(\theta(t))\ge\sin\mathcal{D}(\theta^0)$ is used. Integrating the above differential inequality, we get

$$\mathcal{D}(\theta(t))\le\mathcal{D}(\theta^0)+(\mathcal{D}(\Omega)-\mathcal{C}\sin\mathcal{D}(\theta^0))t,$$

from which it follows that $\mathcal{D}(\theta(t))\le\mathcal{D}^\infty$, for $t\ge T$ with

$$T=\frac{\mathcal{D}(\theta^0)-\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}(\theta^0)-\mathcal{D}(\Omega)},$$

which completes the proof of Lemma 3.4. ∎

Proof of Theorem 3.1.

Proof. Differentiating system (3.1) with respect to t, we obtain

$$\dot{\omega}_i=\sum_{j=1}^N a_{ij}\cos(\theta_j-\theta_i)(\omega_j-\omega_i).$$

By Lemma 3.4, we know $\mathcal{D}(\theta(t))\le\mathcal{D}^\infty<\frac{\pi}{2}$ for $t\ge T$, which implies that $\cos\mathcal{D}(\theta(t))\ge\cos\mathcal{D}^\infty>0$ for $t\ge T$. Using this fact, similar to $D^+\mathcal{D}(\theta(t))$, for $D^+\mathcal{D}(\omega(t))$ one has, for $t\ge T$,

$$
\begin{aligned}
D^+\mathcal{D}(\omega(t)) &= \sum_{k=1}^{N} a_{\bar{M}k}\cos(\theta_k(t)-\theta_{\bar{M}}(t))(\omega_k(t)-\omega_{\bar{M}}(t)) \\
&\qquad - \sum_{k=1}^{N} a_{\bar{m}k}\cos(\theta_k(t)-\theta_{\bar{m}}(t))(\omega_k(t)-\omega_{\bar{m}}(t)) \\
&\le -\left[(a_{\bar{M}\bar{m}}+a_{\bar{m}\bar{M}})\cos(\theta_{\bar{M}}(t)-\theta_{\bar{m}}(t))\right](\omega_{\bar{M}}(t)-\omega_{\bar{m}}(t)) \\
&\qquad - \sum_{\substack{k=1\\ k\neq \bar{M},\bar{m}}}^{N} \min\{a_{\bar{M}k}\cos(\theta_k(t)-\theta_{\bar{M}}(t)), a_{\bar{m}k}\cos(\theta_k(t)-\theta_{\bar{m}}(t))\} \\
&\qquad\qquad \times\left[(\omega_{\bar{M}}(t)-\omega_k(t))+(\omega_k(t)-\omega_{\bar{m}}(t))\right] \\
&\le -\left[(a_{\bar{M}\bar{m}}+a_{\bar{m}\bar{M}})\cos\mathcal{D}^{\infty}\right](\omega_{\bar{M}}(t)-\omega_{\bar{m}}(t)) \\
&\qquad - \sum_{\substack{k=1\\ k\neq \bar{M},\bar{m}}}^{N} \min\{a_{\bar{M}k}, a_{\bar{m}k}\}\cos\mathcal{D}^{\infty}\left[(\omega_{\bar{M}}(t)-\omega_k(t))+(\omega_k(t)-\omega_{\bar{m}}(t))\right] \\
&\le -\min_{i\neq j}\left\{a_{ij}+a_{ji}+\sum_{\substack{k=1\\ k\neq i,j}}^{N}\min\{a_{ik},a_{jk}\}\right\}\cos\mathcal{D}^{\infty}(\omega_{\bar{M}}(t)-\omega_{\bar{m}}(t)) \\
&= -\mathcal{C}\cos\mathcal{D}^{\infty}\mathcal{D}(\omega(t)),
\end{aligned} \tag{3.5}
$$

which leads to

$$
\mathcal{D}(\omega(t)) \le \mathcal{D}(\omega(T))e^{-\mathcal{C}\cos\mathcal{D}^{\infty}(t-T)}, \quad \text{for } t\ge T.
$$

This completes the proof of Theorem 3.1.□

Since $\{\theta_i(t)\}_{i=1}^N$ with $\mathcal{D}(\theta(t))=0$ is not the solution of the Kuramoto model for non-identical oscillators, there is no phase synchronization for system (3.1). The following proposition indicates that we can force the phase diameter to be smaller than any given positive constant ε by enlarging the value of the coupling strength $\mathcal{C}$.

