NON-UNIQUENESS AND PRESCRIBED ENERGY FOR THE CONTINUITY EQUATION*

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Abstract. In this note, we provide new non-uniqueness examples for the continuity equation by constructing infinitely many weak solutions with prescribed energy.

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1. Introduction

In this paper we consider the continuity equation for a bounded scalar function $u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ with a bounded divergence-free vector field $b: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$:

\[
\partial_t u + \text{div}(ub) = 0, \\
\text{div } b = 0.
\]

This equation appears in various problems of mathematical physics, in particular fluid mechanics and kinetic theory. In the smooth setting (and assuming suitable integrability) the energy,

\[
E(t) := \int_{\mathbb{R}^d} u^2(t,x) \, dx
\]

of the solution $u$ is conserved:

\[
\frac{d}{dt} E(t) = 0.
\]

Indeed, since $b$ is divergence-free, by multiplying (1.1) with $u$, using the chain rule, and integrating over $\mathbb{R}^d$, one immediately obtains (1.3).

In many applications, one has to study (1.1) in a nonsmooth setting. Roughly speaking, since (1.1) is linear, the conservation of energy (1.3) implies uniqueness of weak solutions to the corresponding initial-value problem for (1.1). In fact, conservation of energy is a consequence of the so-called renormalization property which was proved in [14] for any vector field $b$ with Sobolev regularity and later extended by Ambrosio in [6] to the case when $b$ has bounded variation. We refer to [15, 3] for a detailed review of recent results in this direction.

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On the other hand, when the regularity of the vector field $b$ is too low, the conservation of energy (1.3) fails in general. In a nonsmooth setting, several counterexamples to the uniqueness, and therefore to the conservation of energy, are known; see [5, 12, 13, 2, 1]. A similar phenomenon occurs in the context of the Euler equations. For example, in the papers [21, 22, 16], weak solutions of the Euler equations were constructed with compact support in space time.

In particular, the example in [13] gives a bounded vector field $b$ and a bounded scalar field $u$, which satisfy (1.1) and (1.2), such that

$$\mathcal{E}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0. \end{cases} \quad (1.4)$$

In this paper, we provide a more general profile for the energy. Our results are also connected to the chain rule problem for the divergence operator; see [4, 7, 11].

We construct such pairs $(b, u)$ using the method of convex integration, and our techniques are similar to the ones used in [16, 24]. The latter reference contains an appendix giving a general framework for convex integration, but, for the problem at hand, we need to consider a nonlinear constraint that depends on the points in the domain (as was the case, e.g., in [17], albeit in a different functional setting). For this reason we adapt the framework from [24] to this more general situation (see §2). We then apply this abstract framework to the specific situation of the continuity equation (see §3).

Finally, let us mention [10, 23, 8], where results were obtained by convex integration, respectively, that yield as a byproduct counterexamples to the energy conservation for continuity equations. However, in these works, the energy profile is always piecewise constant.

2. Differential inclusions with non-constant nonlinear constraint

We start with the so-called Tartar framework (cf. e.g. [16]). Consider a system of $m$ linear partial differential equations

$$\sum_{i=1}^{D} A_i \partial_i z = 0 \quad (2.1)$$

in an open set $\mathcal{D} \subset \mathbb{R}^D$ where $A_i$ are constant $m \times n$ matrices and $z: \mathcal{D} \to \mathbb{R}^n$. Consider a nonlinear constraint

$$z(y) \in K_y \quad (2.2)$$

for a.e. $y$ in $\mathcal{D}$ where $K_y \subset \mathbb{R}^n$ is a compact set for any $y \in \mathcal{D}$. For any $y \in \mathcal{D}$, let $U_y := \text{int conv } K_y$, where with conv we denote the convex hull of the set $K_y$ and with int we denote its interior. Let $\mathcal{W} \subset \mathcal{D}$ be a bounded open set.

**Definition 2.1 (Subsolutions).** We say that $z \in L^2(\mathcal{D})$ is a subsolution of (2.1) and (2.2) if $z$ is a weak solution of (2.1) in $\mathcal{D}$, $z$ is continuous on $\mathcal{W}$, (2.2) holds for a.e. $y \in \mathcal{D} \setminus \mathcal{W}$, and

$$z(y) \in U_y \quad (2.3)$$
for any $y \in \mathcal{U}$.

