ON THE 2-ADIC K-LOCALIZATIONS OF H-SPACES

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Abstract

We determine the 2-adic $K$-localizations for a large class of $H$-spaces and related spaces. As in the odd primary case, these localizations are expressed as fibers of maps between specified infinite loop spaces, allowing us to approach the 2-primary $v_1$-periodic homotopy groups of our spaces. The present $v_1$-periodic results have been applied very successfully to simply-connected compact Lie groups by Davis, using knowledge of the complex, real, and quaternionic representations of the groups. We also functorially determine the united 2-adic $K$-cohomology algebras (including the 2-adic $KO$-cohomology algebras) for all simply-connected compact Lie groups in terms of their representation theories, and we show the existence of spaces realizing a wide class of united 2-adic $K$-cohomology algebras with specified operations.

1. Introduction

In [20], Mahowald and Thompson determined the $p$-adic $K$-localizations of the odd spheres at an arbitrary prime $p$, expressing these localizations as homotopy fibers of maps between specified infinite loop spaces. Then, working at an odd prime $p$ in [8], we generalized this result to give the $p$-adic $K$-localizations for a large class of $H$-spaces and related spaces. In the present paper, we obtain similar results for 2-adic $K$-localizations of such spaces, using our preparatory work in [10] and [11]. By a 2-adic $K$-localization, we mean a $K/2_\ast$-localization (see [2], [3]), which is the same as a $K^\ast(\cdot; \hat{\mathbb{Z}}_2)$-localization, since the $K/2_\ast$-equivalences of spaces or spectra are the same as the $K^\ast(\cdot; \hat{\mathbb{Z}}_2)$-equivalences. Our localization results in this paper will apply to many (but not all) simply-connected finite $H$-spaces and to related spaces such as the spheres $S^{4k-1}$ for $k \geq 1$. We show that these results allow computations of the $v_1$-periodic homotopy groups (see [13], [15]) of our spaces from their united 2-adic $K$-cohomologies, and thus allow computations of the $v_1$-periodic homotopy groups for a large class of simply-connected compact Lie groups from their complex, real, and quaternionic representation theories. The present results will be extended in a

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subsequent paper to cover the remaining simply-connected compact Lie groups and various spaces related to the remaining odd spheres. This work has been applied very successfully by Davis [14] to complete his 13-year program (with Bendersky) of calculating the $v_1$-periodic homotopy groups of all simply-connected compact Lie groups, and has also been applied by Bendersky, Davis, and Mahowald [1].

Throughout this paper, we work at the prime 2 and rely on the **united 2-adic $K$-cohomology**

$$K^*_CR(X; \mathbb{Z}_2) = \{K^*(X; \mathbb{Z}_2), KO^*(X; \mathbb{Z}_2)\}$$

of a space or spectrum $X$ as in [10]. This combines the usual periodic cohomologies with certain operations between them, such as complexification and realification.

For our $H$-spaces and related spaces $X$, the cohomology $K^*_CR(X; \mathbb{Z}_2)$ is essentially determined by the 2-adic Adams $\Delta$-module

$$\tilde{K}^{-1}_\Delta(X; \mathbb{Z}_2) = \{\tilde{K}^{-1}(X; \mathbb{Z}_2), \tilde{KO}^{-1}(X; \mathbb{Z}_2), \tilde{KO}^{-5}(X; \mathbb{Z}_2)\}$$

which combines the specified cohomologies with the additive operations among them (see Definition 6.1). In fact, for most simply-connected finite $H$-spaces $X$, we expect to have an isomorphism $K^*_CR(X; \mathbb{Z}_2) \cong \tilde{L}(M)$ where $M = \{M_C, M_R, M_H\}$ is the submodule of primitives in $\tilde{K}^{-1}_\Delta(X; \mathbb{Z}_2)$ and where $\tilde{L}$ is a functor that we introduce in Lemma 4.5, extending the 2-adic exterior algebra functor on complex components.

For a simply-connected compact Lie group $G$, the required 2-adic Adams $\Delta$-module may be obtained as the indecomposables $QR\Delta G = \{QRG, QR_RG, QR_HG\}$ of the complex, real, and quaternionic representation ring $R\Delta G = \{RG, R_RG, R_HG\}$ (see Definition 10.1), and we have:

**Theorem 1.1.** For a simply-connected compact Lie group $G$, there is a natural isomorphism $K^*_CR(G; \mathbb{Z}_2) \cong \tilde{L}(QR\Delta G)$ of algebras.

