

GODUNOV SCHEME FOR MAXWELL'S EQUATIONS WITH KERR NONLINEARITY*

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Abstract. We study the Godunov scheme for a nonlinear Maxwell model arising in nonlinear optics, the Kerr model. This is a hyperbolic system of conservation laws with some eigenvalues of variable multiplicity, that are neither genuinely nonlinear nor linearly degenerate. The solution of the Riemann problem for the full-vector 6×6 system is constructed and proven to exist for all data. This solution is compared to the one of the reduced transverse magnetic model. The scheme is implemented in one and two space dimensions. The results are very close to the ones obtained by a Kerr–Debye relaxation approximation.

Key words. Godunov, Riemann problem, finite volumes, relaxation, Kerr model, Kerr–Debye model.

AMS subject classifications. 65M08, 35L65, 35L67, 78-04.

1. Introduction

In nonlinear optics, the propagation of electromagnetic waves in a crystal can be modeled by the so-called Kerr and Kerr–Debye models. Denoting by E and H the electric and magnetic fields, and by D and B the electric and magnetic displacements, one writes the three dimensional Maxwell's equations as

$$\begin{cases} \partial_t D - \operatorname{curl} H = 0, \\ \partial_t B + \operatorname{curl} E = 0, \end{cases}$$

with $\operatorname{div} D = \operatorname{div} B = 0$, and the constitutive relations as

$$\begin{cases} B = \mu_0 H, \\ D = \epsilon_0 E + P, \end{cases}$$

where P is the nonlinear polarization and μ_0 and ϵ_0 are the free space permeability and permittivity.

If the medium exhibits an instantaneous response, then one can use a Kerr model, denoting by ϵ_r the relative permittivity,

$$P = P_K = \epsilon_0 \epsilon_r |E|^2 E.$$

See, for example, [16] for further details. In that case, Maxwell's equations read as a 6×6 quasilinear system of conservation laws:

$$\begin{cases} \partial_t D - \operatorname{curl} H = 0, \\ \partial_t H + \mu_0^{-1} \operatorname{curl}(\mathbf{P}(D)) = 0, \end{cases} \quad (1.1)$$

where \mathbf{P} is the reciprocal function of \mathbf{D} ,

$$\mathbf{D}(E) = \epsilon_0(1 + \epsilon_r |E|^2)E.$$

*Received: June 3, 2014; accepted (in revised form): September 20, 2014. Communicated by Richard Tsai.

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Denoting

$$q(e) = \epsilon_0(e + \epsilon_r e^3), \quad e \in \mathbb{R}, \quad p = q^{-1}, \tag{1.2}$$

we have

$$E = \mathbf{P}(D) = \frac{D}{\epsilon_0(1 + \epsilon_r p^2(|D|))}, \quad |E| = p(|D|). \tag{1.3}$$

If (D, H) is solution of (1.1), then $\partial_t(\operatorname{div} D) = \partial_t(\operatorname{div}(\mu_0 H)) = 0$, so at the theoretical level, the divergence conditions have to be satisfied for the initial data only.

Our aim here is to construct an accurate and efficient scheme for the 6×6 Kerr model (1.1). In particular, we have to be able to approximate the shocks which, even with smooth initial data, can appear in finite time; see [5]. Hence, well-known finite difference schemes such as the FDTD method introduced by K.S. Yee [19] (see [21, 20, 2, 3, 14]) become problematic in this case. Very few papers deal with the approximation of this model. One of the only contributions is that of A. de la Bourdonnaye [7] where a third order Roe solver is constructed. More recently, a relaxation scheme was developed in [10] by using the so-called Kerr–Debye model: assuming that the medium exhibits a finite response time $\tau > 0$ and setting

$$P = P_{KD} = \epsilon_0 \chi E, \quad \partial_t \chi + \frac{1}{\tau} \chi = \frac{1}{\tau} \epsilon_r |E|^2,$$

one deals with a quasilinear hyperbolic system with source

$$\begin{cases} \partial_t D_\tau - \operatorname{curl} H_\tau = 0, \\ \partial_t H_\tau + \mu_0^{-1} \operatorname{curl} E_\tau = 0, \quad D_\tau = \epsilon_0(1 + \chi_\tau) E_\tau, \\ \partial_t \chi_\tau = \frac{1}{\tau} (\epsilon_r |E_\tau|^2 - \chi_\tau). \end{cases} \tag{1.4}$$

Let $U_\tau = (D_\tau, H_\tau, \chi_\tau)$ be a solution of (1.4). Formally, if $U_\tau \rightarrow U = (D, H, \chi)$ when τ tends to zero, then $U \in \mathcal{V}$ where \mathcal{V} is the equilibrium manifold for the Kerr–Debye model:

$$\mathcal{V} = \{(D, H, \chi); \frac{\epsilon_r |D|^2}{\epsilon_0^2(1 + \chi)^2} - \chi = 0\} = \{(D, H, \chi); \chi = \epsilon_r p^2(|D|)\}.$$

Therefore, $u = (D, H)$ is a solution of the Kerr system (1.1).

The Kerr–Debye model is a relaxation approximation of the Kerr model, and τ is the relaxation parameter. The Kerr system is the reduced system for the Kerr–Debye one in the sense of [6]; see also [13] for a survey on hyperbolic relaxation problems. In [8, 5], some rigorous existence and convergence results are proven for the Kerr–Debye system. In particular, for $\tau \neq 0$, at least in certain configurations with smooth data, no shock is created. This system has been numerically studied for fixed $\tau > 0$ by the FDTD method in [21] and with finite elements in [9]. Taking advantage of the fact that all the characteristic fields of (1.4) are linearly degenerate, one can design a finite volume scheme which has a relaxed limit when $\tau = 0$, and this limit is a consistent entropic approximation of (1.1). This method has been developed in [10] for the 2×2 and 3×3 cases. It is easy to compute the general case using the same ideas; see the appendix. This gives us an explicit scheme based on a physical model. In the sequel, we call it the Kerr–Debye (relaxation) scheme.

Here, we want to construct the Godunov scheme for the Kerr system (1.1) and to compare it to the Kerr–Debye scheme in one and two space dimensions.

As is well known, the solution of the Riemann problem is the cornerstone of Godunov scheme. Consider a system of conservation laws

$$\partial_t u_j + \operatorname{div} \vec{F}_j(u) = 0, \quad 1 \leq j \leq N.$$

Let $\{C_\alpha, \alpha \in A\}$ be an admissible mesh of the computational domain, and let us denote by $\Gamma_{\alpha\beta}$ the common edge of C_α and C_β and by $\vec{n}_{\alpha\beta}$ the unitary normal vector to $\Gamma_{\alpha\beta}$ pointing from C_α to C_β . The approximation u_α^{n+1} of $u(\cdot, t_{n+1})$ on C_α is computed as follows:

$$u_\alpha^{n+1} = u_\alpha^n - \frac{\Delta t}{|C_\alpha|} \sum_{\beta, C_\alpha \cap C_\beta \neq \emptyset} \Phi(u_\alpha^n, u_\beta^n, \vec{n}_{\alpha\beta}) |\Gamma_{\alpha\beta}|, \tag{1.5}$$

where the numerical flux function Φ is defined by

$$\Phi_j(u, v, \vec{n}) = \vec{F}_j(\bar{w}(0)) \cdot \vec{n}, \quad 1 \leq j \leq N, \tag{1.6}$$

and $\bar{w}(\frac{y}{t}) = w(y, t)$ is the solution of the one-dimensional Riemann problem

$$\begin{cases} \partial_t w_j + \partial_y (\vec{F}_j(w) \cdot \vec{n}) = 0, & 1 \leq j \leq N, \\ w(y, 0) = \begin{cases} u & \text{if } y < 0, \\ v & \text{if } y > 0. \end{cases} \end{cases}$$

Therefore, we have to solve the Riemann problem for the Kerr system (1.1). As detailed hereafter, we have 4 linearly degenerate fields and 2 others that are neither genuinely nonlinear, nor linearly degenerate, and the related eigenvalues have variable multiplicity. Hence the classical Lax existence results do not apply. In [7], a first existence result has been established for a reduced 4×4 case with the assumption that $D \cdot \vec{n} = 0$ and $H \cdot \vec{n} = 0$. In particular, the two-dimensional TM case, which is very important for applications, does not enter this framework. Here, we deal with the full vector system, and we implement the exact solution of the Riemann problem.

In this article, we pay particular attention to the 2D transverse magnetic (TM) case: for solutions depending on $x = (x_1, x_2)$, if one assumes that the data is such that $D_3 = 0$ and $H_1 = H_2 = 0$, then so is the solution. Denoting $D = (D_1, D_2, 0)$, (1.1) reduces to a 3×3 system:

$$\begin{cases} \partial_t D_1 - \partial_2 H_3 = 0, \\ \partial_t D_2 + \partial_1 H_3 = 0, \\ \partial_t H_3 + \mu_0^{-1} (\partial_1(\mathbf{P}_2(D)) - \partial_2(\mathbf{P}_1(D))) = 0. \end{cases} \tag{1.7}$$

An even more particular case is the 1D setting with $D_1 = 0$ and $x = x_1$:

$$\begin{cases} \partial_t D_2 + \partial_x H_3 = 0, \\ \partial_t H_3 + \mu_0^{-1} \partial_x p(D_2) = 0. \end{cases} \tag{1.8}$$

It turns out that the 1D Kerr system (1.8) is a so-called p-system. Since $p' > 0$, it is strictly hyperbolic, but the properties of the function p differ from the ones which appear in the general framework of gas dynamics or viscoelasticity [17]. Here,

$$p(0) = 0, \quad p' > 0,$$

and p is strictly convex on $]-\infty, 0]$ and strictly concave on $[0, +\infty[$.

The plan of this paper is the following. In Section 2, we solve the Riemann problem for the 6×6 system (1.1): the Lax solution is constructed, and its existence and uniqueness are proven. Moreover, if the data is TM, so is the solution.

In Section 3, we focus on the 2D TM system (1.7). Here, we have to use Liu’s condition (E) ([11]–[12]) for the admissibility of shocks and the solution of the Riemann problem. This solution is compared to the one obtained in Section 2. Since the mathematical entropy is the physical electromagnetic energy, it is proven that two distinct entropy solutions of (1.7) (and (1.1)) can exist. This may be surprising, but we recall that no general uniqueness result is available for weak entropy solutions of systems of conservation laws. In the particular case of the Riemann problem, uniqueness theorems are proven only within a prescribed class of solutions; see [15, 11] and theorems 2.16 and 3.8 here below.

Section 4 is devoted to numerical experiments. The 6×6 Riemann solver is implemented in one space dimension and then in a two-dimensional cartesian setting. Comparisons with exact solutions are performed. In case of non-uniqueness, the computed solution is the Liu’s one. Finally, a physically realistic case inspired from [21] is analyzed. In each case, numerical comparison is done with the Kerr–Debye relaxation scheme. Both methods give similar results.

