REAL OPTION MODEL OF DYNAMIC GROWTH PROCESSES WITH CONSUMPTION

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Abstract. A real option model is built upon a set of stochastic processes for some real investment decision making in incomplete markets. Typically, the optimal consumption level is obtained under a logarithmic utility constraint, and a partial integro-differential equation (PIDE) of the real option is deduced by martingale methods. Analytical formulation of the PIDE is solved by Fourier transformation. Two types of decision making strategies, i.e. option price and IRP (inner risk premium) comparisons, are provided. Finally, the Monte Carlo simulation and numerical computation are illustrated to verify the conclusions.

Key words. Real option, asset pricing, jump diffusion, optimal consumption strategy, risk premium.

AMS subject classifications. 35Q91, 60G20.

1. Introduction

Life circle theory [1, 2, 3] indicates that a representative entity accumulates its endowment continuously and stochastically. Meanwhile, the entity must consume part of the endowment to survive during its lifetime. Usually, the endowment can be traded in an invisible market, and real investment decision making concerning an entity and its endowment deserves to be researched.

Plenty of daily phenomena and decision making satisfy the above description, for example, mate choice, corporate merger, employment, forest exploitation, and ocean development, etc. These investment decisions are often difficult to make since the underlying dynamics of growth and consumption are endogenous. Traditional discount cash flow (DCF) and net present value (NPV) methods are not flexible enough to deal with these problems [4, 5]. The real option approach, which is an extension of financial option theory, is highly recommended and widely applied to assess these project investments and natural resource explorations [6, 7, 8].

With the motivation above, we design a mathematical model to help make real investment decisions in this paper. We first use a set of stochastic differential equations with jumps to depict the dynamic process of a representative entity’s endowment, and then find the optimal consumption level for the entity itself under a logarithmic utility constraint.
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constraint. Under the optimal consumption level, we succeed in getting a partial integro-differential equation for the real option by martingale methods and finding its analytical solution by Fourier transform. Given a premium of the real option, one can find an implied option risk premium from the analytical solution. The implied risk premium is helpful for decision making in the field of real investment.

The dynamic of the endowment for the entity is ruled by a stochastic system which looks like the two types of equilibrium models in finance. One is the consumption-based asset pricing model in an exchange economy where a representative investor maximizes its expected utility by choosing an optimal level of consumption at each period [9, 10, 11, 12, 13, 14]. The other is the production-based asset pricing model in a production economy where a representative investor chooses an optimal level of consumption at each period and leaves the rest in the production to grow for the future consumption [15, 16, 17, 18, 19, 20, 21]. However, there are some essential differences in our model. First, no risk-free asset exists in the incomplete real investment market; besides, the entity’s consumption is endogenous in the underlying process; moreover, the growth rate is a stochastic process as well as the volatility process. Thus, the dynamic process of the endowment is complicated but more close to economic reality. From the angle of mathematics, the process is no longer a Levy process as is usual. These differences are a great challenge since they imply that the real option model cannot be obtained and priced in a risk-neutral world as most of the classical financial option models are, e.g. the Black–Scholes model [22], Merton’s jump-diffusion model [23], Heston’s stochastic volatility option model [24], the alternative option with stochastic interested rate and jump-diffusion [25], and Kou’s double exponential jump-diffusion model [26]. Our contribution is to build and solve a real option model in the physical world and to propose two investment criteria: the comparison of the real option price or the IRP (inner risk premium) with the market ones.

The remainder of this paper is organized as follows. In Section 2, we develop the underlying endowment process and find the optimal consumption strategy. In Section 3, we establish the pricing kernel and the real option model. In Section 4, we give the analytical expression of the option. In Section 5, we illustrate some numerical results and leave the code in the appendix for reference. Discussion and a conclusion are given last.

2. The endowment process and optimal consumption level

We develop in a continuous-time framework the following process of a representative entity’s endowment with stochastic growth, fluctuation, and jumps from the realistic perspective. Moreover, we take the endogenous consumption into consideration.

Assume the endowments is valued\(^4\) as \(S(t)\) at time \(t > 0\), with growth rate \(g(t)\) and variance \(v(t)\). Let the accumulated consumption be \(c(t)\), and let the Poisson jump process be \(N(t)\) with constant jump density \(\lambda\) and log jump size \(x\). The process is written as follows:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= g(t)dt - c(t)dt + \sqrt{v(t)}dW_S(t) + S(t)(e^x - 1)dN(t) - \lambda S(t)E(e^x - 1)dt, S(0) = S_0 > 0, \\
\frac{dg(t)}{g(t)} &= \kappa_g(\theta_g - g(t))dt + \varepsilon_g \sqrt{v(t)}dW_g(t), g(0) = g_0, \\
\frac{dv(t)}{v(t)} &= \kappa_v(\theta_v - v(t))dt + \varepsilon_v \sqrt{v(t)}dW_v(t), v(0) = v_0 > 0.
\end{align*}
\]  

\(^4\)This is a fundamental assumption that the endowment can be measured in some way, but not necessarily in the form of money.

\(^5\)Sometimes we use \(S_t\), or omit the subscript \(t\) for simplicity, when there is no ambiguity, other variables are treated likewise in the paper.
Here $E[\cdot]$ is the expectation under some information filter $\{\mathcal{F}_t\}_{t \geq 0}$ in the probability space denoted by $(\Omega, \mathcal{F}, P)$. The variables $\kappa_g$, $\theta_g$, $\varepsilon_g$, $\kappa_v$, $\theta_v$, $\varepsilon_v$, $S_0$, $g_0$, and $v_0$ are different constants. Here, $dW_S(t), dW_g(t), dW_v(t)$ are standard Brownian motions with constant correlation matrix $\Sigma$, but they are all independent of the Poisson jump. Denote

$$\Sigma := \begin{bmatrix} 1 & \rho_{Sg} & \rho_{Sv} \\ \rho_{Sg} & 1 & \rho_{gv} \\ \rho_{Sv} & \rho_{gv} & 1 \end{bmatrix}.$$ 

From Equation (2.1), the endowment market is incomplete since the process contains three Brownian motions and one Poisson jump process. Besides, there is not risk-free asset to use as a benchmark. Moreover, the spot growth rate, which can have a positive or negative value\(^6\), is dependent upon the variance and, in the long run, has a mean-reverse level up to the external economic situation. The customized endowment process is different from some classical financial dynamics of assets, but to some extent is closer to a realistic situation in real investment and management fields.

The expected rate of $g(t)$ and $v(t)$ are directly calculated as

$$\bar{g} = e^{-\kappa_g T} (g_0 - \theta_g) + \theta_g, \quad \bar{v} = e^{-\kappa_v T} v_0 + \theta_v (1 - e^{-\kappa_v T}).$$

(2.2)

For tractability, we assume a logarithmic utility function for the representative entity

$$U(c) = \ln c.$$ (2.3)

Assume the entity’s objective is to maximize the expected utility as follows: \(^7\)

$$\max_c E \int_t^\infty e^{-\alpha(s-t)} U(c_s) ds,$$ (2.4)

where $\alpha$ is a constant discount rate.