PROPOSITION 3.5. *Let the conditions of Theorem 3.1 be satisfied. For any given $\varepsilon>0$, if*

$$
\mathcal{C} > \frac{\mathcal{D}(\Omega)\mathcal{D}^{\infty}}{\varepsilon\sin\mathcal{D}^{\infty}},
$$

then there exists a time $T'>0$ such that

$$
\mathcal{D}(\theta(t)) \le \varepsilon, \quad \text{for } t\ge T'.
$$

Proof. First, recalling Lemma 3.4, we know that there exists some time $T>0$ such that

$$
\mathcal{D}(\theta(t)) \le \mathcal{D}^{\infty}, \quad \text{for } t\ge T.
$$

If $\varepsilon \geq \mathcal{D}^\infty$, then the desired result follows. We next assume that $\varepsilon < \mathcal{D}^\infty$. Similar to (3.3), for $t \geq T$ we get

$$\begin{aligned} D^+\mathcal{D}(\theta(t)) &\leq \mathcal{D}(\Omega) - \mathcal{C}\sin\mathcal{D}(\theta(t)) \\ &\leq \mathcal{D}(\Omega) - \mathcal{C}\frac{\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}\mathcal{D}(\theta(t)), \end{aligned} \tag{3.6}$$

where the above second inequality follows from the fact that

$$\frac{\sin x}{x} \geq \frac{\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}, \quad \text{for } x \in [0, \mathcal{D}^\infty].$$

From (3.6), it follows that

$$\begin{aligned} \mathcal{D}(\theta(t)) &\leq \left(\mathcal{D}(\theta(T)) - \frac{\mathcal{D}(\Omega)\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}^\infty}\right) e^{-\frac{\mathcal{C}\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-T)} + \frac{\mathcal{D}(\Omega)\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}^\infty} \\ &\leq \varepsilon - \frac{\mathcal{D}(\Omega)\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}^\infty} + \frac{\mathcal{D}(\Omega)\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}^\infty} \\ &= \varepsilon \quad \text{for } t \geq T'. \end{aligned}$$

Here we use the fact

$$\lim_{t\to\infty} e^{-\frac{\mathcal{C}\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-T)} = 0$$

to find a time $T' > T$ satisfying

$$e^{-\frac{\mathcal{C}\sin\mathcal{D}^\infty}{\mathcal{D}^\infty}(t-T)} \leq \varepsilon - \frac{\mathcal{D}(\Omega)\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}^\infty}, \quad \text{for } t \geq T'.$$

This completes the proof of Proposition 3.5. □

3.2. Switching topology. We consider the generalized Kuramoto model with switching topology for non-identical oscillators:

$$\dot{\theta}_i = \Omega_i + \sum_{j=1}^N a_{ij}^{\sigma(t)} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N, \tag{3.7}$$

subject to initial conditions

$$\theta_i(0) = \theta_i^0,$$

where Ω_i is the natural frequency of the ith oscillator and $a_{ij}^{\sigma(t)} \geq 0$ is the coupling strength. The switching law $\sigma(t)\colon [0,\infty) \to \mathcal{P} = \{1, \ldots, p\}$ is a piecewise constant function which is continuous from the right. The switching times are $\{t_\ell\}_{\ell=1}^\infty$: $0 = t_0 < t_1 < \cdots < t_\ell < \cdots$ and $\lim_{\ell\to\infty} t_\ell = \infty$.

For $s \in \mathcal{P}$, let

$$\mathcal{C}_s = \min_{i\neq j}\left\{ a_{ij}^s + a_{ji}^s + \sum_{\substack{k=1 \\ k\neq i,j}}^N \min\{a_{ik}^s, a_{jk}^s\} \right\}$$

and $\tilde{\mathcal{C}} = \min_{s\in\mathcal{P}} \mathcal{C}_s$. We have the following result for the generalized Kuramoto model with switching topology (3.7).

THEOREM 3.6. *Let* $\{\theta_i(t)\}_{i=1}^N$ *be the solution of system* (3.7) *satisfying*

$$0 < \mathcal{D}(\theta^0) < \pi, \ \tilde{\mathcal{C}} > \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}(\theta^0)}.$$

Then we have, for some time $T > 0$,

$$\mathcal{D}(\omega(t)) \le \mathcal{D}(\omega(T)) e^{-\mathcal{C} \cos \mathcal{D}^\infty (t-T)}, \quad \text{for } t \ge T,$$

where $\mathcal{D}^\infty$ *satisfies*

$$\sin \mathcal{D}(\theta^0) = \sin \mathcal{D}^\infty, \ \mathcal{D}^\infty \in \left(0, \frac{\pi}{2}\right).$$

To show Theorem 3.6 we need the following lemma, which is similar to Lemma 3.4.

