ON THE COBAR CONSTRUCTION OF A BIALGEBRA

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Abstract

We show that the cobar construction of a DG-bialgebra is a homotopy G-algebra. This implies that the bar construction of this cobar is a DG-bialgebra as well.

1. Introduction

The cobar construction $\Omega C$ of a DG-coalgebra $(C, d : C \to C, \Delta : C \to C \otimes C)$ is, by definition, a DG-algebra. Suppose now that $C$ is additionally equipped with a multiplication $\mu : C \otimes C \to C$ turning $(C, d, \Delta, \mu)$ into a DG-bialgebra. How does this multiplication reflect on the cobar construction $\Omega C$? It was shown by Adams [1] that in the mod 2 situation in this case, the multiplication of $\Omega C$ is homotopy commutative: there exists a $\grave{\smile}_1$ product $\grave{\smile}_1 : \Omega C \otimes \Omega C \to \Omega C$ which satisfies the standard condition

$$d(a \grave{\smile}_1 b) = da \grave{\smile}_1 b + a \grave{\smile}_1 db + a \cdot b + b \cdot a,$$

(since we work mod 2 the signs are ignored in the whole paper). In this note we show that this $\grave{\smile}_1$ gives rise to a sequence of operations $E_{1,k} : \Omega C \otimes (\Omega C)^{\otimes k} \to \Omega C$, $k = 1, 2, 3, ...$ which form on the cobar construction $\Omega C$ of a DG-bialgebra, a structure of homotopy G-algebra (hGa) in the sense of Gerstenhaber and Voronov [8].

There are two remarkable examples of homotopy G-algebras. The first one is the cochain complex of a 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod’s $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(U, U)$ of an associative algebra $U$. The operations $E_{1,k}$ here were defined in [11] with the purpose of describing $A(\infty)$-algebras in terms of Hochschild cochains although the properties of those operations which were used as defining ones for the notion of homotopy G-algebra in [8] did not appear there. These operations were defined also in [9]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber’s circle product which is sort of a $\smile_1$-product in the Hochschild complex.

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In this paper we present a third example of a homotopy G-algebra: we construct the operations $E_{1,k}$ on the cobar construction $\Omega C$ of a DG-bialgebra $C$, and the starting operation $E_{1,1}$ is again classical, it is Adams’s $\sim_1$-product.

The notion of hGa was introduced in [8] as an additional structure on a DG-algebra $(A,d,\cdot)$ that induces a Gerstenhaber algebra structure on homology. The source of the defining identities and the main example was the Hochschild cochain complex $C^\ast(U,U)$. Another point of view is that hGa is a particular case of $B(\infty)$-algebra. This is an additional structure on a DG-algebra $(A,d,\cdot)$ that induces a DG-bialgebra structure on the bar construction $BA$.

We emphasize the third aspect of hGa: this is a structure which measures the noncommutativity of $A$. There exists the classical tool which measures the noncommutativity of a DG-algebra $(A,d,\cdot)$, namely the Steenrod’s $\sim_1$ product, satisfying the condition (1). The existence of such $\sim_1$ guarantees the commutativity of $H(A)$, but the $\sim_1$ product satisfying just the condition (1) is too poor for most applications. In many constructions some deeper properties of $\sim_1$ are needed, for example the compatibility with the dot product of $A$ (the Hirsch formula)

$$(a \cdot b) \sim_1 c + a \cdot (b \sim_1 c) + (a \sim_1 c) \cdot b = 0. \tag{2}$$

For a hGa $(A,d,\cdot,\{E_{1,k}\})$ the starting operation $E_{1,1}$ is a kind of $\sim_1$ product: it satisfies the conditions (1) and (2). As for the symmetric expression

$$a \sim_1 (b \cdot c) + b \cdot (a \sim_1 c) + (a \sim_1 b) \cdot c,$$

it is just homotopical to zero and the appropriate homotopy is the operation $E_{1,2}$. The defining conditions of a hGa which satisfy higher operations $E_{1,k}$ can be regarded as generalized Hirsch formulas. So we can say that a hGa is a DG-algebra with a ”good” $\sim_1$ product.

2. Notation and preliminaries

We work over $\mathbb{Z}_2$. For a graded $\mathbb{Z}_2$-module $M$ we denote by $sM$ the suspension of $M$, i.e. $(sM)^i = M^{i+1}$. Respectively $s^{-1}M$ denotes the desuspension of $M$, i.e. $(s^{-1}M)^i = M^{i-1}$. A differential graded algebra (DG-algebra) is a graded $R$-module $C = \{C^i\}, i \in \mathbb{Z}$, with an associative multiplication $\mu : C^i \otimes C^j \to C^{i+j}$ and a homomorphism (a differential) $d : C^i \to C^{i+1}$ with $d^2 = 0$ and satisfying the Leibniz rule $d(x \cdot y) = dx \cdot y + x \cdot dy$, where $x \cdot y = \mu(x \otimes y)$. We assume that a DG-algebra contains a unit $1 \in C^0$. A non-negatively graded DG-algebra $C$ is connected if $C^0 = \mathbb{Z}_2$. A connected DG-algebra $C$ is n-reduced if $C^i = 0, 1 \leqslant i \leqslant n$. A DG-algebra is commutative if $\mu = \mu T$, where $T(x \otimes y) = y \otimes x$.