This will follow from Theorem 10.3. It extends results of Hodgkin [17], Seymour [23], Minami [21], and others on $K^*(G; \mathbb{Z}_2)$ and $KO^*(G; \mathbb{Z}_2)$. Our main result on $K/2^\star$-localizations will apply to a space $X$ with $K^*_CR(X; \mathbb{Z}_2) \cong \tilde{L}M$ for a 2-adic Adams $\Delta$-module $M$ that is **strong** (see Definition 7.11). This technical algebraic condition seems relatively mild and holds for $QR\Delta G$ when $G$ is a simply-connected compact simple Lie group **other than $E_6$ or Spin$(4k + 2)$ with $k$ not a 2-power** by work of Davis (see Lemma 10.5).

For a strong 2-adic Adams $\Delta$-module $M$, we obtain two stable 2-adic Adams $\Delta$-modules $M = \{M_C, M_R, M_H\}$ and $\bar{p}M = \{\bar{p}M_C, \bar{p}M_R, \bar{p}M_H\}$ where $M_C = M_C, M_R = \text{im}(M_R \to M_C)$, and $M_H = \text{im}(M_H \to M_C)$; and we obtain two corresponding $K/2^\star$-local spectra $E\bar{p}M$ and $E\bar{p}M$ such that $K^{-1}_\Delta(E\bar{p}M; \mathbb{Z}_2) = \bar{M}, K^0(E\bar{p}M; \mathbb{Z}_2) = 0, K^{-1}_\Delta(E\bar{p}M; \mathbb{Z}_2) = \bar{p}M,$ and $K^0(E\bar{p}M; \mathbb{Z}_2) = 0$ (see Definition 8.1). Stated briefly, our main localization result is:

**Theorem 1.2.** If $X$ is a connected space with $K^*_CR(X; \mathbb{Z}_2) \cong \tilde{L}M$ for a strong 2-adic Adams $\Delta$-module $M$, then its $K/2^\star$-localization $X_{K/2}$ is the homotopy fiber of a map from $\Omega^\infty E\bar{p}M$ to $\Omega^\infty E\bar{p}M$ with low dimensional modifications.

This will follow from Theorem 8.6. It will apply to simply-connected compact Lie groups with the above-mentioned exceptions, and it should apply to many
other simply-connected finite $H$-spaces and related spaces; in fact, there must exist a great diversity of spaces with the required united 2-adic $K$-cohomology algebras by:

**Theorem 1.3.** For each strong 2-adic Adams $\Delta$-module $M$, there exists a simply-connected space $X$ with $K_1^{CR}(X;\mathbb{Z}_2) \cong LM$.

This will follow from Theorem 8.5. For our spaces $X$, we also obtain results on the 2-primary $v_1$-periodic homotopy groups $v_1^{-1}\pi_*X$, which are naturally isomorphic to stable homotopy groups $\pi_*\tau_2\Phi_1X$, where $\tau_2\Phi_1X$ is the 2-torsion part of the spectrum $\Phi_1X$ obtained using the $v_1$-stabilization functor $\Phi_1$ constructed in [4, 9, 16, and 18]. From this standpoint, the homotopy $v_1^{-1}\pi_*X$ is essentially determined by the cohomology $KO^*(\Phi_1X;\mathbb{Z}_2)$, since there is an exact sequence

$$\cdots \to KO^{n-3}(\Phi_1X;\mathbb{Z}_2) \xrightarrow{\psi^3-9} KO^{n-3}(\Phi_1X;\mathbb{Z}_2) \to (v_1^{-1}\pi_*X)^\#$

where $(-)^\#$ gives the Pontrjagin dual (see Theorem 9.2). A space $X$ is called $K/2_*$-durable when the $K/2_*$-localization induces an isomorphism $v_1^{-1}\pi_*X \cong v_1^{-1}\pi_*X_{K/2}$ or equivalently $\Phi_1X \cong \Phi_1X_{K/2}$. This condition holds for all connected $H$-spaces (and many other spaces), and our $K/2_*$-localization result implies:

**Theorem 1.4.** If $X$ is a connected $K/2_*$-durable space (e.g. $H$-space) with $K_1^{CR}(X;\mathbb{Z}_2) \cong LM$ for a strong 2-adic Adams $\Delta$-module $M$, then there is a (co)fiber sequence of spectra $\Phi_1X \to \mathcal{E}M \to \mathcal{E}\rho M$ with a $KO^*(-;\mathbb{Z}_2)$ cohomology exact sequence

$$0 \to KO^{-8}(\Phi_1X;\mathbb{Z}_2) \to \tilde{M}_C/(\tilde{M}_R + \tilde{M}_H) \xrightarrow{\lambda^2} \tilde{M}_C/\tilde{M}_R \to KO^{-7}(\Phi_1X;\mathbb{Z}_2)$$