**Theorem 2.1.** The representative entity’s expected utility is positively related to the endowment $S(t)$ and the growth rate $g(t)$, and it is negatively related to the jump and the variance $v(t)$ for a given discount rate $\alpha$. The optimal cumulative consumption level $c^*(t)$ is

$$c^*(t) = \alpha S(t).$$ (2.5)

**Proof.** Denote the objective value function by

$$J(S_t, g_t, v_t) = \max_c E \int_t^\infty e^{-\alpha(s-t)} U(c_s) ds.$$ 

The condition of optimality is given by the following Bellman equation:

$$\alpha J - \max_c \{ \mathcal{L}(J) + U \} = 0.$$ (2.6)

---

\(^6\) We set $g(t)$ to be the form of the Visacek model, and it can be negative as well. However, we failed to get an analytical expression for the real option since the ODE for $v$ is fractional. See footnote 9.

\(^7\) The upper bound $\infty$ is represented formally in mathematics, which is usually a finite physical bound but sufficiently far from the time $t$. 
Here,
\[
\mathcal{L}(J) = [Sg - c - \lambda SE(e^x - 1)] \frac{\partial J}{\partial S} + \frac{1}{2} vS^2 \frac{\partial^2 J}{\partial S^2} + \kappa_g (\theta_g - g) \frac{\partial J}{\partial g} + \frac{1}{2} \varepsilon v^2 \frac{\partial^2 J}{\partial g^2} \\
+ \kappa_v (\theta_v - v) \frac{\partial J}{\partial v} + \frac{1}{2} v\varepsilon v^2 \frac{\partial^2 J}{\partial v^2} + \varepsilon gSvS \frac{\partial J}{\partial g} + \varepsilon \varepsilon v^2 \frac{\partial^2 J}{\partial g^2} \\
+ \varepsilon v \rho S_v vS \frac{\partial^2 J}{\partial S \partial v} + \varepsilon g \varepsilon v \rho S_v vS \frac{\partial^2 J}{\partial g \partial v} + \lambda E \left[ J(S e^x, g, v) - J \right].
\]

Thus we get the following HJB equation which is an elliptic PIDE:
\[
0 = \max \left\{ \left[ Sg - \lambda SE(e^x - 1) \right] \frac{\partial J}{\partial S} + \frac{1}{2} vS^2 \frac{\partial^2 J}{\partial S^2} + \kappa_g (\theta_g - g) \frac{\partial J}{\partial g} + \frac{1}{2} \varepsilon v^2 \frac{\partial^2 J}{\partial g^2} \\
+ \kappa_v (\theta_v - v) \frac{\partial J}{\partial v} + \frac{1}{2} v\varepsilon v^2 \frac{\partial^2 J}{\partial v^2} + \varepsilon gSvS \frac{\partial J}{\partial g} + \varepsilon \varepsilon v^2 \frac{\partial^2 J}{\partial g^2} \\
+ \varepsilon v \rho S_v vS \frac{\partial^2 J}{\partial S \partial v} + \varepsilon g \varepsilon v \rho S_v vS \frac{\partial^2 J}{\partial g \partial v} + \lambda E \left[ J(S e^x, g, v) - J \right] \right\}.
\]

In order to get the optimal consumption level, we take a partial derivative of the HJB Equation (2.7) with respect to \( c \) and obtain the first-order condition
\[
0 = \frac{\partial U}{\partial c} - \frac{\partial J}{\partial S}.
\]

Enlightened by Cox, Ingersoll, and Ross [17], we guess that the HJB Equation (2.7) has a variables separated solution form of
\[
J(S_t, g_t, v_t) = \frac{1}{\alpha} \ln S_t + Ag_t + Bv_t + D,
\]
where \( A, B, D \) are constants independent of \( g_t \) and \( v_t \). Then plugging equations (2.3) and (2.9) into (2.8), we get
\[
c^*(t) = \alpha S_t,
\]
and
\[
0 = \frac{1}{\alpha} \left[ g - \lambda E(e^x - 1) \right] - \frac{1}{2\alpha} v + \kappa_g (\theta_g - g) A \\
+ \kappa_v (\theta_v - v) B + \frac{\lambda}{\alpha} E[x] - \alpha [Ag + Bv + D] + \ln \alpha - 1.
\]

By virtue of the arbitrariness of \( g_t \) and \( v_t \), we have
\[
\begin{cases}
0 = \frac{1}{\alpha} - \kappa_g A - \alpha A, \\
0 = -\frac{1}{2\alpha} - \kappa_v B - \alpha B, \\
0 = -\frac{\lambda}{\alpha} E(e^x - 1) + \kappa_g \theta_g A + \kappa_v \theta_v B + \frac{\lambda}{\alpha} E[x] - \alpha D + \ln \alpha - 1,
\end{cases}
\]
and get
\[
\begin{align*}
A &= \frac{1}{(\kappa_g + \alpha)^\alpha}, \\
B &= -\frac{1}{2\alpha (\kappa_v + \alpha)}, \\
D &= \frac{\lambda}{\alpha^2} E[x] - \frac{\lambda}{\alpha^2} E(e^x - 1) + \frac{\kappa_g \theta_g}{(\kappa_g + \alpha)^{\alpha^2}} - \frac{\kappa_v \theta_v}{2\alpha^2 (\kappa_v + \alpha)} + \frac{\ln \alpha - 1}{\alpha}.
\end{align*}
\]
The theorem results are immediately obtained from equations (2.9), (2.10), and (2.11).

**Remark 2.2.** In economics, \( \alpha > 0 \) means that one prefers consuming today with certainty to consuming the same quantity tomorrow with uncertainty. One can assume that the entity is constrained to consume \( 0 < c_t \leq S(t) \), i.e., \( 0 < \alpha < 1 \) to preclude starvation or overdraft in its lifetime. However, advance by overdraft can be seen everywhere, thus we just assume \( \alpha > 0 \) here to satisfy the preferred hypothesis in economics. In fact, it is unnecessary in mathematics.

