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Random Attractor for Stochastic Wave Equation with Arbitrary Exponent and Additive Noise on $\mathbb{R}^n$

Hongyan Li and Yuncheng You

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Abstract. Asymptotic random dynamics of weak solutions for a damped stochastic wave equation with the nonlinearity of arbitrarily large exponent and the additive noise on $\mathbb{R}^n$ is investigated. The existence of a pullback random attractor is proved in a parameter region with a breakthrough in proving the pullback asymptotic compactness of the cocycle with the quasi-trajectories defined on the integrable function space of arbitrary exponent and on an unbounded domain of arbitrary space dimension.

Contents

1. Introduction

In this paper, we study the asymptotic dynamics of solutions of a damped stochastic wave equation with nonlinearity of arbitrarily large exponent and additive

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noise defined on the entire Euclidean space $\mathbb{R}^n$ of arbitry dimension,

$$u_{tt} - \Delta u + \beta u_t + f(x,u) + \alpha u = g(x) + \varepsilon \sum_{j=1}^{m} h_j(x) \frac{dW_j}{dt}, \tag{1.1}$$

for $t \geq \tau$, with the initial condition

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x), \tag{1.2}$$

where α, β and ε are positive constants, g and h_j $(j = 1, 2, \cdots, m)$ are given functions defined on $\mathbb{R}^n$, βu_t is a damping term, $f(x,u)$ is a nonlinear interaction function satisfying some dissipative conditions, and $\{W_j\}_{j=1}^m$ are independent, two-sided, real-valued Wiener processes on a probability space which will be specified later.

Asymptotic dynamics of solutions for deterministic nonlinear wave equations and nonlinear hyperbolic evolutionary equations with linear or nonlinear or localized damping terms have been studied in last three decades by many authors, e.g. [**2**]-[**4**], [**8**]-[**11**], [**20**]-[**24**], [**26**]-[**27**], [**30, 31**], [**33**]-[**34**], [**37, 40**]. The established results naturally focus on the existence of global attractors by showing the absorbing property and the asymptotic compactness of the solution semigroups for autonomous system [**4, 22, 34, 40**] or the skew-product flow for nonautonomous system [**10, 11, 27**].

For stochastic wave equations, the solution mapping defines a random dynamical system or called a cocycle, which is defined on a state space with a parametric base space. Pullback random attractor (which is simply called random attractor) is the appropriate object for describing the asymptotic random dynamics [**5**]-[**6**], [**12**]-[**17**], [**19, 25, 29, 32, 38**]. The topics of random attractors for stochastic damped wave equations have been studied by several authors [**12, 15, 18, 19, 28, 29, 32, 36, 38, 39**]. In regard to stochastic nonlinear wave equations driven by additive noise, the existence of the random attractor has been established for bounded domains [**15, 19, 29, 32, 39**] and with the critical exponents on the unbounded domain $\mathbb{R}^3$ recently in [**35**]. However, the existence problem of random attractors remains open for the stochastic wave equations with *a nonlinearity of arbitrarily large exponents* and *on the unbounded domain* $\mathbb{R}^n$ with arbitrary dimension n. This is the topic as well as the main contribution in this work.

In case of high growth-order nonlinearity and high dimensional unbounded domain, the issue of pullback asymptotic compactness for a random dynamical system becomes much more difficult to handle due to not only the lack of compactness of the Sobolev embeddings but also the necessarily involved high-order integrable function spaces, in addition to the cumbersome effect by the additive noise. In this work we shall resolve this challenging problem and accomplish the proof of random attractor by means of

1) the uniform estimates for absorbing property and norm-smallness of solutions outside a large ball;
2) the grouping esimates of the extended energy functional for the compactness in the space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$;
3) the convergence criterion of Vitali type proved in Theorem 5.1 for the function space $L^p(\mathbb{R}^n)$ with an arbitrary exponent and arbitrary space dimension n,

to circumvent the crucial difficulties.

The new insight of this work is a new approach to apply the Vitali-type convergence criterion initially proved in this paper for the function space $L^p(\mathbb{R}^n)$ combined with the grouping uniform estimates in the energy Hilbert space and the bootstrap method to directly prove the pullback asymptotic compactness for the quasi-trajectories of random dynamical systems. This new approach will provide great potential applications to many other stochastic PDEs and much further fields, which will no longer be barricaded by high growth nonlinearity and high space dimension of any unbounded domain.

In Section 2, we recall basic concepts and results related to random attractors and random dynamical systems. We shall transform the stochastic wave equation with additive noise to a pathwise random wave equation through Ornstein-Uhlenbeck processes and define the associated cocycle. In Section 3, we shall conduct uniform estimates of solutions for random absorbing sets and tail parts. In Section 4, we shall establish the intricate asymptotic compactness of the cocycle with respect to the Hilbert energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In Section 5, we prove a necessary and sufficient convergence criterion of Vitai-type for space $L^p(\mathbb{R}^n)$ and the pullback asymptotic compactness of the first component of the cocycle in $L^p(\mathbb{R}^n)$, which is crucial. Then the existence of a random attractor for this stochastic wave equation with unlimited growth rate and additive noise on the unbounded domain with unlimited dimension is finally proved.

In this paper, we shall use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote the norm and inner product of $L^2(\mathbb{R}^n)$, respectively. The norm of $L^r(\mathbb{R}^n)$ or a Banach space X will be denoted by $\|\cdot\|_r$ or $\|\cdot\|_X$. We use c, C or C_i to denote generic or specific positive constants.

2. Preliminaries and the Random Dynamical System

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X, \|\cdot\|_X)$ be a real Banach space whose Borel σ-algebra is denoted by $\mathscr{B}(X)$.

Definition 2.1. Let a mapping $\theta_t : \mathbb{R} \times \Omega \to \Omega$ be $(\mathscr{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable such that θ_0 is the identity on Ω, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}, \omega \in \Omega$, and $P\theta_t = P$ for all $t \in \mathbb{R}$. Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is called a *parametric dynamical system.*

Definition 2.2. A mapping $\Phi : \mathbb{R}^+ \times \Omega \times X \to X$ is called a *random dynamical system* or *cocycle* on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, if Φ is $(\mathscr{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathscr{B}(X), \mathscr{B}(X))$-measurable and for all $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$ the following conditions are satisfied:

(i) $\Phi(0, \omega, \cdot)$ is the identity on X.

(ii) $\Phi(t + s, \omega, \cdot) = \Phi(t, \theta_s\omega, \Phi(s, \omega, \cdot))$.

(iii) $\Phi(t, \omega, \cdot) : X \to X$ is strongly continuous.

Such a random dynamical system can be denoted by (Φ, θ) or simply by Φ.

Definition 2.3. A set-valued mapping $D(\omega) : \Omega \to 2^X$ is called a *random set* in X, if the mapping $\omega \longmapsto dist_X(x, D(\omega))$ is measurable with respect to $\mathcal{F}$ for any given $x \in X$, cf. [**1, 13, 14**].

1) A *bounded* random set $B(\omega) \subset X$ means that there is a random variable $r(\omega) \geq 0$ such that $\|B(\omega)\| = \sup_{x\in B(\omega)} \|x\| \leq r(\omega)$, $\omega \in \Omega$.

2) A random set $S(\omega) \subset X$ is called *compact* (reps. *precompact*) if for all $\omega \in \Omega$ the set $S(\omega)$ is a compact (reps. precompact) set in X.

3) If a bounded random set $B(\omega)$ satisfies the condition that, for any constant $\kappa > 0$,

$$\lim_{t\to\infty} e^{-\kappa t}\|B(\theta_{-t}\omega)\| = 0, \quad \omega \in \Omega,$$

then it is called *tempered* with respect to $\{\theta_t\}_{t\in\mathbb{R}}$. A random variable $R : (\Omega, \mathcal{F}, P) \to (0, \infty)$ is called *tempered* if

$$\lim_{t\to-\infty} \frac{1}{t} \log^+ R(\theta_t\omega) = 0, \quad \omega \in \Omega.$$

We shall denote by $\mathscr{D}_X$ or simply $\mathscr{D}$ the family of tempered random subsets of X, which is inclusion-closed and called a *universe*.

Definition 2.4. Let Φ be a random dynamical system on X over the parametric dynamical system $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Let $\mathscr{D}$ be a given universe of tempered random subsets of X.

1) A random set $K = \{K(\omega)\}_{\omega\in\Omega}$ is a *$\mathscr{D}$-pullback absorbing set* for Φ if for any $B \in \mathscr{D}$ there exists $t_B(\omega) \geq 0$ such that

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \text{for all } t \geq t_B(\omega), \text{ a.e. } \omega \in \Omega.$$

2) Φ is called *$\mathscr{D}$-pullback asymptotically compact* if for any $\omega \in \Omega$,

$$\{\Phi(t_m, \theta_{-t_m}\omega, x_m)\}_{m=1}^{\infty} \text{ has a convergent subsequence in } X,$$

whenever $t_m \to \infty$ and $x_m \in B(\theta_{-t_m}\omega)$ for any given $B \in \mathscr{D}$.

Definition 2.5. A random set $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ is called a *random attractor* for a random dynamical system (Φ, θ) with the attraction basin $\mathscr{D}$, if the following conditions are satisfied:
(i) $\mathcal{A}$ is a compact random set.
(ii) $\mathcal{A}$ is invariant, $\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega)$, for all $t \geq 0$ and $\omega \in \Omega$.
(iii) $\mathcal{A}$ pullback attracts every set $B \in \mathscr{D}$ in the sense

$$\lim_{t\to\infty} dist_X(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \quad \omega \in \Omega,$$

where $dist_X(\cdot, \cdot)$ is the Hausdorff semi-distance with respect to the X-norm.

The following result on the existence of random attractor for a random dynamical system has been established in [**1, 6, 13, 15, 35, 36**].

Theorem 2.6. *Let $\mathscr{D}$ be a universe of nonempty tempered random subsets of a Banach space X and let Φ be a random dynamical system on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that $K = \{K(\omega)\}_{\omega\in\Omega}$ is a closed pullback absorbing set for Φ with respect to $\mathscr{D}$ and Φ is $\mathscr{D}$-pullback asymptotically compact in X. Then Φ has a unique random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in X with the attraction basin $\mathscr{D}$, which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \Omega. \tag{2.1}$$

However, for random dynamical systems generated by PDEs with nonlinearity of higher growth exponents and on an unbounded domain, such as the problem of stochastic wave equations in this work, it is very difficult to show the pullback asymptotic compactness of cocycles. It is the advancing contribution of this paper to prove the $\mathscr{D}$-pullback asymptotic compactness in the space $(H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)) \times L^2(\mathbb{R}^n)$ by a new approach shown in Section 4 and Section 5.

Now we formulate the original problem (1.1)-(1.2). Let $\xi = u_t + \delta u$, where δ is a positive number to be determined. Then (1.1)-(1.2) can be rewritten as

$$u_t + \delta u = \xi,$$

$$\xi_t - \delta\xi + \delta^2 u + \alpha u - \Delta u + \beta(\xi - \delta u) + f(x,u) = g(x) + \varepsilon \sum_{j=1}^{m} h_j \frac{dW_j}{dt}, \tag{2.2}$$

$$u(x,\tau) = u_0(x), \quad \xi(x,\tau) = \xi_0(x) = u_1(x) + \delta u_0(x).$$

The autonomous wave equation driven by a stochastic perturbation with white noise or colored noise has the nonautonomos nature as the parametric stochastic processes $W_j(t)$ evolve. Hence the initial condition can be given at any initial time $\tau \in \mathbb{R}$.

Assumption I Assume that $g \in H^1(\mathbb{R}^n)$ and for each $j = 1, 2, \cdots, m$, the function $h_j \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$, where $p > 2$ is arbitrarily given.

Assumption II Assume that the nonlinear term $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ and its antidrivative $F(x,u) = \int_0^u f(x,s)ds$ satisfy the following conditions:

$$|f(x,u)| \le C_1|u|^{p-1} + \phi_1(x), \quad \phi_1(x) \in H^1(\mathbb{R}^n), \tag{2.3}$$

$$f(x,u)u - C_2F(x,u) \ge \phi_2(x), \quad \phi_2(x) \in L^1(\mathbb{R}^n), \tag{2.4}$$

$$F(x,u) \ge C_3|u|^p - \phi_3(x), \quad \phi_3(x) \in L^1(\mathbb{R}^n), \tag{2.5}$$

where C_1, C_2 and C_3 are positive constants and the arbitrarily given $p > 2$ is the same as in Assumption I.

The Assumption II on the heterogeneous nonlinearity is standard for deterministic or stochastic wave equations and reaction-diffusion equations on an unbounded domain.

Assume that $\{W_j\}_{j=1}^m$ are independent, two-sided, real-valued Wiener processes on the canonical probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega = (\omega_1, \omega_2, \cdots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. We will identify a $\omega \in \Omega$ with a sample path

$$\omega(t) = (\omega_1(t), \omega_2(t), \cdots, \omega_m(t)), \quad t \in \mathbb{R}.$$

Define the time-shift operartor θ_t by

$$(\theta_t\omega)(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a parametric dynamical system.

To define a random dynamical system for the formulated problem (2.2), we now convert the stochastic wave equation with the additive noise to a deterministic equation with a time-dependent random parameter $\theta_t\omega$ and corresponding random initial data. Given $j = 1, 2, \cdots, m$, introduce one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \delta z_j dt = dW_j(t), \quad t \in \mathbb{R}. \tag{2.6}$$

Here δ is the same constant as in (2.2). A solution to (2.6) is given by

$$z_j(t,\omega_j) = \int_{-\infty}^{t} e^{-\delta(t-s)}\, dW_j(s,\omega) = -\delta \int_{-\infty}^{0} e^{\delta s}(\theta_t\omega_j)(s)\, ds$$
$$= z_j(0,\theta_t\omega_j) := z_j(\theta_t\omega_j), \quad t\in\mathbb{R}.$$

The random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is continuous in t. It follows from Proposition 4.3.3 in [**1**] that there exists a tempered function $r_0(\omega)>0$ such that

$$\sum_{j=1}^{m}(|z_j(\omega_j)|^2 + |z_j(\omega_j)|^p) \le r_0(\omega).$$

For the positive constant σ specified later in (3.16), the random variable $r_0(\omega)$ satisfies

$$r_0(\theta_t\omega) \le e^{(\sigma/2)|t|} r_0(\omega), \quad t\in\mathbb{R}, \ \ a.s. \tag{2.7}$$

It follows from (2.7) that

$$\sum_{j=1}^{m}(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^p) \le e^{(\sigma/2)|t|} r_0(\omega), \quad t\in\mathbb{R}, \ \ a.s. \tag{2.8}$$

The abstract-valued Ornstein-Uhlenbeck process $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ satisfies the stochastic differential equation

$$dz + \delta z dt = \sum_{j=1}^{m} h_j dW_j. \tag{2.9}$$

We make a transformation

$$v(x,t) = \xi(x,t) - \varepsilon z(\theta_t\omega). \tag{2.10}$$

and convert the problem (2.2) to the following initial value problem:

$$u_t = v + \varepsilon z(\theta_t\omega) - \delta u,$$
$$v_t - \delta v + (\delta^2+\alpha+A)u + f(x,u) = g - \beta(v+\varepsilon z(\theta_t\omega) - \delta u) + 2\varepsilon\delta z(\theta_t\omega), \tag{2.11}$$
$$u(x,\tau) = u_0(x), \quad v(x,\tau) = v_0(x) = u_1(x) + \delta u_0(x) - \varepsilon z(\theta_\tau\omega),$$

where $A=-\Delta$. Define the phase space

$$E = \big(H^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\big)\times L^2(\mathbb{R}^n)$$

endowed with the norm

$$\|(u,v)\|_{(H^1\cap L^p)\times L^2} = \big(\|\nabla u\|^2 + \|u\|^2 + \|v\|^2\big)^{\frac12} + \|u\|_{L^p}, \ \text{ for } (u,v)\in E. \tag{2.12}$$

Lemma 2.7. *Under the Assumptions* I *and* II, *for every* $\omega\in\Omega$ *and any given* $g_0=(u_0,v_0)\in E$, *the initial value problem* (2.11) *has a unique global weak solution*

$$(u(\cdot,\omega,\tau,u_0), v(\cdot,\omega,\tau,v_0)) \in C([\tau,\infty),E).$$

Moreover,

1) *The solution* $(u(t,\omega,\tau,u_0), v(t,\omega,\tau,v_0))$ *is* $(\mathscr{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathscr{B}(\mathbb{R}))$*-measurable in* (t,ω,τ) *for any given* $g_0=(u_0,v_0)\in E$.