2. The Riemann problem for the full vector Kerr system

In this part, we solve the Riemann problem for system (1.1). We denote $u = (D, H)$ and $E = \mathbf{P}(D)$. For a given $\omega \in \mathbb{R}^3$, $|\omega| = 1$, and $u_-, u_+ \in \mathbb{R}^6$, we fix the initial data

$$u(x, 0) = \begin{cases} u_- & \text{if } x \cdot \omega < 0, \\ u_+ & \text{if } x \cdot \omega > 0. \end{cases} \tag{2.1}$$

We look for a selfsimilar entropy solution $u(x, t) = V(\frac{x \cdot \omega}{t})$ of (1.1), (2.1). Denoting $y = x \cdot \omega$, we therefore have to solve the Riemann problem for the one-dimensional 6×6 system

$$\begin{cases} \partial_t D - \partial_y(\omega \times H) = 0, \\ \partial_t H + \mu_0^{-1} \partial_y(\omega \times \mathbf{P}(D)) = 0. \end{cases} \tag{2.2}$$

The admissible shocks of the Kerr system have already been studied in [1]. For the sake of completeness, those results are briefly recalled here. Then we construct the rarefaction waves, and we solve the whole Riemann problem.

2.1. Characteristic fields of the Kerr system, admissible shocks. Using the results of [1] we can state the following proposition.

PROPOSITION 2.1 ([1]). *The Kerr system (1.1) is hyperbolic diagonalizable: for all $\omega \in \mathbb{R}^3$, $|\omega| = 1$, the eigenvalues of system (2.2) are given by*

$$\lambda_1 \leq \lambda_2 = -\lambda < \lambda_3 = \lambda_4 = 0 < \lambda_5 = \lambda \leq \lambda_6 = -\lambda_1 \tag{2.3}$$

where $c = \sqrt{\epsilon_0 \mu_0}^{-1}$ is the light velocity and

$$\lambda_1^2 = \frac{c^2}{1 + \epsilon_r |E|^2}, \quad \lambda^2 = c^2 \frac{1 + \epsilon_r (|E|^2 + 2(E \cdot \omega)^2)}{(1 + \epsilon_r |E|^2)(1 + 3\epsilon_r |E|^2)}. \tag{2.4}$$

The inequalities in (2.3) are strict if and only if $\omega \times D \neq 0$.

PROPOSITION 2.2 ([1]). *The characteristic fields 1,3,4,6 are linearly degenerate. If $\omega \times D \neq 0$, the eigenvectors for λ_2 and λ_5 are:*

$$r_i(u, \omega) = \begin{pmatrix} \text{sgn}(\lambda_i)\omega \times (\omega \times D) \\ -\lambda\omega \times D \end{pmatrix}, \quad i = 2, 5.$$

The characteristic fields 2 and 5 are genuinely nonlinear in the direction ω in the open set

$$\Omega(\omega) = \{(D, H) \in \mathbb{R}^6; \omega \times D \neq 0\},$$

and for all $u \in \Omega(\omega)$ and $i \in \{2, 5\}$,

$$\lambda'_i(u, \omega) r_i(u, \omega) > 0. \tag{2.5}$$

We point out the fact that the fields 2 and 5 are neither genuinely nonlinear, nor linearly degenerate, so that the general theory about the resolution of the Riemann problem does not apply here. The characterization of admissible plane discontinuities is now briefly recalled.

The Rankine–Hugoniot conditions for a discontinuity (u_-, u_+) propagating with velocity σ are

$$\sigma[D] = -\omega \times [H], \quad \sigma\mu_0[H] = \omega \times [E] \tag{2.6}$$

where for a given quantity v , $[v] = v_+ - v_-$.

The divergence free conditions are

$$\omega \cdot [D] = 0, \tag{2.7}$$

$$\omega \cdot [H] = 0. \tag{2.8}$$

If $\sigma \neq 0$, they are fulfilled as soon as (2.6) is satisfied.

PROPOSITION 2.3. **Stationary contact discontinuities.** *([1]) Stationary contact discontinuities are characterized by*

$$\omega \times [H] = 0, \quad \omega \times [E] = 0. \tag{2.9}$$

The divergence free ones are constant.

The fields 1 and 6 are linearly degenerate. The associated contact discontinuities are characterized as follows.

PROPOSITION 2.4. *([1]) A discontinuity σ , u_+ , u_- is a contact discontinuity associated to λ_1 or λ_6 if and only if*

$$\begin{cases} |E_+| = |E_-|, \\ \sigma^2 = c^2(1 + \epsilon_r|E_+|^2)^{-1} = c^2(1 + \epsilon_r|E_-|^2)^{-1}, \end{cases} \tag{2.10}$$

condition (2.7) is satisfied, and

$$[H] = \sigma\omega \times [D]. \tag{2.11}$$

Moreover, the only discontinuities satisfying the Rankine–Hugoniot conditions (2.6) and such that $|E_-| = |E_+|$ are the above contact discontinuities.

At this point, it remains to study the discontinuities that are not contact discontinuities. From now on, we call these discontinuities shocks.

For a fixed left state u_- , the Hugoniot set of u_- , denoted $\mathcal{H}(u_-)$, is the set of the right states u_+ such that there exists a shock connecting u_- and u_+ . We then denote the shock velocity by $\sigma = \sigma(u_+, u_-)$. One can give a similar definition by fixing the right state. Moreover, we impose Lax admissibility conditions, which on the one hand ensure entropy dissipation, and on the other hand ensure that one can construct the solution of the Riemann problem as a superposition of simple waves.

DEFINITION 2.5. *A discontinuity σ, u_-, u_+ is a Lax k -shock if*

$$\begin{cases} \lambda_k(u_+) \leq \sigma \leq \lambda_{k+1}(u_+) \\ \lambda_{k-1}(u_-) \leq \sigma \leq \lambda_k(u_-). \end{cases} \tag{2.12}$$

The following property holds:

PROPOSITION 2.6. *The Lax-admissible shocks are 2-shocks or 5-shocks;*

$\sigma, u_- = (D_-, H_-), u_+ = (D_+, H_+)$ is a Lax 2-shock if and only if $-\sigma, \bar{u}_- = (-D_+, H_+), \bar{u}_+ = (-D_-, H_-)$ is a Lax 5-shock.

Therefore we just give the results for Lax 2-shocks with a fixed left state u_- . Towards that goal, we define two functions:

$$f(d, d_0) = \frac{c^2 d}{1 + \epsilon_r p^2 \left(\sqrt{d_0^2 + d^2} \right)}, \quad d, d_0 \in \mathbb{R}. \tag{2.13}$$

When d_0 is fixed, $f(\cdot, d_0)$ is an increasing function; see [1]. Hence we can define

$$S(d_+, d_-, d_0) = ((f(d_+, d_0) - f(d_-, d_0))(d_+ - d_-))^{\frac{1}{2}}, \quad d_+, d_-, d_0 \in \mathbb{R}. \tag{2.14}$$

Two cases are under consideration.

PROPOSITION 2.7. **Case $D_- \times \omega \neq 0$.** ([1]) *Let $u_- = (D_-, H_-)$ be a fixed left state such that $D_- \times \omega \neq 0$. We denote*

$$\zeta = -\frac{\omega \times (\omega \times D_-)}{|\omega \times (\omega \times D_-)|}. \tag{2.15}$$

Then

$$D_- = d_0 \omega + d_- \zeta, \quad d_- = |\omega \times (\omega \times D_-)| > 0.$$

The set $\mathcal{H}_2(u_-)$ of the right states u_+ connected to u_- by a Lax 2-shock is a curve parametrized by $d_+ \in \mathbb{R}$. It is the set of $(D_+, H_+) \in \mathbb{R}^6$ such that

$$D_+ = d_0 \omega + d_+ \zeta, \quad H_+ - H_- = S(d_+, d_-, d_0) \omega \times \zeta, \quad 0 \leq d_+ \leq d_-.$$

The shock velocity σ satisfies $\sigma < 0$ and

$$\sigma^2 = \frac{f(d_+, d_0) - f(d_-, d_0)}{d_+ - d_-}. \tag{2.16}$$

PROPOSITION 2.8. **Case $D_- \times \omega = 0$.** ([1]) *Let $u_- = (D_-, H_-)$ be a fixed left state such that $D_- \times \omega = 0$. Then the set $\mathcal{H}(u_-)$ of the right states connected to u_- by a shock is the set of $u_+ = (D_+, H_+)$ satisfying (2.7), (2.11), and*

$$\sigma^2 = \lambda_1^2(u_+) = c^2 (1 + \epsilon_r |E_+|^2)^{-1}. \tag{2.17}$$

There is no nontrivial Lax 2-shock connecting u_- to a right state $u_+ \in \mathcal{H}(u_-)$.

2.2. Rarefaction waves. We first determine the 2-rarefactions. The rarefaction waves are computed by using the integral curves of the eigenvectors. As the 2-characteristic field is genuinely nonlinear in $\Omega(\omega) = \{u = (D, H), \omega \times D \neq 0\}$, the integral curves of r_2 allow us to determine a rarefaction only in this open set. Those curves are the solutions of the following differential system:

$$\begin{cases} D'(\xi) = -\omega \times (\omega \times D(\xi)) \\ H'(\xi) = -\lambda(D(\xi))\omega \times D(\xi). \end{cases} \tag{2.18}$$

If $U = (D, H)$ is a solution of this system, then $D(\xi) \cdot \omega$ and $H(\xi) \cdot \omega$ are constant:

$$D(\xi) \cdot \omega = d_0, \quad H(\xi) \cdot \omega = h_0.$$

Using the identity

$$D(\xi) = d_0\omega - \omega \times (\omega \times D(\xi)),$$

one finds that for all ξ, ξ_+ ,

$$\omega \times (\omega \times D(\xi)) = \omega \times (\omega \times D(\xi_+))e^{\xi - \xi_+}.$$

Therefore $D(\xi), D(\xi_+)$ and ω are coplanar. Here, it is convenient to fix $U(\xi_+) = u_+$ in $\Omega(\omega)$. Then $U(\xi) \in \Omega(\omega)$ for all ξ . Let us define ζ as

$$\zeta = -\frac{\omega \times (\omega \times D_+)}{|\omega \times (\omega \times D_+)|}, \tag{2.19}$$

and set $d_+ = |\omega \times (\omega \times D_+)| > 0$. We have

$$D(\xi) = d_0\omega + d_+e^{\xi - \xi_+}\zeta,$$

and for $\xi_- < \xi_+$:

$$H(\xi_-) = H_+ + \left(\int_{\xi_-}^{\xi_+} \lambda(d_0\omega + d_+e^{\xi - \xi_+}\zeta)d_+e^{\xi - \xi_+}d\xi \right) \omega \times \zeta.$$

Going into details, we remark that since ω is fixed, for $D \in \mathbb{R}^6$, if $D \cdot \omega = d_0$ and $|\omega \times (\omega \times D)| = d$, then $|D| = \sqrt{d_0^2 + d^2}$, and by (1.3), $\lambda(D)$ is a function of d and d_0 only, that we still denote by λ :

$$\lambda^2(d, d_0) = c^2 \frac{1 + \epsilon_r \left(|E|^2 + 2 \left(\frac{d_0}{\epsilon_0(1 + \epsilon_r|E|^2)} \right)^2 \right)}{(1 + \epsilon_r|E|^2)(1 + 3\epsilon_r|E|^2)}, \quad |E| = p \left(\sqrt{d_0^2 + d^2} \right).$$

Therefore, denoting $d_- = d_+e^{\xi_- - \xi_+}$ and

$$R(d_1, d_2, d_0) = \int_{d_1}^{d_2} \lambda(s, d_0)ds, \quad 0 \leq d_1 \leq d_2, \tag{2.20}$$

we have

$$H(\xi_-) = H_+ + R(d_-, d_+, d_0)\omega \times \zeta.$$

The function $\psi = \lambda_2 \circ U$ is strictly increasing by Proposition 2.2. For $\xi_- < \xi_+$, $u_{\pm} = U(\xi_{\pm})$, one defines

$$u(y) = \begin{cases} u_- & \text{if } y \leq \lambda_2(u_-), \\ U(\psi^{-1}(y)) & \text{if } \lambda_2(u_-) \leq y \leq \lambda_2(u_+), \\ u_+ & \text{if } y \geq \lambda_2(u_+). \end{cases} \tag{2.21}$$

Then $u(\frac{x \cdot \omega}{t})$ is a centred rarefaction wave for system (1.1); see [4, 15].