$$\to 0 \to \tilde{M}_H/(\tilde{M}_R \cap \tilde{M}_H) \to KO^{-6}(\Phi_1X;\mathbb{Z}_2) \to \tilde{M}_R \cap \tilde{M}_H \xrightarrow{\lambda^2} \tilde{M}_H \to KO^{-5}(\Phi_1X;\mathbb{Z}_2) \to 0 \to KO^{-4}(\Phi_1X;\mathbb{Z}_2) \to \tilde{M}_C/(\tilde{M}_R \cap \tilde{M}_H) \xrightarrow{\lambda^2} \tilde{M}_C/\tilde{M}_H$$

$$\to KO^{-3}(\Phi_1X;\mathbb{Z}_2) \to (\tilde{M}_R + \tilde{M}_H)/(\tilde{M}_R \cap \tilde{M}_H) \xrightarrow{\lambda^2} \tilde{M}_R \to KO^{-2}(\Phi_1X;\mathbb{Z}_2) \to \tilde{M}_R + \tilde{M}_H \xrightarrow{\lambda^2} \tilde{M}_R \to KO^{-1}(\Phi_1X;\mathbb{Z}_2) \to 0.$$

This will follow from Theorem 9.5. It allows effective computations of 2-primary $v_1$-periodic homotopy groups as shown by Davis [14], and its complex analogue implies that our spaces $X$ are usually $\tilde{K}\Phi_1$-good, which means that $\bar{Q}K^n(X;\mathbb{Z}_2)/\lambda^2 \cong K^n(\Phi_1X;\mathbb{Z}_2)$ for $n = -1, 0$.

**Theorem 1.5.** If $X$ is as in Theorem 1.4 with $\lambda^2: \tilde{M}_C \to \tilde{M}_C$ monic, then $X$ is $\tilde{K}\Phi_1$-good.

This will follow from Theorem 9.7. It will be used in a subsequent paper to show that all simply-connected compact Lie groups (and many other spaces) are $\tilde{K}\Phi_1$-good, which is useful because the $v_1$-periodic homotopy groups of $\tilde{K}\Phi_1$-good spaces are often accessible by [10], even when our $K/2_*$-localization theorems do not
apply. From the perspective of [10], the present work verifies important examples of $\hat{K}$/$\Phi_1$-good spaces beyond the odd spheres.

Throughout the paper, spaces and spectra will belong to the usual pointed simplicial or CW homotopy categories. To provide a suitably precise setting for our main theorems and proofs, we must devote considerable attention to developing the algebraic infrastructure of united 2-adic $K$-cohomology theory. The paper is divided into the following sections:

1. Introduction
2. The united 2-adic $K$-cohomologies of spectra and spaces
3. The 2-adic $\phi CR$-algebras
4. The universal 2-adic $\phi CR$-algebra functor $\hat{L}$
5. Stable 2-adic Adams operations and $K/2_\ast$-local spectra
6. On the united 2-adic $K$-cohomologies of infinite loop spaces
7. Strong 2-adic Adams $\Delta$-modules
8. On the $K/2_\ast$-localizations of our spaces
9. On the $v_1$-periodic homotopy groups of our spaces
10. Applications to simply-connected compact Lie groups
11. Proofs of basic lemmas for $\hat{L}$
12. Proof of the Bott exactness lemma for $\hat{L}$
13. Proofs for regular modules
14. Proof of the realizability theorem for $\hat{LM}$

Although we have long been interested in the $K$-localizations and $v_1$-periodic homotopy groups of spaces, we were prompted to develop the present results by Martin Bendersky and Don Davis. We thank them for their questions and comments.

2. The united 2-adic $K$-cohomologies of spectra and spaces

We now consider the united 2-adic $K$-cohomologies

$$K_{CR}^\ast(X; \hat{\mathbb{Z}}_2) = \{K^\ast(X; \hat{\mathbb{Z}}_2), KO^\ast(X; \hat{\mathbb{Z}}_2)\}$$

of spectra and spaces $X$, focusing on their basic structures as 2-adic $CR$-modules or $CR$-algebras. We first recall:

**Definition 2.1** (The 2-adic $CR$-modules). By a 2-adic $CR$-module, we mean a $CR$-module over the category of 2-profinite abelian groups (see [10, 4.1]). Thus, a 2-adic $CR$-module $M = \{M_C, M_R\}$ consists of $\mathbb{Z}$-graded 2-profinite abelian groups $M_C$ and $M_R$ with continuous additive operations

- $B: M_C^\ast \cong M_C^{\ast -2}$,
- $t: M_C^\ast \cong M_C^\ast$,
- $B_R: M_R^\ast \cong M_R^{\ast -8}$,
- $\eta: M_R^\ast \rightarrow M_R^{\ast -1}$,
- $c: M_R^\ast \rightarrow M_C^\ast$,
- $\rho: M_C^\ast \rightarrow M_R^\ast$. 