2) *For any* $\omega\in\Omega$ *and* $t\ge\tau\in\mathbb{R}$, $(u(t,\omega,\tau,u_0), v(t,\omega,\tau,v_0))$ *is weakly continuous with respect to* $g_0=(u_0,v_0)$ *in* E *in the sense that*

$$(u(t,\omega,\tau,u_{0,m}), v(t,\omega,\tau,v_{0,m})) \rightharpoonup (u(t,\omega,\tau,u_0), v(t,\omega,\tau,v_0))$$

weakly in E, provided that $g_{0,m} = (u_{0,m}, v_{0,m}) \rightharpoonup g_0 = (u_0, v_0)$ weakly in E.

Proof. The local existence and uniqueness of a weak solution for this ω-parametrized PDE problem (2.11) in the phase space $E = (H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)) \times L^2(\mathbb{R}^n)$ and its weakly continuous dependence on the initial data can be established by the Galerkin approximation method as in [**10**, Chapter XV] with some adaptions and the result in Lemma 3.1 of [**35**]. Also see [**34, 36, 38**]. The proof of the global existence of weak solutions will be included in the proof of Lemma 3.1 below. □

Definition 2.8. A family of mappings $S(t, \tau; \omega) : X \longrightarrow X$ on a Banach space X for $t \geq \tau \in \mathbb{R}$ and $\omega \in \Omega$ is called a *stochastic semiflow*, if it satisfies the following properties:

(i) $S(t, s; \omega)S(s, \tau; \omega) = S(t, \tau; \omega)$, for all $\tau \leq s \leq t$ and $\omega \in \Omega$;
(ii) $S(t, \tau; \omega) = S(t - \tau, 0; \theta_\tau \omega)$, for all $\tau \leq t$ and $\omega \in \Omega$;
(iii) The mapping $S(t, \tau; \omega)x$ is measurable in (t, τ, ω) and continuous in $x \in X$.

Here for the formulated problem (2.2) and the converted version (2.11) we define

$$\begin{aligned} S(t, \tau; \omega)(u_0, v_0) &= (u, v)(t, \omega, \tau, (u_0, v_0)) \\ &= (u(t, \omega, \tau, u_0), v(t, \omega, \tau, u_1 + \delta u_0 - \varepsilon z(\theta_\tau \omega))), \end{aligned} \tag{2.13}$$

where $(u, v)(t, \omega, \tau, (u_0, v_0))$ is the weak solution of the initial value problem (2.11), shown in Lemma 2.7. This mapping $S(t, \tau; \omega) : E \to E$ is a stochastic semiflow. Then define a mapping $\Phi : \mathbb{R}^+ \times \Omega \times E \to E$ by

$$\Phi(t - \tau, \theta_\tau \omega, (u_0, v_0)) = S(t, \tau; \omega)(u_0, v_0), \tag{2.14}$$

which is equivalent to

$$\begin{aligned} \Phi(t, \omega, (u_0, v_0)) &= S(t, 0; \omega)(u_0, v_0) \\ &= (u(t, \omega, 0, u_0), v(t, \omega, 0, u_1 + \delta u_0 - \varepsilon z(\omega))). \end{aligned} \tag{2.15}$$

Lemma 2.9. *The mapping $\Phi : \mathbb{R}^+ \times \Omega \times E \to E$ defined by* (2.14) *is a random dynamical system* (*or called a cocycle*) *on E over the canonical parametric dynamical system $(\Omega, \mathcal{F}, P, \{\theta_t\})$. Moreover, for any given $g_0 = (u_0, v_0) \in E$,*

$$\begin{aligned} \Phi(t, \theta_{-t}\omega, (u_0, v_0)) &= (u(0, \omega, -t, u_0), v(0, \omega, -t, v_0)) \\ &= (u(0, \omega, -t, u_0), \xi(0, \omega, -t, \xi_0) - \varepsilon z(\omega)), \quad t \geq 0, \end{aligned} \tag{2.16}$$

will be called a pullback quasi-trajectory.

3. Uniform Estimates of Pullback Quasi-Trajectories

In this section, we shall derive uniform estimates on the solutions of the random wave equation (2.11) defined on $\mathbb{R}^n$ in a long run as $t \to \infty$. These *a priori* estimates pave the way to proving the existence of pullback absorbing sets and the pullback asymptotic compactness of the cocycle Φ. In particular, we will show that tails of the solutions for large spatial variables are uniformly small when time is sufficiently large.

Define a new norm of E by

$$\|(u, v)\|_E = \left(\|v\|^2 + (\alpha + \delta^2 - \beta\delta)\|u\|^2 + \|\nabla u\|^2\right)^{\frac{1}{2}} + \|u\|_{L^p}, \tag{3.1}$$

in which and hereafter let δ be a fixed positive constant satisfying

$$\alpha + \delta^2 - \beta\delta > 0 \qquad \text{and} \qquad \beta - 3\delta > 0. \tag{3.2}$$

Obviously the norm $\| \cdot \|_E$ in (3.1) and the Sobolev norm $\| \cdot \|_{(H^1 \cap L^p) \times L^2}$ in (2.12) are equivalent. We make an assumption on the intensity of stochastic perturbation.

Assumption III. Let the intensity $\varepsilon > 0$ of the additive noise satisfy

$$0 < \varepsilon < \frac{\delta C_2 C_3 p}{C_1(p-1)}, \tag{3.3}$$

where $p > 2$ and $C_i (i = 1, 2, 3)$ are the positive constants in Assumptions I and II and δ is the fixed constant in (3.2).

3.1. Pullback Absorbing Set. The next lemma shows that there exists a random absorbing set in the universe $\mathscr{D} = \mathscr{D}_E$ for the random dynamical system Φ associated with the problem (2.11).

LEMMA 3.1. *Under the Assumptions* I, II *and* III*, there exists a $\mathscr{D}$-pullback absorbing set $K = \{K(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$ for the cocycle Φ associated with the problem* (2.11). *For any $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$ and P-a.e. $\omega \in \Omega$, there exists a finite $T_B(\omega) > 0$, such that*

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \text{for all } t \geq T_B(\omega).$$

PROOF. Take the inner product of the second equation of (2.11) with v in $L^2(\mathbb{R}^n)$ to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|^2 - \delta\|v\|^2 + (\alpha + \delta^2)\langle u, v\rangle + \langle Au, v\rangle + \langle f(x,u), v\rangle \\ &= -\langle \beta(v - \delta u + \varepsilon z(\theta_t\omega)), v\rangle + 2\delta\varepsilon\langle z(\theta_t\omega), v\rangle + \langle g(x), v\rangle. \end{aligned} \tag{3.4}$$

By the first equation of (2.11), we have

$$v = u_t - \varepsilon z(\theta_t\omega) + \delta u. \tag{3.5}$$

and

$$-\langle \beta(v + \varepsilon z(\theta_t\omega) - \delta u), v\rangle \leq -\beta\|v\|^2 + \beta\varepsilon\|z(\theta_t\omega)\|\|v\| + \beta\delta\langle u, v\rangle. \tag{3.6}$$

Substituting (3.5) into the third and fourth terms on the left-hand side of (3.4), we find that

$$\langle u, v\rangle = \langle u, u_t + \delta u - \varepsilon z(\theta_t\omega)\rangle \geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - \varepsilon\|z(\theta_t\omega)\|\|u\| \tag{3.7}$$

and

$$\begin{aligned} \langle Au, v\rangle &= \langle \nabla u, \nabla v\rangle = \langle \nabla u, \nabla u_t + \delta\nabla u - \varepsilon\nabla z(\theta_t\omega)\rangle \\ &\geq \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \delta\|\nabla u\|^2 - \varepsilon\|\nabla z(\theta_t\omega)\|\|\nabla u\|. \end{aligned} \tag{3.8}$$

For the last term on the left-hand side of (3.4), we have

$$\begin{aligned} &\langle f(x,u), v\rangle = \langle f(x,u), u_t + \delta u - \varepsilon z(\theta_t\omega)\rangle \\ &= \frac{d}{dt}\int_{\mathbb{R}^n} F(x,u)dx + \delta\langle f(x,u), u\rangle - \varepsilon\langle f(x,u), z(\theta_t\omega)\rangle. \end{aligned} \tag{3.9}$$

By Assumption II, we get

(3.10)
$$
\begin{aligned}
&\delta\langle f(x,u),u\rangle - \langle f(x,u),\varepsilon z(\theta_t\omega)\rangle \\
&\geq \delta C_2 \int_{\mathbb{R}^n} F(x,u)dx + \delta \int_{\mathbb{R}^n} \phi_2\, dx - \varepsilon C_1 \int_{\mathbb{R}^n} |u|^{p-1}|z(\theta_t\omega)|dx - \varepsilon \int_{\mathbb{R}^n} |\phi_1||z(\theta_t\omega)|dx \\
&\geq \delta C_2 \int_{\mathbb{R}^n} F(x,u)dx + \delta\|\phi_2\|_{L^1} - \frac{\varepsilon C_1(p-1)}{p}\|u\|_{L^p}^p - \frac{\varepsilon C_1}{p}\|z(\theta_t\omega)\|_{L^p}^p \\
&\quad - \frac{\varepsilon}{2}\|\phi_1\|^2 - \frac{\varepsilon}{2}\|z(\theta_t\omega)\|^2 \\
&\geq \delta C_2 \int_{\mathbb{R}^n} F(x,u)dx + \delta\|\phi_2\|_{L^1} - \frac{\varepsilon C_1(p-1)}{C_3\, p}\int_{\mathbb{R}^n} F(x,u)\, dx - \frac{\varepsilon C_1(p-1)}{C_3\, p}\|\phi_3\|_{L^1} \\
&\quad - \frac{\varepsilon C_1}{p}\|z(\theta_t\omega)\|_{L^p}^p - \frac{\varepsilon}{2}\|\phi_1\|^2 - \frac{\varepsilon}{2}\|z(\theta_t\omega)\|^2 \\
&\geq \left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right)\int_{\mathbb{R}^n} F(x,u)\, dx - \left(\frac{\varepsilon C_1}{p}\|z(\theta_t\omega)\|_{L^p}^p + \frac{\varepsilon}{2}\|z(\theta_t\omega)\|^2\right) - C_4,
\end{aligned}
$$

where

$$
C_4 = \frac{\varepsilon}{2}\|\phi_1\|^2 - \delta\|\phi_2\|_{L^1} + \frac{\varepsilon C_1(p-1)}{C_3\, p}\|\phi_3\|_{L^1}.
$$

For the last term on the right-hand side of (3.4),

$$
\langle g,v\rangle \leq \|g\|\|v\| \leq \frac{\|g\|^2}{2(\beta-\delta)} + \frac{\beta-\delta}{2}\|v\|^2. \tag{3.11}
$$

Substitute (3.6)-(3.11) into (3.4). Then we obtain

(3.12)
$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 - \delta\|v\|^2 + \frac{1}{2}\left(\alpha+\delta^2-\beta\delta\right)\frac{d}{dt}\|u\|^2 + \delta\left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 \\
&- \varepsilon\left(\alpha+\delta^2-\beta\delta\right)\|z(\theta_t\omega)\|\|u\| + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \delta\|\nabla u\|^2 \\
&- \varepsilon\|\nabla z(\theta_t\omega)\|\|\nabla u\| + \frac{d}{dt}\int_{\mathbb{R}^n} F(x,u)\, dx + \left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right)\int_{\mathbb{R}^n} F(x,u)\, dx \\
&- \left(\frac{\varepsilon C_1}{p}\|z(\theta_t\omega)\|_{L^p}^p + \frac{\varepsilon}{2}\|z(\theta_t\omega)\|^2\right) - C_4 + \beta\|v\|^2 - \beta\varepsilon\|z(\theta_t\omega)\|\|v\| \\
&\leq \frac{\|g\|^2}{2(\beta-\delta)} + \frac{\beta-\delta}{2}\|v\|^2 + 2\varepsilon\delta\|z(\theta_t\omega)\|\|v\|.
\end{aligned}
$$

Grouping some terms together in (3.12), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n} F(x,u)\,dx\right] \\
&+ \frac{\delta}{2}\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2\right] \\
&+ \left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right)\int_{\mathbb{R}^n} F(x,u)\,dx \\
\le& \frac{3\delta-\beta}{2}\|v\|^2 + \frac{\|g\|^2}{2(\beta-\delta)} + \frac{\varepsilon^2(\alpha+\delta^2-\beta\delta)}{2\delta}\|z(\theta_t\omega)\|^2 + \frac{\varepsilon^2}{2\delta}\|\nabla z(\theta_t\omega)\|^2 \\
&+ \frac{\varepsilon^2(2\delta+\beta)^2}{2\delta}\|z(\theta_t\omega)\|^2 + \frac{\varepsilon C_1}{p}\|z(\theta_t\omega)\|^p_{L^p} + \frac{\varepsilon}{2}\|z(\theta_t\omega)\|^2 + C_4 \\
\le& \frac{\|g\|^2}{2(\beta-\delta)} + C_0\left(\|z(\theta_t\omega)\|^2 + \|\nabla z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|^p_{L^p}\right) + C_4,
\end{aligned} \tag{3.13}$$

where $C_0 > 0$ is a constant and by (3.2) the term $(3\delta-\beta)\|v\|^2/2 \le 0$. Since $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j \in H^2(\mathbb{R}^n)\cap W^{2,p}(\mathbb{R}^n)$, by (2.7) and (2.8) there is a constant $C_5 > 0$ such that for P-a.e. $\omega\in\Omega$,

$$\begin{aligned}
&\Gamma_1(\theta_t\,\omega) := C_0\left(\|z(\theta_t\omega)\|^2 + \|\nabla z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|^p_{L^p}\right) \\
&\le c\sum_{j=1}^m\left(\|z_j(\theta_t\omega_j)\|^2 + \|z_j(\theta_t\omega_j)\|^p_{L^p}\right) \le C_5\, e^{\frac{1}{2}\sigma|t|} r_0(\omega), \quad \text{for all } t\in\mathbb{R}.
\end{aligned} \tag{3.14}$$

It follows from (3.13)-(3.14) that

$$\begin{aligned}
&\frac{d}{dt}\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n} F(x,u)\,dx\right] \\
&+ \delta\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2\right] + 2\left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right)\int_{\mathbb{R}^n} F(x,u)\,dx \\
&\le \frac{\|g\|^2}{\beta-\delta} + 2\Gamma_1(\theta_t\omega) + 2C_4, \quad t\in\mathbb{R},\ \omega\in\Omega.
\end{aligned}$$