Moreover, if $\xi_- \rightarrow -\infty$, then $d_- \rightarrow 0$, and $H(\xi_-)$ also has a limit, so we can extend the definition to left states $u_- \notin \Omega(\omega)$. As a consequence, the following proposition holds.

PROPOSITION 2.9. *Let $u_+ = (D_+, H_+) \in \Omega(\omega)$ be a given right state. Using notation (2.19),*

$$D_+ = d_0\omega + d_+\zeta, \quad d_0 \in \mathbb{R}, \quad d_+ > 0.$$

For $0 \leq d_- \leq d_+$, let u_- be defined by

$$D_- = d_0\omega + d_-\zeta, \quad H_- = H_+ + R(d_-, d_+, d_0)\omega \times \zeta.$$

Then u_- and u_+ are connected by a 2-rarefaction wave.

By symmetry we deduce the equivalent proposition for 5-rarefaction waves.

PROPOSITION 2.10. *Let $u_- = (D_-, H_-) \in \Omega(\omega)$ be a given left state. Using notation (2.15):*

$$D_- = d_0\omega + d_-\zeta, \quad d_0 \in \mathbb{R}, \quad d_- > 0.$$

For $0 \leq d_+ \leq d_-$, let u_+ be defined by

$$D_+ = d_0\omega + d_+\zeta, \quad H_+ = H_- - R(d_+, d_-, d_0)\omega \times \zeta.$$

Then u_- and u_+ are connected by a 5-rarefaction wave.

2.3. Wave curves. As a conclusion to this section, we define the 2 and 5 wave curves. Let ϕ be the function defined for $d_1 \geq 0$, $d_2 \geq 0$, and $d_0 \in \mathbb{R}$ by

$$\phi(d_1, d_2, d_0) = \begin{cases} S(d_1, d_2, d_0) & \text{if } d_2 \leq d_1, \\ -R(d_1, d_2, d_0) & \text{if } d_1 \leq d_2. \end{cases} \tag{2.22}$$

PROPOSITION 2.11. *The function ϕ is a decreasing C^1 -function with respect to d_2 , and for all $d > 0$ and $d_0 \in \mathbb{R}$,*

$$\phi(d, 0, d_0) = \frac{cd}{\sqrt{1 + \epsilon_r p^2 (\sqrt{d_0^2 + d^2})}}, \quad \lim_{d_2 \rightarrow +\infty} \phi(d, d_2, d_0) = -\infty. \tag{2.23}$$

Proof. We have

$$\partial_2 \phi(d_1, d_2, d_0) = \begin{cases} \frac{f(d_2, d_0) - f(d_1, d_0) - \partial_1 f(d_2, d_0)(d_1 - d_2)}{2S(d_1, d_2, d_0)} & \text{if } d_1 > d_2, \\ -\lambda(d_2, d_0) & \text{if } d_1 < d_2. \end{cases}$$

In [1], we proved that f is a twice differentiable, concave, increasing function with respect to $d_2 \geq 0$, with

$$\partial_1 f(d_2, d_0) = \lambda^2(d_2, d_0). \tag{2.24}$$

Therefore, one obtains that

$$\lim_{d_2 \rightarrow d_1^\pm} \partial_2 \phi(d_1, d_2, d_0) = -\lambda(d_1, d_0),$$

which proves that ϕ is C^1 and $\partial_2 \phi(d_1, d_2, d_0) < 0$ for all d_2 .

The first equality in (2.23) is immediate. To prove the second one, we first remark that

$$\lambda^2(s, d_0) \geq \frac{c^2}{1 + \epsilon_r p^2 (\sqrt{s^2 + d_0^2})},$$

and we perform the change of variable $w = p^2 (\sqrt{s^2 + d_0^2})$:

$$R(d, d_2, d_0) \geq \frac{\epsilon_0^2}{2} \int_{w(d)}^{w(d_2)} \frac{c}{\sqrt{1 + \epsilon_r w}} \frac{1 + 4\epsilon_r w + 3\epsilon_r^2 w^2}{\sqrt{\epsilon_0^2 w (1 + \epsilon_r w)^2 - d_0^2}} dw.$$

When d_2 tends to $+\infty$, so does $w(d_2)$. Hence the result follows. □

If $u_- \neq u_+$ are connected by a Lax k-shock or a k-rarefaction wave, u_- and u_+ are said to be connected by a k-wave. In such a case, $D_- \neq D_+$ and $D_- \cdot \omega = D_+ \cdot \omega$. Moreover $\omega \times (\omega \times D_-)$ and $\omega \times (\omega \times D_+)$ are colinear.

PROPOSITION 2.12. *Let us consider u_- and u_+ such that $D_- \neq D_+$ and $D_- \cdot \omega = D_+ \cdot \omega = d_0$. If $\omega \times D_+ \neq 0$, we define ζ by (2.19). Else, $\omega \times D_- \neq 0$, and we define ζ by (2.15).*

The states u_- and u_+ are connected by a 2-wave if there exist two distinct nonnegative real numbers d_- and d_+ such that

$$D_\pm = d_0 \omega + d_\pm \zeta, \quad H_+ = H_- + \phi(d_-, d_+, d_0) \omega \times \zeta. \tag{2.25}$$

The states u_- and u_+ are connected by a 5-wave if there exist two distinct nonnegative real numbers d_- and d_+ such that

$$D_\pm = d_0 \omega + d_\pm \zeta, \quad H_+ = H_- + \phi(d_+, d_-, d_0) \omega \times \zeta. \tag{2.26}$$

2.4. Solution of the Riemann problem. Suppose that $u_\pm = (D_\pm, H_\pm)$ and $\omega \in R^3, |\omega| = 1$, are given. We look for intermediate states u_1, u_*, u_{**}, u_2 such that:

- u_- and u_1 are connected by a 1-contact discontinuity,
- u_1 and u_* are connected by a 2-wave,
- u_* and u_{**} are connected by a stationary contact discontinuity,
- u_{**} and u_2 are connected by a 5-wave,
- u_2 and u_+ are connected by a 6-contact discontinuity.

In the following, we shall denote $d_0^\pm = D_\pm \cdot \omega$.

2.4.1. Necessary conditions. Suppose that a solution exists. For the contact discontinuities 1 and 6, the following conditions have to be fulfilled:

$$\begin{cases} D_1 \cdot \omega = D_- \cdot \omega = d_0^-, \\ D_2 \cdot \omega = D_+ \cdot \omega = d_0^+, \\ |D_1| = |D_-|, \\ |D_2| = |D_+|, \end{cases} \tag{2.27}$$

$$\begin{cases} H_1 - H_- = \sigma_- \omega \times (D_1 - D_-), \\ H_+ - H_2 = \sigma_+ \omega \times (D_+ - D_2), \end{cases} \tag{2.28}$$

with

$$\sigma_- = \lambda_1(u_-) = \lambda_1(u_1), \quad \sigma_+ = \lambda_6(u_+) = \lambda_6(u_2).$$

That is

$$\sigma_{\pm} = \pm c(1 + \epsilon_r |E_{\pm}|)^{-\frac{1}{2}}. \tag{2.29}$$

For the 2 and 5 waves, we know that D_1, D_*, ω are coplanar and that D_2, D_{**}, ω are coplanar. Moreover, $[D] \cdot \omega = 0$. There exist unitary vectors ζ_1 and ζ_2 , orthogonal to ω such that

$$D_1 = d_0^- \omega + d_1 \zeta_1, \quad D_* = d_0^- \omega + d_* \zeta_1,$$

$$D_2 = d_0^+ \omega + d_2 \zeta_2, \quad D_{**} = d_0^+ \omega + d_{**} \zeta_2,$$

and d_1, d_*, d_{**}, d_2 are non-negative.

The stationary contact discontinuity is defined by conditions (2.9). One has

$$E_* = e_0^* \omega + e_* \zeta_1, \quad E_{**} = e_0^{**} \omega + e_{**} \zeta_2,$$

where

$$e_* = \frac{d_*}{\epsilon_0(1 + \epsilon_r p^2(|D_*|))}, \quad e_{**} = \frac{d_{**}}{\epsilon_0(1 + \epsilon_r p^2(|D_{**}|))}.$$

Therefore $e_* \omega \times \zeta_1 = e_{**} \omega \times \zeta_2$. Hence, either $e_* = e_{**} = 0$, or those quantities are both positive and $\zeta_1 = \zeta_2$. The first case occurs if and only if $\omega \times D_* = \omega \times D_{**} = 0$. In the second case, we have $e_* = e_{**}$, which also reads as

$$f(d_*, d_0^-) = f(d_{**}, d_0^+). \tag{2.30}$$

First case: $\omega \times D_* = \omega \times D_{**} = 0$.

In this case, $D_* = d_0^- \omega$ and $D_{**} = d_0^+ \omega$. We have that u_1 and u_* are the left and right states of a 2-shock propagating with speed

$$\sigma_2 = -\sqrt{\frac{f(d_1, d_0^-) - f(0, d_0^-)}{d_1}} = \sigma_-.$$

In the same way, u_{**} and u_2 are the left and right states of a 5-shock propagating with speed σ_+ . Consequently, the contact discontinuities merge with the shocks; see Figure 2.1.

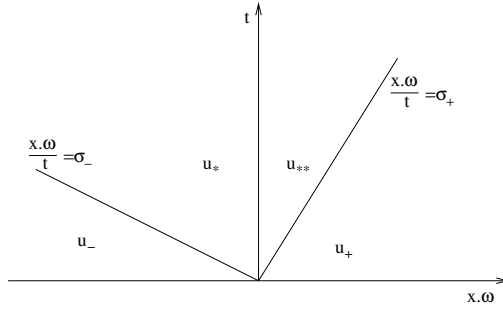


FIGURE 2.1. Case $\omega \times D_* = \omega \times D_{**} = 0$.

We have the following relations:

$$\begin{cases} H_1 - H_- = \sigma_- \omega \times (D_1 - D_-), \\ H_* - H_1 = -\sigma_- \omega \times D_1 \\ H_2 - H_{**} = \sigma_+ \omega \times D_2 \\ H_+ - H_2 = \sigma_+ \omega \times (D_+ - D_2). \end{cases}$$

Let us denote

$$V = \omega \times (H_+ - H_- - \omega \times (\sigma_+ D_+ - \sigma_- D_-)). \tag{2.31}$$

Using the second relation of (2.9),

$$V = 0, \tag{2.32}$$

and

$$H_* = H_- - \omega \times \sigma_- D_-, \quad H_{**} = H_+ - \omega \times \sigma_+ D_+. \tag{2.33}$$

If $D_- \times \omega = 0$, then $u_- = u_*$. Else, one has $D_- = d_0^- \omega + d_- \zeta$ with ζ defined by (2.15), so

$$H_* = H_- - \sigma_- d_- \omega \times \zeta = H_- + \phi(d_-, 0, d_0^-) \omega \times \zeta.$$

This proves that u_- and u_* are connected by a Lax 2-shock.