**Definition 2.2 (Localized plane waves/wave cone).** A set $\Lambda \subset \mathbb{R}^n$ is called a wave cone if there exists a constant $C > 0$ such that for any $\bar{z} \in \Lambda$ there exists a sequence $w_k \in C_0^\infty(B_1(0);\mathbb{R}^n)$ solving (2.1) in $\mathbb{R}^D$ such that

- $\text{dist}(w_k(x),[-\bar{z},\bar{z}]) \to 0$ for all $x \in B_1(0)$ uniformly as $k \to \infty$,
- $w_k \to 0$ in $L^2$ as $k \to \infty$,
- $\int |w_k|^2 \, dy > C|\bar{z}|^2$ for all $k \in \mathbb{N}$.

In the above definition we denoted the segment with endpoints $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $[x,y] := \text{conv}\{x,y\}$. The functions $w_k$ are called localized plane waves. We make the following assumptions:

**Assumption 2.1 (Existence of the wave cone).** There exists a wave cone $\Lambda$ that is dense in $\mathbb{R}^n$.

Let $\mathcal{K}$ denote the set of all compact subsets of $\mathbb{R}^n$, endowed with the Hausdorff metric $d_H$. It is well-known that $\mathcal{K}$ is a complete metric space.

**Assumption 2.2 (Continuity of the nonlinear constraint).** The map $f: \mathcal{U} \ni y \mapsto K_y \in \mathcal{K}$ is continuous and bounded in the Hausdorff metric.

Our main abstract result is the following:

**Theorem 2.1.** Suppose that assumptions 2.1 and 2.2 hold. Suppose that $z_0$ is a subsolution of (2.1) and (2.2). Then there exist infinitely many weak solutions $z \in L^2(\mathcal{D})$ of (2.1) which agree with $z_0$ a.e. on $\mathcal{D} \setminus \mathcal{U}$ and satisfy (2.2) for a.e. $y \in \mathcal{D}$.

### 2.1. Geometric preliminaries.

The next lemma shows that compact subsets of the interior of the convex hull of a compact set $K$ are stable with respect to sufficiently small perturbations of $K$ in the Hausdorff metric.

**Lemma 2.1.** Let $K \subset \mathbb{R}^n$ be a compact set. Then for any compact set $C \subset \text{int conv} K$, there exists $\varepsilon > 0$ such that for any compact set $K' \subset \mathbb{R}^n$ with $d_H(K,K') < \varepsilon$ we have

$$C \subset \text{int conv} K'.$$

![Figure 2.1. An illustration of Lemma 2.1 in the case when $K = \{L,M,N\}$ and $K' = \{L',M',N'\}$.](image)
Proof. Since $\text{int conv } K$ is open, for any point $x \in C$ there exists a simplex $S_x$ with vertices $\{v_i\}_{i=1..n+1} \subset \text{conv } K$ such that $x$ belongs to the inner open simplex,

$$I_x := \left\{ \sum_{i=1}^{n+1} \lambda_i v_i : \lambda_i \in \left( \frac{1}{2(n+1)} \right), \sum_{i=1}^{n+1} \lambda_i = 1, i = 1..n+1 \right\}.$$  

Since $C$ is a compact set and the inner simplices $\{I_x\}_{x \in C}$ cover $C$, we can extract a finite subcover $\{I_{x_k}\}_{k=1..m}$ of $C$. Let us fix $k \in 1..m$ and consider the simplex $S := S_{x_k}$ with vertices $\{v_i\}_{i=1..n+1} \subset \text{conv } K$. Let $I := I_{x_k}$ denote the corresponding inner simplex. If $\varepsilon < \text{dist}(\partial I, \partial S)$, then for any points $v'_i \in B_\varepsilon(v_i), i = 1..n+1$, one has

$$I \subset \text{conv}\{v'_1,v'_2,\ldots,v'_{n+1}\}.$$  

(2.4)