LEMMA 3.7. *Let the conditions of Theorem 3.6 be satisfied. Then we have, for some time* $T > 0$,

$$\mathcal{D}(\theta(t)) \le \mathcal{D}^\infty, \ t \ge T. \tag{3.8}$$

Proof. Let $\sigma(t) = s_0$ for $t \in [0, t_1)$. Then system (3.7) can be written as

$$\dot{\theta}_i = \Omega_i + \sum_{j=1}^N a_{ij}^{s_0} \sin(\theta_j - \theta_i), \ t \in [0, t_1),$$

subject to initial conditions

$$\theta_i(0) = \theta_i^0.$$

Similar to (3.3), by the assumption that $\tilde{\mathcal{C}} > \frac{\mathcal{D}(\Omega)}{\sin \theta^0}$, we have

$$\begin{aligned} D^+ \mathcal{D}(\theta^0) &\le \mathcal{D}(\Omega) - \mathcal{C}_{s_0} \sin \mathcal{D}(\theta^0) \\ &\le \mathcal{D}(\Omega) - \tilde{\mathcal{C}} \sin \mathcal{D}(\theta^0) < 0. \end{aligned}$$

This implies that $\mathcal{D}(\theta(t))$ is strictly decreasing for $t \in [0, \varepsilon)$ when ε is sufficiently small. We first claim that

$$\mathcal{D}(\theta(t)) < \mathcal{D}(\theta^0) \quad \text{for } t \in (0, t_1). \tag{3.9}$$

Assume the contrary, so that there exists a first time $t' \in (0, t_1)$ such that

$$\mathcal{D}(\theta(t)) < \mathcal{D}(\theta^0) \quad \text{for } t \in (0, t') \quad \text{and } \mathcal{D}(\theta(t')) = \mathcal{D}(\theta^0).$$

On the other hand, we note that

$$\begin{aligned} D^- \mathcal{D}(\theta(t_0)) &\le \mathcal{D}(\Omega) - \mathcal{C}_{s_0} \sin \mathcal{D}(\theta(t_0)) \\ &= \mathcal{D}(\Omega) - \tilde{\mathcal{C}} \sin \mathcal{D}(\theta^0) < 0. \end{aligned}$$

This is in contradiction with the fact that $\mathcal{D}(\theta(t)) < \mathcal{D}(\theta(t'))$ for $t \in [0, t')$. Hence we prove (3.9). Now we show

$$\mathcal{D}(\theta(t)) < \mathcal{D}(\theta^0) \quad \text{for } t \in [t_1, t_2).$$

Let $\sigma(t)=s_1$ for $t\in[t_1,t_2)$. Then system (3.7) can be written as

$$\dot{\theta}_i=\Omega_i+\sum_{j=1}^{N}a_{ij}^{s_1}\sin(\theta_j-\theta_i),\ t\in[t_1,t_2).$$

We now proceed as for $t\in[0,t_1)$ to arrive at

$$\mathcal{D}(\theta(t))<\mathcal{D}(\theta(t_1))<\mathcal{D}(\theta^0)\quad\text{for } t\in(t_1,t_2).$$

By induction, we can obtain that

$$\mathcal{D}(\theta(t))<\mathcal{D}(\theta^0)\quad\text{for all } t>0. \tag{3.10}$$

We now show (3.8).

Case (i) If $\mathcal{D}(\theta^0)\in\left(0,\frac{\pi}{2}\right)$, then one has $\mathcal{D}(\theta^0)=\mathcal{D}^\infty$. The desired result (3.8) then follows from (3.10).