In the same way, if $D_+ \times \omega = 0$, then $u_+ = u_{**}$. Else u_+ and u_{**} are connected by a Lax 5-shock.

Second case: $D_* \times \omega \neq 0$ and $D_{**} \times \omega \neq 0$.

In this case, $\zeta_1 = \zeta_2 = \zeta$ and

$$D_1 = d_0^- \omega + d_1 \zeta, \quad D_2 = d_0^+ \omega + d_2 \zeta, \tag{2.34}$$

$$D_* = d_0^- \omega + d_* \zeta, \quad D_{**} = d_0^+ \omega + d_{**} \zeta, \tag{2.35}$$

with $d_1 \geq 0, d_* > 0, d_{**} > 0, d_2 \geq 0$. Let us denote

$$d = D \cdot \zeta, \quad h = H \cdot (\omega \times \zeta). \tag{2.36}$$

By (2.27)–(2.28),

$$d_1 = |\omega \times (\omega \times D_-)|, \quad d_2 = |\omega \times (\omega \times D_+)|, \tag{2.37}$$

and

$$h_1 = h_- + \sigma_-(d_1 - d_-), \quad h_2 = h_+ + \sigma_+(d_2 - d_+).$$

By Proposition 2.12, for the 2-wave curve connecting u_1 and u_* ,

$$H_* - H_1 = \phi(d_1, d_*, d_0^-) \omega \times \zeta. \tag{2.38}$$

In the same way,

$$H_2 - H_{**} = \phi(d_2, d_{**}, d_0^+) \omega \times \zeta. \tag{2.39}$$

By (2.9), $h_* = h_{**}$ and

$$h_* = h_1 + \phi(d_1, d_*, d_0^-) = h_2 - \phi(d_2, d_{**}, d_0^+). \tag{2.40}$$

Therefore, using (2.30), we see that d_* and d_{**} are solutions of the 2×2 system,

$$\begin{cases} f(d_*, d_0^-) = f(d_{**}, d_0^+), \\ h_1 + \phi(d_1, d_*, d_0^-) = h_2 - \phi(d_2, d_{**}, d_0^+). \end{cases} \tag{2.41}$$

Since ϕ is decreasing and d_*, d_{**} are positive,

$$\phi(d_1, d_*, d_0^-) + \phi(d_2, d_{**}, d_0^+) < \phi(d_1, 0, d_0^-) + \phi(d_2, 0, d_0^+) = \sigma_+ d_2 - \sigma_- d_1. \tag{2.42}$$

This inequality is useful to determine ζ . As a matter of fact, using (2.28), we have also

$$\begin{cases} H_* = H_- + \sigma_- \omega \times (D_1 - D_-) + \phi(d_1, d_*, d_0^-) \omega \times \zeta, \\ H_{**} = H_+ - \sigma_+ \omega \times (D_+ - D_2) - \phi(d_2, d_{**}, d_0^+) \omega \times \zeta. \end{cases}$$

Again by (2.9), using notation (2.31),

$$V = (\sigma_+ d_2 - \sigma_- d_1 - \phi(d_1, d_*, d_0^-) - \phi(d_2, d_{**}, d_0^+)) \zeta.$$

Therefore $V \neq 0$ and

$$\zeta = \frac{V}{|V|}. \tag{2.43}$$

We sum up the results in the following proposition.

PROPOSITION 2.13. *Consider u_-, u_+ such that the Riemann problem for system (1.1) has a solution which is a superposition of simple waves. Let V be defined by (2.31). Then only the following two cases occur:*

1) $V = 0$, u_- and u_* are connected by a Lax 2-shock propagating with velocity σ_- , u_+ and u_{**} are connected by a Lax 5-shock propagating with velocity σ_+ , $D_* = (D_- \cdot \omega) \omega$, $D_{**} = (D_+ \cdot \omega) \omega$, and H_* and H_{**} are given by (2.33).

2) $V \neq 0$, ζ is defined by (2.43), u_1 and u_2 are determined by conditions (2.28), (2.29), (2.34), and (2.37), and u_* and u_{**} are determined by (2.35), (2.38)–(2.39), and the solution of system (2.41).

2.4.2. Sufficient conditions. Consider initial Riemann data. We consider two cases according to whether or not $V = 0$.

First case: $V = 0$. We define $D_* = (D_- \cdot \omega)\omega$, $D_{**} = (D_+ \cdot \omega)\omega$, and H_* and H_{**} by (2.33). It is easy to see that u_- and u_* are connected by a Lax 2-shock, u_* and u_{**} are connected by a stationary contact discontinuity, and that u_{**} and u_+ are connected by a Lax 5-shock, so we have constructed the solution of the problem.

Second case: $V \neq 0$. We define ζ by (2.43). Then we set

$$D_{\pm} = d_0^{\pm}\omega + d_{\pm}\zeta + d'_{\pm}\omega \times \zeta, \quad H_{\pm} = h_0^{\pm}\omega + h'_{\pm}\zeta + h_{\pm}\omega \times \zeta,$$

so that

$$V = (-h_+ + h_- + \sigma_+d_+ - \sigma_-d_-)\zeta + (h'_+ - h'_- + \sigma_+d'_+ - \sigma_-d'_-)\omega \times \zeta.$$

Hence we have the following lemma.

LEMMA 2.14. *The two following properties hold:*

$$-h_+ + h_- + \sigma_+d_+ - \sigma_-d_- > 0, \tag{2.44}$$

$$h'_+ - h'_- + \sigma_+d'_+ - \sigma_-d'_- = 0. \tag{2.45}$$

Using notation (2.29) and (2.36), we define u_1 and u_2 by (2.34), (2.37), and (2.28). Clearly, u_- and u_1 are connected by a 1-contact discontinuity and u_+ and u_2 are connected by a 6-contact discontinuity. Then we solve system (2.41).

LEMMA 2.15. *The system (2.41) has a unique solution $(d_*, d_{**}) \in \mathbb{R}_+^2$.*

Proof. The values of d_0^{\pm} , d_1 , d_2 , h_1 , and h_2 are fixed. Denoting $f_{\pm} = f(\cdot, d_0^{\pm})$, we know that f_+ and f_- are increasing, C^1 -diffeomorphisms from \mathbb{R} to \mathbb{R} such that $f_-(0) = f_+(0) = 0$; see [1]. Hence we can define $G = f_+^{-1} \circ f_-$ which is a C^1 , increasing, one-to-one function such that $G(0) = 0$. We only need to define G on \mathbb{R}_+ : $G(\mathbb{R}_+) = \mathbb{R}_+$.

Solving system (2.41) is equivalent to finding $d_* \in \mathbb{R}_+$ such that

$$h_2 - h_1 = \phi(d_1, d_*, d_0^-) + \phi(d_2, G(d_*), d_0^+). \tag{2.46}$$

We have

$$h_2 - h_1 = h_+ - h_- - \sigma_+d_+ + \sigma_-d_- + \phi(d_1, 0, d_0^-) + \phi(d_2, 0, d_0^+).$$

By using (2.44), we deduce

$$h_2 - h_1 < \phi(d_1, 0, d_0^-) + \phi(d_2, 0, d_0^+).$$

Hence by Proposition 2.11, the solution d_* of (2.46) exists and is unique. Setting $d_{**} = G(d_*)$, (d_*, d_{**}) is the unique solution of system (2.41). \square

Let (d_*, d_{**}) be the solution of system (2.41). We define u_* and u_{**} by (2.35), (2.38), and (2.39). By construction, u_1 and u_* are connected by a 1-wave and u_2 and u_{**} are connected by a 5-wave.

It remains to verify that u_* and u_{**} are connected by a stationary contact discontinuity. First it is easy to see that $[H \times \omega] = 0$ if and only if $|V| = -h_+ + h_- + \sigma_+d_+ - \sigma_-d_-$, which is a consequence of Lemma 2.14. Moreover, $[E \times \omega] = 0$ if and only if (2.30) is satisfied, which is true because (d_*, d_{**}) is a solution of system (2.41).

We sum up the results in the following theorem.

THEOREM 2.16. *Let u_-, u_+ be a Riemann data for system (1.1) in the direction ω . The Riemann problem has a unique solution in the class of the functions which are superpositions of simple waves as detailed at the beginning of Section 2.4. Let V be the vector defined in (2.31).*

If $V = 0$, then the solution is the superposition of a Lax 2-shock, a stationary contact discontinuity, and a Lax 5-shock.

If $V \neq 0$, then the solution is the superposition of a 1-contact discontinuity, a 2-wave (Lax shock or rarefaction), a stationary contact discontinuity, a 5-wave (Lax shock or rarefaction), and a 6-contact discontinuity.

In each case, the solution is constructed as in Proposition 2.13.

2.5. The transverse magnetic case: 6×6 viewpoint. Let us detail the solution of the Riemann problem (1.1), (2.1) in that case; that is $\omega = (\omega_1, \omega_2, 0)$, and u_{\pm} is such that $D_{\pm} = (D_{1,\pm}, D_{2,\pm}, 0)$, and $H_{\pm} = (0, 0, H_{3,\pm})$. Then the vector V defined in (2.31) takes the form

$$V = v \begin{pmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix}, \quad v = H_{3,+} - H_{3,-} - \omega_1(\sigma_+ D_{2,+} - \sigma_- D_{2,-}) + \omega_2(\sigma_+ D_{1,+} - \sigma_- D_{1,-}).$$

If $v = 0$, the intermediate states u_*, u_{**} are clearly transverse magnetic: the electric field is colinear to ω and the magnetic one is given by (2.33).

Else, the vector ζ defined by (2.43) reads as

$$\frac{v}{|v|} \begin{pmatrix} \omega_2 \\ -\omega_1 \\ 0 \end{pmatrix}.$$

Therefore, all the intermediate states are TM, see formulas (2.28), (2.34)–(2.35), (2.38)–(2.39).

Hence, the solution of the Riemann problem (1.1), (2.1) with transverse magnetic data is transverse magnetic. Moreover, it is easy to see that if the Riemann data is divergence free, so is the solution.

Particular case of the p-system. Here we consider the Riemann problem (1.1), (2.1) with $\omega = (1, 0, 0)$, and u_{\pm} is such that $D_{\pm} = (0, D_{2,\pm}, 0)$ and $H_{\pm} = (0, 0, H_{3,\pm})$. This field is divergence free. Then

$$V = v \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad v = H_{3,+} - H_{3,-} - (\sigma_+ D_{2,+} - \sigma_- D_{2,-}).$$

If $v = 0$, we find $D_* = D_{**} = 0$ and $H_* = H_{**} = (0, 0, H_3)$ with

$$H_3 = H_{3,+} - \sigma_+ D_{2,+} = H_{3,-} - \sigma_- D_{2,-}.$$

Else, the vector ζ defined by (2.43) reads as

$$\zeta = \frac{v}{|v|} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

To avoid confusion, we denote $u_1 = (D^{(1)}, H^{(1)})$ and $u_2 = (D^{(2)}, H^{(2)})$ for the intermediate states 1 and 2. We have $D^{(1)} = d_1\zeta$, $D^{(2)} = d_2\zeta$, and $D_{\pm} = d_{\pm}\zeta$ with $d_{\pm} = -\frac{v}{|v|}D_{2,\pm}$, $d_1 = |d_-|$, and $d_2 = |d_+|$. Hence by (2.28), $H^{(1)} = (0, 0, H_3^{(1)})$ and $H^{(2)} = (0, 0, H_3^{(2)})$. Moreover, $D_* = D_{**} = d_*\zeta$, and by (2.38)–(2.39), $H_* = H_{**} = (0, 0, H_{3,*})$. Thus, the solution of the 6×6 Riemann problem is a solution of the 2×2 p-system (1.8).