Observe that for any $\varepsilon > 0$ and $i = 1..n+1$, the ball $B_\varepsilon(v_i)$ contains a point $v'_i \in \text{conv } K'$. Indeed, by Caratheodory’s theorem, $v_i = \sum_{j=1}^{n+1} \mu_j z_j$ for some $z_j \in K$ and $\mu_j \in [0,1]$ with $\sum_{j=1}^{n+1} \mu_j = 1$. Since $d_H(K,K') < \varepsilon$ there exist points $z'_j \in K'$ such that $z'_j \in B_\varepsilon(z_j)$ where $j = 1..n+1$. Let

$$v'_i := \sum_{j=1}^{n+1} \mu_j z'_j,$$

then $|v_i - v'_i| \leq \sum_{j=1}^{n+1} \mu_j |z_j - z'_j| < \varepsilon$. Hence by (2.4) we have $I \subset \text{conv}\{v'_1,v'_2,\ldots,v'_{n+1}\}$ provided that $\varepsilon$ is small enough. But $v'_i \in \text{conv } K'$, hence $I \subset \text{conv } K'$. Since $I$ is open, we can also write $I \subset \text{int conv } K'$. Since we have finitely many simplices, we can choose $\varepsilon > 0$ in such a way that the inclusion $I_{x_k} \subset \text{int conv } K'$ holds for any $k = 1..m$ (provided that $d_H(K,K') < \varepsilon$). Then

$$C \subset \bigcup_{k=1..m} I_{x_k} \subset \text{int conv } K'.$$

We will also need the following elementary lemma:

**Lemma 2.2.** Suppose that $z \in C(\mathcal{U};\mathbb{R}^n)$ where $\mathcal{U} \subset \mathbb{R}^D$ is an open set. Suppose that for any $y \in \mathcal{U}$ we have a compact set $K_y \subset \mathbb{R}^n$ and the function $y \mapsto K_y$ is continuous in the Hausdorff metric. Then the function $F : y \mapsto \text{dist}(z(y),K_y)$ is continuous on $\mathcal{U}$.

Proof. One can prove directly that the function $(z,K) \mapsto \text{dist}(z,K)$ is continuous on $\mathbb{R}^n \times \mathcal{U}$. The function $y \mapsto (z(y),K_y)$ is continuous in view of the assumptions. Hence the function $F$ is continuous as a composition of continuous functions.

In general, the distance from a point $z$ to a compact set $K$ does not control from below the distance from $z$ to the boundary of $\text{conv } K$. However the following lemma shows that there exists a segment inside $\text{int conv } K$ with midpoint $z$ and length controlled from below by $\text{dist}(z,K)$:

**Lemma 2.3 (Geometric lemma).** Let $K \subset \mathbb{R}^n$ be a compact set. For any $z \in \text{int conv } K$, there exists $\bar{z} \in \mathbb{R}^n$ such that

- $[z - \bar{z}, z + \bar{z}] \subset \text{int conv } K$
- $|\bar{z}| \geq \frac{1}{2n} \text{dist}(z,K)$

(This is exactly Lemma 5.3 from [18].)
2.2. Convex integration. The following lemma is the main building block of the convex integration scheme.

**Lemma 2.4 (Perturbation lemma).** Suppose that assumptions 2.1 and 2.2 hold and that \( z \) is a subsolution of (2.1) and (2.2) such that

\[
\int_{\mathcal{W}} \mathrm{dist}^2(z(y), K_y) dy = \varepsilon > 0.
\]

Then there exists \( \delta = \delta(\varepsilon) > 0 \) and a sequence \( \{z_k\}_{k \in \mathbb{N}} \) of subsolutions of (2.1) and (2.2) such that

- \( z_k = z \) on \( \mathcal{D} \setminus \mathcal{W} \) for any \( k \in \mathbb{N} \)
- \( \int_{\mathcal{W}} |z - z_k|^2 dy \geq \delta \) for any \( k \in \mathbb{N} \)
- \( z_k \to z \) in \( L^2(\mathcal{W}) \) as \( k \to \infty \).