### 3. Pricing kernel and real option model

Assume the representative entity consumes its endowment according to the optimal strategy (2.5). Then process (2.1) is rewritten as

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (g(t) - \alpha) dt + \sqrt{v(t)} dW_S(t) + (e^x - 1) dN(t) - \lambda E(e^x - 1) dt, S_0 > 0, \\
g(t) &= \kappa g(\theta_g - g(t)) dt + \varepsilon_g \sqrt{v(t)} dW_g(t), g(0) = g_0, \\
v(t) &= \kappa_v (\theta_v - v(t)) dt + \varepsilon_v \sqrt{v(t)} dW_v(t), v_0 > 0.
\end{align*}
\]  

Thus we integrate process (3.1) and get

\[
\ln S(T) = \int_t^T \left( g(s) - \alpha - \frac{1}{2} v(s) - \lambda E(e^x - 1) \right) ds + \int_t^T \sqrt{v(s)} dW_S(s) + \sum_{i=1}^{N(T-t)} x_i.
\]  

**Theorem 3.1.** The pricing kernel for the endowment process is given by

\[
\frac{d\pi_t}{\pi_t} = -\mu(t) dt - \sqrt{v(t)} dW_S(t) + (e^y - 1) dN(t) - \lambda E(e^y - 1) dt.
\]  

Integrate to get

\[
\frac{\pi_T}{\pi_t} = \exp \left\{ \int_t^T \left( -\mu(s) - \frac{1}{2} v(s) - \lambda E(e^y - 1) \right) ds - \int_t^T \sqrt{v(s)} dW_S(s) + \sum_{i=1}^{N(T-t)} y_i \right\}.
\]  

where

\[
\mu(t) = g(t) - v(t) - \lambda E[(1 - e^{-x})(e^x - 1)],
\]  

and the random jump size \( y \) satisfies the following restriction:

\[
E[(e^y - e^{-x})(e^x - 1)] = 0.
\]  

**Proof.** Plugging equations (2.5), (3.2), and (3.4) into the martingale condition [27]

\[
\pi_t S_t = E \left[ \int_t^T \pi(s) c^*(s) ds + \pi_T S_T | F_t \right], \ \forall t,
\]  

we get

\[
1 = E_t \left[ \int_t^T \frac{\pi(\zeta) c^*(\zeta)}{\pi_t S_t} d\zeta + \frac{\pi_T S_T}{\pi_t S_t} \right].
\]
\[ E_t[\alpha \int_t^T \exp\{\int_t^t [-\mu(\zeta) - \frac{1}{2} v(\zeta) - \lambda E(e^y - 1)]d\zeta - \int_t^t \sqrt{v(\zeta)}dW_S(\zeta) + \sum_{i=1}^{N(t-t)} y_i\}] \]

\[ \exp\{\int_t^T [g(\zeta) - \alpha - \frac{1}{2} v(\zeta) - \lambda E(e^x - 1)]d\zeta + \int_t^T \sqrt{v(\zeta)}dW_S(\zeta) + \sum_{i=1}^{N(t-t)} x_i\}dt \]

\[ + \exp\{\int_t^T [-\mu(\zeta) - \frac{1}{2} v(\zeta) - \lambda E(e^y - 1)]d\zeta - \int_t^T \sqrt{v(\zeta)}dW_S(\zeta) + \sum_{i=1}^{N(t-t)} y_i\} \]

\[ \sum_{i=1}^{N(t-t)} (x_i + y_i)\} ] \]

\[ = E_t[\alpha \int_t^T \exp\{\int_t^t [g(\zeta) - \mu(\zeta) - \alpha - v(\zeta) - \lambda E(e^y - 1) - \lambda E(e^x - 1)]d\zeta \]

\[ + \sum_{i=1}^{N(t-t)} (x_i + y_i)\} ] \]

\[ = E_t[\alpha \int_t^T \exp\{\int_t^t [g(\zeta) - \mu(\zeta) - \alpha - v(\zeta) - \lambda E[(1 - e^y)](e^x - 1)]d\zeta \]

\[ + \exp\{\int_t^T [g(\zeta) - \mu(\zeta) - \alpha - v(\zeta) - \lambda E[(1 - e^y)](e^x - 1)]d\zeta\}. \]

\[ (3.9) \]

Remark 3.2. An extreme case of constraint (3.6) is that the jump \( y \) in the pricing kernel perfectly synchronizes with the jump \( x \) and \( y = -x \). Since we should not have \( y \) in the growth rate \( \mu(t) \) of the process \( S(t) \), we put \( y = -x \) into the definition of \( \mu(t) \) which can be viewed as a pure discount rate which has subtracted the variance premium and jump premium formally. It is worth noting that the pricing kernel is not unique due to the incomplete market, because there is only one underlying process, but there are four risk sources-three Brownian motions and a jump.

Define a new probability measure \( Q \) by the Radon–Nikodym derivative

\[ \frac{dQ}{dP} = \exp\{-\lambda E(e^y - 1)T + \sum_{i=1}^{N_T} y_i\}. \]

Then it holds true that (Lemma C.1 in [21])

\[ E[f(x)e^y] = E^Q[f(x)E(e^y)]. \]

(3.11)
4. The real option model and the pricing formula

Many real investment and management decisions are, in fact, some options of the underlying entity and can be considered in the frame of real options. To be exact, the decision relies upon the payment of the investor and the payoff gained from the entity. Take a European real investment option of endowment process (3.1), for example. Assume the strike price is \( K \) and the payoff at the expiry date \( T \) is \([ (S(T) - K)\varpi ]^+\), where \( \varpi = 1 \) for the call option and \( \varpi = -1 \) for the put option. Therefore, we can evaluate a price of the real option by virtue of the formal pricing kernel (3.4), and we get

\[
V(S(t),K,g(t),v(t),t) = E\left[ \frac{\pi_T}{\pi_t} [ (S_T - K)\varpi ]^+ | \mathcal{F}_t \right],
\]

which satisfies a PIDE given in Theorem 4.1.

**Theorem 4.1.** The European option prices \( V(S(t),K,g(t),v(t),t) \) satisfy

\[
0 = -\mu V + \frac{\partial V}{\partial t} + \left[ \mu - \alpha - \lambda^Q E^Q(e^x - 1) \right] S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2}
\]

\[
+ \left[ \kappa_g(\theta_g - g) + \epsilon_g \rho_g v \right] \frac{\partial V}{\partial g} + \frac{1}{2} \epsilon_g^2 \frac{\partial^2 V}{\partial g^2}
\]

\[
+ \left[ \kappa_v(\theta_v - v) - \epsilon_v \rho_v S v \right] \frac{\partial V}{\partial v} + \frac{1}{2} \epsilon_v^2 \frac{\partial^2 V}{\partial v^2} + \epsilon_g \rho_g v S \frac{\partial^2 V}{\partial S \partial g}
\]

\[
+ \epsilon_v \rho_v S v S \frac{\partial^2 V}{\partial S \partial v} + \lambda^Q \left[ E^Q\{ V(S(e^x,g,v,t)) - V \} \right],
\]

with boundary condition

\[
V(S(T),K,g(T),v(T),T) = [(S(T) - K)\varpi]^+,
\]

where \( \varpi = \pm 1 \) and \( \lambda^Q = \lambda E[e^y] \) and \( E^Q[\cdot] \) are defined in (3.11).