A differential graded coalgebra (DG-coalgebra) is a graded $\mathbb{Z}_2$-module $C = \{C_i\}, i \in \mathbb{Z}$, with a coassociative comultiplication $\Delta : C \to C \otimes C$ and a homomorphism (a differential) $\delta : C_i \to C_{i+1}$ with $\delta^2 = 0$ and satisfying $\Delta \delta = (d \otimes id + id \otimes d)\Delta$. A DG-coalgebra $C$ is assumed to have a counit $\epsilon : C \to Z_2$, $(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id$. A non-negatively graded dgc $C$ is connected if $C_0 = \mathbb{Z}_2$. A connected DG-coalgebra $C$ is n-reduced if $C_i = 0, 1 \leqslant i \leqslant n$. A differential graded bialgebra (DG-bialgebra) $(C,d,\mu,\Delta)$ is a DG-coalgebra $(C,d,\Delta)$ with a morphism of DG-coalgebras $\mu : C \otimes C \to C$ turning $(C,d,\mu)$ into a DG-algebra.
2.1. Cobar and Bar constructions

Let $M$ be a graded $\mathbb{Z}_2$-vector space with $M^{i \leq 0} = 0$ and let $T(M)$ be the tensor algebra of $M$, i.e. $T(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$.

$T(M)$ is a free graded algebra: for a graded algebra $A$ and a homomorphism $\alpha : M \rightarrow A$ of degree zero there exists its multiplicative extension, a unique morphism of graded algebras $f_\alpha : T(M) \rightarrow A$ such that $f_\alpha(a) = \alpha(a)$. The map $f_\alpha$ is given by $f_\alpha(a_1 \otimes \ldots \otimes a_n) = \alpha(a_1) \cdot \ldots \cdot \alpha(a_n)$. Dually, let $T^c(M)$ be the tensor coalgebra of $M$, i.e. $T^c(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$, and the comultiplication $\nabla : T^c(M) \rightarrow T^c(M) \otimes T^c(M)$ is given by

$$\nabla(a_1 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \ldots \otimes a_k) \otimes (a_{k+1} \otimes \ldots \otimes a_n).$$

$(T^c(M), \nabla)$ is a cofree graded coalgebra: for a graded coalgebra $C$ and a homomorphism $\beta : C \rightarrow M$ of degree zero there exists its comultiplicative extension, a unique morphism of graded coalgebras $g_\beta : C \rightarrow T^c(M)$ such that $p_1g_\beta = \beta$, here $p_1 : T^c(M) \rightarrow M$ is the clear projection. The map $g_\beta$ is given by

$$g_\beta(c) = \sum_{n} \beta(c^{(1)}) \otimes \ldots \otimes \beta(c^{(n)}),$$

where $\Delta^n(c) = c^{(1)} \otimes \ldots \otimes c^{(n)}$ and $\Delta^n : C \rightarrow C^{\otimes n}$ is $n$-th iteration of the diagonal $\Delta : C \rightarrow C \otimes C$, i.e. $\Delta^1 = id$, $\Delta^2 = \Delta$, $\Delta^n = (\Delta^{n-1} \otimes id)\Delta$.

Let $(C, d, \Delta)$ be a connected DG-coalgebra and $\Delta = id \otimes 1 + 1 \otimes id + \Delta'$. The (reduced) cobar construction $\Omega C$ on $C$ is a DG-algebra whose underlying graded algebra is $T(sC^{>0})$. An element $(s_{c_1} \otimes \ldots \otimes s_{c_n}) \in (sC)^{\otimes n} \subset T(sC^{>0})$ is denoted by $[c_1, \ldots, c_n] \in \Omega C$. The differential on $\Omega C$ is the sum $d = d_1 + d_2$ which for a generator $[c] \in \Omega C$ is defined by $d_1[c] = [d_C(c)]$ and $d_2[c] = \sum c' \otimes c''$ for $\Delta'(c) = \sum c' \otimes c''$, and extended as a derivation. Let $(A, d_A, \mu)$ be a 1-reduced DG-algebra. The (reduced) bar construction $BA$ on $A$ is a DG-coalgebra whose underlying graded coalgebra is $T^c(s^{-1}A^{>0})$. Again an element $(s^{-1}a_1 \otimes \ldots \otimes s^{-1}a_n) \in (s^{-1}A)^{\otimes n} \subset T^c(s^{-1}A^{>0})$ we denote as $[a_1, \ldots, a_n] \in BA$. The differential of $BA$ is the sum $d = d_1 + d_2$ which for an element $[a_1, \ldots, a_n] \in BA$ is defined by

$$d_1[a_1, \ldots, a_n] = \sum_{i=1}^{n} [a_1, \ldots, d_Aa_i, \ldots, a_n], d_2[a_1, \ldots, a_n] = \sum_{i=1}^{n-1} [a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_n].$$

2.2. Twisting cochains

Let $(C, d, \Delta)$ be a dgc, $(A, d, \mu)$ a dga. A twisting cochain $[5]$ is a homomorphism $\tau : C \rightarrow A$ of degree +1 satisfying the Browns’ condition

$$d\tau + \tau d = \tau - \tau,$$

where $\tau \sim \tau' = \mu_A(\tau \otimes \tau')\Delta$. We denote by $T(C, A)$ the set of all twisting cochains $\tau : C \rightarrow A$.

There are universal twisting cochains $C \rightarrow \Omega C$ and $BA \rightarrow A$ being clear inclusion and projection respectively. Here are essential consequences of the condition (3):

(i) The multiplicative extension $f_\tau : \Omega C \rightarrow A$ is a map of DG-algebras, so there is a bijection $T(C, A) \leftrightarrow \text{Hom}_{DG-\text{Alg}}(\Omega C, A)$;
(ii) The comultiplicative extension $g_{τ} : C \to BA$ is a map of DG-coalgebras, so there is a bijection $T(C, A) \leftrightarrow \text{Hom}_{\text{DG-Coalg}}(C, BA)$.