**Proof.**

**Step 1.** Let \( y \in \mathcal{W} \). Since \( z(y) \in U_y \), we can apply Lemma 2.3 to obtain \( \tilde{z}_+(y) \) such that

\[
[z(y) - \tilde{z}_+(y), z(y) + \tilde{z}_+(y)] \subset U_y,
\]

\[
|\tilde{z}_+(y)| \geq \frac{1}{2n} \text{dist}(z(y), K_y).
\]

Since \( \Lambda \) is dense in \( \mathbb{R}^n \) and \( U_y \) is open, we can find \( \tilde{z}(y) \in \Lambda \) such that

\[
[z(y) - \tilde{z}(y), z(y) + \tilde{z}(y)] \subset U_y, \quad (2.5)
\]

\[
|\tilde{z}(y)| \geq \frac{1}{4n} \text{dist}(z(y), K_y). \quad (2.6)
\]

Due to (2.5), there exists \( \rho(y) > 0 \) such that

\[
[z(y) - \tilde{z}(y), z(y) + \tilde{z}(y)] + B_{2\rho(y)}(0) \subset U_y.
\]

Hence, using Assumption 2.2, Lemma 2.1, and the continuity of \( z \), we can find \( R(y) > 0 \) such that

\[
[z(x) - \tilde{z}(y), z(x) + \tilde{z}(y)] + B_{\rho(y)}(0) \subset U_x \quad (2.7)
\]

for all \( x \in B_{R(y)}(y) \subset \mathcal{W} \). Moreover, in view of Lemma 2.2, we can choose \( R(y) \) in such a way that

\[
\text{dist}(z(x), K_x) \leq 2 \text{dist}(z(y), K_y) \quad (2.8)
\]

for all \( x \in B_{R(y)}(y) \). Using Assumption 2.1 for any fixed \( y \in \mathcal{W} \), we can construct a sequence \( \{w_{y,k}\}_{k \in \mathbb{N}} \subset C^\infty_0(B_1(0)) \) such that

- \( w_{y,k}(x) \in [-\tilde{z}(y), \tilde{z}(y)] + B_{\rho(y)}(0) \) for all \( x \in B_1(0) \) and \( k \in \mathbb{N} \),
- \( w_{y,k} \to 0 \) in \( L^2 \) as \( k \to \infty \),
- \( \int |w_{y,k}|^2 dx > C|\tilde{z}(y)|^2 \) for all \( k \in \mathbb{N} \).

**Step 2.** Let \( \varepsilon := \int_{\mathcal{W}} \text{dist}^2(z(y), K_y) dy \). The balls \( \{B_r(y) : y \in \mathcal{W}, r \in (0, R(y))\} \) cover \( \mathcal{W} \), so using Vitali’s covering theorem (see e.g. [9], Theorem 5.5.2) and the absolute continuity of the Lebesgue integral, we can find finitely many points \( \{y_i\}_{i=1}^N \subset \mathcal{W} \) and radii \( r_i \in (0, R(y_i)) \) such that

\[
\sum_{i=1}^N \int_{B_{r_i}} \text{dist}^2(z(y), K_y) dy > \frac{1}{2} \varepsilon, \quad (2.9)
\]
where the balls \( B_i := B_{r_i}(y_i) \) are pairwise disjoint.

For each \( i = 1..N \), let us introduce the scaled and translated perturbations \( w_{i,k}(x) := w_{y_i,k}(\frac{x-y_i}{r_i}) \). These functions belong to \( C_0^\infty(B_i) \) and satisfy
(i) \( w_{i,k}(x) \in [-z(y_i), z(y_i)] + B_{\rho(y_i)}(0) \) for all \( x \in B_i, k \in \mathbb{N}, i = 1..N \);
(ii) \( w_{i,k} \to 0 \) in \( L^2 \) as \( k \to \infty \) (for each fixed \( i = 1..N \));
(iii) \( \int |w_{i,k}|^2 \, dx > C|z(y_i)|^2 \mathcal{L}^D(B_i) \) for all \( k \in \mathbb{N} \).

In view of (i) and (2.7), we have \( z(x) + w_{i,k}(x) \in U_x \) for all \( x \in \mathcal{W} \) and \( i = 1..N \), hence \( z + w_{i,k} \in X_0 \). Since the balls \( B_i \) are pairwise disjoint, the function
\[
z_k := z + \sum_{i=1}^N w_{i,k}
\]
also belongs to \( X_0 \).