It leads to the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n} (F(x,u)+\phi_3(x))\,dx\right] \\
&+ \delta\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2\right] \\
&+ 2\left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3 p}\right)\int_{\mathbb{R}^n} (F(x,u)+\phi_3(x))\,dx \\
\le& \frac{\|g\|^2}{\beta-\delta} + 2\Gamma_1(\theta_t\omega) + 2\left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right)\|\phi_3\|_{L^1} + 2C_4.
\end{aligned} \tag{3.15}$$

In view of (3.3) in the Assumption III, let σ be a fixed positive constant:

$$\sigma = \min\left\{\delta,\ \delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right\} > 0. \tag{3.16}$$

Note that $\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))\,dx \geq 0$ due to (2.5) in the Assumption II. It follows from (3.15) and (3.16) that

$$\begin{aligned} &\frac{d}{dt}\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))\,dx\right] \\ &+\sigma\left[\|v\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))\,dx\right] \\ &\leq \frac{\|g\|^2}{\beta-\delta} + 2\Gamma_1(\theta_t\omega) + C_6, \quad t\in\mathbb{R},\ \omega\in\Omega, \end{aligned} \tag{3.17}$$

where

$$C_6 = 2\left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\,p}\right)\|\phi_3\|_{L^1} + 2C_4.$$

Thus we can apply Gronwall inequality to (3.17) to confirm that for any $\omega\in\Omega$ and $t\geq\tau$ the weak solution of (2.11) satisfies

$$\begin{aligned} &\|v(t,\omega,\tau,v_0)\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u(t,\omega,\tau,u_0)\|^2 \\ &+\|\nabla u(t,\omega,\tau,u_0)\|^2 + 2\int_{\mathbb{R}^n}(F(x,u(t,\omega,\tau,u_0))+\phi_3(x))\,dx \\ &\leq e^{-\sigma(t-\tau)}\left[\|v_0\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u_0\|^2 + \|\nabla u_0\|^2 + 2\int_{\mathbb{R}^n}F(x,u_0)\,dx\right] \\ &+2e^{-\sigma(t-\tau)}\|\phi_3\|_{L^1} + 2\int_\tau^t e^{\sigma(s-t)}\Gamma_1(\theta_s\omega)\,ds + \frac{1}{\sigma}\left(C_6+\frac{\|g\|^2}{\beta-\delta}\right). \end{aligned} \tag{3.18}$$

Replace the time interval $[\tau,t]$ by $[-t,0]$ and consider any given tempered set $B\in\mathscr{D}$. For $(u_0(\theta_{-t}\omega),v_0(\theta_{-t}\omega))\in B(\theta_{-t}\omega)$, we have

$$\begin{aligned} &\|v(0,\,\omega,-t,v_0(\theta_{-t}\omega))\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u(0,\,\omega,-t,u_0(\theta_{-t}\omega))\|^2 \\ &+\ \|\nabla u(0,\,\omega,-t,u_0(\theta_{-t}\omega))\|^2 + 2\int_{\mathbb{R}^n}(F(x,u(0,\,\omega,-t,u_0(\theta_{-t}\omega)))+\phi_3(x))\,dx \\ &\leq e^{-\sigma t}\left[\|v_0(\theta_{-t}\omega)\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u_0(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2\right] \\ &+2e^{-\sigma t}\left[\int_{\mathbb{R}^n}F(x,u_0(\theta_{-t}\omega))\,dx + \|\phi_3\|_{L^1}\right] + 2\int_{-t}^0 e^{\sigma s}\Gamma_1(\theta_s\omega)\,ds + \frac{1}{\sigma}\left(C_6+\frac{\|g\|^2}{\beta-\delta}\right) \\ &\leq e^{-\sigma t}\left(\|v_0(\theta_{-t}\omega)\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u_0(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2 + 2\|\phi_3\|_{L^1}\right) \\ &+\ 2e^{-\sigma t}\int_{\mathbb{R}^n}F(x,u_0(\theta_{-t}\omega))\,dx + 2C_5\int_{-t}^0 e^{\sigma s+\frac{1}{2}\sigma|s|}r_0(\omega)\,ds + \frac{1}{\sigma}\left(C_6+\frac{\|g\|^2}{\beta-\delta}\right) \\ &= e^{-\sigma t}\left(\|v_0(\theta_{-t}\omega)\|^2 + \left(\alpha+\delta^2-\beta\delta\right)\|u_0(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2\right) \\ &+\ 2e^{-\sigma t}\int_{\mathbb{R}^n}F(x,u_0(\theta_{-t}\omega))\,dx + 2e^{-\sigma t}\|\phi_3\|_{L^1} + \frac{1}{\sigma}\left(4C_5\,r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta-\delta}\right), \end{aligned} \tag{3.19}$$

for $t\geq 0$ and P-a.e. $\omega\in\Omega$. Note that (2.3) and (2.4) imply that there is a constant $c=c(C_1,C_2,\phi_1,\phi_2)>0$ such that

$$\int_{\mathbb{R}^n}F(x,u_0(\theta_{-t}\omega))\,dx \leq c\left(1+\|u_0(\theta_{-t}\omega)\|^2 + \|u_0(\theta_{-t}\omega)\|_{L^p}^p\right). \tag{3.20}$$

For any set $B \in \mathscr{D}$, which is a tempered set in E, since $(u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \in B(\theta_{-t}\omega)$, there exists a constant $C > 0$ and a finite $T_B(\omega) > 0$ such that for all $t \geq T_B(\omega)$ one has

$$\begin{aligned} & e^{-\sigma t}\left[\|v_0(\theta_{-t}\omega)\|^2 + \left(\alpha + \delta^2 - \beta\delta\right)\|u_0(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2\right] \\ & + 2e^{-\sigma t}\left[\int_{\mathbb{R}^n} F(x, u_0(\theta_{-t}\omega))\,dx + \|\phi_3\|_{L^1}\right] \\ \leq\, & Ce^{-\sigma t}\left(1 + \|v_0(\theta_{-t}\omega)\|^2 + \|u_0(\theta_{-t}\omega)\|_{H^1}^2 + \|u_0(\theta_{-t}\omega)\|_{L^p}^p\right) \leq 1. \end{aligned} \tag{3.21}$$

Substitute (3.21) into the right-hand side of the last equality of (3.19) and note that (2.5) implies

$$2\int_{\mathbb{R}^n} (F(x, u(0, \omega, -t, u_0(\theta_{-t}\omega)) + \phi_3(x))\,dx \geq 2C_3\|u(0, \omega, -t, u_0(\theta_{-t}\omega))\|_{L^p}^p.$$

Then it results in

$$\begin{aligned} & \|v(0, \omega, -t, v_0(\theta_{-t}\omega))\|^2 + \left(\alpha + \delta^2 - \beta\delta\right)\|u(0, \omega, -t, u_0(\theta_{-t}\omega))\|^2 \\ & + \|\nabla u(0, \omega, -t, u_0(\theta_{-t}\omega))\|^2 + 2C_3\|u(0, \omega, -t, u_0(\theta_{-t}\omega))\|_{L^p}^p \\ \leq\, & 1 + \frac{1}{\sigma}\left(4C_5\, r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right), \quad \text{for } t \geq T_B(\omega) \text{ and a.e. } \omega \in \Omega. \end{aligned} \tag{3.22}$$

By (2.16) and (3.22), we conclude that $\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) = B_E(0, R(\omega))$ for $t \geq T_B(\omega)$, a.e. $\omega \in \Omega$, where the radius of the closed ball $B_E(0, R(\omega))$ in the space E is

$$\begin{aligned} R(\omega) = & \left(\frac{1}{\min\{1, (\alpha + \delta^2 - \beta\delta)\}}\left[1 + \frac{1}{\sigma}\left(4C_5\, r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right)\right]\right)^{\frac{1}{2}} \\ & + \left(\frac{1}{2C_3}\left[1 + \frac{1}{\sigma}\left(4C_5\, r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right)\right]\right)^{\frac{1}{p}}. \end{aligned} \tag{3.23}$$

Since $r_0(\omega)$ is a tempered random variable, $K = \{K(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$. Therefore, $K = \{K(\omega)\}_{\omega \in \Omega}$ is a $\mathscr{D}$-pullback absorbing set for the cocycle Φ. □

3.2. Tail Estimates. Next we conduct uniform estimates on the tail parts of the weak solutions for large spatial and time variables. These estimates play key roles in proving the pullback asymptotic compactness of the random dynamical systems Φ generated by the random wave equation (2.11) on the unbounded domain $\mathbb{R}^n$.

LEMMA 3.2. *Under the Assumptions* I, II *and* III, *for every* $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathscr{D}$, $0 < \eta \leq 1$ *and* P*-a.e.* $\omega \in \Omega$, *there exists* $T = T(\omega, B, \eta) > 0$ *and* $V = V(\omega, \eta) \geq 1$ *such that the cocycle* Φ *associated with the problem* (2.11) *satisfies*

$$\|\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))\|_{E(\mathbb{R}^n \setminus B_r)} = \max_{g_0 \in B(\theta_{-t}\omega)} \|\Phi(t, \theta_{-t}\omega, g_0)\chi_{B_r^c}\|_E < \eta, \tag{3.24}$$

for all $t \geq T$ *and every* $r \geq V$, *where* $\chi_{B_r^c}(x)$ *is the characteristic function of the set* $\{x \in \mathbb{R}^n : |x| > r\}$.

PROOF. Choose a smooth nondecreasing function ρ such that $0 \leq \rho(s) \leq 1$ for all $s \in [0, \infty)$ and

$$\rho(s) = \begin{cases} 0, & if \;\; 0 \leq s < 1, \\ 1, & if \;\; s > 2, \end{cases} \tag{3.25}$$

with $0 \le \rho'(s) \le 2$ for $s \ge 0$. Taking the inner product of the second equation of (2.11) with $\rho(|x|^2/r^2)v$ in $L^2(\mathbb{R}^n)$, we get

(3.26)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\,dx - \delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\,dx + (\alpha+\delta^2)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)uv\,dx\\
&+\int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{r^2}\right)v\,dx + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)f(x,u)v\,dx\\
&=\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)(g+2\delta\varepsilon z(\theta_t\omega))\,v\,dx - \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\beta(v+\varepsilon z(\theta_t\omega)-\delta u)v\,dx.
\end{aligned}$$

Substitute

$$-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\beta(v+\varepsilon z(\theta_t\omega)-\delta u)v\,dx$$

$$\le -\beta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\,dx + \beta\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)uv\,dx + \varepsilon\beta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|z(\theta_t\omega)||v|\,dx.$$

into (3.26) to obtain

(3.27)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\,dx + (\alpha+\delta^2-\beta\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)uv\,dx\\
&+(\beta-\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|v|^2\,dx + \int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{r^2}\right)v\,dx + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)f(x,u)\,v\,dx\\
&\le \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(\frac{\delta}{2}|v|^2 + \frac{\varepsilon^2\beta^2}{2\delta}|z(\theta_t\omega)|^2\right)dx + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)(g+2\delta\varepsilon z(\theta_t\omega))\,v\,dx.
\end{aligned}$$

For the second term on the left-hand side of (3.27), by (2.11) we have

$$\begin{aligned}
&(\alpha+\delta^2-\beta\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)uv\,dx\\
&=(\alpha+\delta^2-\beta\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)u(u_t+\delta u-\varepsilon z(\theta_t\omega))\,dx\\
&\ge(\alpha+\delta^2-\beta\delta)\left(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|u|^2\,dx + \delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)|u|^2\,dx\right)\\
&\quad-(\alpha+\delta^2-\beta\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)\left(\frac{\delta}{2}|u|^2+\frac{\varepsilon^2}{2\delta}|z(\theta_t\omega)|^2\right)dx.
\end{aligned} \tag{3.28}$$

For the fourth term on the left-hand side of (3.27),

$$\begin{aligned}
&\int_{\mathbb{R}^n}(Au)\,\rho\left(\frac{|x|^2}{r^2}\right)v\,dx = \int_{\mathbb{R}^n}(Au)\,\rho\left(\frac{|x|^2}{r^2}\right)(u_t+\delta u-\varepsilon z(\theta_t\omega))\,dx\\
&=\int_{\mathbb{R}^n}(\nabla u)\nabla\left(\rho\left(\frac{|x|^2}{r^2}\right)(u_t+\delta u-\varepsilon z(\theta_t\omega))\right)dx\\
&=\int_{\mathbb{R}^n}(\nabla u)\left(\frac{2x}{r^2}\rho'\left(\frac{|x|^2}{r^2}\right)(u_t+\delta u-\varepsilon z(\theta_t\omega)) + \rho\left(\frac{|x|^2}{r^2}\right)\nabla(u_t+\delta u-\varepsilon z(\theta_t\omega))\right)dx\\
&=\int_{\mathbb{R}^n}(\nabla u)\frac{2x}{r^2}\rho'\left(\frac{|x|^2}{r^2}\right)v\,dx + \int_{\mathbb{R}^n}(\nabla u)\rho\left(\frac{|x|^2}{r^2}\right)\nabla(u_t+\delta u-\varepsilon z(\theta_t\omega))\,dx
\end{aligned}$$

$$= \int_{\mathbb{R}^n} (\nabla u) \frac{2x}{r^2} \rho' \left(\frac{|x|^2}{r^2} \right) v \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla u|^2 \, dx + \delta \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla u|^2 \, dx$$
$$- \varepsilon \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) (\nabla u)(\nabla z(\theta_t \omega)) \, dx.$$

Since $0 \le \rho'(s) \le 2$, it follows that

(3.29)
$$\int_{\mathbb{R}^n} (Au) \, \rho \left(\frac{|x|^2}{r^2} \right) v \, dx \ge - \int_{r \le |x| \le \sqrt{2} r} \frac{4|x|}{r^2} |(\nabla u) v| dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla u|^2 \, dx$$
$$+ \delta \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla u|^2 \, dx - \varepsilon \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) (\nabla u)(\nabla z(\theta_t \omega)) \, dx$$
$$\ge - \frac{2\sqrt{2}}{r} \int_{r \le |x| \le \sqrt{2} r} (|\nabla u|^2 + |v|^2) \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla u|^2 \, dx$$
$$+ \frac{\delta}{2} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla u|^2 \, dx - \frac{\varepsilon^2}{2\delta} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\nabla z(\theta_t \omega))|^2 \, dx.$$