Case (ii) For $\mathcal{D}(\theta^0)\in\left(\frac{\pi}{2},\pi\right)$, we have $\mathcal{D}^\infty<\mathcal{D}(\theta^0)$. If $\mathcal{D}^\infty\le\mathcal{D}(\theta(t))\le\mathcal{D}(\theta^0)$, then $\sin\mathcal{D}(\theta(t))\ge\sin\mathcal{D}(\theta^0)$. Let $\sigma(t)=s_k$ for $t\in[t_k,t_{k+1})$. As a result, for $t\in[t_k,t_{k+1})$,

$$\begin{aligned}D^+\mathcal{D}(\theta(t))&\le\mathcal{D}(\Omega)-\mathcal{C}_{s_k}\sin\mathcal{D}(\theta(t))\\&\le\mathcal{D}(\Omega)-\tilde{\mathcal{C}}\sin\mathcal{D}(\theta^0).\end{aligned}$$

This implies that

$$D^+\mathcal{D}(\theta(t))\le\mathcal{D}(\Omega)-\tilde{\mathcal{C}}\sin\mathcal{D}(\theta^0),\quad\text{for all } t\ge0.$$

Integrating the above differential inequality, we get

$$\mathcal{D}(\theta(t))\le\mathcal{D}(\theta^0)+(\mathcal{D}(\Omega)-\tilde{\mathcal{C}}\sin\mathcal{D}(\theta^0))t,$$

from which it follows that $\mathcal{D}(\theta(t))\le\mathcal{D}^\infty$, for $t\ge T$ with

$$T=\frac{\mathcal{D}(\theta^0)-\mathcal{D}^\infty}{\mathcal{C}\sin\mathcal{D}(\theta^0)-\mathcal{D}(\Omega)}.$$

This completes the proof of Lemma 3.7. ∎

Proof of Theorem 3.6. By Lemma 3.7, we know $\mathcal{D}(\theta(t))\le\mathcal{D}^\infty<\frac{\pi}{2}$ for $t\ge T$, which implies that $\cos\mathcal{D}(\theta(t))\ge\cos\mathcal{D}^\infty>0$ for $t\ge T$. Let $[t_k,t_{k+1})$ be such that $T\in[t_k,t_{k+1})$. Note that $\sigma(t)=s_k$ when $t\in[t_k,t_{k+1})$. For $t\in[T,t_{k+1})$, differentiate system (3.7) with respect to t to obtain

$$\dot{\omega}_i=\sum_{j=1}^{N}a_{ij}^{s_k}\cos(\theta_j-\theta_i)(\omega_j-\omega_i).$$

Similar to (3.5), one has

$$D^+\mathcal{D}(\omega(t))\le-\mathcal{C}_{s_k}\cos\mathcal{D}^\infty\mathcal{D}(\omega(t))\le-\tilde{\mathcal{C}}\cos\mathcal{D}^\infty\mathcal{D}(\omega(t)),\quad\text{for } t\in[T,t_{k+1}),$$

which implies that

$$D^+\mathcal{D}(\omega(t))\le-\tilde{\mathcal{C}}\cos\mathcal{D}^\infty\mathcal{D}(\omega(t)),\quad\text{for } t\ge T.$$

This leads to

$$\mathcal{D}(\omega(t)) \leq \mathcal{D}(\omega(T)) e^{-\tilde{\mathcal{C}} \cos \mathcal{D}^{\infty}(t-T)}, \quad \text{for } t \geq T.$$

This completes the proof of Theorem 3.6.

Analogous to Proposition 3.5, the following proposition shows that we can enlarge the value of the coupling strength $\tilde{\mathcal{C}}$ so as to make the phase diameter be smaller than any given positive constant ε. We omit its proof since it is very similar to the proof of Proposition 3.5.

PROPOSITION 3.8. *Let the conditions of Theorem 3.6 be satisfied. For any given $\varepsilon > 0$, if*

$$\tilde{\mathcal{C}} > \frac{\mathcal{D}(\Omega)\mathcal{D}^{\infty}}{\varepsilon \sin \mathcal{D}^{\infty}},$$

then there exists a time $T' > 0$ such that

$$\mathcal{D}(\theta(t)) \leq \varepsilon, \quad \text{for } t \geq T'.$$

Appendix A. Łojasiewicz inequality. In the sixties, Łojasiewicz [17] (see also [18]) proved the following fundamental inequality for gradient systems of analytic functions.

THEOREM A.1. *Suppose that $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ is analytic in the open set D. Let $\bar{x}$ be a critical point of f, i.e., $\nabla f(\bar{x}) = 0$. Then there exist $r > 0$, $c > 0$, and $\theta \in (0,1)$ such that*

$$\|\nabla f(x)\| \geq c|f(x) - f(\bar{x})|^{\theta} \qquad \text{for all } \; x \in B(\bar{x}, r).$$

Based on the above Łojasiewicz inequality, Łojasiewicz obtained the following result. For the convenience of the reader, we give the proof.