Using successively (iii), (2.6), (2.8), and (2.9) we obtain:
\[
\int_{\mathcal{W}} |z - z_k|^2 \, dy = \sum_{i=1}^N \int_{B_i} |w_{i,k}(y)|^2 \, dy
\]
\[
\overset{(iii)}{>} C \sum_{i=1}^N |z(y_i)|^2 \mathcal{L}^D(B_i)
\]
\[
\overset{(2.6)}{\geq} \frac{C}{16n^2} \sum_{i=1}^N \text{dist}^2(z(y_i), K_{y_i}) \mathcal{L}^D(B_i)
\]
\[
= \frac{C}{16n^2} \sum_{i=1}^N \int_{B_i} \text{dist}^2(z(y_i), K_{y_i}) \, dx
\]
\[
\overset{(2.8)}{>} \frac{C}{32n^2} \sum_{i=1}^N \int_{B_i} \text{dist}^2(z(x), K_x) \, dx
\]
\[
\overset{(2.9)}{> \frac{C}{64n^2} \varepsilon}.
\]

It remains to observe that, since \( N \) is finite and the points \( y_i \) are fixed, we have \( z_k \rightharpoonup z \) in \( L^2 \) as \( k \to \infty \).

\[\square\]

2.3. Proof of Theorem 2.1. We are now ready to prove our main abstract theorem.

Proof of Theorem 2.1. Let \( X_0 \) denote a set of all subsolutions of (2.1) and (2.2) which agree with \( z_0 \) on \( \mathcal{D} \setminus \mathcal{W} \). Let \( X \) be the closure of \( X_0 \) in the weak topology of \( L^2(\mathcal{W}) \) endowed with the corresponding induced weak topology. Clearly any \( z \in X \) solves (2.1) and satisfies (2.2) a.e. on \( \mathcal{D} \setminus \mathcal{W} \).

For any \( z \in X \), let us define
\[
I(z) := \int_{\mathcal{W}} |z(y)|^2 \, dy.
\]

This functional is a Baire-1 function on \( X \). Indeed, for any \( j \in \mathbb{N} \) let
\[
I_j(z) := \int_{\{y \in \mathcal{W} : \text{dist}(y, \partial U) > 1/j\}} |(\omega_{1/j} * z)(y)|^2 \, dy
\]
where for any $\varepsilon > 0$ we denote by $\omega_\varepsilon(\cdot) = \varepsilon^{-D} \omega(\cdot/\varepsilon)$ the standard convolution kernel. Then for any $j \in \mathbb{N}$, the functional $I_j$ is continuous on $X$, and for any $z \in X$, we have $I_j(z) \to I(z)$ as $j \to \infty$.

In view of Assumption 2.2, $X$ is a bounded subset of $L^2(\mathcal{W})$. Since the weak topology is metrizable on the norm-bounded subsets of $L^2(\mathcal{W})$, we can consider $X$ as a complete metric space with some metric $d_X$.

Then by Baire category theorem (see also Theorem 7.3 from [20]), the set

$$Y := \{ z \in X : I \text{ is continuous at } z \}$$

is residual in $X$ (and hence is infinite). We claim that $z \in Y$ implies $J(z) = 0$ where

$$J(z) := \int_{\mathcal{W}} \text{dist}^2(z(y), K_y) dy.$$ 

Indeed, suppose that $J(z) = \varepsilon > 0$ for some $z \in Y$. Let $z_j \in X_0$ be a sequence such that $z_j \to z$ in $L^2(\mathcal{W})$ as $j \to \infty$. Since $I$ is continuous at $z$, this implies that $I(z_j) \to I(z)$ and consequently $z_j \to z$ in $L^2(\mathcal{W})$ as $j \to \infty$.

Then in view of Assumption 2.2, we have $J(z_j) \to J(z)$ as $j \to \infty$, and hence, without loss of generality, we can assume that $J(z_j) > \varepsilon/2$ for all $j \in \mathbb{N}$.

Applying Lemma 2.4 to $z_j$ for each $j \in \mathbb{N}$, we can find $\tilde{z}_j \in X_0$ such that

$$d_X(\tilde{z}_j, z_j) < 2^{-j} \quad \text{and} \quad \int_{\mathcal{W}} |\tilde{z}_j - z_j|^2 dy \geq \delta > 0 \text{ where } \delta = \delta(\varepsilon) \text{ is independent of } j.$$

Since $d_X(\tilde{z}_j, z) \leq d_X(\tilde{z}_j, z_j) + d_X(z_j, z) \to 0$ as $j \to \infty$ we also have $\tilde{z}_j \to z$ in $L^2$. Since $z$ is a point of continuity of $I$, we also have $z_j \to z$ in $L^2(\mathcal{W})$ as $j \to \infty$. But then $\tilde{z}_j - z_j \to 0$ in $L^2(\mathcal{W})$ which contradicts the construction of $\tilde{z}_j$.