For the fifth term on the left-hand side of (3.27), by (2.3)-(2.5), we have

(3.30)
$$\int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) f(x, u) v \, dx = \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) f(x, u)(u_t + \delta u - \varepsilon z(\theta_t \omega)) \, dx$$
$$\ge \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) F(x, u) \, dx + \delta \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) (C_2 F(x, u) + \phi_2(x)) \, dx$$
$$- \varepsilon C_1 \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |u|^{p-1} |z(\theta_t \omega)| dx - \varepsilon \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\phi_1(x)| |z(\theta_t \omega)| dx$$
$$\ge \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) F(x, u) \, dx + \delta \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) (C_2 F(x, u) + \phi_2(x)) \, dx$$
$$- \varepsilon C_1 \frac{p-1}{p} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |u|^p \, dx - \frac{\varepsilon C_1}{p} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |z(\theta_t \omega)|^p \, dx$$
$$- \frac{\varepsilon}{2} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |z(\theta_t \omega)|^2 dx - \frac{\varepsilon}{2} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |\phi_1|^2 dx$$
$$\ge \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) F(x, u) \, dx + \left(\delta C_2 - \frac{\varepsilon C_1 (p-1)}{C_3 \, p} \right) \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) F(x, u) \, dx$$
$$- C_7 \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) (|z(\theta_t \omega)|^p + |z(\theta_t \omega)|^2) dx$$
$$- \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) \left(\frac{\varepsilon C_1 (p-1)}{C_3 \, p} |\phi_3| - \delta |\phi_2| + \frac{\varepsilon}{2} |\phi_1|^2 \right) dx,$$

where $C_7 = C_7(\varepsilon) > 0$ is a constant and we used the Hölder inequality in the second inequality as well as

$$-\varepsilon C_1 \frac{p-1}{p} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) |u|^p \, dx \ge - \frac{\varepsilon C_1 (p-1)}{C_3 p} \int_{\mathbb{R}^n} \rho \left(\frac{|x|^2}{r^2} \right) (F(x, u) + \phi_3(x)) \, dx$$

in the third inequality. For the last term on the right-hand side of (3.27), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(g + 2\delta\varepsilon z(\theta_t\omega))\, v\, dx \\
&\le \frac{1}{2(\beta-\delta)}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)\left(|g| + 2\delta\varepsilon|z(\theta_t\omega)|\right)^2 dx + \frac{\beta-\delta}{2}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)|v|^2\, dx.
\end{aligned} \tag{3.31}$$

Now substitute (3.28)-(3.31) into (3.27), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)\big(|v|^2 + (\alpha+\delta^2-\beta\delta)|u|^2 + |\nabla u|^2 + 2F(x,u)\big)\, dx \\
&\quad + \frac{\delta}{2}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)|v|^2\, dx + \frac{\delta}{2}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)((\alpha+\delta^2-\beta\delta)|u|^2 + |\nabla u|^2)\, dx \\
&\quad + \left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3\, p}\right)\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)F(x,u)\, dx \\
&\le \frac{\varepsilon^2}{2\delta}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|\nabla z(\theta_t\omega)|^2 + (\alpha+\delta^2-\beta\delta)|z(\theta_t\omega)|^2 + \beta^2|z(\theta_t\omega)|^2)\, dx \\
&\quad + \frac{2\sqrt{2}}{r}\int_{r\le|x|\le\sqrt{2}r}(|\nabla u|^2 + |v|^2)\, dx + C_7\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2)\, dx \\
&\quad + \frac{1}{\beta-\delta}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|g|^2 + 4\delta^2\varepsilon^2|z(\theta_t\omega)|^2)\, dx \\
&\quad + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)\left(\frac{\varepsilon C_1(p-1)}{C_3\, p}|\phi_3| - \delta|\phi_2| + \frac{\varepsilon}{2}|\phi_1|^2\right) dx \\
&\le \frac{2\sqrt{2}}{r}\int_{r\le|x|\le\sqrt{2}r}(|\nabla u|^2 + |v|^2)\, dx \\
&\quad + C_8\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|\nabla z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p)\, dx \\
&\quad + C_9\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|g|^2 + |\phi_1|^2 + |\phi_2| + |\phi_3|)\, dx,
\end{aligned} \tag{3.32}$$

where $C_8 = C_8(\varepsilon) > 0$ and $C_9 = C_9(\varepsilon) > 0$ are constants. In grouping the coefficients of the terms $\int_{\mathbb{R}^n} \rho(|x|^2/r^2)|v|^2\, dx$ appearing on both sides of (3.32), we used (3.2) to get $(\beta-\delta)/2 = (\beta-3\delta)/2 + \delta \ge \delta$.

Since $g, \phi_1 \in L^2(\mathbb{R}^n)$ and $\phi_2, \phi_3 \in L^1(\mathbb{R}^n)$, for any given $\eta > 0$, there exists $K_0 = K_0(\eta) \ge 1$ such that for all $r \ge K_0$,

$$\begin{aligned}
&C_9\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)\big(|g|^2 + |\phi_1|^2 + |\phi_2| + |\phi_3|\big)\, dx + 2\sigma\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)|\phi_3(x)|\, dx \\
&\le C_9\int_{|x|\ge r}\big(|g|^2 + |\phi_1|^2 + |\phi_2| + |\phi_3|\big)\, dx + 2\sigma\int_{\mathbb{R}^n}|\phi_3(x)|\, dx \le \eta.
\end{aligned} \tag{3.33}$$

By (3.16) and (3.32)-(3.33), there exists $K_1 = K_1(\eta) \geq 1$ such that for all $r \geq K_1$,

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)\left(|v|^2 + (\alpha + \delta^2 - \beta\delta)|u|^2 + |\nabla u|^2 + 2(F(x,u) + \phi_3)\right)dx$$
$$+\sigma\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)\left(|v|^2 + (\alpha + \delta^2 - \beta\delta)|u|^2 + |\nabla u|^2 + 2(F(x,u) + \phi_3)\right)dx$$

$$\leq \eta\left[1 + \int_{r\leq|x|\leq\sqrt{2}r}(|\nabla u|^2 + |v|^2)\,dx\right]$$
$$+ C_8\int_{|x|\geq r}\left(|\nabla z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p\right)dx.$$

Integrating the above inequality over the time interval $[-t, 0]$, we see that for any $t > 0, \omega \in \Omega$ and $r \geq K_1$,

(3.34)
$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|v(0,\omega,-t,v_0(\theta_{-t}\omega))|^2 + \left(\alpha + \delta^2 - \beta\delta\right)|u(0,\omega,-t,u_0(\theta_{-t}\omega))|^2)\,dx$$
$$+\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)|\nabla u(0,\omega,-t,u_0(\theta_{-t}\omega))|^2 dx$$

$$+ 2\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(F(x,u(0,\omega,-t,u_0(\theta_{-t}\omega)) + \phi_3(x))\,dx$$
$$\leq e^{-\sigma t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(|v_0(\theta_{-t}\omega)|^2 + (\alpha + \delta^2 - \beta\delta)|u_0(\theta_{-t}\omega)|^2 + |\nabla u_0(\theta_{-t}\omega)|^2)\,dx$$

$$+ 2e^{-\sigma t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right)(F(x,u_0(\theta_{-t}\omega)) + \phi_3(x))\,dx + \frac{\eta}{\sigma}$$
$$+ \eta\int_{-t}^{0} e^{\sigma s}\int_{r\leq|x|\leq\sqrt{2}r}(|\nabla u(s,\omega,-t,u_0(\theta_{-t}\omega))|^2 + |v(s,\omega,-t,v_0(\theta_{-t}\omega))|^2)\,dx\,ds$$

$$+ C_8\int_{-t}^{0} e^{\sigma s}\int_{|x|\geq r}(|\nabla z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^p)\,dx\,ds.$$

Next we conduct estimate of the terms on the right-hand side of in (3.34). For the first two terms we have

(3.35)

$$
\begin{aligned}
& e^{-\sigma t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)(|v_0(\theta_{-t}\omega))|^2+(\alpha+\delta^2-\beta\delta)|u_0(\theta_{-t}\omega))|^2+|\nabla u_0(\theta_{-t}\omega)|^2)\,dx \\
&\quad + 2e^{-\sigma t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{r^2}\right)(F(x,u_0(\theta_{-t}\omega))+\phi_3(x))\,dx \\
&\le e^{-\sigma t}\int_{\mathbb{R}^n}\left(|v_0(\theta_{-t}\omega)|^2+(\alpha+\delta^2-\beta\delta)|u_0(\theta_{-t}\omega)|^2+|\nabla u_0(\theta_{-t}\omega)|^2\right)dx \\
&\quad + 2e^{-\sigma t}\int_{\mathbb{R}^n}\left[\frac{1}{C_2}\left(C_1|u_0(\theta_{-t}\omega)|^p+|u_0(\theta_{-t}\omega)||\phi_1(x)|+|\phi_2(x)|\right)+|\phi_3(x)|\right]dx \\
&\le e^{-\sigma t}\left(\|v_0(\theta_{-t}\omega)\|^2+(\alpha+\delta^2-\beta\delta)\|u_0(\theta_{-t}\omega)\|^2+\|\nabla u_0(\theta_{-t}\omega)\|^2\right) \\
&\quad + 2e^{-\sigma t}\left[\frac{1}{C_2}\left(\|u_0(\theta_{-t}\omega)\|^2+C_1\|u_0(\theta_{-t}\omega)\|^p_{L^p}+\|\phi_1\|^2+\|\phi_2\|_{L^1}\right)+\|\phi_3\|_{L^1}\right] \\
&\le C_{10}\,e^{-\sigma t}\left(\|v_0(\theta_{-t}\omega)\|^2+\|u_0(\theta_{-t}\omega)\|^2+\|\nabla u_0(\theta_{-t}\omega)\|^2+\|u_0(\theta_{-t}\omega)\|^p_{L^p}\right) \\
&\quad + C_{10}\,e^{-\sigma t}\left(\|\phi_1\|^2+\|\phi_2\|_{L^1}+\|\phi_3\|_{L^1}\right)<\eta, \quad \text{for all } t\ge T_1,
\end{aligned}
$$

where $C_{10} > 0$ and $T_1 = T_1(B, \omega, \eta) > 0$ are constants, and the last step follows from the tempered property of $B \in \mathscr{D}$.

Note that $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$. Thus there is a constant $K_2 = K_2(\omega, \eta) \ge 1$ such that for all $r \ge K_2$,

$$
\max_{1\le j\le m}\int_{|x|\ge r}\left(|h_j(x)|^2+|h_j(x)|^p+|\nabla h_j(x)|^2\right)dx \le \frac{\sigma\,\eta}{2C_8\,r_0(\omega)}, \tag{3.36}
$$

where $r_0(\omega)$ is the tempered function in (2.8). By (2.8) and (3.36) we obtain the estimate of the last integral term in (3.34),

(3.37)

$$
\begin{aligned}
& C_8\int_{-t}^0 e^{\sigma s}\int_{|x|\ge r}\left(|\nabla z(\theta_s\omega)|^2+|z(\theta_s\omega)|^2+|z(\theta_s\omega)|^p\right)dx\,ds \\
&\le C_8\int_{-t}^0 e^{\sigma s}\sum_{j=1}^m\int_{|x|\ge r}\left(|\nabla h_j|^2|z_j(\theta_s\omega_j)|^2+|h_j|^2|z_j(\theta_s\omega_j)|^2+|h_j|^p|z_j(\theta_s\omega_j)|^p\right)dx\,ds \\
&\le \frac{\sigma\eta}{2r_0(\omega)}\int_{-t}^0 e^{\sigma s}\sum_{j=1}^m(|z_j(\theta_s\omega_j)|^2+|z_j(\theta_s\omega_j)|^p)\,ds \le \frac{\sigma\eta}{2r_0(\omega)}\int_{-t}^0 e^{\frac{1}{2}\sigma s}\,r_0(\omega)\,ds<\eta.
\end{aligned}
$$

Finally we estimate the third integral term on the right-hand side of (3.34). Applying the Gronwall inequality to (3.17) while taking the spatial integral over the region $r \le |x| \le \sqrt{2}r$, with (3.16) in mind, we can get

(3.38)

$$\begin{aligned}
&\int_{-t}^{0} e^{\sigma s} \int_{r\le |x|\le \sqrt{2}r} (|\nabla u(s,\omega,-t,u_0(\theta_{-t}\omega))|^2 + |v(s,\omega,-t,v_0(\theta_{-t}\omega))|^2)\,dx\,ds \\
&\le e^{-\sigma(s+t)} \left(\|v_0(\theta_{-t}\omega)\|^2 + (\alpha+\delta^2-\beta\delta)\|u_0(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2\right) \\
&\quad + 2e^{-\sigma(s+t)} \int_{\mathbb{R}^n} \left(F(x,u_0(\theta_{-t}\omega)) + \phi_3(x)\right)\,dx \\
&\quad + \frac{1}{\sigma}\left(C_6 + \frac{1}{\beta-\delta}\|g\|^2\right) + 2\int_{-t}^{s} e^{-\sigma(s-\tau)}\Gamma_2(\theta_\tau\omega)\,d\tau.
\end{aligned}$$

In the last integral of (3.38), due to (3.36) and (2.8), similar to (3.14) we find that for $r \ge K_2$,

$$\begin{aligned}
\Gamma_2(\theta_t\,\omega) &= C_0 \int_{r\le|x|\le\sqrt{2}r} \left(|z(\theta_t\omega)|^2 + |\nabla z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p\right)\,dx \\
&\le C_0 \sum_{j=1}^{m} \int_{|x|\ge r} \left(|\nabla h_j|^2|z_j(\theta_t\omega_j)|^2 + |h_j|^2|z_j(\theta_t\omega_j)|^2 + |h_j|^p|z_j(\theta_t\omega_j)|^p\right)dx \\
&\le \frac{C_0\,\sigma\eta}{2C_8\,r_0(\omega)} \sum_{j=1}^{m} \left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^p\right) \le \frac{C_0\,\sigma\eta}{2C_8}\,e^{(\sigma/2)|t|}.
\end{aligned}$$

Based on (3.38) and the above inequality as well as the tempered property of $B \in \mathscr{D}$, there exists $T_2 = T_2(B,\omega,\eta) > 0$ such that

(3.39)

$$\begin{aligned}
&\int_{-t}^{0} e^{\sigma s} \int_{r\le|x|\le\sqrt{2}r} \left(|\nabla u(s,\omega,-t,u_0(\theta_{-t}\omega))|^2 + |v(s,\omega,-t,v_0(\theta_{-t}\omega))|^2\right)dx\,ds \\
&\le Ct\,e^{-\sigma t}\left[\|(u_0,v_0)(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2 + \int_{\mathbb{R}^n} \left(F(x,u_0(\theta_{-t}\omega)) + \phi_3(x)\right)dx\right] \\
&\quad + \frac{1}{\sigma}\left(C_6 + \frac{1}{\beta-\delta}\|g\|^2\right) + \frac{C_0\,\sigma\eta}{C_8}\int_{-t}^{0}\int_{-t}^{s} e^{\sigma\tau - \frac{1}{2}\sigma\tau}\,d\tau\,ds \\
&\le Ct\,e^{-\sigma t}\left(\|(u_0,v_0)(\theta_{-t}\omega)\|^2 + \|\nabla u_0(\theta_{-t}\omega)\|^2 + \|\phi_3\|_{L^1}\right) \\
&\quad + \frac{C}{C_2}t\,e^{-\sigma t}\left(C_1\|u_0(\theta_{-t}\omega)\|_{L^p}^p + \|u_0(\theta_{-t}\omega)\|^2 + \|\phi_1\|^2 + \|\phi_2\|_{L^1}\right) \\
&\quad + \frac{1}{\sigma}\left(C_6 + \frac{1}{\beta-\delta}\|g\|^2\right) + \frac{4C_0\eta}{C_8\sigma} \le M, \quad \text{for all } t \ge T_2,
\end{aligned}$$

where the constant

$$M = 1 + \frac{1}{\sigma}\left(C_6 + \frac{1}{\beta-\delta}\|g\|^2 + \frac{4C_0\eta}{C_8\sigma}\right).$$

Now assemble all these estimates and substitute (3.35), (3.37) and (3.39) into (3.34). It shows that for any $B \in \mathscr{D}, 0 < \eta \le 1$ and a.e. $\omega \in \Omega$, as long as $r \ge V =$

$\max\{K_0, K_1, K_2\}$ and $t \geq \max\{T_1, T_2\}$ one has

(3.40)
$$\begin{aligned}
&\int_{|x|\geq\sqrt{2}r} (|v(0,\omega,-t,v_0(\theta_{-t}\omega))|^2 + (\alpha+\delta^2-\beta\delta)|u(0,\omega,-t,u_0(\theta_{-t}\omega))|^2)\,dx \\
&\quad + \int_{|x|\geq\sqrt{2}r} (|\nabla u(0,\omega,-t,u_0(\theta_{-t}\omega))|^2 + |u(0,\omega,-t,u_0(\theta_{-t}\omega))|^p)\,dx \\
&\leq \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) \big(|v(0,\omega,-t,v_0(\theta_{-t}\omega))|^2 + (\alpha+\delta^2-\beta\delta)|u(0,\omega,-t,u_0(\theta_{-t}\omega))|^2\big)\,dx \\
&\quad + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\nabla u(0,\omega,-t,u_0(\theta_{-t}\omega))|^2\,dx \\
&\quad + \frac{2}{C_3}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (F(x,u(0,\omega,-t,u_0(\theta_{-t}\omega))) + \phi_3)\,dx \leq \left(1+\frac{1}{C_3}\right)(2+M)\eta.
\end{aligned}$$

By (3.1), the above inequality (3.40) demonstrates that for any $B \in \mathscr{D}$ and a.e. $\omega \in \Omega$, it holds that

(3.41)
$$\begin{aligned}
\|\Phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\|_{E(\mathbb{R}^n\setminus B_R)} &= \max_{g_0\in B(\theta_{-t}\omega)} \|\Phi(t,\theta_{-t}\omega,g_0)\|_{E(\mathbb{R}^n\setminus B_R)} \\
&\leq \left[\left(1+\frac{1}{C_3}\right)(2+M)\eta\right]^{1/2} + \left[\left(1+\frac{1}{C_3}\right)(2+M)\eta\right]^{1/p},
\end{aligned}$$

where $R = \sqrt{2}r$. (3.41) implies (3.24) according to (2.16) by renaming r to be R and η to be $((1+1/C_3)(2+M)\eta)^{1/2} + ((1+1/C_3)(2+M)\eta)^{1/p}$. The proof is completed. □

4. Pullback Asymptotic Compactness in Space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$

In this section, we shall prove the pullback asymptotic compactness in the space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ of the random dynamical sytem Φ associated with the stochastic damped wave equation (2.2) which has been converted to (2.11).