THEOREM A.2. *Consider the gradient system*

$$\dot{x}(t) = -\nabla f(x) \tag{A.1}$$

where $x(t) \in \mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ is real analytic function. If $x(t)$ has a limit point x_0, i.e., $x(t_n) \to x_0$ for some sequence $t_n \to \infty$, then we have $x(t) \to x_0$ as $t \to \infty$. Moreover, $x_0 \in M = \{x\colon \nabla f(x) = 0\}$, and therefore $\dot{x}(t) \to 0$ as $t \to \infty$.

Proof. For the solution $x(t)$ of (A.1), we have

$$\frac{d}{dt} f(x(t)) = \langle \nabla f, \dot{x}(t) \rangle = -\|\nabla f(x(t))\|^2 \leq 0. \tag{A.2}$$

Thus, the function $f(x(t))$ is decreasing with respect to t. Since $x(t_n) \to x_0$, we have $f(x(t_n)) \to f(x_0)$, which implies that $f(x(t)) \to f(x_0)$ as $t \to \infty$. Without loss of generality, we assume $f(x_0) = 0$ (if not, replace $f(x(t))$ by $f(x(t)) - f(x_0)$). That is, $\lim_{t\to\infty} f(x(t)) = 0$. Next, we prove $\lim_{t\to\infty} x(t) = x_0$.

By Theorem A.1, there exist $r > 0$, $c > 0$ and $\theta \in (0,1)$ such that

$$\|\nabla f(x)\| \geq c|f(x)|^{\theta}, \quad \text{for } x \in B(x_0, r). \tag{A.3}$$

Let $h(t)=[f(x(t))]^{1-\theta}$. Then $h(t)$ is decreasing function of time t and $\lim_{t\to 0}h(t)=0$. As a result, we can find a time t_N such that for any $t\geq t_N$,

$$\frac{|h(t)-h(t_N)|}{c(1-\theta)}\leq\frac{r}{3} \tag{A.4}$$

and

$$\|x(t_N)-x_0\|<\frac{r}{3}.$$

Define

$$T=\inf\{t\geq t_N:\ x(t)\notin B(x_0,r)\}.$$

Then $T>t_N$. Next we prove that $T=+\infty$.

For $t\in[t_N,T)$, by (A.2) and (A.3) we have

$$\begin{aligned}h'(t)&=(1-\theta)[f(x(t))]^{-\theta}\frac{d}{dt}f(x(t))\\&=-(1-\theta)[f(x(t))]^{-\theta}\|\nabla f(x(t))\|^2\\&\leq -c(1-\theta)\|\nabla f(x(t))\|,\end{aligned}$$

which implies that

$$\int_{t_N}^{t}\|\nabla f(x(s))\|ds\leq-\frac{1}{c(1-\theta)}\int_{t_N}^{t}h'(s)ds=\frac{1}{c(1-\theta)}(h(t_N)-h(t))\leq\frac{r}{3}.$$

Therefore,

$$\int_{t_N}^{t}\|\dot{x}(s)\|ds=\int_{t_N}^{t}\|\nabla f(x(s))\|ds\leq\frac{r}{3}\quad\text{for } t\in[t_N,T). \tag{A.5}$$

Suppose that $T<+\infty$. Then

$$\begin{aligned}\|x(T)-x_0\|&=\left\|x(t_N)+\int_{t_N}^{T}\dot{x}(s)ds-x_0\right\|\\&\leq\|x(t_N)-x_0\|+\int_{t_N}^{T}\|\dot{x}(s)\|ds\\&\leq\frac{2}{3}r.\end{aligned}$$

This is in contradiction with the definition of T. Hence we have $T=+\infty$. It follows from (A.5) that $\|\dot{x}(t)\|\in L^1(0,+\infty)$, which implies that $\lim_{t\to+\infty}x(t)$ exists. By $\lim_{n\to+\infty}x(t_n)=x_0$, we have $\lim_{t\to+\infty}x(t)=x_0$. From the fact

$$\int_{t_N}^{\infty}\|\nabla f(x(s))\|ds<+\infty,$$

it follows that

$$\liminf_{t\to+\infty}\|\nabla f(x(t))\|=0.$$

Thus, the continuity of $\nabla f(x)$ implies that $\nabla f(x_0)=0$, i.e., $x_0\in M$. This completes the proof. ∎

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