\section*{3. Application to the continuity equation}

In this section we apply our abstract framework to the case of the continuity equation.

\textbf{Theorem 3.1.} Suppose that $d \geq 2$. Let $E : \mathbb{R} \to \mathbb{R}$ be a non-negative bounded function which is continuous on some bounded open interval $I \subset \mathbb{R}$ and vanishes on $\mathbb{R} \setminus I$. Then there exist infinitely many bounded, compactly supported $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy (1.1) and (1.2) in the sense of distributions and are such that

$$\int_{\mathbb{R}^2} u^2(t, x) dx = E(t) \quad \text{for a.e. } t \in I.$$

\textbf{Remark 3.1.} It is well-known that a representative of $u$ can be chosen such that the map $t \mapsto u(t, \cdot)$ is continuous with values in $L^2$ equipped with the weak topology. Then the question arises whether the assertion in the theorem holds for every, and not just almost every, time $t$. We expect this to be true; indeed this should follow by methods similar to those of [17]. We will, however, not pursue this question further in this article.

\textbf{Remark 3.2.} When $d = 2$ and $f$ is a characteristic function of an interval, the statement of Theorem 3.1, essentially, follows from the example constructed in [13]. This particular case of Theorem 3.1 was also proved in [19] using the convex integration method.

\textbf{Remark 3.3.} A similar problem can be addressed for more general equation of the form $\text{div}(u \mathbf{B}) = 0$ instead of (1.1). For this equation the problem is stated as follows:
given a distribution \( g \), is it possible to construct compactly supported bounded functions 
\( u : \mathbb{R}^n \to \mathbb{R}, \ B : \mathbb{R}^n \to \mathbb{R}^n \) such that 
\[
\text{div}(uB) = 0, \quad \text{div}B = 0, \quad \text{div}(u^2B) = g?
\]

This is related to the so-called \textit{chain rule problem} for the divergence operator [4]. When 
\( n = 2 \) such a construction is not possible for \( g \neq 0 \) in view of [7], but for \( n \geq 3 \) it is possible 
and is obtained using convex integration and rank-2 laminates in [11].

Let us put the continuity equation in the framework of the previous section. Fix a 
bounded open set \( \Omega \subset \mathbb{R}^d \). Let \( \mathcal{H} := I \times \Omega \) and 
\[
F(t,x) := \frac{E(t)}{\mathcal{D}^d(\Omega)} \mathbf{1}_\Omega(x),
\]

where \( \mathbf{1}_\Omega \) denotes the characteristic function of \( \Omega \).

We consider equations (1.1) and (1.2) as a linear system 
\[
\begin{align*}
\partial_t u + \text{div}_x \mathbf{m} &= 0, \\
\text{div}_x \mathbf{b} &= 0
\end{align*}
\]
in \( \mathcal{D} := \mathbb{R} \times \mathbb{R}^d \) with \( u : \mathcal{D} \to \mathbb{R}, \ \mathbf{m} : \mathcal{D} \to \mathbb{R}^d \) and \( \mathbf{b} : \mathcal{D} \to \mathbb{R}^d \) such that 
\( z := (u, \mathbf{m}, \mathbf{b}) \) satisfies the constraint 
\[
z(y) \in K_y := \begin{cases} 
\{(u, \mathbf{m}, \mathbf{b}) : \mathbf{m} = u\mathbf{b}, \ |\mathbf{b}| = 1, \ u^2 = F(y) \} & \text{if } y \in \mathcal{U} \\
0 & \text{if } y \in \mathcal{D} \setminus \mathcal{U}
\end{cases}
\]

for a.e. \( y = (x,t) \in \mathcal{D} \).

Suppose that \( z = (u, \mathbf{m}, \mathbf{b}) \in L^\infty(\mathcal{D}) \) satisfies (3.1) and (3.2) in the sense of distributions and, moreover, (3.3) holds a.e. in \( \mathcal{D} \). Then the couple \( (u, \mathbf{b}) \) satisfies the assertion of Theorem 3.1.

Let us check the assumption of Theorem 3.1.