LEMMA 4.1. *The following statements hold for $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.*

1) *Let $\{\psi_m\}$ be a sequence and ψ be a function in $L^p(\mathbb{R}^n)$ such that $\|\psi_m - \psi\|_{L^p} \to 0$ as $m \to \infty$. Then there exists a subsequence $\{\psi_{m_k}\}$ such that*

$$\lim_{k\to\infty} \psi_{m_k}(x) = \psi(x), \quad \text{a.e. on } \mathbb{R}^n.$$

2) *If a sequence $\{\psi_m\}$ and a function ψ in $L^p(\mathbb{R}^n)$ satisfy the following two conditions:*

$$\lim_{m\to\infty} \psi_m(x) = \psi(x), \ \text{a.e. on } \mathbb{R}^n \quad \text{and} \quad \psi_m \text{ is bounded in } L^p(\mathbb{R}^n), \tag{4.1}$$

then $\psi_m \to \psi$ weakly in $L^p(\mathbb{R}^n)$, as $m \to \infty$.

3) *For $1 < p < \infty$, if a sequence $\{\psi_m\}$ and a function ψ in $L^p(\mathbb{R}^n)$ satisfy the following two conditions:*

$$\lim_{m\to\infty} \psi_m(x) = \psi(x), \ \text{a.e. on } \mathbb{R}^n \quad \text{and} \quad \lim_{m\to\infty} \|\psi_m\|_{L^p} = \|\psi\|_{L^p}, \tag{4.2}$$

then $\lim_{m\to\infty} \|\psi_m - \psi\|_{L^p} = 0$.

Proof. Since $\mathbb{R}^n$ with the Lebesgue measure is a σ_0-finite measure space, the first item is a standard result in Real and Functional Analysis.

For the second item, since $L^p(\mathbb{R}^n)$ is a reflexive Banach space for $1 < p < \infty$, the boundedness of $\{\psi_m\}$ in $L^p(\mathbb{R}^n)$ implies that there is $\varphi \in L^p(\mathbb{R}^n)$ such that $\psi_m \to \varphi$ weakly as $m \to \infty$. By Mazur's lemma, this weak convergence implies there exists a sequence $\{\zeta_m\} \subset L^p(\mathbb{R}^n)$ such that

$$\zeta_m \in \operatorname{conv}\{\psi_m, \psi_{m+1}, \cdots\} \text{ and } \zeta_m \to \varphi \text{ strongly in } L^p(\mathbb{R}^n). \tag{4.3}$$

It follows from the condition $\psi_m \to \psi$ a.e. and $\zeta_m \in \operatorname{conv}(\bigcup_{i=m}^{\infty} \psi_i)$ that

$$\zeta_m \to \psi \text{ a.e. in } \mathbb{R}^n. \tag{4.4}$$

On the other hand, by the first statement in this lemma, the strong convergence in (4.3) implies that there exists a subsequence $\{\zeta_{m_k}\}$ such that $\zeta_{m_k} \to \varphi$ a.e. as $k \to \infty$. Therefore, (4.4) leads to $\psi = \varphi$ a.e. on $\mathbb{R}^n$ and $\psi_m \to \psi$ weakly as $m \to \infty$. The third item is a known result in Functional Analysis, cf. [**7**, Chapter 4]. □

Let us define the following energy functional on E: for $(u, v) \in E$,

$$Q(u,v) = \|v\|^2 + \left(\alpha + \delta^2 - \beta\delta\right)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n} (F(x,u) + \phi_3(x))\, dx. \tag{4.5}$$

Compare (3.1) and (4.5), we see that

$$Q(u,v) \le \|(u,v)\|_E^2 + 2\int_{\mathbb{R}^n} (F(x,u) + \phi_3(x))\, dx. \tag{4.6}$$

Lemma 4.2. *For every $\omega \in \Omega$ and any $B \in \mathscr{D}$ and any integer $k > 0$, there exists a constant $M_1 = M_1(B, \omega, k) > 0$ such that for all $m \ge M_1$ one has $t_m > k$ and*

$$\begin{aligned} &Q\left(u(t,\omega,-t_m,u_{0,m}), v(t,\omega,-t_m,v_{0,m})\right) \le R(\omega) + 1 \\ &\quad + \frac{1}{\sigma} e^{\frac{1}{2}\sigma k}\left[4C_5 r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right], \quad t \in [-k, 0], \end{aligned} \tag{4.7}$$

for all $(u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$, where $R(\omega)$ and $r_0(\omega)$ are the same as in (3.23) *and* (2.7)*, respectively.*

Proof. Integrate the inequality (3.17) over the time interval $[-k, t] \subset [-k, 0]$, where $\delta \ge \sigma$ by (3.16). Similar to (3.22), there exists $M_1 = M_1(B, \omega, k) > 0$ such that for all $m \ge M_1$ one has $t_m > k$ and

$$\begin{aligned} &Q(u(t,\omega,-t_m,u_{0,m}), v(t,\omega,-t_m,v_{0,m})) \\ &\le e^{-\sigma(t+k)} Q(u(-k,\omega,-t_m,u_{0,m}), v(-k,\omega,-t_m,v_{0,m})) \\ &\quad + \int_{-k}^{t} e^{-\sigma(t-s)}\left(2\Gamma_1(\theta_s\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right) ds \\ &\le R(\omega) + 1 + 2C_5 \int_{-k}^{t} e^{-\sigma t + (\sigma - \frac{1}{2}\sigma)s} r_0(\omega)\, ds + \frac{1}{\sigma}\left(C_6 + \frac{\|g\|^2}{\beta - \delta}\right) \\ &\le R(\omega) + 1 + \frac{1}{\sigma}\left(4e^{-\frac{1}{2}\sigma t} C_5 r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right), \quad t \in [-k, 0]. \end{aligned} \tag{4.8}$$

Therefore, (4.7) holds. □

THEOREM 4.3. *Under Assumptions* I, II *and* III, *for every* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ *and for any sequences* $t_m \to \infty$ *and* $g_{0,m} = (u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$, *the sequence*

$$\{\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\}_{m=1}^{\infty}$$

of a pullback quasi-trajectory of the cocycle Φ *associated with the problem* (2.11) *of the stochastic wave equation has a strongly convergent subsequence in* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

PROOF. The proof goes through the following five steps.

STEP 1. First by Lemma 3.1, there exist a constant $M_2 = M_2(B, \omega) > 0$ such that for all $m \geq M_2$ and $g_{0,m} \in B(\theta_{-t_m}\omega)$, we have

$$\|\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\|_E \leq R(\omega) + 1, \quad \omega \in \Omega, \tag{4.9}$$

where $R(\omega) > 0$ is given by (3.23). Then for any $\omega \in \Omega$ there is $(\tilde{u}(\omega), \tilde{v}(\omega)) \in E$ such that, up to a subsequence relabeled the same,

$$\begin{aligned} &\Phi(t_m, \theta_{-t_m}\omega, g_{0,m}) \longrightarrow (\tilde{u}(\omega), \tilde{v}(\omega)) \quad \text{weakly in } E; \\ &\Phi(t_m, \theta_{-t_m}\omega, g_{0,m}) \longrightarrow (\tilde{u}(\omega), \tilde{v}(\omega)) \quad \text{weakly in } H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n); \\ &\Phi(t_m, \theta_{-t_m}\omega, g_{0,m}) \longrightarrow (\tilde{u}(\omega), \tilde{v}(\omega)) \quad \text{weakly in } L^p(\mathbb{R}^n). \end{aligned} \tag{4.10}$$

Since E is a reflexive and separable Banach space, the weak lower-semicontinuity of the E-norm and the norm of $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ implies that

$$\begin{aligned} &\liminf_{m\to\infty} \|\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\|_E \geq \|(\tilde{u}(\omega), \tilde{v}(\omega))\|_E, \\ &\liminf_{m\to\infty} \|\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\|_{H^1\times L^2} \geq \|(\tilde{u}(\omega), \tilde{v}(\omega))\|_{H^1\times L^2}. \end{aligned} \tag{4.11}$$

Next we shall prove that in the Hilbert space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,

$$\Phi(t_m, \theta_{-t_m}\omega, g_{0,m}) \longrightarrow (\tilde{u}(\omega), \tilde{v}(\omega)) \quad \text{strongly}. \tag{4.12}$$

It suffices to show that

$$\limsup_{m\to\infty} \|\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\|_{H^1\times L^2} \leq \|(\tilde{u}(\omega), \tilde{v}(\omega))\|_{H^1\times L^2}. \tag{4.13}$$

Then (4.11) and (4.13) lead to

$$\lim_{m\to\infty} \|\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\|_{H^1\times L^2} = \|(\tilde{u}(\omega), \tilde{v}(\omega))\|_{H^1\times L^2}.$$

By the item 3 of Lemma 4.1, we shall obtain (4.12).

STEP 2. By Lemma 4.2 and (2.5), there exists a constant $C > 0$ such that for every $\omega \in \Omega$ and any given integer $k > 0$, whenever $m \geq M_1$ one has $t_m > k$ and

$$\begin{aligned} &\|(u(t, \omega, -t_m, u_{0,m}), v(t, \omega, -t_m, v_{0,m}))\|_E \\ &\leq C \left[R(\omega) + 1 + \frac{1}{\sigma} e^{\frac{1}{2}\sigma k} \left(4C_5 r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right)\right]^{1/2} \\ &\quad + C \left[R(\omega) + 1 + \frac{1}{\sigma} e^{\frac{1}{2}\sigma k} \left(4C_5 r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta}\right)\right]^{1/p}, \quad t \in [-k, 0], \end{aligned} \tag{4.14}$$

for any $(u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$. In particular, (4.14) is satisfied for $t = -k$.

Then by the Banach-Alaoglu theorem, there exists a sequence $\{\tilde{u}_k(\omega), \tilde{v}_k(\omega)\}_{k=1}^{\infty}$ in the space E and subsequences of $\{-t_m\}_{m=1}^{\infty}$ and $\{(u_{0,m}, v_{0,m})\}_{m=1}^{\infty}$ again relabeled as the same, such that for all $\omega \in \Omega$ and every integer $k \geq 1$,

$$(u(-k, \omega, -t_m, u_{0,m}), v(-k, \omega, -t_m, v_{0,m})) \longrightarrow (\tilde{u}_k(\omega), \tilde{v}_k(\omega)) \quad \text{weakly in } E, \tag{4.15}$$

as $m \to \infty$, which can be extracted through a diagonal selection procedure.

By the weakly continuous dependence on the initial data of the solutions stated in Lemma 2.7, the weak convergence (4.15) and the fact of concatenation,

$$
\begin{aligned}
&(u(0,\omega,-t_m,u_{0,m}),\ \ v(0,\omega,-t_m,v_{0,m})) \\
&= (u(0,\omega,-k,u(-k,\omega,-t_m,u_{0,m})),\ \ v(0,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))),
\end{aligned}
\tag{4.16}
$$

imply that for all integers $k \geq 1$ and $\omega \in \Omega$, when $m \to \infty$,

$$
(u(0,\omega,-t_m,u_{0,m}), v(0,\omega,-t_m,v_{0,m})) \longrightarrow (u(0,\omega,-k,\tilde{u}_k), v(0,\omega,-k,\tilde{v}_k))
\tag{4.17}
$$

weakly in E. By (2.16), (4.10) and (4.17) we reach the following equality that for every $\omega \in \Omega$ and all positive integers k,

$$
(\tilde{u}(\omega), \tilde{v}(\omega)) = (u(0,\omega,-k,\tilde{u}_k(\omega)), v(0,\omega,-k,\tilde{v}_k(\omega))).
\tag{4.18}
$$

According to (3.4)-(3.9), the weak solutions (u,v) of (2.11) satisfies

$$
\begin{aligned}
\frac{d}{dt} Q(u,v) + 2\sigma Q(u,v) &\leq -2(\beta-\delta-\sigma)\|v\|^2 - 2(\delta-\sigma)\left(\alpha+\delta^2-\beta\delta\right)\|u\|^2 \\
&- 2(\delta-\sigma)\|\nabla u\|^2 + 4\sigma \int_{\mathbb{R}^n} (F(x,u)+\phi_3(x))\,dx - 2\delta\langle f(x,u),u\rangle \\
&+ 2\varepsilon\left(\alpha+\delta^2-\beta\delta\right)\langle z(\theta_t\omega),u\rangle + 2\varepsilon\langle \nabla z(\theta_t\omega), \nabla u\rangle + 2\varepsilon\langle z(\theta_t\omega), f(x,u)\rangle \\
&+ (4\delta\varepsilon - 2\beta\varepsilon)\langle z(\theta_t\omega), v\rangle + 2\langle g,v\rangle := G(u(t,\omega,\tau,u_0), v(t,\omega,\tau,v_0)).
\end{aligned}
\tag{4.19}
$$

From (4.18) and (4.19), for any integer $k \geq 1$ we have

$$
\begin{aligned}
Q(\tilde{u}(\omega),\tilde{v}(\omega)) &\leq e^{-2\sigma k} Q(\tilde{u}_k(\omega),\tilde{v}_k(\omega)) \\
&\quad + \int_{-k}^{0} e^{2\sigma\xi}\, G(u(\xi,\omega,-k,\tilde{u}_k), v(\xi,\omega,-k,\tilde{v}_k))\,d\xi.
\end{aligned}
\tag{4.20}
$$

STEP 3. From the concatenation (4.16) and (4.19), on the other hand, we have

$$
\begin{aligned}
&Q(u(0,\omega,-t_m,u_{0,m}),v(0,\omega,-t_m,v_{0,m}))\\
&\le e^{-2\sigma k}Q(u(-k,\omega,-t_m,u_{0,m}),v(-k,\omega,-t_m,v_{0,m}))\\
&-2(\beta-\delta-\sigma)\int_{-k}^{0}e^{2\sigma\xi}\|v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\|^2d\,\xi\\
&-2(\delta-\sigma)\left(\alpha+\delta^2-\beta\delta\right)\int_{-k}^{0}e^{2\sigma\xi}\|u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\|^2d\xi\\
&-2(\delta-\sigma)\int_{-k}^{0}e^{2\sigma\xi}\|\nabla u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\|^2\,d\xi\\
&+4\sigma\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}(F(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))+\phi_3(x))\,dx\,d\xi\\
&-2\delta\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}f(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))\cdot\\
&\cdot u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\,dx\,d\xi\\
&+2\varepsilon\left(\alpha+\delta^2-\beta\delta\right)\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}z(\theta_\xi\omega)u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\,dx\,d\xi\\
&+2\varepsilon\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}\nabla z(\theta_\xi\omega)\nabla u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\,dx\,d\xi\\
&+2\varepsilon\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}z(\theta_\xi\omega)f(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))\,dx\,d\xi\\
&+(4\delta\varepsilon-2\beta\varepsilon)\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}z(\theta_\xi\omega)v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\,dx\,d\xi\\
&+2\int_{-k}^{0}e^{2\sigma\xi}\int_{\mathbb{R}^n}g(x)\,v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\,dx\,d\xi.
\end{aligned}
\tag{4.21}
$$

Below we treat all the terms on the right-hand side of (4.21).