Let us remark that the stationary contact discontinuity is trivial, but that if $d_- < 0$ (respectively $d_+ < 0$), the contact discontinuity 1 (respectively 6) is not.

3. The Riemann problem for the 3×3 transverse magnetic case

Here, we study the reduced TM system (1.7), which is important for applications.

In this section, we use two components vectors: $D = (D_1, D_2)$, $\omega = (\omega_1, \omega_2)$, $E = \mathbf{P}(D)$. We denote

$$\omega \times D = \omega_1 D_2 - \omega_2 D_1, \quad \omega^\perp = (-\omega_2, \omega_1).$$

3.1. Wave curves. Using the methods of the 6×6 case, we can prove the following proposition.

PROPOSITION 3.1. *The TM Kerr system (1.7) is hyperbolic diagonalizable: for all $\omega \in \mathbb{R}^2$, $|\omega| = 1$, the eigenvalues are given by*

$$\lambda_1 = -\lambda < \lambda_3 = 0 < \lambda_3 = \lambda \tag{3.1}$$

where

$$\lambda^2 = c^2 \frac{1 + \epsilon_r (|E|^2 + 2(E \cdot \omega)^2)}{(1 + \epsilon_r |E|^2)(1 + 3\epsilon_r |E|^2)}. \tag{3.2}$$

If $\omega \times D \neq 0$, the eigenvectors for λ_1 and λ_3 are

$$r_1 = \begin{pmatrix} (\omega \times D)\omega^\perp \\ -\lambda(\omega \times D) \end{pmatrix}, \quad r_3 = \begin{pmatrix} -(\omega \times D)\omega^\perp \\ -\lambda(\omega \times D) \end{pmatrix}.$$

The characteristic fields related to λ_1 and λ_3 are genuinely nonlinear in the domain

$$\Omega(\omega) = \{(D, H_3) \in \mathbb{R}^3; \omega \times D \neq 0\},$$

and for all $u \in \Omega(\omega)$

$$\lambda_i'(u, \omega) r_i(u, \omega) > 0, \quad i = 1, 3. \tag{3.3}$$

When we compare the 6×6 and 3×3 situations, we observe that the reduction to TM fields makes the eigenvalues related to non-stationary contact discontinuities disappear. This is easily understandable since we have seen that those waves induce a 3D rotation of the electric field, namely a rotation around the direction of ω .

The stationary contact discontinuities are characterized as in Proposition 2.3.

For a fixed left state $u_- = (D_-, H_{3,-})$, we can proceed as in [1] to determine the related Hugoniot set, that is the set of all right states u_+ satisfying the Rankine–Hugoniot relations, which read here as

$$\sigma[D] = [H_3]\omega^\perp, \quad \sigma\mu_0[H_3] = [\omega \times E].$$

Therefore, non-stationary shocks are divergence free. As $D = (D \cdot \omega)\omega + (\omega \times D)\omega^\perp$, we have $D_- \cdot \omega = D_+ \cdot \omega$ and

$$\sigma[\omega \times D] = [H_3], \quad \sigma\mu_0[H_3] = [\omega \times E].$$

Consequently,

$$\sigma^2 = \frac{[\omega \times E]}{\mu_0[\omega \times D]} = \frac{f(\omega \times D_+, d_0) - f(\omega \times D_-, d_0)}{(\omega \times D_+) - (\omega \times D_-)}$$

where f is the function defined in (2.13). The vector ζ defined in Section 2 is not useful here. Instead, we denote

$$D_- = d_0\omega + d_-\omega^\perp.$$

PROPOSITION 3.2. *Let $u_- = (D_-, H_{3,-})$ be a fixed left state.*

The set $\mathcal{H}(u_-)$ of the states u_+ connected to u_- by a non-stationary shock is the set of u_+ such that

$$\begin{cases} D_+ = d_0\omega + d_+\omega^\perp, & d_+ \in \mathbb{R}, \\ H_{3,+} = H_{3,-} + \sigma(d_+ - d_-), \end{cases}$$

and the shock speed $\sigma(u_-, u_+) = \sigma$ satisfies (2.16).

Let us now study the Lax entropy conditions (2.12). For a 1-shock, they read as

$$-\lambda(u_+) \leq \sigma \leq -\lambda(u_-).$$

Those conditions are very different from the 6×6 case where for a 2-shock the requirement $\lambda_1(u_-) \leq \sigma$ imposes a sign condition on d_+ ; see [1]. Here, this sign condition no longer exists. Instead, we see that $\sigma < 0$, and using (2.24):

$$\partial_1 f(d_-, d_0) \leq \frac{f(d_+, d_0) - f(d_-, d_0)}{d_+ - d_-} \leq \partial_1 f(d_+, d_0). \tag{3.4}$$

If $d_- = 0$, since $\partial_1 f(\cdot, d_0)$ is maximal for $d = 0$, we have $d_+ = 0$, and hence $u_- = u_+$.

Else, if $d_- > 0$, (respectively $d_- < 0$), since $f(\cdot, d_0)$ is strictly concave (respectively convex) on \mathbb{R}^+ (respectively \mathbb{R}^-), the formula is true if $0 \leq |d_+| \leq |d_-|$ and $d_- d_+ \geq 0$, but this is not necessary. Thus, we have to go beyond the point where $\omega \times D = 0$, and the characteristic field 1 is not genuinely nonlinear. The relevant condition in that case is Liu’s entropy condition; see [11, 12].

DEFINITION 3.3. *Let u_- be a given left state, and consider $u_+ \in \mathcal{H}(u_-)$. The discontinuity is Liu-admissible if*

$$(E) \quad \sigma(u_+, u_-) \leq \sigma(u, u_-), \quad \text{for all } u \in \mathcal{H}(u_-), u \text{ between } u_- \text{ and } u_+.$$

PROPOSITION 3.4. Liu’s 1-shocks. *Let u_- be a given left state and consider $u_+ \in \mathcal{H}(u_-)$, written as in Proposition 3.2, with $\sigma < 0$.*

If $\omega \times D_- = 0$, the discontinuity is neither Liu-admissible, nor Lax-admissible.

If $\omega \times D_- \neq 0$, let $d_ = d_*(d_-)$ be the unique real number such that $d_- d_* < 0$ and*

$$\partial_1 f(d_*, d_0) = \frac{f(d_-, d_0) - f(d_*, d_0)}{d_- - d_0}.$$

The shock is Liu-admissible if and only if $d_+ \in [d_*, d_-]$. When the shock is Liu-admissible, it is also Lax-admissible.

We point out the fact that $[d_*, d_-]$ is to be understood as the segment having d_* and d_- as extreme points. This result is proved in [1] for the particular case of system (1.8). The proof of Proposition 3.4 follows the same lines, so we omit it. In the following, we shall denote

$$D_* = d_0\omega + d_*(d_-)\omega^\perp, \quad H_{3,*} = H_{3,-} + \sigma(d_*(d_-) - d_-), \quad u_*(u_-) = (D_*, H_{3,*}).$$

The 3-shocks are deduced from the 1-shocks by symmetry, as in Proposition 2.6.

We compute the rarefaction waves as in the 6×6 case, using again the function R defined in (2.20). We give the result for the 1-rarefactions; the 3-rarefactions are deduced by symmetry.

PROPOSITION 3.5. 1-rarefactions. *Let $u_+ = (D_+, H_{3,+}) \in \Omega(\omega)$ be a given right state:*

$$D_+ = d_0\omega + d_+\omega^\perp, \quad d_0 \in \mathbb{R}, \quad d_+ \neq 0.$$

For $0 \leq |d_-| \leq |d_+|$, $d_- d_+ \geq 0$, let u_- be defined by

$$D_- = d_0\omega + d_-\omega^\perp, \quad H_{3,-} = H_{3,+} + \text{sgn}(d_+)R(|d_-|, |d_+|, d_0).$$

Then u_- and u_+ are connected by a 1-rarefaction wave.

As a particular case, if we fix $u_- = (d_0\omega, H_{3,-})$, we can define a global 1-wave curve parametrized by $d_+ \in \mathbb{R}$ which consists of rarefactions only:

$$D_+ = d_0\omega + d_+\omega^\perp, \quad H_{3,+} = H_{3,-} - \text{sgn}(d_+)R(0, |d_+|, d_0), \quad d_+ \in \mathbb{R}.$$

Otherwise, if we fix u_- such that $D_- = d_0\omega + d_-\omega^\perp$, $d_- \neq 0$, putting together Liu’s 1-shocks and 1-rarefactions gives a wave curve which is defined for a parameter $d_+ \in [d_*(d_-), +\infty[$ if $d_- > 0$, and a parameter $d_+ \in]-\infty, d_*(d_-)]$ if $d_- < 0$. We complete this curve by using composed waves as explained in [18] and [12], that is by the 1-rarefaction curve related to the left state $u_*(u_-)$.

Finally, we define the wave function. For $d \neq 0$, $d_*(d)$ is defined as in Proposition 3.4. For $d = 0$, we set $d_*(0) = 0$. Then for all d_1, d_2 , and d_0 , we define $\varphi(d_1, d_2, d_0)$ as

$$\varphi(d_1, d_2, d_0) = \begin{cases} -\text{sgn}(d_1)R(|d_1|, |d_2|, d_0) & \text{if } |d_1| \leq |d_2|, d_1 d_2 \geq 0, \\ \text{sgn}(d_1)S(d_1, d_2, d_0) & \text{if } d_2 \in [d_*(d_1), d_1], \\ \text{sgn}(d_1)(S(d_1, d_*(d_1), d_0) + R(d_*(d_1), |d_2|, d_0)) & \text{else.} \end{cases} \tag{3.5}$$

As in Proposition 2.11, we can prove that φ is a decreasing C^1 -function with respect to d_2 and for all $d_1, d_0 \in \mathbb{R}$:

$$\lim_{d_2 \rightarrow \pm\infty} \varphi(d_1, d_2, d_0) = \mp\infty. \tag{3.6}$$

PROPOSITION 3.6. *Let us consider u_- and u_+ such that $D_- \neq D_+$ and $D_- \cdot \omega = D_+ \cdot \omega = d_0$.*

The states u_- and u_+ are connected by a 1-wave if there exist two distinct real numbers d_-, d_+ such that

$$D_\pm = d_0\omega + d_\pm\omega^\perp, \quad H_+ = H_- + \varphi(d_-, d_+, d_0). \tag{3.7}$$

The states u_- and u_+ are connected by a 3-wave if there exist two distinct real numbers d_-, d_+ such that

$$D_\pm = d_0\omega + d_\pm\omega^\perp, \quad H_+ = H_- + \varphi(d_+, d_-, d_0). \tag{3.8}$$

3.2. Solution of the Riemann problem. Suppose that $u_{\pm} = (D_{\pm}, H_{3,\pm})$ and $\omega \in R^3, |\omega| = 1$, are given. We look for intermediate states $u^{(1)}, u^{(2)}$ such that:

- u_- and $u^{(1)}$ are connected by a 1-wave,
- $u^{(1)}$ and $u^{(2)}$ are connected by a stationary contact discontinuity,
- $u^{(2)}$ and u_+ are connected by a 3-wave.