3. Homotopy G-algebras

3.1. Products in the bar construction

Let $(A, d, \cdot)$ be a 1-reduced DG-algebra and $BA$ its bar construction. We are interested in the structure of a multiplication

$$\mu : BA \otimes BA \to BA,$$

turning $BA$ into a DG-bialgebra, i.e. we require that

(i) $\mu$ is a DG-coalgebra map;

(ii) is associative;

(iii) has the unit element $1_A \in \Lambda \subset BA$.

Because of the cofreeness of the tensor coalgebra $BA = Tc(s^{-1}A)$, a map of graded coalgebras $\mu : BA \otimes BA \to BA$ is uniquely determined by the projection of degree +1 $E = pr \cdot \mu : BA \otimes BA \to BA \to A$.

Conversely, a homomorphism $E : BA \otimes BA \to A$ of degree +1 determines its coextension, a graded coalgebra map $\mu_E : BA \otimes BA \to BA$ given by

$$\mu_E = \sum_{k=0}^{\infty} (E \otimes \ldots \otimes E) \nabla_{BA \otimes BA}^k,$$

where $\nabla_{BA \otimes BA}^k : BA \otimes BA \to (BA \otimes BA)^{\otimes k}$ is the k-fold iteration of the standard coproduct of tensor product of coalgebras

$$\nabla_{BA \otimes BA} = (id \otimes T \otimes id)(\nabla \otimes \nabla) : BA \otimes BA \to (BA \otimes BA)^{\otimes 2}.$$

The map $\mu_E$ is a chain map (i.e. it is a map of DG-coalgebras) if and only if it is a twisting cochain in the sense of E. Brown, i.e. satisfies the condition

$$dE + Ed_{BA \otimes BA} = E \cdot E.$$  \hfill (4)

Indeed, again because of the cofreeness of the tensor coalgebra $BA = Tc(s^{-1}A)$ the condition $d_{BA\mu_E} = \mu_E d_{BA \otimes BA}$ is satisfied if and only if it is satisfied after the projection on $A$, i.e. if $pr \cdot d_{BA} \mu_E = pr \cdot \mu_E d_{BA \otimes BA}$ but this condition is nothing else than the Brown’s condition (4).

The same argument shows that the product $\mu_E$ is associative if and only if $pr \cdot \mu_E (\mu_E \otimes id) = pr \cdot \mu_E (id \otimes \mu_E)$, or, having in mind $E = pr \cdot \mu_E$

$$E(\mu_E \otimes id) = E(id \otimes \mu_E).$$  \hfill (5)

A homomorphism $E : BA \otimes BA \to A$ consists of components

$$\{E_{p,q} : (s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q} \to A, \ p, q = 0, 1, 2, \ldots \},$$
where $E_{pq}$ is the restriction of $E$ on $(s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q}$. Each component $E_{p,q}$ can be regarded as an operation

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A, \ p, q = 0, 1, 2, \ldots$$

The value of $E_{p,q}$ on the element $(a_1 \otimes \ldots \otimes a_p) \otimes (b_1 \otimes \ldots \otimes b_q)$ we denote by $E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q)$.

It is not hard to check that the multiplication $\mu_E$ induced by $E$ (or equivalently by a collection of multioperations \{E_{p,q}\}) has the unit $1_A \in A \subseteq BA$ if and only if

$$E_{0,1} = E_{1,0} = \text{id}; \ E_{0,k} = E_{k,0} = 0, \ k > 1. \quad (6)$$

So we can summarize:

**Proposition 1.** The multiplication $\mu_E$ induced by a collection of multioperations \{E_{p,q}\} turns $BA$ into a DG-bialgebra, i.e. satisfies (i-iii), if and only if the conditions (4), (5), and (6) are satisfied.

Let us interpret the condition (4) in terms of the components $E_{pq}$. The restriction of (4) on $A \otimes A$ gives

$$dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a. \quad (7)$$

This condition coincides with the condition (1), i.e. the operation $E_{1,1}$ is sort of a $\sim_1$ product, which measures the noncommutativity of $A$. Below we denote $E_{1,1}(a; b) = a \sim_1 b$.

The restriction on $A^{\otimes 2} \otimes A$ gives

$$dE_{2,1}(a, b; c) + E_{2,1}(da, b; c) + E_{2,1}(a, db; c) + E_{2,1}(a, b; dc) = (a \cdot b) \sim_1 c + a \cdot (b \sim_1 c) + (a \sim_1 c) \cdot b, \quad (8)$$

this means, that this $\sim_1$ satisfies the left Hirsch formula (2) up to homotopy and the appropriate homotopy is the operation $E_{2,1}$.

The restriction on $A \otimes A^{\otimes 2}$ gives:

$$dE_{1,2}(a; b, c) + E_{1,2}(da; b, c) + E_{1,2}(a; db, c) + E_{1,2}(a; b, dc) =$$

$$a \sim_1 (b \cdot c) + (a \sim_1 b) \cdot c + b \cdot (a \sim_1 c). \quad (9)$$

this means, that this $\sim_1$ satisfies the right Hirsch formula (2) up to homotopy and the appropriate homotopy is the operation $E_{1,2}$.