**Proof.** We apply the Feynman–Kac Theorem to the dynamics (3.1) to get the PIDE. Typically, applying the Itô formulation, we get

\[
dV = \frac{\partial V}{\partial t} dt + \left[ g - \alpha - \lambda E(e^x - 1) \right] S \frac{\partial V}{\partial S} dt + \frac{1}{2} \sigma^2 S \frac{\partial^2 V}{\partial S^2} dt
\]

\[
+ \left[ \kappa_g(\theta_g - g) \right] \frac{\partial V}{\partial g} dt + \frac{1}{2} \epsilon_g^2 \frac{\partial^2 V}{\partial g^2} dt
\]

\[
+ \left[ \kappa_v(\theta_v - v) - \epsilon_v \rho_v S v \right] \frac{\partial V}{\partial v} dt + \frac{1}{2} \epsilon_v^2 \frac{\partial^2 V}{\partial v^2} dt
\]

\[
+ \epsilon_g \rho_g v S \frac{\partial^2 V}{\partial S \partial g} dt + \epsilon_v \rho_v S v S \frac{\partial^2 V}{\partial S \partial v} dt + \epsilon_g \rho_g v v \frac{\partial^2 V}{\partial g \partial v} dt + S \sqrt{v} \frac{\partial V}{\partial S} dW_S(t)
\]

\[
+ \epsilon_g \sqrt{v} \frac{\partial V}{\partial g} dW_g(t) + \epsilon_v \sqrt{v} \frac{\partial V}{\partial v} dW_v(t) + [V(S(e^x,g,v,t) - V)] dN(t),
\]

and

\[
d(\pi V) = V d\pi + \pi dV + d\pi dV
\]

\[
= V \pi \{ -\mu dt - \sqrt{v} dW_S dt - \lambda E(e^y - 1) dt \}
\]

\[8\text{Usually, } K \text{ may not be money in the real investment. It might be some counterpart of the endowment, e.g. the investors’ endowment or something like opportunity cost, etc.}\]
Theorem 4.2. Given an expected rate 

we employ the standard Fourier transformation methods to get an exact expression of Fourier transformation methods to obtain the transform-based solution of option prices. Carr and Madan [30], Sepp [31], and Lewis [32, 33] summarize the transform analysis to price the valuation of options for affine jump-diffusions with stochastic volatility. Duffie, Pan, and Singleton [28] and Chacko and Das [29] present a transform analysis to price the valuation of options for affine jump-diffusions with stochastic volatility.

Due to the martingale condition \( E[d(\pi(t)V(t))] = 0 \), we can obtain

\[
0 = -[\mu + \lambda E(e^y - 1)]V + \frac{\partial V}{\partial t} + [g - \alpha - \lambda E(e^x - 1) - v]S \frac{\partial V}{\partial S} + \frac{1}{2}vS^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \varepsilon_g^2 V \frac{\partial^2 V}{\partial g^2} + \frac{1}{2} \varepsilon_g \varepsilon_v S \frac{\partial^2 V}{\partial g \partial v} + \frac{1}{2} \varepsilon_v \varepsilon_g S \frac{\partial^2 V}{\partial v^2} + \varepsilon_g \varepsilon_v v S \frac{\partial^2 V}{\partial g \partial v} + \varepsilon_v \varepsilon_g v S \frac{\partial^2 V}{\partial v^2} + \varepsilon_g \varepsilon_v v S \frac{\partial^2 V}{\partial g \partial v} + \lambda E[e^y V(S^x, g_v, v, t) - \lambda V].
\]

Making use of (3.5), (3.6), and (3.11), we can get (4.2) immediately. \( \square \)

Although the PIDE (4.2) is a little complicated, it can be successfully solved by Fourier methods. Duffie, Pan, and Singleton [28] and Chacko and Das [29] present a transform analysis to price the valuation of options for affine jump-diffusions with stochastic volatility. Carr and Madan [30], Sepp [31], and Lewis [32, 33] summarize the Fourier transformation methods to obtain the transform-based solution of option prices. We employ the standard Fourier transformation methods to get an exact expression of the options as follows.

**Theorem 4.2.** Given an expected rate \( \omega(t) \) of the real option, the analytical expression of the solution to the PIDE (4.2) is

\[
V(S, K, g, v, t) = \frac{1 + \varpi}{2} S e^{-\alpha \tau} + \frac{1 - \varpi}{2} K e^{-i \int_0^T \mu(s) ds} - W,
\]

where

\[
W = \frac{e^{-i \int_0^T \varpi(s) ds}}{2\pi} \int_{i z_i - \infty}^{i z_i + \infty} e^{-i z\ln S} \frac{K^{i z + 1}}{z^2 - iz} e^{i \Gamma(\tau) g + \Lambda(\tau) v + \Pi(\tau)} dz,
\]

\( i = \sqrt{-1}, \quad \tau = T - t, \quad z = z_i + i z_i, \quad 0 < z_i < 1, \)

\( \varpi = \pm 1 \) for the call/put option,

\[
\Gamma(\tau) = -\frac{1 + iz}{\kappa_g} (1 - e^{-\kappa_g \tau}),
\]

\[
\Pi(\tau) = \delta \tau - (1 + iz) \theta_g (\frac{e^{-\kappa_g \tau} - 1}{\kappa_g}) + \kappa_v \theta_v \int_0^T \Lambda(s) ds,
\]
\[ \delta = (1 + iz)\lambda E[(1 - e^{-x})(e^{x} - 1)] + iz\alpha + iz\lambda Q E^{Q}(e^{x} - 1) + \lambda Q E^{Q}[e^{-izx} - 1], \]

and \( \Lambda(\tau) \) is determined by the ODE
\[
\begin{cases}
\lambda (\tau) - \frac{1}{2}\varepsilon_{\tau}^{2} \Lambda(\tau) + \eta \Lambda(\tau) + \vartheta = 0, \\
\Lambda(0) = 0,
\end{cases}
\]

where
\[ \eta = \kappa_{v} + (1 + iz)\varepsilon_{v} \rho_{v} + \frac{1 + iz}{\kappa_{g}} \varepsilon_{g} \rho_{gv} (1 - e^{-\kappa_{\tau}}), \]
\[ \vartheta = -1 - \frac{3}{2}iz + \frac{1}{2}z^{2} - (1 + iz)^{2} \varepsilon_{g} \rho_{g} \frac{1}{\kappa_{g}} (1 - e^{-\kappa_{\tau}}) - \frac{1}{2} \varepsilon_{g}^{2} \frac{(1 + iz)^{2}}{\kappa_{g}^{2}} (1 - e^{-\kappa_{\tau}})^{2}. \]

**Remark 4.3.** The difference of formulations (4.1) and (4.5) is that the former is an abstract backward pricing principle which needs the information of \( T \), while the latter is an explicit forward pricing given at time \( t \). The numerical computation speed of the latter is much faster than the former. The option price (4.5) is expressed in the physical world, thus the expected rate \( \mu(t) \) of the underlying endowment and \( \omega(t) \) of the real option are different. This is a major difference from classic financial options whose worlds exist risk-free with a rate equal to \( \mu(t) \) and \( \omega(t) \) in risk neutral pricing methods. For this reason, the real option prices are dependent upon \( \omega(t) \) which is related to investors’ aversion attitudes.