In the following, we shall denote $D_{\pm} = d_{\pm}^{\pm}\omega + d_{\pm}\omega^{\perp}$.

3.2.1. Necessary conditions. Suppose that a solution exists. There exist real numbers d_1, d_2 such that

$$D^{(1)} = d_0^- \omega + d_1 \omega^{\perp}, \quad D^{(2)} = d_0^+ \omega + d_2 \omega^{\perp}, \tag{3.9}$$

$$H_3^{(1)} = H_{3,-} + \varphi(d_-, d_1, d_0^-), \quad H_3^{(2)} = H_{3,+} - \varphi(d_+, d_2, d_0^+), \tag{3.10}$$

$$H_3^{(1)} = H_3^{(2)}, \quad f(d_1, d_0^-) = f(d_2, d_0^+).$$

Therefore, (d_1, d_2) is solution of a 2×2 system which is similar to (2.41):

$$\begin{cases} f(d_1, d_0^-) = f(d_2, d_0^+), \\ H_{3,-} + \varphi(d_-, d_1, d_0^-) = H_{3,+} - \varphi(d_+, d_2, d_0^+). \end{cases} \tag{3.11}$$

3.2.2. Sufficient conditions.

LEMMA 3.7. *The system (3.11) has a unique solution $(d_1, d_2) \in \mathbb{R}^2$.*

Proof. The values of $d_0^{\pm}, d_{\pm}, H_{3,\pm}$ are fixed. We define f_{\pm} as in the proof of Lemma 2.15, and $G = f_+^{-1} \circ f_-$, which is a C^1 , increasing, one-to-one function such that $G(0) = 0$. Here, we need to define G on \mathbb{R} , and $G(\mathbb{R}) = \mathbb{R}$.

Solving system (3.11) is equivalent to find d_1 such that

$$H_{3,+} - H_{3,-} = \varphi(d_-, d_1, d_0^-) + \varphi(d_+, G(d_1), d_0^+).$$

We end the proof by using the properties of φ . □

In the following theorem, we sum up those considerations, and we make the link between 2×2 and 3×3 solutions.

THEOREM 3.8. *Let u_-, u_+ be a Riemann data for system (1.7) in the direction ω . The Riemann problem has a unique solution in the class of the functions which are superpositions of a 1-wave, a stationary contact discontinuity, and a 3-wave.*

The intermediate states $u^{(1)}$ and $u^{(2)}$ are defined by (3.9), (3.10), and the solution of system (3.11).

For Riemann data of the form $u_{\pm} = (0, D_{2,\pm}, H_{3,\pm})$, the solution has the form $(0, D_2, H_3)$, the stationary contact discontinuity is trivial, and (D_2, H_3) is the Liu's solution of the p-system (1.8) for the data $(D_{2,\pm}, H_{3,\pm})$.

3.3. Comparison of the 6×6 solution with the 3×3 and 2×2 ones. For $u = (D_1, D_2, H_3)$, we denote $\bar{u} = (\bar{D}, \bar{H})$ where $\bar{D} = (D_1, D_2, 0)$ and $\bar{H} = (0, 0, H_3)$.

As observed in paragraph 2.5, if the Riemann data is transverse magnetic, so is the solution \bar{u} of system (1.1) provided by Theorem 2.16. The related u is a weak solution of the 3×3 system (1.7). But if the non stationary contact discontinuities are not trivial for \bar{u} , u is not the solution of (1.7) given by Theorem 3.8.

For example, we can find a nontrivial transverse magnetic 6-contact discontinuity for system (1.1). We choose $\omega = (1, 0, 0)$,

$$\bar{D}_- = \begin{pmatrix} 0 \\ D_{2,-} \\ 0 \end{pmatrix}, \quad \bar{H}_- = \begin{pmatrix} 0 \\ 0 \\ H_{3,-} \end{pmatrix}, \quad \bar{D}_+ = -\bar{D}_-,$$

and, σ_+ defined by (2.29):

$$\bar{H}_+ = \bar{H}_- + \sigma_+ \omega \times (\bar{D}_+ - \bar{D}_-) = \begin{pmatrix} 0 \\ 0 \\ H_{3,-} + \sigma_+(D_{2,+} - D_{2,-}) \end{pmatrix}.$$

The Liu’s solution of the 3×3 Riemann problem for system (1.7) with data $u_{\pm} = (0, D_{2,\pm}, H_{3,\pm})$ cannot be such a contact discontinuity. The solution consists of a 1-wave and a 3-wave. Such solutions are compared in Figure 4.5; see Section 4 for the numerical details. Here, the 1-wave is a rarefaction, and the 3-wave is composed of a shock connecting u_+ and $u^*(u_+)$ and a rarefaction connecting $u^*(u_+)$ and $u^{(2)} = u^{(1)}$.

Consequently, one has two distinct solutions of the problem. This is not contrary to known results. In particular, we point out the fact that, as is usual for such problems, uniqueness in theorems 2.16 and 3.8 holds only in a definite class of solutions.

In order to choose the physical solution, we study the electromagnetic energy of each of them. For the reduced case (1.7), still denoting $E = \mathbf{P}(D)$, the energy density reads as ([5])

$$\eta(D, H_3) = \mathcal{E}(D) + \frac{1}{2} \mu_0 H_3^2, \quad \mathcal{E}(D) = \epsilon_0 (|E|^2 + \frac{3\epsilon_r}{2} |E|^4).$$

Actually, η is the mathematical entropy for the Kerr system, with entropy flux

$$Q(D, H_3) = H_3(E_2, -E_1).$$

As is well known, contact discontinuities and rarefactions preserve entropy; see [15], for example. Let us study what happens for Liu’s shocks.

A shock (σ, u_-, u_+) is entropy dissipative if

$$\partial_t \eta(u) + \operatorname{div} Q(u) \leq 0$$

in a weak sense. This inequality also reads as

$$-\sigma[\eta(D, H_3)] + [H_3 \omega \times \mathbf{P}(D)] \leq 0. \tag{3.12}$$

THEOREM 3.9. Entropy dissipation for Liu’s shocks. *Let (σ, u_-, u_+) be a Liu’s shock. The entropy dissipation inequality (3.12) holds.*

In the particular case $D_{\pm} \cdot \omega = 0$, denoting $e = p(d)$ and $D = d\omega^{\perp}$, the amount of entropy dissipation is

$$-\sigma[\eta(D, H_3)] + [H_3 e] = -\frac{c \epsilon_0 \epsilon_r}{4\sqrt{1 + \epsilon_r(e_+^2 + e_+ e_- + e_-^2)}} [e]^2 |[e^2]| \leq 0. \tag{3.13}$$

Proof. We write the proof for a 1-shock with $D_- \cdot \omega^\perp > 0$. The other cases are similar. The Liu’s 1-shock curve for given u_- is parametrized by $d \in [d_*(d_-), d_-]$ as

$$u(d) = \left(\begin{array}{c} d_0\omega + d\omega^\perp \\ H_{3,-} + \sigma(d, d_-)(d - d_-) \end{array} \right), \quad \sigma < 0, \quad \sigma^2 = \frac{f(d, d_0) - f(d_-, d_0)}{d - d_-}. \quad (3.14)$$

For such a u , $\omega \times \mathbf{P}(D) = \mu_0 f(d, d_0)$, so that (3.12) reads as

$$-\sigma(\eta(u(d)) - \eta(u(d_-))) + \mu_0 f(d, d_0)(H_{3,-} + \sigma(d - d_-)) - \mu_0 f(d_-, d_0)H_{3,-} \leq 0.$$

We denote by $-\sigma \mathcal{D}(d)$ the left-hand-side of this inequality. We have that $\mathcal{D}(d_-) = 0$ and

$$\mathcal{D}(d) = \mathcal{E}(d_0\omega + d\omega^\perp) - \mathcal{E}(d_0\omega + d_-\omega^\perp) - \frac{1}{2}\mu_0(d - d_-)(f(d, d_0) + f(d_-, d_0)).$$

Using the fact that $\mathcal{E}'(D, H_3) = (E_1, E_2, \mu_0 H_3)$, we find

$$\mathcal{D}'(d) = \frac{1}{2}\mu_0(d_- - d) \left(\partial_1 f(d, d_0) - \frac{f(d_-, d_0) - f(d, d_0)}{d_- - d} \right).$$

The properties of f and the definition of $d_*(d_-)$ allow us to conclude that $\mathcal{D}'(d) \geq 0$ for $d \in [d_*(d_-), d_-]$, and this proves the entropy dissipation property.

In the case where $d_0 = 0$, $\mathcal{E}(D) = \epsilon_0 e^2 (1 + \frac{3\epsilon_r}{2} e^2)$ and $\mu_0 f(d, 0) = p(d) = e$, and hence the result follows. \square

As Lax’ shocks are also Liu’s shocks, we conclude that we have found two distinct selfsimilar entropy (or energy) solutions of the Riemann problem for (1.7). In the case of the above example, the 6×6 solution conserves the electromagnetic energy, and Liu’s solution dissipates this energy by the presence of a shock. Numerical experiments will bring more information about this problem; see Section 4.

4. Numerical experiments

We present one and two-dimensional computations with the Godunov scheme for the 6×6 Kerr system. The one-dimensional tests are concerned with comparisons to exact solutions of the Riemann problem. As a particular case, we investigate numerically the problem of the nonuniqueness of selfsimilar entropy TM solutions.

The two-dimensional experiments are performed on a cartesian grid. We take transverse magnetic data (D_1, D_2, H_3) , but we use the 6×6 solver; see paragraph 2.5. The first case is concerned with piecewise constant initial data for which one-dimensional waves remain visible. Then we study an ultrashort optical pulse proposed in [21].

In all cases, we also compare our results with those obtained by the Kerr–Debye relaxation scheme.

The relative permittivity is $\epsilon_r = 2 \times 10^{-18}$.

All the computations have been performed with a CFL number of 0.3. An important remark is that all the characteristic velocities are bounded by the light velocity $c = \sqrt{\epsilon_0 \mu_0}^{-1}$, so that we are able to fix a constant time step. All the results are obtained with a second order extension, in space by affine reconstructions with min/mod limiters and in time by a second order Runge–Kutta scheme.

4.1. One-dimensional cases. We fix the computation domain as $[-X, X]$ with $X = cT$, T being the maximal time, so that if N is the number of cells, then the number of time steps is $p = N/0.6$.