Generally the restriction of (4) on $A^{\otimes m} \otimes A^{\otimes n}$ gives:

$$dE_{m,n}(a_1, \ldots, a_m; b_1, \ldots, b_n) + \sum_i E_{m,n}(a_1, \ldots, da_i, \ldots, a_m; b_1, \ldots, b_n) + \sum_i E_{m,n}(a_1, \ldots, a_m; b_1, \ldots, db_i, \ldots, b_n) =$$

$$a_1 \cdot E_{m-1,n}(a_2, \ldots, a_m; b_1, \ldots, b_n) + \cdots + b_1 \cdot E_{m,n-1}(a_1, \ldots, a_m; b_2, \ldots, b_n) + E_{m,n-1}(a_1, \ldots, a_m; b_1, \ldots, b_{n-1}) \cdot b_m +$$

$$\sum_{i=1}^{m-1} E_{m,n-1}(a_1, \ldots, a_i; a_{i+1}, \ldots, a_m; b_1, \ldots, b_{n-1}) +$$

$$\sum_{p=1}^{m-1} \sum_{q=1}^{n-1} E_{p,q}(a_1, \ldots, a_p; b_1, \ldots, b_q) \cdot E_{m-p,n-q}(a_{p+1}, \ldots, a_m; b_{q+1}, \ldots, b_n). \quad (10)$$

Now let us interpret the associativity condition (5) in terms of the components
The restriction of (5) on $A \otimes A \otimes A$ gives
\[
(a \sim_1 b) \sim_1 c + a \sim_1 (b \sim_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b) + E_{2,1}(a; b; c) + E_{2,1}(b; a; c).
\] (11)

Generally the restriction of (5) on $A^k \otimes A^l \otimes A^m$ gives
\[
\sum_{r=1}^{l+m} \sum_{i_1+\ldots+i_r = l, m_1+\ldots+m_r = m} E_{k,r}(a_1, \ldots, a_k; E_{1,m_1}(b_1, \ldots, b_{l_1}; c_1, \ldots, c_{m_1}), \ldots,
E_{l,r,m_1}(b_1, \ldots, b_{l_1}; c_1+\ldots+m_{r-1} + 1, \ldots, b_l; c_{m_1}+\ldots+m_{r-1} + 1, \ldots, c_m)
\sum_{s+r+1}^{k+l} \sum_{k_1+\ldots+k_r = k, l_1+\ldots+l_s = l} E_{s,m}(E_{k_1,l_1}(a_1, \ldots, a_{k_1}; b_1, \ldots, b_{l_1}), \ldots,
E_{k_r,l_s}(a_{k_1+\ldots+k_{r-1} + 1}, \ldots, a_k; b_1+\ldots+l_s - 1 + 1, \ldots, b_l; c_1, \ldots, c_m)
\]
\] (12)

We define a Hirsch algebra as a DG-algebra $(A, d, \cdot)$ endowed with a sequence of multoperations $\{E_{p,q}\}$ satisfying (6), (10). This name is inspired by the fact that the defining condition (10) can be regarded as generalizations of classical Hirsch formula (2). This notion was used in [12], [13].

A Hirsch algebra we call associative if in addition the condition (12) is satisfied. This structure is a particular case of a $B_\infty$-algebra, see below. Moreover, the notion of homotopy $G$-algebra, described below, is a particular case of an associative Hirsch algebra.

### 3.2. Some particular cases

For a Hirsch algebra $(A, d, \cdot, \{E_{p,q}\})$ the operation $E_{1,1} = \sim_1$ satisfies (1), so this structure can be considered as a tool which measures the noncommutativity of the product $a \cdot b$ of $A$. We distinguish various levels of "noncommutativity" of $A$ according to the form of $\{E_{p,q}\}$.

**Level 1.** Suppose for the collection $\{E_{p,q}\}$ all the operations except $E_{0,1} = id$ and $E_{1,0} = id$ are trivial. Then it follows from (7) that in this case $A$ is a strictly commutative DG-algebra.

**Level 2.** Suppose all operations except $E_{0,1} = id$, $E_{1,0} = id$ and $E_{1,1}$ are trivial. In this case $A$ is endowed with a "strict" $\sim_1$ product $a \sim_1 b = E_{1,1}(a; b)$: the condition (10) here degenerate to the following 4 conditions
\[
d(a \sim_1 b) = da \sim_1 b + a \sim_1 db + a \cdot b + b \cdot a,
(a \cdot b) \sim_1 c + a \cdot (b \sim_1 c) + (a \sim_1 c) \cdot b = 0,
a \sim_1 (b \cdot c) + b \cdot (a \sim_1 c) + (a \sim_1 b) \cdot c = 0,
(a \sim_1 c) \cdot (b \sim_1 d) = 0.
\]
The condition (12) degenerates to the associativity $\sim_1$
\[
a \sim_1 (b \sim_1 c) = (a \sim_1 b) \sim_1 c.
\]
As we see in this case we have very strong restrictions on the $\sim_1$-product. An example of a DG-algebra with such strict $\sim_1$ product is $(H^*(SX, Z_2), d = 0)$ with
a \sim_1 b = 0 \text{ if } a \neq b \text{ and } a \sim_1 a = Sq^{|a|-1}a; \text{ another example is } C^{\ast}(SX, CX), \text{ where } SX \text{ is the suspension and } CX \text{ is the cone of a space } X (\text{see [18]}).