**Lemma 3.1.** Suppose that \( A, B \subset \mathbb{R}^n \) are compact sets and \( r > 0 \) is such that 
\begin{itemize}
    \item for any \( z \in A \) there exists \( z' \in B \cap B_r(z) \)
    \item for any \( z \in B \) there exists \( z' \in A \cap B_r(z) \)
\end{itemize}

Then \( d_H(A, B) < r \).

**Proof.** Suppose that \( d_H(A, B) \geq r \). Then without loss of generality, we can assume 
that there exists \( z \in A \) such that for any \( z' \in B \) we have \( z \notin B_r(z') \). But then the ball 
\( B_r(z) \) cannot contain any point of \( B \) which leads to a contradiction. \( \square \)

**Lemma 3.2.** If \( F : \mathcal{U} \to \mathbb{R} \) is continuous, bounded, and non-negative then the map 
\( y \to K_y \) is continuous and bounded (w.r.t. \( d_H \)) on \( \mathcal{U} \).

**Proof.** Let \( f(y) := \sqrt{F(y)} \). Let us fix \( y \in \mathcal{U} \). For any \( \varepsilon > 0 \) let \( \delta > 0 \) be such that 
\( |f(y) - f(y')| < \varepsilon \) for any \( y' \in B_\delta(y) \subset \mathcal{U} \). Let us prove that 
\( d_H(K_y, K_{y'}) < 2\varepsilon \) for all \( y' \in B_\delta(y) \). For any \( z \in K_y \), there exist \( \sigma \in \{\pm 1\} \) and \( \mathbf{b} \in \mathbb{R}^d \) with \( |\mathbf{b}| = 1 \) such that 
\( z = (\sigma f(y), \sigma f(y)\mathbf{b}, \mathbf{b}) \). Then \( z' := (\sigma f(y'), \sigma f(y')\mathbf{b}, \mathbf{b}) \) belongs to \( K_{y'} \) and 
\( |z - z'| \leq 2|f(y) - f(y')| \). Hence there exists \( z' \in K_{y'} \cap B_{2\varepsilon}(z) \). Similarly, for any \( z' \in K_{y'} \) 
there exist \( \sigma \in \{\pm 1\} \) and \( \mathbf{b} \in \mathbb{R}^d \) with \( |\mathbf{b}| = 1 \) such that 
\( z' = (\sigma f(y'), \sigma f(y')\mathbf{b}, \mathbf{b}) \). Then 
\( z := (\sigma f(y), \sigma f(y)\mathbf{b}, \mathbf{b}) \) belongs to \( K_y \) and \( |z - z'| \leq 2|f(y) - f(y')| \). Hence there exists 
\( z \in K_y \cap B_{2\varepsilon}(z') \). Therefore by Lemma 3.1 we have \( d_H(K_y, K_{y'}) < 2\varepsilon \). \( \square \)
Lemma 3.3. Assumption 2.1 holds for the system (3.1)–(3.3).

Proof. Let \( \phi : \mathcal{D} \to \mathbb{R} \) be a non-negative smooth function such that \( 0 \leq \phi \leq 1 \) on \( \mathcal{D} \), \( \phi = 0 \) on \( \mathcal{D} \setminus B_1(0) \), and \( \phi = 1 \) on \( B_{1/2}(0) \).

Part 1. Suppose that \( d > 2 \). Let us show that Assumption 2.1 holds with \( \Lambda = \mathbb{R}^{2d+1} \).

Fix \( \bar{u} \in \mathbb{R}, \bar{m} \in \mathbb{R}^d \), and \( \bar{b} \in \mathbb{R}^d \) and let \( \bar{z} = (\bar{u}, \bar{m}, \bar{b}) \). Since \( d > 2 \), there exists a unit vector \( \bar{n} \in \mathbb{R}^d \) such that \( \bar{n} \cdot \bar{m} = \bar{n} \cdot \bar{b} = 0 \). Denote \( \bar{a} = (\bar{u}, \bar{m}) \). For any \( k \in \mathbb{N} \), define \( \bar{a}_k : \mathcal{D} \to \mathbb{R}^{d+1} \) by

\[
\bar{a}_k(y) := \bar{a}(\bar{n} \cdot \nabla_y (\phi \Pi_k)) - \bar{n}(\bar{a} \cdot \nabla_y (\phi \Pi_k))
\]

where \( y = (t, x) \) and

\[
\Pi_k(y) := \frac{\sin(k\bar{n} \cdot y)}{k}.
\]