**Proof.** Notice that the call/put option payoff can be rewritten as
\[ \left( (S_{T} - K) \omega \right)^{+} = \frac{1 + \omega}{2} S_{T} + \frac{1 - \omega}{2} K - \min \{ S_{T}, K \}, \tag{4.6} \]

and by virtue of equations (4.1), (3.2), and (3.4), we get
\[
E \left[ \frac{\pi_{t}}{\pi_{t}} \left( \frac{1 + \omega}{2} S_{T} + \frac{1 - \omega}{2} K \right) \right] = \frac{1 + \omega}{2} S_{t} e^{-\alpha_{\tau}} + \frac{1 - \omega}{2} e^{-f_{T} \mu(s) ds} K.
\]

Thus one may solve the call/put European options by the PIDE (4.2) with terminal payoff \( V(S(T), K, g(T), v(T), T) = \min \{ S_{T}, K \} \).

With some variable substitutions, \( X = \ln S, \ \tau = T - t, \) and \( f(X, K; g, v, t) := V(S, K; g, v, t) \), the PIDE (4.2) can be rewritten as
\[
\begin{cases}
\frac{\partial f}{\partial \tau} = -\mu f + \left[ \mu - \alpha - \lambda Q E^{Q}(e^{x} - 1) - \frac{1}{2} v \right] \frac{\partial f}{\partial x} + \frac{1}{2} v \frac{\partial^{2} f}{\partial x^{2}} + \left( \kappa_{g}(\theta_{g} - g) - \varepsilon_{g} \rho_{sg} v \right) \frac{\partial f}{\partial g} \\
\quad + \frac{1}{2} \varepsilon_{g}^{2} \frac{\partial^{2} f}{\partial g^{2}} + \left( \kappa_{v}(\theta_{v} - v) - \varepsilon_{v} \rho_{sv} v \right) \frac{\partial f}{\partial v} + \frac{1}{2} v \varepsilon_{g} \frac{\partial^{2} f}{\partial g \partial v} + \varepsilon_{g} \rho_{sv} v \frac{\partial^{2} f}{\partial x \partial v} + \lambda Q E^{Q} [f(x + X, g, v, t)] - \lambda Q f, \\
\quad + \varepsilon_{g} \rho_{sv} v \frac{\partial^{2} f}{\partial x \partial g} + \varepsilon_{v} \rho_{vg} v \frac{\partial^{2} f}{\partial g \partial v} + \lambda Q E^{Q} [f(x + X, g, v, t)] - \lambda Q f,
\end{cases}
\]

where \( f(X, K; g, v, 0) = \min \{ e^{X}, K \} \).

Let \( F(z, K; g, v, \tau) \) be the Fourier transform of \( f(X, K; g, v, \tau) \):
\[
F(z, K; g, v, \tau) = \int_{-\infty}^{+\infty} e^{izX} f(X, K; g, v, \tau) dX.
\]

The initial condition is a simple integral
\[ F(z, K; g, v, 0) = \int_{-\infty}^{+\infty} e^{izX} \min \{ e^{X}, K \} dX. \]
\[
= \lim_{{U \to \infty}} \int_{{-U}}^{ln K} e^{izX} e^X dX + \lim_{{U \to \infty}} \int_{{ln K}}^{U} K e^{izX} dX
\]
\[
= \frac{K^{iz+1}}{z^2 - iz}, 0 < \Im Z < 1. \tag{4.8}
\]

Denote \(z = z_r + iz_i\). The inverse Fourier transform is given by
\[
f(X, K; g, v, \tau) = \frac{1}{2\pi} \int_{{iz_i + \infty}}^{iz_i - \infty} e^{-izX} F(z, K; g, v, \tau) dz.
\]

The option price is an integral along a straight line in the complex \(z\)-plane parallel
to the real axis. By introducing the transform (4.6), in the cases of call and put option
expiration, this line can lie anywhere in the region \(0 < \Im z < 1\). The integral is indeed
independent of the choice of \(z_i\), and we usually take \(z_i = \frac{1}{2}\) as the value is halfway between
the poles of the integrand which is continuous vs \(z_r = \Re z\). The integral converges fastest
along that contour if \(z_{max} = \max z_r\) is chosen large enough [33].

Then, the PIDE (4.7) can be rewritten as
\[
\begin{align*}
\frac{\partial F}{\partial \tau} &= -\mu F - iz[\mu - \alpha - \lambda^Q E^Q(e^x - 1) - \frac{1}{2}v]F - \frac{1}{2} vz^2 F \\
&\quad + [\kappa_\psi(\theta - g) - \varepsilon g \rho_S g v] \frac{\partial F}{\partial g} + \frac{1}{2} v^2 g v \frac{\partial^2 F}{\partial g^2} + [\kappa_v(\theta - v) - \varepsilon v \rho_S v] \frac{\partial F}{\partial v} + \frac{1}{2} v^2 \rho_v \frac{\partial^2 F}{\partial v^2} \\
&\quad - i z \varepsilon g \rho_S g v \frac{\partial F}{\partial g} - i z \varepsilon v \rho_S v \frac{\partial F}{\partial v} + i z^2 \varepsilon v \rho_S v \frac{\partial^2 F}{\partial g \partial v} + \lambda^Q E^Q [e^{-izx} - 1] F;
\end{align*}
\]
\[
F(z, K; g, v, 0) = \frac{K^{iz+1}}{z^2 - iz}. \tag{4.9}
\]

It is enough to solve the same PIDE with unitary initial value and then scale the solution. Hence,
\[
V(S, K; g, v, t) = \frac{e^{-\int_0^t \omega(s) ds}}{2\pi} \int_{{iz_i + \infty}}^{iz_i - \infty} e^{-izln S} K^{iz+1} \frac{z^2 - iz}{z^2 - iz} F(z, K; g, v, \tau) dz, \tag{4.10}
\]
and \(\hat{F}(z, K; g, v, \tau)\) satisfies the following equation:
\[
\begin{align*}
\frac{\partial \hat{F}}{\partial \tau} &= -\mu \hat{F} - iz[\mu - \alpha - \lambda^Q E^Q(e^x - 1) - \frac{1}{2}v] \hat{F} - \frac{1}{2} vz^2 \hat{F} \\
&\quad + [\kappa_\psi(\theta - g) - \varepsilon g \rho_S g v] \frac{\partial \hat{F}}{\partial g} + \frac{1}{2} v^2 g v \frac{\partial^2 \hat{F}}{\partial g^2} + [\kappa_v(\theta - v) - \varepsilon v \rho_S v] \frac{\partial \hat{F}}{\partial v} + \frac{1}{2} v^2 \rho_v \frac{\partial^2 \hat{F}}{\partial v^2} \\
&\quad - i z \varepsilon g \rho_S g v \frac{\partial \hat{F}}{\partial g} - i z \varepsilon v \rho_S v \frac{\partial \hat{F}}{\partial v} + i z^2 \varepsilon v \rho_S v \frac{\partial^2 \hat{F}}{\partial g \partial v} + \lambda^Q E^Q [e^{-izx} - 1] \hat{F},
\end{align*}
\]
\[
\hat{F}(z, K; g, v, 0) = 1. \tag{4.11}
\]

According to [31, 33], we guess that Equation (4.11) has an exponential solution of the form
\[
\hat{F}(z, K; g, v, \tau) = e^{\Gamma(\tau)g + \Lambda(\tau)v + \Pi(\tau)}, \tag{4.12}
\]
with \(\Gamma(0) = 0\), \(\Lambda(0) = 0\), and \(\Pi(0) = 0\).