**Level 3.** Suppose all operations except } E_{0,1} = id, E_{1,0} = id \text{ and } E_{1,k}, k = 1, 2, 3, \ldots \text{ are trivial. In this case the condition (10) degenerates into two conditions: at } A \otimes A \otimes k

\begin{align}
    dE_{1,k}(a; b_1, \ldots, b_k) + E_{1,k}(da; b_1, \ldots, b_k) + \sum_i E_{1,k-1}(a; b_1, \ldots, db_i, \ldots, b_k) &= \\
    b_1 \cdot E_{1,k-1}(a; b_2, \ldots, b_k) + E_{1,k-1}(a; b_1, \ldots, b_{k-1}) \cdot b_k &+ \\
    \sum_i E_{1,k-1}(a; b_1, \ldots, b_i \cdot b_{i+1}, \ldots, b_k),
\end{align}

\text{and at } A^{\otimes 2} \otimes A^{\otimes k}

\begin{align}
    E_{1,k}(a_1 \cdot a_2; b_1, \ldots, b_k) &= a_1 \cdot E_{1,k}(a_2; b_1, \ldots, b_k) + E_{1,k}(a_1; b_1, \ldots, b_k) \cdot a_2 + \\
    \sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, \ldots, b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, \ldots, b_k);
\end{align}

\text{moreover at } A^{\otimes n > 2} \otimes A^{\otimes k} \text{ the condition is trivial. In particular the condition (8) here degenerates to Hirsch formula (2).}

The associativity condition (12) in this case looks like

\begin{align}
    E_n(E_{1,m}(a; b_1, \ldots, b_m); c_1, \ldots, c_n) &= \sum_{0 \leq i_1 \leq \ldots \leq i_m \leq n} \sum_{0 \leq n_1 + \ldots + n_r \leq n} c_{i_2, E_{1,n_1}(b_1; c_{i_1+1}, \ldots, c_{i_1+n_1}), c_{i_1+n_1+1}, \ldots, \\
    &c_{i_m, E_{1,n_m}(b_m; c_{i_m+1}, \ldots, c_{i_m+n_m}), c_{i_m+n_m+1}, \ldots, c_n)},
\end{align}

\text{In particular the condition (11) here degenerates to}

\begin{align}
    (a \sim_1 b) \sim_1 c + a \sim_1 (b \sim_1 c) &= E_{1,2}(a; b, c) + E_{1,2}(a; c, b).
\end{align}

\text{The structure of this level coincides with the notion of Homotopy G-algebra, see below.}

**Level 4.** As the last level we consider a Hirsch algebra structure with no restrictions. An example of such structure is the cochain complex of a 1-reduced cubical set. Note that it is a non-associative Hirsch algebra.

**3.3. B\infty-algebra**

The notion of a $B_{\infty}$-algebra was introduced in [2], [10] as an additional structure on a DG-algebra $(A, \cdot, d)$ which turns the tensor coalgebra $T^{\ast}(s^{-1}A) = BA$ into a DG-bialgebra. So it requires a new differential

$$
\tilde{d} : BA \to BA
$$

\text{(which should be a coderivation with respect to standard coproduct of } BA \text{) and a new associative multiplication}

$$
\tilde{\mu} : (BA, \tilde{d}) \otimes (BA, \tilde{d}) \to (BA, \tilde{d})
$$

\text{which should be a map of DG-coalgebras, with } 1_A \in A \subset BA \text{ as the unit element.}

\text{It is known that such } \tilde{d} \text{ specifies on } A \text{ a structure of } A_{\infty}\text{-algebra in the sense of Stasheff [19], namely a sequence of operations } \{m_i : \otimes^i A \to A, i = 1, 2, 3, \ldots\} \text{ subject of appropriate conditions.}

\text{As for the new multiplication } \tilde{\mu}, \text{ it follows from the above considerations, that it is induced by a sequence of operations } \{E_{pq}\} \text{ satisfying (6), (12) and the modified condition (10) with involved } A_{\infty}\text{-algebra structure } \{m_i\}.\]
Thus the structure of associative Hirsch algebra is a particular $B_{\infty}$-algebra structure on $A$ when the standard differential of the bar construction $d_B : BA \to BA$ does not change, i.e. $\bar{d} = d_B$ (in this case the corresponding $A_{\infty}$-algebra structure is degenerate: $m_1 = d_A, m_2 = 0, m_3 = 0, m_4 = 0, ...$).

Let us mention that a twisting cochain $E$ satisfying (6) and (4), (but not (5) i.e. the induced product in the bar construction is not strictly associative), was constructed in [14] for the singular cochain complex of a topological space $C^* (X)$ using acyclic models. The condition (6) determines this twisting cochain $E$ uniquely up to standard equivalence (homotopy) of twisting cochains in the sense of N. Berikashvili [4].

3.4. Strong homotopy commutative algebras

The notion of strong homotopy commutative algebra (shc-algebra), as a tool for measuring of noncommutativity of DG-algebras, was used in many papers: [17], [20], etc.

A shc-algebra is a DG-algebra $(A, d, \cdot)$ with a given twisting cochain $\Phi : B (A \otimes A) \to A$ which satisfies appropriate up to homotopy conditions of associativity and commutativity. Compare with the Hirsch algebra structure which is represented by a twisting cochain $E : BA \otimes BA \to A$. Standard contraction of $B (A \otimes A)$ to $BA \otimes BA$ allows one to establish a connection between these two notions.

3.5. DG-Lie algebra structure in a Hirsch algebra

A structure of an associative Hirsch algebra on $A$ induces on the homology $H (A)$ a structure of Gerstenhaber algebra (G-algebra) (see [6], [8], [21]) which is defined as a commutative graded algebra $(H, \cdot)$ together with a Lie bracket of degree -1 $[\cdot, \cdot] : H^p \otimes H^q \to H^{p+q-1}$ (i.e. a graded Lie algebra structure on the desuspension $s^{-1}H$) that is a biderivation: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$.