1) For the first term on the right-hand side of (4.21), by (4.7) in Lemma 4.2, for every $\omega\in\Omega$ and all $m\ge M_1(B,\omega,k)$ we have

$$
\begin{aligned}
&e^{-2\sigma k}Q(u(-k,\omega,-t_m,u_{0,m}),v(-k,\omega,-t_m,v_{0,m}))\\
&\le e^{-2\sigma k}\left(R(\omega)+1+\frac{1}{\sigma}e^{\frac{1}{2}\sigma k}\left[4C_5r_0(\omega)+C_6+\frac{\|g\|^2}{\beta-\delta}\right]\right)\\
&\le e^{-\sigma k}\left(R(\omega)+1+\frac{1}{\sigma}\left[4C_5r_0(\omega)+C_6+\frac{\|g\|^2}{\beta-\delta}\right]\right).
\end{aligned}
\tag{4.22}
$$

2) For the second term on the right-hand side of (4.21), by (4.15) and the weakly continuous dependence of solutions on the initial data stated in Lemma 2.7, we find that for every $\omega\in\Omega$ and all $\xi\in[-k,0]$, when $m\to\infty$,

$$v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\longrightarrow v(\xi,\omega,-k,\tilde v_k(\omega))\quad\text{weakly in }L^2(\mathbb{R}^n),$$

which implies that for all $\xi\in[-k,0]$,

$$\liminf_{m\to\infty}\|v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\|^2\ge\|v(\xi,\omega,-k,\tilde v_k(\omega))\|^2. \tag{4.23}$$

By (4.23) and Fatou's lemma we obtain

$$\begin{aligned}
&\liminf_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\|v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\|^2\,d\xi\\
(4.24)\qquad &\geq \int_{-k}^{0} e^{2\sigma\xi}\liminf_{m\to\infty}\|v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\|^2\,d\xi\\
&\geq \int_{-k}^{0} e^{2\sigma\xi}\|v(\xi,\omega,-k,\tilde v_k(\omega))\|^2\,d\xi.
\end{aligned}$$

Therefore, since (3.2) and (3.16) implies $\beta-\delta-\sigma\geq\beta-2\delta>0$, (4.24) leads to

$$\begin{aligned}
&\limsup_{m\to\infty} -2(\beta-\delta-\sigma)\int_{-k}^{0} e^{2\sigma\xi}\|v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\|^2\,d\xi\\
(4.25)\qquad &= -2(\beta-\delta-\sigma)\liminf_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\|v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\|^2\,d\xi\\
&\leq -2(\beta-\delta-\sigma)\int_{-k}^{0} e^{2\sigma\xi}\|v(\xi,\omega,-k,\tilde v_k(\omega))\|^2\,d\xi.
\end{aligned}$$

Similarly for the third and fourth terms, by (4.15) and Fatou's lemma we obtain

(4.26)

$$\begin{aligned}
&\limsup_{m\to\infty} -2(\delta-\sigma)\left(\alpha+\delta^2-\beta\delta\right)\int_{-k}^{0} e^{2\sigma\xi}\|u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\|^2\,d\xi\\
&\qquad\leq -2(\delta-\sigma)\left(\alpha+\delta^2-\beta\delta\right)\int_{-k}^{0} e^{2\sigma\xi}\|u(\xi,\omega,-k,\tilde u_k(\omega))\|^2\,d\xi,\\
&\limsup_{m\to\infty} -2(\delta-\sigma)\int_{-k}^{0} e^{2\sigma\xi}\|\nabla u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\|^2\,d\xi\\
&\qquad\leq -2(\delta-\sigma)\int_{-k}^{0} e^{2\sigma\xi}\|\nabla u(\xi,\omega,-k,\tilde u_k(\omega))\|^2\,d\xi.
\end{aligned}$$

3) For the fifth term on the right-hand side of (4.21), we have

(4.27)

$$\begin{aligned}
&\left|\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n}(F(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))-F(x,u(\xi,\omega,-k,\tilde u_k)))dxd\xi\right|\\
\leq&\int_{-k}^{0} e^{2\sigma\xi}\int_{|x|>r}|F(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))-F(x,u(\xi,\omega,-k,\tilde u_k))|dxd\xi\\
+&\int_{-k}^{0} e^{2\sigma\xi}\int_{|x|\leq r}|F(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))-F(x,u(\xi,\omega,-k,\tilde u_k))|dxd\xi.
\end{aligned}$$

A) For any given $\eta>0$, by the proof of Lemma 3.2 adapted to the time interval $(-\infty,-k]$, there exist $M_3=M_3(B,\omega,\eta)>M_2$ and $K=K(B,\omega,\eta)\geq 1$ such that for $\omega\in\Omega$ and $\xi\in[-k,0]$, whenever $r\geq K$ and $m\geq M_3$ one has

(4.28)

$$\int_{|x|>r}\left(|u(\xi,\omega,-t_m,u_{0,m})|^2+|u(\xi,\omega,-t_m,u_{0,m})|^p+|\phi_1|^2+|\phi_2|+|\phi_3|\right)dx<\eta.$$

In view of the Assumption II, there exists a constant $L_1 > 0$ such that for all $\omega \in \Omega$ and $\xi \in [-k, 0]$, one has

$$\int_{|x|>r} |F(x, u(\xi, \omega, -t_m, u_{0,m}))|\, dx$$
$$\leq \int_{|x|>r} L_1(|u(\xi, \omega, -t_m, u_{0,m})|^2 + |u(\xi, \omega, -t_m, u_{0,m})|^p + |\phi_1|^2 + |\phi_2| + |\phi_3|)dx$$
$$< L_1\eta, \qquad \text{for all } r \geq K,\ m \geq M_3.$$

B) Since (4.15) shows that

$$\tilde{u}_k(\omega) = (\text{weak}) \lim_{m\to\infty} u(-k, \omega, -t_m, u_{0,m}) \quad \text{in} \quad L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),$$

by the weakly continuous dependence of solutions on intial data stated in Lemma 2.7 and the weak lower-semicontinuity of the L^2 and L^p norms, it follows from (4.28) that

$$\int_{-k}^{0} e^{2\sigma\xi} \int_{|x|>r} |F(x, u(\xi, \omega, -k, \tilde{u}_k))|\, dx\, d\xi$$
$$\leq \int_{-k}^{0} e^{2\sigma\xi} \int_{|x|>r} L_1(|u(\xi, \omega, -k, \tilde{u}_k)|^2 + |u(\xi, \omega, -k, \tilde{u}_k)|^p + |\phi_1|^2 + |\phi_2| + |\phi_3|)\, dxd\xi$$
$$= \int_{-k}^{0} e^{2\sigma\xi} L_1 \left(\|u(\xi, \omega, -k, \tilde{u}_k)\|^2_{L^2(\mathbb{R}^n\setminus B_r)} + \|u(\xi, \omega, -k, \tilde{u}_k)\|^p_{L^p(\mathbb{R}^n\setminus B_r)} \right) d\xi$$
$$+ \int_{-k}^{0} e^{2\sigma\xi} L_1 \int_{|x|>r} (|\phi_1|^2 + |\phi_2| + |\phi_3|)\, dx\, d\xi$$
$$\leq \int_{-k}^{0} e^{2\sigma\xi} L_1 \liminf_{m\to\infty} \|u(\xi, \omega, -k, \tilde{u}_k)\|^2_{L^2(\mathbb{R}^n\setminus B_r)}\, d\xi$$
$$+ \int_{-k}^{0} e^{2\sigma\xi} L_1 \liminf_{m\to\infty} \|u(\xi, \omega, -k, \tilde{u}_k)\|^p_{L^p(\mathbb{R}^n\setminus B_r)}\, d\xi$$
$$+ \int_{-k}^{0} e^{2\sigma\xi} L_1 \int_{|x|>r} (|\phi_1|^2 + |\phi_2| + |\phi_3|)\, dx\, d\xi \leq \frac{L_1}{2\sigma}\eta, \text{ for } \omega \in \Omega, r \geq K_1, m \geq M_3.$$

The above two inequalities show that there exists a constant $L_2 = L_1(1+1/(2\sigma)) > 0$ such that the first term on the right-hand side of (4.27) satisfies

(4.29)
$$\int_{-k}^{0} e^{2\sigma\xi} \int_{|x|>r} |F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))) - F(x, u(\xi, \omega, -k, \tilde{u}_k))|\, dxd\xi$$
$$\leq \int_{-k}^{0} e^{2\sigma\xi} \int_{|x|>r} (|F(x, u(\xi, \omega, -t_m, u_{0,m}))| + |F(x, u(\xi, \omega, -k, \tilde{u}_k))|)\, dx\, d\xi \leq L_2\eta,$$

for all $\omega \in \Omega, r \geq K$ and $m \geq M_3$.

C) For the second term on the right-hand side of (4.27), by (4.15) we have

$$u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m})) \longrightarrow u(\xi, \omega, -k, \tilde{u}_k) \text{ weakly in } H^1(\mathbb{B}_r) \cap L^p(\mathbb{B}_r).$$

Since $H^1(\mathbb{B}_r)$ is compactly embedded in $L^2(\mathbb{B}_r)$, it follows that for any $\omega \in \Omega$ and $\xi \in [-k, 0]$,

$$u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m})) \longrightarrow u(\xi, \omega, -k, \tilde{u}_k) \text{ strongly in } L^2(\mathbb{B}_r). \tag{4.30}$$

Then by the first item of Lemma 4.1 and the continuity of $F(x,u)$, as $m \to \infty$,

$$F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))) \longrightarrow F(x, u(\xi, \omega, -k, \tilde{u}_k)) \text{ in } \mathbb{B}_r. \tag{4.31}$$

On the other hand, by the Assumption II and Lemma 4.2, we have a uniform bound that there exists a constant $L_3 > 0$ such that

(4.32)
$$\begin{aligned}
&\int_{|x|<r} |F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m})))| \, dx \\
&\le L_1 \Big(\|u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))\|^2_{L^2(B_r)} \\
&\quad + \|u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))\|^p_{L^p(B_r)} + \|\phi_1\|^2 + \|\phi_2\|_{L^1(\mathbb{R}^n)} + \|\phi_3\|_{L^1(\mathbb{R}^n)} \Big) \\
&\le L_3 \left[R(\omega) + 1 + \frac{1}{\sigma} e^{\frac{1}{2}\sigma k} \left(4C_5 r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta} \right) + \|\phi_1\|^2 + \|\phi_2\|_{L^1} + \|\phi_3\|_{L^1} \right]
\end{aligned}$$

for all $\omega \in \Omega$, $\xi \in [-k, 0]$ and $m \ge M_1$. By the second item of Lemma 4.2, it follows from (4.31) and (4.32) that as $m \to \infty$,

$$F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))) \longrightarrow F(x, u(\xi, \omega, -k, \tilde{u}_k)) \text{ weakly in } L^1(\mathbb{B}_r).$$

Consequently, when $m \to \infty$,

(4.33)
$$\int_{|x|<r} F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m})))dx \longrightarrow \int_{|x|<r} F(x, u(\xi, \omega, -k, \tilde{u}_k)) \, dx.$$

Furthermore, by (4.32) we have

(4.34)
$$\begin{aligned}
&\left| \int_{|x|<r} [F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))) - F(x, u(\xi, \omega, -k, \tilde{u}_k))] \, dx \right| \\
&\le L_3 \left[R(\omega) + 1 + \frac{1}{\sigma} e^{\frac{1}{2}\sigma k} \left(4C_5 r_0(\omega) + C_6 + \frac{\|g\|^2}{\beta - \delta} \right) + \|\phi_1\|^2 + \|\phi_2\|_{L^1} + \|\phi_3\|_{L^1} \right] \\
&\quad + \|F(\cdot, u(\xi, \omega, -k, \tilde{u}_k))\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

According to the Lebesgue dominated convergence theorem, (4.33) and (4.34) imply that for every $\omega \in \Omega$, integer $k \ge 1$ and any given $r \ge K$,

$$\begin{aligned}
&\lim_{m \to \infty} \int_{-k}^{0} e^{2\sigma\xi} \int_{|x|<r} F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))) \, dx \, d\xi \\
&\qquad = \int_{-k}^{0} e^{2\sigma\xi} \int_{|x|<r} F(x, u(\xi, \omega, -k, \tilde{u}_k)) \, dx \, d\xi.
\end{aligned} \tag{4.35}$$

Combine (4.27), (4.29) and (4.35), we obtain

$$\begin{aligned}
&\lim_{m \to \infty} \int_{-k}^{0} e^{2\sigma\xi} \int_{\mathbb{R}^n} \left(F(x, u(\xi, \omega, -k, u(-k, \omega, -t_m, u_{0,m}))) + \phi_3(x) \right) \, dx \, d\xi \\
&\qquad = \int_{-k}^{0} e^{2\sigma\xi} \int_{\mathbb{R}^n} \left(F(x, u(\xi, \omega, -k, \tilde{u}_k)) + \phi_3(x) \right) \, dx \, d\xi.
\end{aligned} \tag{4.36}$$

4) By an argument similar to the proof of (4.36) shown above, we can also prove the convergence of the sixth term on the right-hand side of (4.21). Namely,

$$\begin{aligned}
&\lim_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} f(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))\cdot\\
&\qquad\cdot u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\,dx\,d\xi \\
&= \int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} f(x,u(\xi,\omega,-k,\tilde{u}_k(\omega)))u(\xi,\omega,-k,\tilde{u}_k(\omega))\,dx\,d\xi.
\end{aligned} \tag{4.37}$$