Observe that

\[
\text{div}_y \bar{a}_k = (\bar{a} \cdot \nabla_y)(\bar{n} \cdot \nabla_y (\phi \Pi_k)) - (\bar{n} \cdot \nabla_y)(\bar{a} \cdot \nabla_y (\phi \Pi_k)) = 0.
\]

Let \( (u_k, m_k) \) denote the components of \( \bar{a}_k \); then by the equation above we have \( \partial_t u_k + \text{div}_x m_k = 0 \).

Similarly, let

\[
b_k(t, x) := \bar{b}(\bar{n} \cdot \nabla_x (\phi \Pi_k)) - \bar{n}(\bar{b} \cdot \nabla_x (\phi \Pi_k)).
\]

Then arguing as above, \( \text{div} b_k = 0 \).

Now we introduce \( w_k := (u_k, m_k, b_k) \). Then

\[
w_k(y) = \bar{z}\phi \cos(k\bar{n} \cdot y) + f\Pi_k
\]

where \( f \) does not depend on \( k \) and vanishes on \( B_{1/2}(0) \).

On the other hand,

\[
\int_{\mathcal{D}} |w_k|^2 \, dy \geq \int_{B_{1/2}(0)} |w_k|^2 \, dy = \int_{B_{1/2}(0)} |\bar{z}|^2 \cos^2(k\bar{n} \cdot y) \, dy
\]

\[
= \int_{B_{1/2}(0)} |\bar{z}|^2 \frac{1 + \cos(2k\bar{n} \cdot y)}{2} \, dy \geq \frac{|\bar{z}|^2}{4} |B_{1/2}(0)|
\]

provided that \( k \) is sufficiently large.

Part 2. Suppose that \( d = 2 \) and fix \( \bar{z} = (\bar{u}, \bar{m}, \bar{b}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) with \( \bar{u} \neq 0 \). Let us look for a localized plane wave in the following form:

\[
w_k = (a_k, b_k)
\]

with

\[
a_k(y) = \nabla_y \times \left( \phi A \frac{\sin(kn \cdot y)}{k} \right)
\]

\[
b_k(t, x) = \nabla^\perp_x \left( \phi \frac{\sin(kn \cdot (t, x))}{k} \right)
\]
where \( n = (n_t, n_x) \in \mathbb{R} \times \mathbb{R}^2 \) and \( A \in \mathbb{R}^3 \) are to be chosen and \( k \in \mathbb{N} \). Then, by construction,

\[
\text{div}_y a_k = 0, \quad \text{div}_x b_k = 0.
\]

Then, we get

\[
w_k = \hat{z} \phi \cos(kn \cdot y) + f \frac{\sin(kn \cdot y)}{k}
\]

where \( \hat{z} = (A \times n, n^\perp_x) \) and \( f \) does not depend on \( k \) and vanishes on \( B_{1/2}(0) \).

In order to have \( \hat{z} = \bar{z} \), the vectors \( A \) and \( n \) must satisfy

\[
A \times n = (\bar{u}, \bar{m}),
\]

\[
n_x^\perp = \bar{b}.
\]

From the second equation we immediately obtain that \( n_x = -\bar{b}^\perp \). Since \( \bar{u} \neq 0 \) there exists \( n_y \) such that \( n \perp (\bar{u}, \bar{m}) \). Then, we can always find \( A \) such that the first equation is satisfied. It remains to observe that the estimate (3.4) also holds in the considered case. We thus have verified Assumption 2.1 for \( \Lambda = \mathbb{R}^5 \setminus \{ \bar{u} = 0 \} \).

**Proof of Theorem 3.1.** By symmetry of \( K_y \) for any \( y \in \mathcal{W} \), we have \( 0 \in \text{int} \text{conv} K_y \). On the other hand, \( K_y = \{ 0 \} \) for any \( y \in \mathcal{D} \setminus \mathcal{W} \). Therefore \( u \equiv 0 \), \( m \equiv 0 \), and \( b \equiv 0 \) is a subsolution of (3.1)–(3.3). Then the result follows from Lemma 2.2, Lemma 3.3, and Theorem 2.1.

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