The existence of this structure in the homology $H (A)$ is seen by the following argument.

Let $(A, d, \cdot, \{E_{p,q}\})$ be an associative Hirsch algebra, then in the desuspension $s^{-1}A$ there appears a structure of DG-Lie algebra: although the $\sim_1 = E_{1,1}$ is not associative, the condition (11) implies the pre-Jacobi identity

$$a \sim_1 (b \sim_1 c) + (a \sim_1 b) \sim_1 c = a \sim_1 (c \sim_1 b) + (a \sim_1 c) \sim_1 b$$

This condition guarantees that the commutator $[a, b] = a \sim_1 b + b \sim_1 a$ satisfies the Jacobi identity. Besides, condition (7) implies that $[\cdot, \cdot] : A^p \otimes A^q \to A^{p+q-1}$ is a chain map. Thus on $s^{-1}H (A)$ there appears the structure of graded Lie algebra. The up to homotopy Hirsh formulae (8) and (9) imply that the induced Lie bracket is a biderivation.

3.6. Homotopy G-algebra

An associative Hirsch algebra of level 3 in the literature is known as Homotopy $G$-algebra.
A \textit{Homotopy G-algebra} in [8] and [21] is defined as a DG-algebra \((A, d, \cdot)\) with a given sequence of multibraces \(a\{a_1, \ldots, a_k\}\) which, in our notation, we regard as a sequence of operations

\[ E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 0, 1, 2, 3, \ldots \]

which, together with \(E_{01} = id\) satisfies the conditions (6), (13), (14) and (15).

The name \textit{Homotopy G-algebra} is motivated by the fact that this structure induces on the homology \(H(A)\) the structure of \(G\)-algebra (as we have seen in the previous section such a structure appears even on the homology of an associative Hirsch algebra).

The conditions (13), (14), and (15) in \([8]\) are called \textit{higher homotopies, distributivity and higher pre-Jacobi identities} respectively. As we have seen the first two conditions mean that \(E : BA \otimes BA \to A\) is a twisting cochain, or equivalently \(\mu_E : BA \otimes BA \to BA\) is a chain map, and the third one means that this multiplication is associative.

### 3.7. Operadic description

Appropriate language to describe such huge sets of operations is the operadic language. Here we use the \textit{surjection operad} \(\chi\) and the \textit{Barratt-Eccles operad} \(E\) which are the most convenient \(E_\infty\) operads. For definitions we refer to [3].

The operations \(E_{1,k}\) forming \(hGa\) have nice description in the \textit{surjection operad}, see [15], [16], [3]. Namely, to the dot product corresponds the element \((1, 2) \in \chi_0(2)\), to \(E_{1,1} = -1\) product corresponds \((1, 2, 1) \in \chi_1(2)\), to the operation \(E_{1,2}\) the element \((1, 2, 1, 3) \in \chi_2(3)\), etc. Generally to the operation \(E_{1,k}\) corresponds the element

\[ E_{1,k} = (1, 2, 1, 3, \ldots, 1, k, 1, k + 1, 1) \in \chi_k(k + 1). \quad (17) \]

We remark here that the defining conditions of a \(hGa\) (13), (14), (15) can be expressed in terms of operadic structure (differential, symmetric group action and composition product) and the elements (17) satisfy these conditions \textit{already in the operad} \(\chi\). This in particular implies that any \(\chi\)-\textit{algebra} is automatically a \(hGa\).

Note that the elements (17) together with \((1, 2)\) generate the suboperad \(F_2\chi\) which is equivalent to the little square operad. This fact and a \(hGa\) structure on the Hochschild cochain complex \(C^*(U, U)\) of an algebra \(U\) are used by many authors to prove so called Deligne conjecture about the action of the little square operad on \(C^*(U, U)\).

Now look at the operations \(E_{p,q}\) which define a structure of Hirsch algebra. They \textit{can not live} in \(\chi\); it is enough to mention that the Hirsch formula (2), as a part of defining conditions of \(hGa\), is satisfied in \(\chi\), but for a Hirsch algebra this condition is satisfied up to homotopy \(E_{2,1}\), see (8). We believe that \(E_{p,q}\)-s live in the Barratt-Eccles operad \(E\). In particular direct calculation shows that

\begin{align*}
E_{1,1} &= ((1, 2), (2, 1)) \in \mathcal{E}_1(2); \\
E_{1,2} &= ((1, 2, 3), (2, 1, 3), (2, 3, 1)) \in \mathcal{E}_2(3); \\
E_{2,1} &= ((1, 2, 3), (1, 3, 2), (3, 1, 2)) \in \mathcal{E}_2(3); \\
E_{1,3} &= ((1, 2, 3, 4), (2, 1, 3, 4), (2, 3, 1, 4), (2, 3, 4, 1)) \in \mathcal{E}_3(4); \\
E_{3,1} &= ((1, 2, 3, 4), (1, 2, 4, 3), (1, 4, 2, 3), (4, 1, 2, 3)) \in \mathcal{E}_3(4); 
\end{align*}
and in general
\[ E_{1,k} = ((1, 2, \ldots, k + 1), \ldots, (2, 3, \ldots, i, 1, i + 1, \ldots, k + 1), \ldots, (2, 3, \ldots, k + 1, 1)); \]
\[ E_{k,1} = ((1, 2, \ldots, k + 1), \ldots, (1, 2, \ldots, i, k + 1, i + 1, \ldots, k), \ldots, (k + 1, 1, 2, \ldots, k)). \]