5) The convergence of the remaining terms on the right-hand side of (4.21) can be shown even simpler:

$$\begin{aligned}
&\lim_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\,dx\,d\xi\\
&\qquad= \int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)u(\xi,\omega,-k,\tilde{u}_k(\omega))\,dx\,d\xi,\\
&\lim_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} \nabla z(\theta_\xi\omega)\nabla u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m}))\,dx\,d\xi\\
&\qquad= \int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} \nabla z(\theta_\xi\omega)\nabla u(\xi,\omega,-k,\tilde{u}_k(\omega))\,dx\,d\xi,\\
&\lim_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)f(x,u(\xi,\omega,-k,u(-k,\omega,-t_m,u_{0,m})))\,dx\,d\xi\\
&\qquad= \int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)f(x,u(\xi,\omega,-k,\tilde{u}_k(\omega)))\,dx\,d\xi,\\
&\lim_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\,dx\,d\xi\\
&\qquad= \int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)v(\xi,\omega,-k,\tilde{v}_k(\omega))\,dx\,d\xi,\\
&\lim_{m\to\infty}\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} g(x)\,v(\xi,\omega,-k,v(-k,\omega,-t_m,v_{0,m}))\,dx\,d\xi\\
&\qquad= \int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} g(x)\,v(\xi,\omega,-n,\tilde{v}_k(\omega))\,dx\,d\xi.
\end{aligned}$$

STEP 4. Take the limit of (4.21) as $m \to \infty$ and assemble together the results shown above in the items 1) through 5) of Step 3. Then we get

$$
\begin{aligned}
&\limsup_{m\to\infty} Q(u(0,\omega,-t_m,u_{0,m}), v(0,\omega,-t_m,v_{0,m}))\\
&\le e^{-\sigma k}\left(R(\omega)+1+\frac{1}{\sigma}\left[4C_5 r_0(\omega)+C_6+\frac{\|g\|^2}{\beta-\delta}\right]\right)\\
&\quad -2(\beta-\delta-\sigma)\int_{-k}^{0} e^{2\sigma\xi}\|v(\xi,\omega,-k,\tilde v_k(\omega))\|^2\,d\xi\\
&\quad -2(\delta-\sigma)\left(\alpha+\delta^2-\beta\delta\right)\int_{-k}^{0} e^{2\sigma\xi}\|u(\xi,\omega,-k,\tilde u_k(\omega))\|^2\,d\xi\\
&\quad -2(\delta-\sigma)\int_{-k}^{0} e^{2\sigma\xi}\|\nabla u(\xi,\omega,-k,\tilde u_k(\omega))\|^2\,d\xi\\
&\quad +4\sigma\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n}(F(x,u(\xi,\omega,-k,\tilde u_k(\omega)))+\phi_3(x))\,dx\,d\xi\\
&\quad -2\delta\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} f(x,u(\xi,\omega,-k,\tilde u_k(\omega)))u(\xi,\omega,-k,\tilde u_k(\omega))\,dx\,d\xi\\
&\quad +2\varepsilon\left(\alpha+\delta^2-\beta\delta\right)\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)u(\xi,\omega,-k,\tilde u_k(\omega))\,dx\,d\xi\\
&\quad +2\varepsilon\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} \nabla z(\theta_\xi\omega)\nabla u(\xi,\omega,-k,\tilde u_k(\omega))\,dx\,d\xi\\
&\quad +2\varepsilon\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)f(x,u(\xi,\omega,-k,\tilde u_k(\omega)))\,dx\,d\xi\\
&\quad +(4\delta\varepsilon-2\beta\varepsilon)\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} z(\theta_\xi\omega)v(\xi,\omega,-k,\tilde v_k(\omega))\,dx\,d\xi\\
&\quad +2\int_{-k}^{0} e^{2\sigma\xi}\int_{\mathbb{R}^n} g(x)\,v(\xi,\omega,-k,\tilde v_k(\omega))\,dx\,d\xi.
\end{aligned}
\tag{4.38}
$$

It follows from (4.20) and (4.38) that

$$
\begin{aligned}
&\limsup_{m\to\infty} Q(u(0,\omega,-t_m,u_{0,m}), v(0,\omega,-t_m,v_{0,m}))\\
&\le e^{-\sigma k}\left(R(\omega)+1+\frac{1}{\sigma}\left[4C_5 r_0(\omega)+C_6+\frac{\|g\|^2}{\beta-\delta}\right]\right)\\
&\quad +\int_{-k}^{0} e^{2\sigma\xi}\, G(u(\xi,\omega,-k,\tilde u_k), v(\xi,\omega,-k,\tilde v_k))\,d\xi,\\
&= e^{-\sigma k}\left(R(\omega)+1+\frac{1}{\sigma}\left[4C_5 r_0(\omega)+C_6+\frac{\|g\|^2}{\beta-\delta}\right]\right)\\
&\quad + Q(\tilde u(\omega),\tilde v(\omega)) - e^{-2\sigma k}Q(\tilde u_k(\omega),\tilde v_k(\omega))\\
&\le e^{-\sigma k}\left(R(\omega)+1+\frac{1}{\sigma}\left[4C_5 r_0(\omega)+C_6+\frac{\|g\|^2}{\beta-\delta}\right]\right) + Q(\tilde u(\omega),\tilde v(\omega)),
\end{aligned}
\tag{4.39}
$$

because $-e^{-2\sigma k}Q(\tilde u_k(\omega),\tilde v_k(\omega))\le 0$ due to (2.5) and (4.5). Take limit $k\to\infty$. We obtain

$$
\limsup_{m\to\infty} Q\left(u(0,\omega,-t_m,u_{0,m}), v(0,\omega,-t_m,v_{0,m})\right) \le Q\left(\tilde u(\omega),\tilde v(\omega)\right).
\tag{4.40}
$$

On the other hand, from (4.18), (4.31) and (4.32) we get

$$\lim_{m\to\infty}\int_{\mathbb{R}^n} F(x,u(0,\omega,-t_m,u_{0,m}))\,dx = \int_{\mathbb{R}^n} F(x,\tilde{u})\,dx, \tag{4.41}$$

which along with (4.40) shows that

$$\begin{aligned}&\limsup_{m\to\infty}\big(\|v(0,\omega,-t_m,v_{0,m})\|^2 + (\alpha+\delta^2-\beta\delta)\|u(0,\omega,-t_m,u_{0,m})\|^2 \\ &+ \|\nabla u(0,\omega,-t_m,u_{0,m})\|^2\big) \le \|\tilde{v}\|^2 + (\alpha+\delta^2-\beta\delta)\|\tilde{u}\|^2 + \|\nabla\tilde{u}\|^2.\end{aligned} \tag{4.42}$$

STEP 5. Note that the norm of $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ is equivalent to

$$\|(u,v)\|_\Pi \stackrel{\text{def}}{=} Q(u,v) - 2\int_{\mathbb{R}^n}(F(x,u)+\phi_3(x))\,dx = \|v\|^2 + (\alpha+\delta^2-\beta\delta)\|u\|^2 + \|\nabla u\|^2.$$

Same as the second inequality in (4.11), from the weak convergence shown by (4.10), for any $g_{0,m} = (u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$ we have

$$\liminf_{m\to\infty}\|\Phi(t_m,\theta_{-t_m}\omega,g_{0,m})\|_\Pi \ge \|(\tilde{u}(\omega),\tilde{v}(\omega))\|_\Pi.$$

Meanwhile, (4.42) implies that

$$\limsup_{m\to\infty}\|\Phi(t_m,\theta_{-t_m}\omega,g_{0,m})\|_\Pi \le \|(\tilde{u}(\omega),\tilde{v}(\omega))\|_\Pi.$$

It follows that

$$\lim_{m\to\infty}\|\Phi(t_m,\theta_{-t_m}\omega,g_{0,m})\|_\Pi = \|(\tilde{u}(\omega),\tilde{v}(\omega))\|_\Pi. \tag{4.43}$$

Finally, for the Hilbert space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, the weak convergence (4.10) and the norm convergence (4.43) imply the strong convergence. Therefore, up to a finite steps of subsequence selections always relabeled as the same in this proof, we reach the conclusion that

$$\Phi(t_m,\theta_{-t_m}\omega,g_{0,m}) \to (\tilde{u},\tilde{v}) \quad \text{strongly in } H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n).$$

Thus the proof is completed. □

5. The Existence of Random Attractor

In this section we shall first prove an instrumental convergence theorem in the space $L^p(X,\mathcal{M},\mu)$ of Vitali type. It will pave the way to prove pullback asymptotic compactness of the cocycle Φ in the space $L^p(\mathbb{R}^n)$ for $2 < p < \infty$. This is the crcucial and final step to accomplish the proof of the existence of a random attractor for this random dynamical system Φ generated by the stochastic wave equation (1.1)-(1.2).

THEOREM 5.1. *Let $(X,\mathcal{M},\mu)$ be a σ_0-finite measure space and assume that a sequence $\{f_m\}_{m=1}^\infty \subset L^p(X,\mathcal{M},\mu)$ with $1\le p<\infty$ satisfies*

$$\lim_{m\to\infty} f_m(x) = f(x), \quad \text{a.e.} \tag{5.1}$$

Then $f\in L^p(X,\mathcal{M},\mu)$ and

$$\lim_{m\to\infty}\|f_m - f\|_{L^p(X,\mathcal{M},\mu)} = 0 \tag{5.2}$$

if and only if the following two conditions are satisfied:

(a) *For any given $\varepsilon>0$, there exists a set $A_\varepsilon\in\mathcal{M}$ such that $\mu(A_\varepsilon)<\infty$ and*

$$\int_{X\setminus A_\varepsilon}|f_m(x)|^p\,d\mu < \varepsilon, \quad \text{for all } m\ge 1. \tag{5.3}$$

(b) *The absolutely continuous property of the L^p integrals of the functions in the sequence is satisfied uniformly,*

$$\lim_{\mu(Y)\to 0} \int_Y |f_m(x)|^p \, d\mu = 0, \quad \text{uniformly in } m \geq 1. \tag{5.4}$$

Proof. First we prove the necessity. Statement (a): Under the condition (5.2), for an arbitrarily given $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that

$$\|f_m - f\|^p_{L^p(X,\mathcal{M},\,\mu)} < \frac{\varepsilon}{2^p}, \quad \text{for all } m > N. \tag{5.5}$$

Since $f \in L^p(X, \mathcal{M}, \mu)$, there exist measurable sets B_ε and S_ε both of finite measure, such that

$$\int_{X\setminus B_\varepsilon} |f(x)|^p \, d\mu < \frac{\varepsilon}{2^p} \quad \text{and} \quad \int_{X\setminus S_\varepsilon} |f_m(x)|^p \, d\mu < \varepsilon, \text{ for } m = 1, \cdots, N. \tag{5.6}$$

Put $A_\varepsilon = B_\varepsilon \cup S_\varepsilon$. Then $\mu(A_\varepsilon) < \infty$ and we have

$$\int_{X\setminus A_\varepsilon} |f_m(x)|^p \, d\mu = \int_{X\setminus A_\varepsilon} (|f_m(x) - f(x)| + |f(x)|)^p \, d\mu$$

$$\leq 2^{p-1}\left(\int_X |f_m(x) - f(x)|^p \, d\mu + \int_{X\setminus B_\varepsilon} |f(x)|^p \, d\mu\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for } m > N.$$

Besides it follows from the second inequality in (5.6) that

$$\int_{X\setminus A_\varepsilon} |f_m(x)|^p \, d\mu \leq \int_{X\setminus S_\varepsilon} |f_m(x)|^p \, d\mu < \varepsilon, \quad \text{for } m = 1, \cdots, N.$$

Therefore, the statement (a) is valid.

Statement (b): By the absolutely continuous property of Lebesgue integral on a σ_0-finite measure space, for any given $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that whenever $\mu(Y) < \delta_0$ one has

$$\int_Y |f(x)|^p \, d\mu < \frac{\varepsilon}{2^p} \quad \text{and} \quad \int_Y |f_m(x)|^p \, d\mu < \varepsilon, \quad \text{for } m = 1, \cdots, N, \tag{5.7}$$

where $N = N(\varepsilon)$ is the same integer in (5.5). Then for any measurable set $Y \subset X$ with $\mu(Y) < \delta_0$ one also has

$$\int_Y |f_m(x)|^p \, d\mu \leq 2^{p-1}\left(\int_X |f_m(x) - f(x)|^p \, d\mu + \int_Y |f(x)|^p \, d\mu\right) < \varepsilon, \text{ for } m > N.$$

Thus the statement (b) is also valid.

Next we prove the sufficiency. Suppose the two conditions (a) and (b) are satisfied. First of all, by the condition (a) and Fatou's Lemma, for an arbitrarily given $\varepsilon > 0$ there exists a set A_ε of finite maesure with

$$\sup_{m\geq 1} \int_{X\setminus A_\varepsilon} |f_m(x)|^p \, d\mu < \varepsilon,$$

which implies that the limit function f in the assumption (5.1) satisfies

$$\int_{X\setminus A_\varepsilon} |f(x)|^p \, d\mu \leq \liminf_{m\to\infty} \int_{X\setminus A_\varepsilon} |f_m(x)|^p \, d\mu < \varepsilon. \tag{5.8}$$

Hence it follows that

$$f \in L^p(X\setminus A_\varepsilon) \quad \text{and} \quad \|f_m - f\|_{L^p(X\setminus A_\varepsilon)} < 2\varepsilon^{1/p}, \quad \text{for all } m \geq 1. \tag{5.9}$$

Therefore, the proof of $f \in L^p(X, \mathcal{M}, \mu)$ and (5.2) is reduced to proving that

$$f \in L^p(Y) \quad \text{and} \quad \lim_{m\to\infty} \|f_m - f\|_{L^p(U)} = 0, \tag{5.10}$$

for any given measurable set $Y \subset X$ with $\mu(Y) < \infty$.

Then by the condition (b), for any given $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for any $S \subset X$ with $\mu(S) < \delta_1$ one has

$$\int_S |f_m(x)|^p \, d\mu < \varepsilon^p, \quad \text{uniformly in} \quad m \geq 1. \tag{5.11}$$

Consequently, by Fatou's lemma,

$$\int_S |f(x)|^p \, d\mu \leq \liminf_{m\to\infty} \int_S |f_m(x)|^p \, d\mu < \varepsilon^p. \tag{5.12}$$

By Egorov's theorem on Lebesgue integral over such a set Y of finite measure in the space $(X, \mathcal{M}, \mu)$, there exists a measurable subset $B \subset Y$ with $\mu(Y \backslash B) < \delta_1$ such that

$$\lim_{m\to\infty} f_m(x) = f(x), \quad \text{uniformly a.e. on } B,$$

so that there exists an integer $m_0 = m_0(\varepsilon) \geq 1$ such that

$$\|f_m - f\|_{L^p(B)} < \varepsilon, \quad \text{for all } m \geq m_0. \tag{5.13}$$

Combining (5.11), (5.12) and (5.13), we see that

$$\|f_m - f\|_{L^p(Y)} \leq \|f_m\|_{L^p(Y\backslash B)} + \|f\|_{L^p(Y\backslash B)} + \|f_m - f\|_{L^p(B)} < 3\varepsilon, \quad \text{for } m \geq m_0.$$

Therefore, (5.10) is proved. The proof is completed. □

Finally we present and prove the main result of this work on the existence of a pullback random attractor for this random dynamical system Φ associated with the concerned stochastic wave equation on the product Banach space E with arbitrary exponent and arbitrary space dimension.

THEOREM 5.2. *Under the Assumptions* I, II *and* III, *the random dynamical system* Φ *generated by the damped stochastic wave equation* (1.1) *on the Banach space* $E = (H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)) \times L^2(\mathbb{R}^n)$ *over the parametric dynamical space* $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ *has a unique* $\mathscr{D}$*-pullback random attractor* $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$.