Substituting (4.12) into Equation (4.11), we get \(^9\)
\[
\begin{align*}
\hat{\Gamma}(\tau)g + \hat{\Lambda}(\tau)v + \hat{\Pi}(\tau) &= -\mu - iz[\mu - \alpha - \lambda^Q E^Q(e^x - 1) - \frac{1}{2}v] - \frac{1}{2} vz^2 \\
&\quad + [\kappa_\psi(\theta - g) - \varepsilon g \rho_S g v] \Gamma(\tau) + \frac{1}{2} v^2 \rho_v [\Gamma(\tau)]^2
\end{align*}
\]

\(^9\) In order to get a homogeneous ODE in \(v\) and avoid the cross terms, we managed to design the process of \(g(t)\) related to \(\sqrt{v}\) shown in system (2.1).
\[ + [\kappa_v (\theta_v - \nu) - \varepsilon_v \nu \rho S_v] \Lambda (\tau) + \frac{1}{2} \varepsilon_v^2 [\Lambda (\tau)]^2 - iz \varepsilon_g \rho S_g v \Gamma (\tau) - iz \varepsilon_v \rho S_v v \Lambda (\tau) + \varepsilon_g \varepsilon_g \rho g v \Gamma (\tau) \Lambda (\tau) + \lambda^2 Q E^Q [e^{-ix} - 1]. \]

Since \( g \) and \( v \) are arbitrary, taking (3.5) into consideration, we see that the ODEs are

\[
\begin{align*}
\dot{\Gamma} (\tau) &= -1 - iz - \kappa g \Gamma (\tau), \\
\Gamma (0) &= 0,
\end{align*}
\]

(4.13)

\[
\begin{align*}
\dot{\Lambda} (\tau) &= 1 + \frac{3}{2} iz - \frac{1}{2} z^2 - (1 + iz) \varepsilon_g \rho S_g \Gamma (\tau) + \frac{1}{2} \varepsilon_g^2 \Gamma^2 (\tau) - [\kappa_v + (1 + iz) \varepsilon_v \rho S_v] \Lambda (\tau) + \frac{1}{2} \varepsilon_v^2 \Lambda^2 (\tau) + \varepsilon_g \varepsilon_v \rho g v \Lambda (\tau) \Gamma (\tau), \\
\Lambda (0) &= 0,
\end{align*}
\]

(4.14)

\[
\begin{align*}
\dot{\Pi} (\tau) &= \delta + \kappa g \theta g \Gamma (\tau) + \kappa v \theta v \Lambda (\tau), \\
\Pi (0) &= 0,
\end{align*}
\]

(4.15)

where

\[
\delta = (1 + iz) \lambda E [1 - e^{-x}] (e^x - 1) + iz \alpha + iz \lambda^2 Q E^Q (e^x - 1) + \lambda^2 Q E^Q [e^{-ix} - 1].
\]

The ODE (4.13) has a solution

\[
\Gamma (\tau) = - \frac{1 + iz}{\kappa g} (1 - e^{-\kappa g \tau}).
\]

(4.16)

Then, the ODE (4.14) can be written as

\[
\begin{align*}
\dot{\Lambda} (\tau) - \frac{1}{2} \varepsilon_v^2 [\Lambda (\tau)]^2 + \eta \Lambda (\tau) + \vartheta &= 0, \\
\Lambda (0) &= 0,
\end{align*}
\]

(4.17)

where

\[
\eta = \kappa_v + (1 + iz) \varepsilon_v \rho S_v + \frac{1 + iz}{\kappa g} \varepsilon_g \varepsilon_v \rho g v (1 - e^{-\kappa g \tau}),
\]

\[
\vartheta = -1 - \frac{3}{2} iz + \frac{1}{2} z^2 - (1 + iz)^2 \varepsilon_g \rho S_g \frac{1}{\kappa g} (1 - e^{-\kappa g \tau}) - \frac{1}{2} \varepsilon_g^2 \frac{1 + iz)^2}{\kappa_g^2} (1 - e^{-\kappa g \tau})^2.
\]

To solve the Riccati differential equation (4.17), we make the substitution

\[
\Lambda (\tau) = - \frac{2}{\varepsilon_v^2} \frac{I'(\tau)}{I(\tau)}
\]

(4.18)

and obtain a second order differential equation\(^\text{10}\)

\[
\begin{align*}
I'' (\tau) + \eta I' (\tau) - \frac{\varepsilon_v^2}{2} \vartheta I (\tau) &= 0, \\
I'(0) &= 0.
\end{align*}
\]

(4.19)

\(^\text{10}\)The solution \( \Lambda (\tau) \) will blow up in finite time due to the major quadratic term in Equation (4.17). Though the global solution of Equation (4.19) exists, Equation (4.19) is not well-posed and the singularity appears in the denominator of the substitution (4.18). The parameters (5.2) will not blow up for \( \Lambda (\tau), \tau \in [0, 1] \) by virtue of numerical tests.
The general solution of Equation (4.19) can be expressed by the confluent hypergeometric function $U(a, b, z)$ and the generalized Laguerre polynomial $L_n^\alpha(\tau)$, but it is too complicated to show the expression here\textsuperscript{11}. We use the Euler method to solve Equation (4.17) directly by numerical computation.

Then, plugging Equation (4.16) into Equation (4.15), we get

\[
\left\{ \begin{array}{l}
\dot{V}(\tau) = \delta - (1 + iz)\theta_g(1 - e^{-\kappa_g \tau}) + \kappa_v \theta_v \lambda(\tau), \\
V(0) = 0.
\end{array} \right. \tag{4.20}
\]

The theorem follows from equations (4.16), (4.17), (4.20), and (4.10).

\[\square\]

**Remark 4.4.** It is easily to find the following relationship from (4.2) and (4.4):

\[
\omega(t) = E\left[ \frac{dV}{V dt} \right] = \mu(t) + \phi(t), \tag{4.21}
\]

where

\[
\phi(t) = v(S \frac{\partial V}{\partial S} + \varepsilon_g \rho g \frac{\partial V}{\partial g} + \varepsilon_v \rho v \frac{\partial V}{\partial v})/V \tag{4.22}
\]

\[+ \left\{ V(S^e x, g, v, t) - V \right\} - \lambda Q \left[ \frac{V(S^e x, g, v, t) - V)}{V} \right].
\]

The rate $\omega(t)$ of the option equals to the underlying rate $\mu(t)$ plus a risk premium which compensates the risk of Brownian motions and jumps.

### 5. Decision making and numerical computation

With the help of option pricing, one can make investment decisions in two ways. Firstly, given a discount rate $\omega(t)$, the representative investor can compute the real option value through expression (4.5) and compare it directly with the market price. Secondly, if the real option price is given in the form of rates, like ROE, ROA, and so on, one can compute the IRP through (4.1) and (4.5) and make an investment decision by comparing the IRP with the real option rate. The two manners are the same as the famous NVP and IRR (inner rate return) methods in investment. We will perform the decision making process with numerical illustration.