As for other \( p,q \)'s we can indicate just
\[ E_{2,2} = ((1, 2, 3, 4), (1, 3, 4, 2), (3, 1, 4, 2), (3, 4, 1, 2)) + \\
((1, 2, 3, 4), (3, 1, 2, 4), (3, 1, 4, 2), (3, 4, 1, 2)) + \\
((1, 2, 3, 4), (1, 3, 2, 4), (3, 1, 2, 4), (3, 4, 1, 2)). \]

We remark that the operadic table reduction map \( TR : \mathcal{E} \to \chi \), see [3], maps \( E_{k>1,1} \) and \( E_{2,2} \) to zero, and \( E_{1,k} \in \mathcal{E}_k(k + 1) \) to \( E_{1,k} \in \chi_k(k + 1) \).

### 4. Adams \( \sim_1 \)-product in the cobar construction of a bialgebra

Here we present the Adams \( \sim_1 \)-product \( \sim_1 : \Omega A \otimes \Omega A \to \Omega A \) on the cobar construction \( \Omega A \) of a DG-bialgebra \( (A, d, \Delta : A \to A \otimes A, \mu : A \otimes A \to A) \) (see [1]). This will be the first step in the construction of an \( hGa \) structure on \( \Omega A \).

This \( \sim_1 \) product satisfies the Steenrod condition (1) and the Hirsch formula (2).

First we define the \( \sim_1 \)-product of two elements \( x = [a], y = [b] \in \Omega A \) of length 1 as \( [a] \sim_1 [b] = [a \cdot b] \). Extending this definition by (2) we obtain
\[ [a_1, a_2] \sim_1 [b] = ([a_1] \cdot [a_2]) \sim_1 [b] = [a_1] \cdot ([a_2] \sim_1 [b]) + ([a_1] \sim_1 [b]) \cdot [a_2] = \\
[a_1] \cdot [a_2] \cdot [b] + [a_1] \cdot [b] \cdot [a_2] = [a_1, a_2] \cdot [b] + [a_1] \cdot [b, a_2]. \]

Further iteration of this process gives
\[ [a_1, \ldots, a_n] \sim_1 [b] = \sum_i [a_1, \ldots, a_{i-1}, a_i \cdot b, a_{i+1}, \ldots, a_n]. \]

Now let's define \( [a] \sim_1 [b_1, b_2] = [a^{(1)} \cdot b, a^{(2)} \cdot b] \) where \( \Delta a = a^{(1)} \otimes a^{(2)} \) is the value of the diagonal \( \Delta : A \to A \otimes A \) on \([a]\). Inspection shows that the condition (1) for short elements
\[ d([a] \sim_1 [b]) = d[a] \sim_1 [b] + [a] \sim_1 d[b] + [a] \cdot [b] + [b] \cdot [a]. \]

is satisfied.

Generally we define the \( \sim_1 \) product of an element \( x = [a] \in \Omega A \) of length 1 and an element \( y = [b_1, \ldots, b_n] \in \Omega A \) of arbitrary length by
\[ [a] \sim_1 [b_1, \ldots, b_n] = [a^{(1)} \cdot b_1, \ldots, a^{(n)} \cdot b_n]; \]
here \( \Delta^n(a) = a^{(1)} \otimes \ldots \otimes a^{(n)} \) is the \( n \)-fold iteration of the diagonal \( \Delta : A \to A \otimes A \) and \( a \cdot b = \mu(a \otimes b) \) is the product in \( A \).

Extending this definition for the elements of arbitrary lengths \( [a_1, \ldots, a_m] \sim_1 [b_1, \ldots, b_n] \) by the Hirsch formula (2) we obtain the general formula
\[ [a_1, \ldots, a_m] \sim_1 [b_1, \ldots, b_n] = \sum_k [a_1, \ldots, a_{k-1}, a^{(1)}_k \cdot b_1, \ldots, a^{(n)}_k \cdot b_n, a_{k+1}, \ldots, a_m]. \quad (18) \]
Of course, so defined, the $\sim_1$ satisfies the Hirsch formula (2) automatically. It remains to prove the

**Proposition 2.** This $\sim_1$ satisfies Steenrod condition (1)

$$d_\Omega([a_1, ..., a_m] \sim_1 [b_1, ..., b_n]) =$$

$$d_\Omega[a_1, ..., a_m] \sim_1 [b_1, ..., b_n] + [a_1, ..., a_m] \sim_1 d_\Omega[b_1, ..., b_n] +$$

$$[a_1, ..., a_m, b_1, ..., b_n] + [b_1, ..., b_n, a_1, ..., a_m].$$

**Proof.** Let us denote this condition by $Steen_{m,n}$. The first step consists in direct checking of the conditions $Steen_{1,m}$ by induction on $m$. Furthermore, assume that $Steen_{m,n}$ is satisfied. Let us check the condition $Steen_{m+1,n}$ for $[a, a_1, ..., a_m] \sim_1 [b_1, ..., b_n]$. We denote $[a_1, ..., a_m] = x$, $[b_1, ..., b_n] = y$. Using the Hirsch formula (2), $Steen_{m,n}$, and $Steen_{1,n}$ we obtain:

$$d([a, a_1, ..., a_m] \sim_1 [b_1, ..., b_n]) =$$

$$d([a] \cdot (x \sim_1 y) = d([a] \cdot (x \sim_1 y) + ([a] \sim_1 y) \cdot x) =$$

$$= d[a] \cdot (x \sim_1 y) + [a] \cdot (dx \sim_1 y + x \sim_1 dy + x \cdot y + y \cdot x) +$$

$$(d[a] \sim_1 y + [a] \sim_1 dy + [a] \cdot y + y \cdot [a] \cdot x + ([a] \sim_1 y)dx) =$$

$$d[a] \cdot (x \sim_1 y) + [a] \cdot (dx \sim_1 y) + [a] \cdot (x \sim_1 dy) + [a] \cdot x \cdot y + [a] \cdot y \cdot x +$$

$$(d[a] \sim_1 y) \cdot x + ([a] \sim_1 dy) \cdot x + [a] \cdot y \cdot x + y \cdot [a] \cdot x + ([a] \sim_1 y)dx.$$