We first consider the Kerr system (1.1) with the following Riemann data:

$$\begin{aligned}
 D(x,0) &= \begin{pmatrix} 0 \\ 0.03 \\ 0 \end{pmatrix} \quad \text{if } x_1 < 0, & D(x,0) &= \begin{pmatrix} 0.03 \\ 0.04 \\ 0.04 \end{pmatrix} \quad \text{if } x_1 > 0, \\
 \mu_0 H(x,0) &= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \quad \text{if } x_1 < 0, & \mu_0 H(x,0) &= \begin{pmatrix} 0.001 \\ 0 \\ 3 \end{pmatrix} \quad \text{if } x_1 > 0.
 \end{aligned}
 \tag{4.1}$$

This data is not divergence free. The solution only depends on $x = x_1$, $D_1(x,t) = D_1(x,0)$ and $H_1(x,t) = H_1(x,0)$. The scheme (1.5) with (1.6) reads as

$$\begin{aligned}
 D_i^{n+1} &= D_i^n - \frac{\Delta t}{\Delta x} \begin{pmatrix} 0 \\ H_{3,i+\frac{1}{2}}^n - H_{3,i-\frac{1}{2}}^n \\ -H_{2,i+\frac{1}{2}}^n + H_{2,i-\frac{1}{2}}^n \end{pmatrix}, \\
 H_i^{n+1} &= H_i^n - \mu_0^{-1} \frac{\Delta t}{\Delta x} \begin{pmatrix} 0 \\ -E_{3,i+\frac{1}{2}}^n + E_{3,i-\frac{1}{2}}^n \\ E_{2,i+\frac{1}{2}}^n - E_{2,i-\frac{1}{2}}^n \end{pmatrix}.
 \end{aligned}$$

As a consequence, $D_{1,i}^n = D_{1,i}^0$ and $H_{1,i}^n = H_{1,i}^0$, and the error for those components is only due to initial discretization of data. As $x = 0$ is an interface between two cells, this error is zero. Hence, we do not represent D_1 and H_1 .

Figures 4.1–4.2 show respectively the components (D_2, D_3) and (H_2, H_3) at time $T = 10$ femtoseconds, for 400 and 1600 cells. The exact solution consists of a 1-contact discontinuity, a 2-rarefaction, a nontrivial stationary contact discontinuity, a 5-shock, and a 6-contact discontinuity. It is well retrieved by the Godunov scheme. We have also tested the Kerr–Debye relaxation scheme (A.2) with (A.1) presented in the appendix. Both schemes give very close results. In Figure 4.3, L^1 relative errors with respect to the space step are depicted for each of them. We make the number of cells vary from 400 to 1600. The numerical order of accuracy is 0.66.

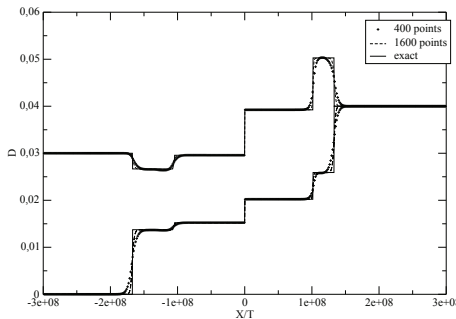


FIGURE 4.1. One space dimension: (D_2, D_3) for Riemann data (4.1) after 10 femtoseconds. Godunov scheme.

In a second series, we try to understand the problem of non uniqueness shown in paragraph 3.3. We take a sequence (D^m, H^m) of 6-contact discontinuities as follows:

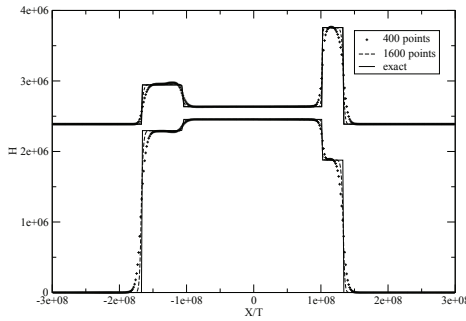


FIGURE 4.2. One space dimension: (H_2, H_3) for Riemann data (4.1) after 10 femtoseconds. Godunov scheme.

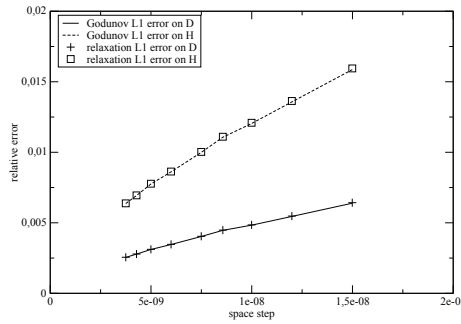


FIGURE 4.3. One space dimension: relative L^1 errors for Godunov and Kerr–Debye relaxation schemes, for Riemann data (4.1), from 400 to 1600 cells.

$\omega = (1, 0, 0)$, and for $m = 1, \dots, 12$, $\theta_m = \frac{m\pi}{12}$,

$$D^m(x, 0) = D_- \text{ if } x_1 < 0, \quad D^m(x, 0) = D_+^m = R_m D_- \text{ if } x_1 > 0, \tag{4.2}$$

$$H^m(x, 0) = H_- \text{ if } x_1 < 0, \quad H^m(x, 0) = H_+^m \text{ if } x_1 > 0,$$

with $H_+^m = H_- + \sigma_+ \omega \times (D_+^m - D_-)$, σ_+ defined in (2.29), and

$$D_- = \begin{pmatrix} 0 \\ 0.03 \\ 0 \end{pmatrix}, \quad \mu_0 H_- = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \quad R_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_m & -\sin \theta_m \\ 0 & \sin \theta_m & \cos \theta_m \end{pmatrix}.$$

When $m = 12$ ($\theta_m = \pi$), we have a transverse magnetic field which is also a weak solution of the p-system (1.8), and the entropy is conserved. Denoting $e = p(0.03)$,

$$-\sigma_+ [\eta(D, H_3)] + [H_3 e] = 0.$$

We have $|D_-| = |D_+^m| = 0.03$,

$$|D_+^m - D_-| = |D_-| \sqrt{2(1 - \cos \theta_m)},$$

and

$$|H_{3,+}^m - H_{3,-}| = \frac{c}{\sqrt{1 + \epsilon_r e^2}} |D_+^m - D_-|.$$

Therefore $|u_+ - u_-|$ is an increasing function of m .

Figure 4.4 shows the evolution of L^1 relative error for D and H respectively. We can observe that this error increases with the rotation angle, but it always converges to zero, except when $\theta_m = \pi$. As shown in Figure 4.5, in this case, convergence holds to Liu's solution which consists of a 1-rarefaction and a 2-wave composed by a 2-rarefaction and a 2-shock. The Kerr–Debye relaxation scheme gives the same results.

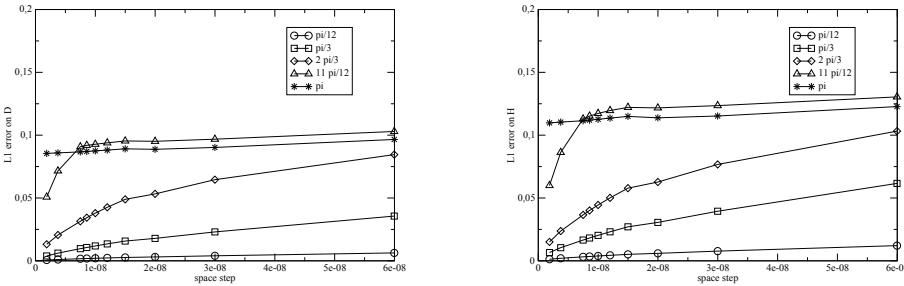


FIGURE 4.4. L^1 relative error for D (left) and H (right) with respect to Δx : convergence holds except when reducing to a TM field.

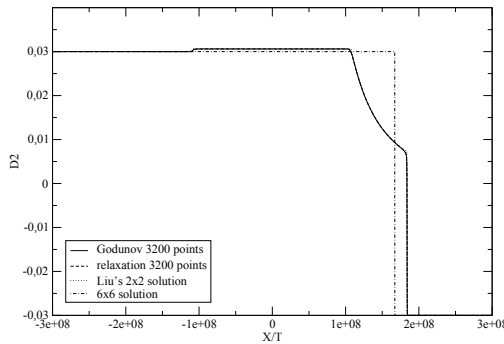


FIGURE 4.5. D -component: Godunov and Kerr–Debye relaxation scheme both compute Liu's solution. H -component is similar.

4.2. Two-dimensional cases. We restrict ourselves to computations of transverse magnetic fields on cartesian grids. As a Riemann solver, we take the 6×6 solution provided by Theorem 2.16. We have also tested the 3×3 solution provided by Theorem 3.8, but, as one can guess in view of one-dimensional tests, this solver gives the same

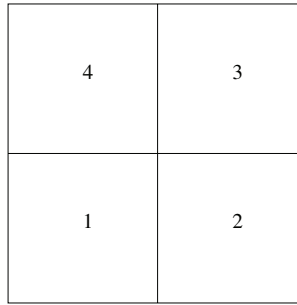


FIGURE 4.6. *Square partition.*

results as the 6×6 one. The scheme can be written as

$$\begin{cases} D_{1,ij}^{n+1} = D_{1,ij}^n + \frac{\Delta t}{\Delta y} \left(H_{3,i,j+\frac{1}{2}}^n - H_{3,i,j-\frac{1}{2}}^n \right) \\ D_{2,ij}^{n+1} = D_{2,ij}^n - \frac{\Delta t}{\Delta x} \left(H_{3,i+\frac{1}{2},j}^n - H_{3,i-\frac{1}{2},j}^n \right) \\ H_{3,ij}^{n+1} = H_{3,ij}^n - \frac{\Delta t}{\mu_0 \Delta x} \left(E_{2,i+\frac{1}{2},j}^n - E_{2,i-\frac{1}{2},j}^n \right) + \frac{\Delta t}{\mu_0 \Delta y} \left(E_{1,i,j+\frac{1}{2}}^n - E_{1,i,j-\frac{1}{2}}^n \right). \end{cases}$$

As a first test, we consider a square divided into four quadrants numbered as in Figure 4.6. On square i , we take $u^{(i)}$ as initial data, with

$$u^{(1)} = \begin{pmatrix} \delta_m \\ \delta_m \\ H_3^{(1)} \end{pmatrix}, u^{(2)} = \begin{pmatrix} \delta_m \\ \delta_p \\ H_3^{(2)} \end{pmatrix}, u^{(3)} = \begin{pmatrix} \delta_p \\ \delta_p \\ H_3^{(3)} \end{pmatrix}, u^{(4)} = \begin{pmatrix} \delta_p \\ \delta_m \\ H_3^{(4)} \end{pmatrix},$$

in such a way that

- $u^{(1)}$ and $u^{(2)}$ are connected by a Lax 5-shock,
- $u^{(4)}$ and $u^{(3)}$ are connected by a 2-rarefaction.

The computation is performed for a time $T = 10$ femtoseconds, on a square $\Omega =]-cT, cT[$ with a 400×400 cartesian mesh, that is about 1320 time steps.

Our data is divergence free, but this property is not preserved by the scheme, even if for each interface the solver is divergence free. This 2D feature has already been reported in the context of MHD where it can lead to a complete blow up of the numerical solution. In our case, the results seem to be correct. The numerical ratio between $\text{div}(D)$ and ∇D is around 10^{-3} :

$$\int_{\Omega} |\text{div} D(x, y, t)| dx dy \leq 0.004 \int_{\Omega} |\nabla D(x, y, t)| dx dy.$$

This ratio remained in the same range for all the performed tests.

In Figure 4.7, the isovalues of D_1 and B are shown. We do not represent those of D_2 since they are in the same spirit. Near the boundaries, the problem is one-dimensional. When y is fixed, we retrieve the 5-shock and the 2-rarefaction; see Figure 4.8-left for a comparison with the exact solution near the top boundary. For fixed x , in view of Figure 4.7, one could think that a single rarefaction and a single shock also occur, but this is not true. The exact solution is composed on the left by a 2-rarefaction and a (small) 5-shock, and on the right, we have a (small) 2-rarefaction and a 5-shock. Our two-dimensional computation retrieves all those waves; see Figure 4.8-right for the right side.