PROOF. Lemma 3.1 shows that there exists a $\mathscr{D}$-pullback random absorbing set, the $K = \{B_E(0, R(\omega))\}_{\omega\in\Omega}$ in the space E for the cocycle Φ. Thus it suffices to prove that the cocycle Φ is $\mathscr{D}$-pullback asymptotically compact in E.

(1) Theorem 4.3 shows that for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ and any sequence

$$\{\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\}_{m=1}^{\infty},$$

where $t_m \to \infty$ and $g_{0,m} = (u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$, along a pullback quasi-trajectory of the cocycle Φ has a subsequence, which is denoted by the same, such that

$$\Phi(t_m, \theta_{-t_m}\omega, g_{0,m}) \longrightarrow (\tilde{u}(\omega), \tilde{v}(\omega)) \quad \text{strongly in } H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \tag{5.14}$$

and consequently

$$\mathbb{P}_u \, \Phi(t_m, \theta_{-t_m}\omega, g_{0,m}) \longrightarrow \tilde{u}(\omega) \quad \text{strongly in } L^2(\mathbb{R}^n). \tag{5.15}$$

Here $\mathbb{P}_u : (u, v) \mapsto u$ is the projection.

(2) Apply the first item in Lemma 4.1 to $L^2(\mathbb{R}^n)$. It follows from (5.14) that there exists a subsequence $\{\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})\}_{k=1}^{\infty}$ of $\{\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\}_{m=1}^{\infty}$, such that

$$\lim_{k\to\infty} \Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x) = (\tilde{u}(\omega)(x), \tilde{v}(\omega)(x)), \text{ a.e. in } \mathbb{R}^n. \tag{5.16}$$

Hence we have

$$\lim_{k\to\infty} \mathbb{P}_u\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x) = \tilde{u}(\omega)(x), \text{ a.e. in } \mathbb{R}^n. \tag{5.17}$$

Therefore, the assumption (5.1) in Theorem 5.1 is satisfied by the sequence of functions $\{\mathbb{P}_u\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)\}_{k=1}^{\infty}$ in $L^p(\mathbb{R}^n)$.

(3) By Lemma 3.2, for a.e. $\omega \in \Omega$ and any $\varepsilon > 0$, there exists an integer $k_0 = k_0(B, \omega, \varepsilon) > 0$ and $V = V(\omega, \varepsilon) \geq 1$ such that for all $k > k_0$ one has

$$\int_{\mathbb{R}^n\setminus B_V} |\mathbb{P}_u\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)|^p\,dx \leq |||\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})|||^p_{E(\mathbb{R}^n\setminus B_V)} < \varepsilon, \tag{5.18}$$

for any $g_{0,m_k} \in B(\theta_{-t_{m_k}}\omega)$, where B_V is the ball centered at the origin with radius V in $\mathbb{R}^n$. Then there exists $V_0 = V_0(\omega, \varepsilon) > 0$ such that

$$\int_{\mathbb{R}^n\setminus B_{V_0}} |\mathbb{P}_u\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)|^p\,dx < \varepsilon, \quad \text{for } k = 1, \cdots, k_0. \tag{5.19}$$

Here (5.18) and (5.19) confirm that with $A_\varepsilon = B_{\max\{V,V_0\}}$ in (5.3) of the condition (a) in Theorem 5.1 is satisfied by the sequence $\{\mathbb{P}_u\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)\}_{k=1}^{\infty}$ in $L^p(\mathbb{R}^n)$.

(4) Finally we show that the uniform absolutely continuous condition (b) of Theorem 5.1 is satisfied by the sequence of functions $\{\mathbb{P}_u\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)\}_{k=1}^{\infty}$ in $L^p(\mathbb{R}^n)$.

According to the Assumption II, for any measurable set $Y \subset \mathbb{R}^n$, we have

$$C_3 \int_Y |u|^p\,dx \leq \int_Y (F(x,u) + \phi_3(x))\,dx \leq Q_Y(u,v), \quad \text{for } (u,v) \in E,$$

where $Q_Y(u,v)$ is analogous to (4.5) and defined by

$$Q_Y(u,v) = \|v\|^2_{L^2(Y)} + \left(\alpha + \delta^2 - \beta\delta\right)\|u\|^2_{L^2(Y)} + \|\nabla u\|^2_{L^2(Y)} + 2\int_Y (F(x,u) + \phi_3(x))dx. \tag{5.20}$$

We integrate the inequality (3.17) over the time interval $[-t_m, 0]$ to get

$$\begin{aligned} &Q_Y(u(0,\omega,-t_m,u_{0,m}), v(0,\omega,-t_m,v_{0,m})) \\ &\leq e^{-\sigma t_m} Q_Y((u_{0,m}, v_{0,m})) + \int_{-t_m}^0 e^{\sigma t}\left(2\Gamma_1^Y(\theta_t\omega) + C_6(Y) + \frac{\|g\|^2_{L^2(Y)}}{\beta - \delta}\right) dt, \end{aligned} \tag{5.21}$$

where, in view of (3.14) and the set-up of the constants C_4 and C_6 in Section 3.1, we have

$$\Gamma_1^Y(\theta_t\omega) = C_0\left(\|z(\theta_t\omega)\|^2_{L^2(Y)} + \|\nabla z(\theta_t\omega)\|^2_{L^2(Y)} + \|z(\theta_t\omega)\|^p_{L^p(Y)}\right)$$

and

(5.22)

$$C_6(Y) = 2\left(\delta C_2 - \frac{\varepsilon C_1(p-1)}{C_3 p}\right)\|\phi_3\|_{L^1(Y)} + \varepsilon\|\phi_1\|^2_{L^2(Y)} - \delta\|\phi_2\|_{L^1(Y)} + \frac{\varepsilon C_1(p-1)}{C_3 p}\|\phi_3\|_{L^1(Y)} \leq \varepsilon\|\phi_1\|^2_{L^2(Y)} + 2\delta C_2\|\phi_3\|_{L^1(Y)}.$$

Note that $z(\theta_t\omega) = \sum_{j=1}^m h_j(x)z_j(\theta_t\omega_j)$. By (2.8), we obtain

$$\begin{aligned}\Gamma_1^Y(\theta_t\omega) &= C\max_{1\leq j\leq m}\left\{\|h_j\|^2_{H^1(Y)}, \|h_j\|^p_{L^p(Y)}\right\}\sum_{j=1}^m\left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^p\right)\\ &\leq C\max_{1\leq j\leq m}\left\{\|h_j\|^2_{H^1(Y)}, \|h_j\|^p_{L^p(Y)}\right\}e^{\frac{\sigma}{2}|t|}r_0(\omega),\end{aligned} \tag{5.23}$$

where $C>0$ is a constant.

Substitute the expression of $Q_Y((u_{0,m}, v_{0,m}))$ for $(u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$ and (5.22), (5.23) into the inequality (5.21). Since (2.3)-(2.4) yield

$$\int_Y (F(x,u) + \phi_3(x))\,dx \leq \frac{1}{C_2}\left[C_1\|u\|^p_{L^p(Y)} + \|u\|^2_{L^2(Y)} + \|\phi_1\|^2_{L^2(Y)} + \|\phi_2\|_{L^1(Y)}\right],$$

for every $\omega\in\Omega$ and $B\in\mathscr{D}$ and any $g_{0,m} = (u_{0,m}, v_{0,m}) \in B(\theta_{-t_m}\omega)$ it holds that

(5.24)

$$\begin{aligned}&C_3\int_Y |u(0,\omega,-t_m,u_{0,m})|^p\,dx \leq Q_Y(u(0,\omega,-t_m,u_{0,m}), v(0,\omega,-t_m,v_{0,m}))\\ &\leq e^{-\sigma t_m}\left[\|v_{0,m}\|^2_{L^2(Y)} + (\alpha+\delta^2-\beta\delta)\|u_{0,m}\|^2_{L^2(Y)} + \|\nabla u_{0,m}\|^2_{L^2(Y)}\right]\\ &+ e^{-\sigma t_m}\frac{1}{C_2}\left[C_1\|u_{0,m}\|^p_{L^p(Y)} + \|u_{0,m}\|^2_{L^2(Y)} + \|\phi_1\|^2_{L^2(Y)} + \|\phi_2\|_{L^1(Y)}\right]\\ &+ \int_{-t_m}^0 2e^{\sigma t}C\max_{1\leq j\leq m}\left\{\|h_j\|^2_{H^1(Y)}, \|h_j\|^p_{L^p(Y)}\right\}e^{-\frac{\sigma}{2}t}r_0(\omega)\,dt\\ &+ (\varepsilon+2\delta C_2)\int_{-t_m}^0 e^{\sigma t}\left(\|\phi_1\|^2_{L^2(Y)} + \|\phi_3\|_{L^1(Y)}\right)dt + \int_{-t_m}^0 \frac{e^{\sigma t}}{\beta-\delta}\|g\|^2_{L^2(Y)}\,dt.\end{aligned}$$

Due to the absolute continuity of the respective Lebesgue integrals of the functions $\phi_1(x), \phi_2(x), \phi_3(x), h_j(x), j = 1,\cdots,m$, and g involved in the above inequality (5.24), for an arbitrarily given $\eta>0$, there exists $\mu_0 = \mu_0(\omega,\eta) > 0$ such that for any measurable set $Y\subset\mathbb{R}^n$ with $\mu(Y)<\mu_0$ one has

(5.25)

$$\begin{aligned}&e^{-\sigma t_m}\frac{1}{C_2}\left(\|\phi_1\|^2_{L^2(Y)} + \|\phi_2\|_{L^1(Y)}\right)\\ &+ \int_{-t_m}^0 2e^{\sigma t}C\max_{1\leq j\leq m}\left\{\|h_j\|^2_{H^1(Y)}, \|h_j\|^p_{L^p(Y)}\right\}e^{-\frac{\sigma}{2}t}r_0(\omega)\,dt\\ &+ (\varepsilon+2\delta C_2)\int_{-t_m}^0 e^{\sigma t}\left(\|\phi_1\|^2_{L^2(Y)} + \|\phi_3\|_{L^1(Y)}\right)dt + \int_{-t_m}^0\frac{e^{\sigma t}}{\beta-\delta}\|g\|^2_{L^2(Y)}\,dt\\ &\leq \frac{1}{C_2}\left(\|\phi_1\|^2_{L^2(Y)} + \|\phi_2\|_{L^1(Y)}\right) + \frac{4C}{\sigma}r_0(\omega)\max_{1\leq j\leq m}\left\{\|h_j\|^2_{H^1(Y)}, \|h_j\|^p_{L^p(Y)}\right\}\\ &+ \frac{1}{\sigma}(\varepsilon+2\delta C_2)\left(\|\phi_1\|^2_{L^2(Y)} + \|\phi_3\|_{L^1(Y)}\right) + \frac{1}{\sigma(\beta-\delta)}\|g\|^2_{L^2(Y)} < \frac{\eta}{2}.\end{aligned}$$

Moreover, since it has been specified in the beginning of Section 3.1 that the universe $\mathscr{D} = \mathscr{D}_E$ and here $B \in \mathscr{D}$, there exists a constant $C^* > 0$ such that

$$\begin{aligned} & e^{-\sigma t_m} \left[\|v_{0,m}\|^2_{L^2(Y)} + (\alpha + \delta^2 - \beta\delta)\|u_{0,m}\|^2_{L^2(Y)} + \|\nabla u_{0,m}\|^2_{L^2(Y)} \right] \\ & + \frac{1}{C_2} e^{-\sigma t_m} \left[C_1 \|u_{0,m}\|^p_{L^p(Y)} + \|u_{0,m}\|^2_{L^2(Y)} \right] \\ \le & \, e^{-\sigma t_m} C^* \left(\|B(\theta_{-t_m}\omega)\|^2_{E(Y)} + \|B(\theta_{-t_m}\omega)\|^p_{E(Y)} \right), \end{aligned}$$

where $\|B(\theta_{-t_m}\omega)\|_{E(Y)} = \max_{g_0 \in B(\theta_{-t_m}\omega)} \|g_0\, \zeta_Y\|_E$ with ζ_Y being the characteristic function for the set Y. Since $\lim_{t\to\infty} e^{-\sigma t}\|B(\theta_{-t}\omega)\|_E = 0$, for the aforementioned arbitrary $\eta > 0$ there exists an integer $m_0 = m_0(B, \omega, \eta) \ge 1$ such that

$$\begin{aligned} & e^{-\sigma t_m} C^* \left(\|B(\theta_{-t_m}\omega)\|^2_{E(Y)} + \|B(\theta_{-t_m}\omega)\|^p_{E(Y)} \right) \\ \le & \, e^{-\sigma t_m} C^* \left(\|B(\theta_{-t_m}\omega)\|^2_{E} + \|B(\theta_{-t_m}\omega)\|^p_{E} \right) < \frac{\eta}{2}, \quad \text{for all } m > m_0. \end{aligned} \tag{5.26}$$

Then there esists $\mu_1 = \mu_1(B, \omega, m_0, \eta) > 0$ such that for any set Y with $\mu(Y) < \mu_1$ one has

$$e^{-\sigma t_j} C^* \left(\|B(\theta_{-t_j}\omega)\|^2_{E(Y)} + \|B(\theta_{-t_j}\omega)\|^p_{E(Y)} \right) < \frac{\eta}{2}, \quad j = 1, \cdots, m_0. \tag{5.27}$$

Put together (5.24), (5.25), (5.26) and (5.27). It shows that, for every $\omega \in \Omega$, whenever a measurable set $Y \subset \mathbb{R}^n$ satisfies $\mu(Y) < \min\{\mu_0, \mu_1\}$ one has

$$C_3 \int_Y |u(0, \omega, -t_m, u_{0,m})|^p \, dx \le \frac{\eta}{2} + \frac{\eta}{2} = \eta, \quad \text{for all } m \ge 1. \tag{5.28}$$

Therefore,

$$\lim_{\mu(Y)\to 0} \int_Y |\mathbb{P}_u \Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m})(x)|^p \, dx = 0, \quad \text{uniformly in } k \ge 1, \tag{5.29}$$

so that the condition (b) of Theorem 5.1 is satisfied by the sequence of functions $\{\mathbb{P}_u \Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)\}_{k=1}^\infty$ in $L^p(\mathbb{R}^n)$.

As checked by the steps (2), (3) and (4) in this proof, all the conditions in Theorem 5.1 are satisfied by the sequence of functions $\{\mathbb{P}_u \Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})(x)\}_{k=1}^\infty$ in $L^p(\mathbb{R}^n)$. Thus we apply Theorem 5.1 to obtain

$$\lim_{k\to\infty} \mathbb{P}_u \Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k}) = \tilde{u}(\omega), \text{ strongly in } L^p(\mathbb{R}^n). \tag{5.30}$$

Finally, combination of (5.14) and (5.30) shows that there exists a convergent subsequence $\{\Phi(t_{m_k}, \theta_{-t_{m_k}}\omega, g_{0,m_k})\}_{k=1}^\infty$ of the sequence $\{\Phi(t_m, \theta_{-t_m}\omega, g_{0,m})\}_{m=1}^\infty$ in the space $E = (H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)) \times L^2(\mathbb{R}^n)$. Therefore, the random dynamical system Φ on E is $\mathscr{D}$-pullback asymptotically compact.

According to Theorem 2.6, we conclude that there exists a $\mathscr{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathscr{D}$ for this random dynamical system Φ on E generated by the original stochastic damped wave equation (1.1). The proof is completed. □

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College of Management, Shanghai University of Engineering Science, Shanghai 201620, China

E-mail address: hongyanlishu@163.com

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

E-mail address: you@mail.usf.edu