Since the option pricing comparison is direct, we mainly illustrate the IRP method. Assume the market price of the real option is given by Equation (4.1); otherwise, the market is in arbitrage, and investment decision making is obvious. Let the market price of the real option be equal to expression (4.21). We can solve the expected inner risk premium $\phi(t)$. By comparing it with the real option rate, an investment decision can be made at once.

We perform Monte Carlo simulations for the endowment dynamics and the real option prices. The discrete scheme of the endowment system (3.1) is as follows:

\[
\left\{ \begin{array}{l}
v(t + \Delta t) = v(t) + \kappa_v (\theta_v - v(t)) \Delta t + \varepsilon_v \sqrt{v(t)} dW_v(t), \quad v_0 > 0, \\
g(t + \Delta t) = g(t) + \kappa_g (\theta_g - g(t)) \Delta t + \varepsilon_g \sqrt{g(t)} dW_g(t), \quad g(0) = g_0, \\
\ln S(t + \Delta t) = \ln S(t) + (g(t) - \alpha - 0.5 v(t)) \Delta t + \sqrt{v(t)} dW_S(t) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad + (e^\alpha - 1)(N(t + \Delta t) - N(t)) - \lambda E(e^\alpha - 1) \Delta t, \quad S_0 > 0,
\end{array} \right. \tag{5.1}
\]

where

\[
\left\{ \begin{array}{l}
dW_S(t) = \varepsilon_1 \sqrt{\Delta t}, \\
dW_g(t) = \rho g dW_S(t) + \sqrt{1 - \rho g^2} \varepsilon_2 \sqrt{\Delta t}, \\
dW_v(t) = \rho v dW_S(t) + \sqrt{1 - \rho v^2} \varepsilon_3 \sqrt{\Delta t},
\end{array} \right.
\]

\[\text{One can rely on the Mathemtica software to get the expression. DSolve[\{y''[t] + (C_1 + C_2 \exp[-k g t]) y'[t] + (C_3 + C_4 \exp[-k g t] + C_5 \exp[-2k g t]) y[t] == 0, y'[0] == 0, y[0] == 0\}, y[t], t].}\]
Δt is the time interval, and ε_i are samples from the standard normal distribution. We have that \( N(t) = j \), if \( S_j \leq t < S_{j+1} \). Furthermore, \( S_j = \sum_{k=1}^{j} \tau_k \) where \( \tau_k \) is the kth jump time interval which obeys an exponential distribution with parameter \( \lambda \). The option prices are calculated by Monte Carlo simulation with (4.1). An antithetic variable technique is used to diminish computation errors. Every simulation trial involves calculating two values of the option price. The first value is calculated in the usual way; the second value is calculated by changing the sign of all the random samples from standard normal distributions.

Parameters are set as follows:

\[
\begin{align*}
\kappa_g &= 0.3, \quad \theta_g = 0.13, \quad \varepsilon_g = 0.26, \quad \kappa_v = 1.2, \quad \theta_v = 0.04, \quad \varepsilon_v = 0.28, \quad \alpha = 0.2, \\
\rho_{Sv} &= -0.54, \quad \rho_{Sg} = 0.60, \quad \rho_{gv} = 0.44, \quad \lambda = 1, \quad x = -0.105, \quad \Delta t = 0.01, \quad T = 1, \quad v_0 = 0.04, \quad g_0 = 0.13, \quad S_0 = 100, \quad \gamma = 1, \quad K = 100, 95, 105.
\end{align*}
\]

Several paths of \( g(t) \), \( v(t) \), and \( S(t) \) are illustrated in Figure 5.1 with parameters (5.2). Black and blue paths are generated by the sample pair \( \varepsilon \) and \( -\varepsilon \) respectively. In the first column, two pairs of arbitrary paths are illustrated, and 20 pairs of arbitrary paths are shown in the second column. One can see directly that \( v(t) \) is always greater than 0, but \( g(t) \) is sometimes negative. These results illustrate that our newly added dynamic of \( g(t) \) works and is in accordance with reality. Numerical results of the option prices are denoted by \( MC \) and listed in Table 5.1.

Then, taking the Monte Carlo option price as the market price, we evaluate the average inner risk premium \( \bar{\phi} \) by the expression (4.5) by virtue of numerical computation. The analytical integration formula of the option price (4.5) can be calculated by Fast Fourier Transform; see Carr and Madan [30], Lewis [32, 33], and Pillay [34] for details. Here, we follow Lewis’ methods and calculate the complex integration directly with Mathematica. We use the average \( \bar{\mu} = \bar{g} - \bar{v} - \lambda E[(1 - e^{-x})(e^{x} - 1)] \) in the calculation, where \( \bar{g} \) and \( \bar{v} \) are calculated in Equation (2.2). Option prices and the average inner risk premium \( \bar{\phi} \) are listed in Table 5.1. Thus, comparing the charge rate of the real option and the IRP, one can easily make investment decisions. For example, for the
at-the-money call option in the first set of parameters, if an investment project charges less than 0.13, the investment is feasible theoretically.

One can see from Table 5.1 that the inner risk premium of the option is negative, which means investors would rather sacrifice some return in order to get the option. Besides, the risk premium of put is larger than call since the put option provides insurance. For the parameters (5.2), \( \bar{\mu} - \alpha_1 < 0 \) which means the average real growth rate after consumption is negative while \( \bar{\mu} - \alpha_2 > 0 \) is the opposite, so the option prices are quite different in the two sets of parameters. Though the risk premium of the option comes from three Brownian motions and a jump, the algorithm shows that \( \bar{\phi} \) in Table 5.1 is relatively stable both in the call and put options. This is not easy even for financial options in Black–Scholes’ world, since the \( \beta \)-risk of the option is more susceptible than the underlying and difficult to capture; see e.g. Coval and Shumway (2001) [35].

<table>
<thead>
<tr>
<th></th>
<th>Call option</th>
<th>Put option</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 )</td>
<td>100 100 100</td>
<td>100 100 100</td>
</tr>
<tr>
<td>( K )</td>
<td>95 100 105</td>
<td>95 100 105</td>
</tr>
<tr>
<td>( MC_{C1} )</td>
<td>8.80 6.61 4.76</td>
<td>5.95 7.96 10.33</td>
</tr>
<tr>
<td>( \bar{\mu}_1 )</td>
<td>0.079 0.079 0.079</td>
<td>0.079 0.079 0.079</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-0.111 -0.130 -0.148</td>
<td>-0.224 -0.245 -0.265</td>
</tr>
<tr>
<td>( \bar{\phi}_1 )</td>
<td>-0.2 0.2 0.2</td>
<td>-0.228 -0.228 -0.228</td>
</tr>
<tr>
<td>( MC_{C2} )</td>
<td>22.84 19.38 16.07</td>
<td>0.26 0.41 0.42</td>
</tr>
<tr>
<td>( \bar{\mu}_2 )</td>
<td>0.229 0.229 0.229</td>
<td>0.229 0.229 0.229</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.1 0.1 0.1</td>
<td>0.1 0.1 0.1</td>
</tr>
<tr>
<td>( \bar{\phi}_2 )</td>
<td>-0.121 -0.121 -0.121</td>
<td>-0.228 -0.228 -0.232</td>
</tr>
</tbody>
</table>

Table 5.1. Prices and IRPs (inner risk premium) of the real options.