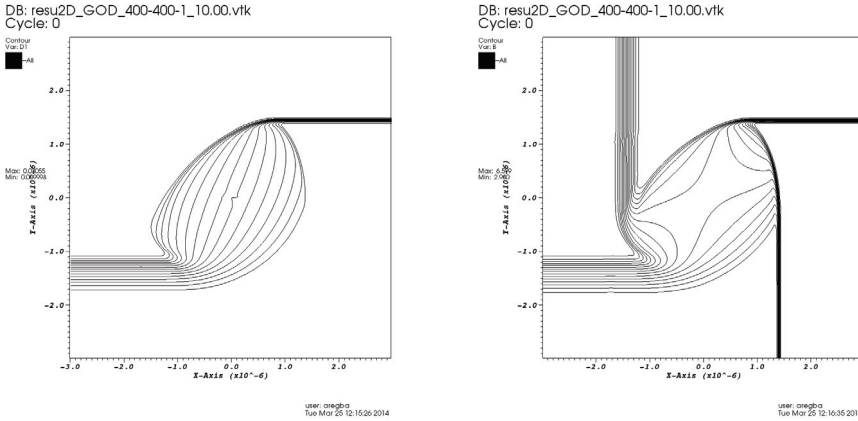


FIGURE 4.7. 2D Riemann problem: isovalues of D_1 (left) and $B = \mu_0 H$ (right). The values of D_2 are not represented.

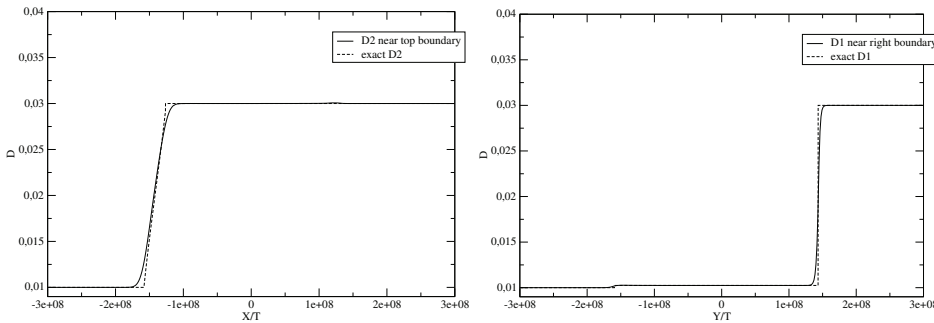


FIGURE 4.8. 2D Riemann problem: near the boundaries, the solution is 1D. Near the top, a single rarefaction for (D_1, H_3) ; near the right boundary, a small rarefaction and a shock for (D_2, H_3) .

The second test is taken from an article by R.-W. Ziolkowski and J.B. Judkins [21]. An ultrashort pulsed optical beam is generated by a Gaussian waisted magnetic field imposed at the left boundary of a rectangular domain $]0, X[\times]-Y, Y[$:

$$H_3(0, y, t) = \mu_0^{-1} B_0 \left(1 - \cos\left(\frac{2\pi t}{T}\right) \right) \exp\left(-\frac{y^2}{w^2}\right) \text{ if } t \in [0, T], \quad 0 \text{ else.}$$

The amplitude $B_0 = 6.087$ Tesla, the period $T = 20$ fs, and the initial waist $w = 10\mu\text{m}$ are fixed. In the cited article, the response time τ of the material is not zero, and the authors solve Kerr–Debye Equation (1.4) by a finite-difference time-domain (FDTD) method. They study self-focusing phenomena occurring in such cases. Those results have been retrieved in [9] by a finite element method and in [10] with a finite volume scheme of which (A.3)–(A.4) is the relaxed Kerr limit. Also in [10], the Kerr limit $\tau = 0$ has been investigated. Here we compare the results obtained by the Godunov scheme with those obtained in [10].

The symmetry of the problem allows us to compute the field only in the domain $\Omega =]0, X[\times]0, Y[$. We use a 800×600 mesh. The self-focusing phenomenon can be detected by studying the time evolution of the maximal electric intensity $I(t) = \max_{\Omega} |E|^2$. After decreasing for a rather long time, by a strong interaction between the components of E , this quantity increases to reach a local maximum and then decreases again. As the creation of shocks dissipates energy, this maximum is less important when the response time is zero (Kerr model) than for the Kerr–Debye model, but one can still observe it. In the present case, the local maximum is reached at time $t_1 = 109.2$ femtoseconds. In Figure 4.9-left, we zoom in on the time evolution of $I(t)$. We remark that the relaxation scheme (A.3)–(A.4) and the Godunov scheme give the same result. We just represent the isolines of $|E(t_1)|^2$ for the Godunov scheme (Figure 4.9-right); they are nearly the same as those obtained by the relaxation scheme. As can be seen on this figure, we have both self-focusing and shock creation which means that physical situations require an efficient computation of solutions with shocks.

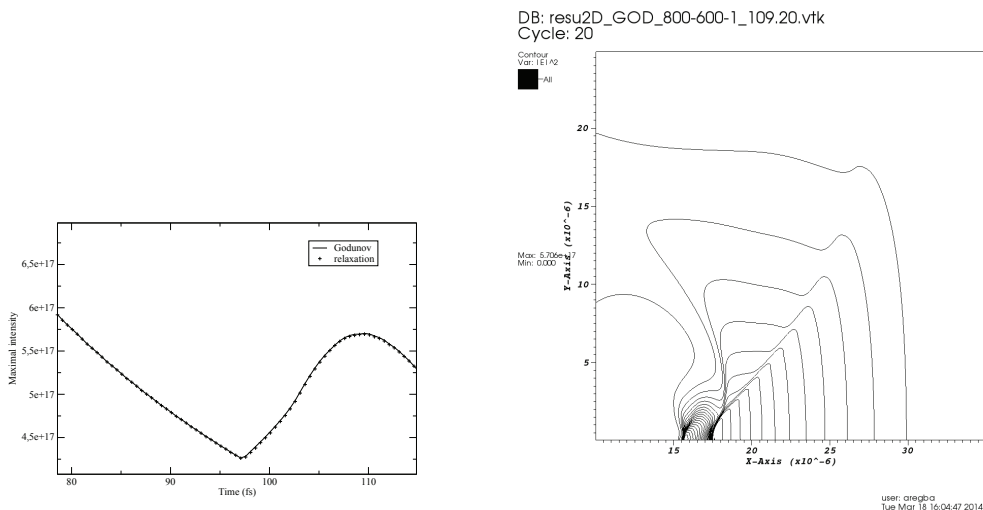


FIGURE 4.9. 2D case. Left: maximal intensity $|E|^2$ in the computational domain with respect to time. Right: intensity at the self-focusing time.

5. Conclusion

We have been able to solve the Riemann problem for the 6×6 Kerr system. The multiplicity of the eigenvalues is not constant, and the characteristic fields 2 and 5 are neither genuinely nonlinear, nor linearly degenerate. Nevertheless, in all cases, we can construct a unique Lax solution. For transverse magnetic data, this solution is transverse magnetic and does not coincide with Liu’s solution of the reduced 3×3 system. This allows us to point out the non uniqueness of selfsimilar weak entropy solutions of the Kerr system.

From the numerical viewpoint, the 6×6 Lax solution has been implemented as an exact Riemann solver for the Godunov scheme. Numerical experiments have been performed in one and two dimensions, including realistic physical cases. The results are very close to those obtained by the Kerr–Debye relaxation scheme coming from the non-zero response time model. In the particular case of coexistence of two entropy solutions, the more dissipative Liu’s solution is always reached by our schemes. This may

be due to numerical viscosity. In the physical case of an ultrashort pulsed optical beam, our results are consistent with those of the literature, and the well known self-focusing phenomenon is retrieved.

The results of the Godunov and Kerr–Debye relaxation schemes are very close. The relaxation scheme is completely explicit. Due to the resolution of a nonlinear algebraic equation for each cell, the Godunov scheme is more expensive in terms of CPU time. By construction, the Godunov scheme allows one to compute weak solutions of the Kerr system, even when they contain shocks. Along with the physically consistent relaxation scheme, we now have two reliable computational methods for the Kerr system.

Appendix A. Kerr–Debye relaxation scheme.

System (1.4) is hyperbolic with eigenvalues

$$\mu_1 = \mu_2 = -\mu, \quad \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = \mu_7 = \mu$$

with $\mu = \frac{c}{\sqrt{1+\chi}}$. Moreover, all the characteristic fields are linearly degenerate. These properties are useful for designing a numerical approximation of (1.1) following the classical projection-transport technique. At every time step, one first projects the solution onto equilibrium by setting $\chi = \epsilon_r p^2(|D|)$; then the homogeneous system related to (1.4) is solved. As we use the finite volume method, we just have to know the solution of the Riemann problem to find the numerical fluxes at each interface, that is the approximation of $-\omega \times H$ and $\mu_0^{-1}\omega \times E$. The Riemann problem is easy to solve because we have only contact discontinuities here. Denoting by U_- and U_+ the left and right initial states, $r_{\pm} = \sqrt{1+\chi_{\pm}}$ and $E_{\pm} = \frac{D_{\pm}}{\epsilon_0(1+\chi_{\pm})}$, we find (see [10] for a proof in the transverse magnetic case):

$$\varphi(U_-, U_+, \omega) = \begin{pmatrix} -\omega \times \frac{r_+ H_- + r_- H_+}{r_+ + r_-} + \omega \times \left(\omega \times \frac{r_+ r_- (E_+ - E_-)}{c\mu_0(r_+ + r_-)} \right) \\ \omega \times \frac{r_+ E_+ + r_- E_-}{\mu_0(r_+ + r_-)} + c\omega \times \left(\omega \times \frac{H_+ - H_-}{r_+ + r_-} \right) \end{pmatrix}.$$

In one space dimension, we set $\overline{U}_i = (u_i^n, \epsilon_r p^2(|D_i^n|))$, $\omega = (1, 0, 0) = e_1$,

$$F_{i+\frac{1}{2}}^n = \varphi(\overline{U}_i, \overline{U}_{i+1}, e_1), \tag{A.1}$$

and

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n). \tag{A.2}$$

Notice that this implies that $D_{i,1}^{n+1} = D_{i,1}^n$ and $H_{i,1}^{n+1} = H_{i,1}^n$ for all i and n , which means that $\text{div}D$ and $\text{div}H$ are constant throughout the computation.

In two space dimensions, for a cartesian mesh, we set

$$F_{i+\frac{1}{2},j}^n = \varphi(\overline{U}_{i,j}, \overline{U}_{i+1,j}, e_1), \quad G_{i,j+\frac{1}{2}}^n = \varphi(\overline{U}_{i,j}, \overline{U}_{i,j+1}, e_2), \tag{A.3}$$

and the scheme reads as

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{\Delta t}{\Delta x} (F_{i+\frac{1}{2},j}^n - F_{i-\frac{1}{2},j}^n) - \frac{\Delta t}{\Delta y} (G_{i,j+\frac{1}{2}}^n - G_{i,j-\frac{1}{2}}^n). \tag{A.4}$$

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