Besides, using Hirsch (2) formula we obtain

$$d([a, a_1, ..., a_m] \sim_1 [b_1, ..., b_n]) =$$

$$d([a] \cdot x) \sim_1 y = (d[a] \cdot x) \sim_1 y + ([a] \cdot dx) \sim_1 y =$$

$$d[a] \cdot (x \sim_1 y) + (d[a] \sim_1 y) \cdot x + [a] \cdot (dx \sim_1 y) + ([a] \sim_1 y) \cdot dx$$

and

$$[a, a_1, ..., a_m] \sim_1 d[b_1, ..., b_n] =$$

$$(d[a] \cdot x) \sim_1 dy = (d[a] \cdot (x \sim_1 dy) + ([a] \sim_1 dy) \cdot x,$$

now it is evident that $Steen_{m+1,n}$ is satisfied. This completes the proof. \square

5. **Homotopy G-algebra structure on the cobar construction of a bialgebra**

Below we present a sequence of operations

$$E_{1,k} : \Omega A \otimes (\Omega A)^{\otimes k} \to \Omega A,$$

which extends the above described $E_{1,1} = \sim_1$ to a structure of a homotopy G-algebra on the cobar construction of a DG-bialgebra. This means that $E_{1,k}$s satisfy the conditions (13), (14) and (15).
For $x = [a] \in \Omega A$ of length 1, $y_i \in \Omega A$ and $k > 1$ we define $E_{1,k}([a]; y_1, ..., y_k) = 0$ and extend for an arbitrary $x = [a_1, ..., a_n]$ by (14). This gives

$$E_{1,k}([a_1, ..., a_n]; y_1, ..., y_k) = 0$$

for $n < k$ and

$$E_{1,k}([a_1, ..., a_k]; y_1, ..., y_k) = [a_1 \circ y_1, ..., a_k \circ y_k],$$

here we use the notation $a \circ (b_1, ..., b_n) = (a^{(1)} \cdot b_1, ..., a^{(s)} \cdot b_s)$, so using this notation $[a] \sim_1 [b_1, ..., b_n] = [a \circ (b_1, ..., b_n)]$. Further iteration by (14) gives the general formula

$$E_{1,k}([a_1, ..., a_n]; y_1, ..., y_k) = \sum [a_1, ..., a_i, a_{i+1}, ..., a_k; y_1, a_{i+1}, ..., a_k \circ y_k, a_{k+1}, ..., a_n],$$

where the summation is taken over all $1 \leq i_1 < ... < i_k \leq n$.

Of course, so defined, the operations $E_{1,k}$ automatically satisfy the condition (14). It remains to prove the

**Proposition 3.** The operations $E_{1,k}$ satisfy the conditions (13) and (15).

**Proof.** The condition (13) is trivial for $x = [a]$ of length 1 and $k > 2$. For $x = [a]$ and $k = 2$ this condition degenerates to

$$E_{1,1}([a]; y_1 \cdot y_2) + y_1 \cdot E_{1,1}([a]; y_2) - E_{1,1}([a]; y_1) \cdot y_2 = 0$$

and this equality easily follows from the definition of $E_{1,1} = \sim_1$. For a long $x = [a_1, ..., a_n]$ the condition (13) can be checked by induction on the length $m$ of $x$ using the condition (14).

Similarly, the condition (15) is trivial for $x = [a]$ of length 1 unless the case $m = n = 1$ and in this case this condition degenerates to

$$E_{1,1}(E_{1,1}(x; y); z) = E_{1,1}(x; E_{1,1}(y; z)) + E_{1,2}(x; y, z) + E_{1,2}(x; z, y).$$

This equality easily follows from the definition of $E_{1,1} = \sim_1$. For a long $x = [a_1, ..., a_n]$ the condition (15) can be checked by induction on the length $m$ of $x$ using the condition (14). \qed

**Remark 1.** For a DG-coalgebra $(A, d, \Delta : A \to A \otimes A)$ there is a standard DG-coalgebra map $g_A : A \to B \Omega A$ from $A$ to the bar of cobar of $A$. This map is the coextension of the universal twisting cochain $\phi_A : A \to \Omega A$ defined by $\phi(a) = [a]$ and is a weak equivalence, i.e. it induces an isomorphism of homology. Suppose $A$ is a DG-bialgebra. Then the constructed sequence of operations $E_{1,k}$ define a multiplication $\mu_E : B \Omega A \otimes B \Omega A \to B \Omega A$ on the bar construction $B \Omega A$ so that it becomes a DG-bialgebra. Direct inspection shows that $g_A : A \to B \Omega A$ is multiplicative, so it is a weak equivalence of DG-bialgebras. Dualizing this statement we obtain a weak equivalence of DG-bialgebras $\Omega BA \to A$ which can be considered as a free (as an algebra) resolution of a DG-bialgebra $A$. 
References


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