In the computation of the integral expression, \( z_i = \frac{1}{2} \) and \( z_{r_{max}} = 100000 \), and in the Monte Carlo simulations, 100000 pairs of paths are set. Option prices \( MC_1 \) and the risk premium \( \bar{\phi}_1 \) are calculated with parameters (5.2). Option prices \( MC_2 \) and the risk premium \( \bar{\phi}_2 \) are calculated with the changed parameters \( \theta_\beta = 0.25, \theta_\gamma = 0.01, g_0 = 0.25, v_0 = 0.01, \) and \( \alpha = 0.1 \).

6. Discussions and conclusions

In the text, we design a real option model to help make decisions in real investment and management fields. To comply with the economic situation, we allow the underlying process to be non-Levy with stochastic growth rate and variance processes. The optimal consumption rate is solved under logarithmic utility and thus can be substituted in the discussion of real options. Since the market is incomplete without risk free assets, we can only price the real option under a pricing kernel, and we give two methods of decision making—option prices and IRP comparisons. Numerical illustrations verify the feasibility.

Mathematically, the endowment process can be trivially extended to slightly more complicated cases by introducing a stochastic jump size \( \lambda \) and time-varying discount rate \( \alpha \). We are temporarily unsure whether the optimal consumption conclusion (2.5) holds true for CRRA (constant relative risk aversion) type utility functions, but in this case, the risk aversion parameter \( \gamma \) will appear in the pricing kernel and the PIDE. This does not affect the concepts of real investment and management decision making in this
Appendix A. Numerical computation of the analytical formulation (Mathematica).

(* The fundamental transform F *)
F[z_, kappa_g_, kappa_v_, tau_, theta_g_, theta_v_, lambda_, varepsilon_g_, varepsilon_v_, rho_g_, rho_v_, x_, alpha_, g0_, v0_] :=
Module[{Fval, theta, eta, delta, Gam, Lam, Pit, dt, i, step},
  step = 8; dt = tau/step;
  theta = Table[0, step]; eta = Table[0, step]; Lam = Table[0, step];
  theta = -1 - 1.5 I z + 0.5 z^2 - (1 + I z)^2*varepsilon_g* rho_g*(1 - E^(-kappa_g Range[0, tau, dt]))/kappa_g -0.5 * varepsilon_g^2*(1 + I z)^2*(1 - E^(-kappa_g Range[0, tau, dt]))^2/kappa_g^2;
  eta = kappa_v + (1 + I z) (varepsilon_v*rho_v + varepsilon_g*rho_g*(1 - E^(-kappa_g Range[0, tau, dt])))/kappa_v;
  delta = (1 + I z)*(lambda*(1 - E^(-x)) (Exp[x] - 1) + alpha) + I z lambda E^(-x) (E^(-I z x) - 1);
  Gam = -(1 + I z)*(1 - E^(-kappa_g tau))/kappa_g;
  For[i = 1, i ≤ step - 1, i++,
    Lam[[i + 1]] = Lam[[i]] + (0.5*varepsilon_v^2*Lam[[i]]^2 - eta[[i]]*Lam[[i]] - theta[[i]])*dt;
  ];
  Pit = delta tau - (1 + I z) thet_g (tau + (E^(-kappa_g tau) - 1)/kappa_g) + kappa_v*theta_v*Sum[Lam[[i]], i, 1, step]/step;
  Fval = E^(Gam*g0 + Lam[[step]]*v0 + Pit);
  Return[Fval]
(* This code is used to find the excess return of the options. cpflag=1 or (-1) for call (put) option. Option prices are prepared by Monte Carlo. To avoid branch cut crossing, imaginary part must satisfy 0<zi<1. *)
Cvalue[S_, K_, zi_, kappa_g_, kappa_v_, tau_, theta_g_, theta_v_, lambda_, varepsilon_g_, varepsilon_v_, rho_g_, rho_v_, x_, alpha_, g0_, v0_, option_, cpflag_] :=
Module[{z, zr, val, mu, kmax, phi}, Clear[z]; z[zr] := z_i I + z_r;
  If[zi ≤ 1 || zi ≤ 0, Print["Illegal zi value"]; Abort[], Null];
  Print["To avoid branch cut crossing, imaginary part 0<zi<1!"];
  Print["Parameter zi=", zi];
  Print["Call option if cpflag=1; Put option if cpflag=-1."];
  Print["Parameter cpflag=", cpflag];
  Print[" S="", S, "; K="", K, "; alpha="", alpha, "; g0="", g0, "; v0="", v0, "; tau="", tau];
  Print[" kappa_g="", kappa_g, "; kappa_v="", kappa_v, "; theta_g="", theta_g, "; theta_v="", theta_v];
  Print[" lambda="", lambda, "; rho_g="", rho_g, "; rho_v="", rho_v, "; rho_gv="", rho_gv];
  kmax = 100000;(* stand for infinity in the integral *)
  mu = Exp[-kappa_v tau] (g0 - theta_v) + theta_v - Exp[-kappa_v tau] evaluating
  - theta_v (1 - Exp[-kappa_v tau] - lambda (1 - Exp[-x]) (Exp[x] - 1));
  val = N[ K /Pi * NIntegrate[ Re[Exp[-I z[zt] (Exp[-kappa_v tau] (g0 - theta_v) + theta_v - Exp[-kappa_v tau] v0 - theta_v (1 - Exp[-kappa_v tau] - lambda (1 - Exp[-x]) (Exp[x] - 1) - alpha) tau] Exp[ I z[zt] Log[K/S]]/(z[zt]^2 - I z[zt])^2 F[z[zt], kappa_g, kappa_v, tau, theta_v, lambda, varepsilon_g, varepsilon_v, rho_g, rho_v, x, alpha, g0, v0]}
}
REAL OPTION MODEL

In[99]:= Cvalue[100, 95, 0.5, 0.3, 1.2, 1, 0.13, 0.04, 1, 0.26, 0.28, 0.60, -0.54, 0.44, -0.105, 0.13, 0.04, 5.95, -1]

Out[99]:= To avoid branch cut crossing, imaginary part 0< zi <1!

Parameter zi=0.5

Call option if cpflag=1; Put option if cpflag=-1.

Parameter cpflag=-1

S=100; K=95; alpha=0.2; g0=0.13; v0=0.04; tau=1

kappag=0.3; kappav=1.2; thetag=0.13; thetav=0.04

lambda=1; rho sg=0.6; rho sv=-0.54; rho gv=0.44

jump=-0.105; thetag=0.13; thetav=0.04

Average return of the underlying mu=0.0789649

Infinity kmax=100000, excess return implied by put option=-0.223924

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REFERENCES