satisfying the relations
\[
\begin{align*}
2\eta &= 0, & \eta^3 &= 0, & \eta B_R &= B_R \eta, & \eta r &= 0, & c \eta &= 0, \\
t^2 &= 1, & t B &= -B t, & t r &= r, & t c &= c, & c B_R &= B^4 c, \\
r B^4 &= B_R r, & cr &= 1 + t, & rc &= 2, & r B c &= \eta^2, & r B^{-1} c &= 0.
\end{align*}
\]

For \( z \in M^*_C \) and \( x \in M^*_R \), the elements \( t z \in M^*_C \) and \( r B^2 cx \in M^*_R \) are sometimes written as \( z^* \) (or \( \psi^{-1} z \)) and \( \xi x \). For a spectrum or space \( X \), the united 2-adic \( K \)-cohomology
\[
K^*_CR(X; \hat{\mathbb{Z}}_2) = \{ K^*(X; \hat{\mathbb{Z}}_2), KO^*(X; \hat{\mathbb{Z}}_2) \}
\]
has a natural 2-adic \( CR \)-module structure with the usual periodicities \( B: K^*(X; \hat{\mathbb{Z}}_2) \cong K^{*+2}(X; \hat{\mathbb{Z}}_2) \), and \( B_R: KO^*(X; \hat{\mathbb{Z}}_2) \cong KO^{*+8}(X; \hat{\mathbb{Z}}_2) \), conjunction \( t: K^*(X; \hat{\mathbb{Z}}_2) \cong K^*(X; \hat{\mathbb{Z}}_2) \), Hopf operation \( \eta: KO^*(X; \hat{\mathbb{Z}}_2) \rightarrow KO^{*-1}(X; \hat{\mathbb{Z}}_2) \), complexification \( c: KO^*(X; \hat{\mathbb{Z}}_2) \rightarrow K^*(X; \hat{\mathbb{Z}}_2) \), and realification \( r: K^*(X; \hat{\mathbb{Z}}_2) \rightarrow KO^*(X; \hat{\mathbb{Z}}_2) \).

**Definition 2.2** (Bott exactness). As in [10, 4.1], we say that a 2-adic \( CR \)-module \( M \) is **Bott exact** when the Bott sequence
\[
\cdots \rightarrow M_R^{*+1} \xrightarrow{\eta} M_R^* \xrightarrow{c} M_C^* \xrightarrow{r B^{-1}} M_R^{*+2} \xrightarrow{\eta} \cdots
\]
is exact, and we note that the 2-adic \( CR \)-module \( K^*_CR(X; \hat{\mathbb{Z}}_2) \) is always Bott exact for a spectrum or space \( X \). To compare \( CR \)-modules, we shall often use:

**Lemma 2.3.** For Bott exact 2-adic \( CR \)-modules \( M \) and \( N \), a map \( f: M \rightarrow N \) is an isomorphism if and only if \( f: M_C \rightarrow N_C \) is an isomorphism.

**Proof.** For the “if” part, we treat the Bott sequences of \( M \) and \( N \) as exact couples, and we note that \( f \) induces an isomorphism of the associated spectral sequences since \( f: M_C \cong N_C \). Using the map of second derived couples with \( f: M^{(2)}_C \cong N^{(2)}_C \), we easily see that \( f: \eta^2 M_R \cong \eta^2 N_R \); then using the map of first derived couples with \( f: M^{(1)}_C \cong N^{(1)}_C \), we easily see that \( f: \eta M_R \cong \eta N_R \); and finally using the original map of exact couples, we easily see that \( f: M_R \cong N_R \).

**Definition 2.4** (The free 2-adic \( CR \)-modules). For each integer \( n \) and \( L = C, R \), there is a **monogenic free 2-adic \( CR \)-module** \( F^L(g, n) \) on a generator \( g \in F^L(g, n)_n \) having the universal property that, for each 2-adic \( CR \)-module \( M \) and \( y \in M^n \), there is a unique map \( f: F^L(g, n) \rightarrow M \) with \( f(g) = y \). The 2-adic \( CR \) modules \( F^C(g, n) \) and \( F^R(g, n) \) are given more explicitly by
\[
\begin{align*}
F^C(g, n)^{n-2i} &= \hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2 = (B^i g) \oplus (B^i g^*), & F^C(g, n)^{n-2i-1} &= 0, \\
F^C(g, n)_R^{n-2i} &= \hat{\mathbb{Z}}_2 = (r B^i g), & F^C(g, n)_R^{n-2i-1} &= 0, \\
F^R(g, n)_C^{n-2i} &= \hat{\mathbb{Z}}_2 = (B^i g), & F^R(g, n)_C^{n-2i-1} &= 0, \\
F^R(g, n)_R^{n-8i} &= \hat{\mathbb{Z}}_2 = (B^i R g), & F^R(g, n)_R^{n-8i-1} &= \mathbb{Z}/2 = (B^i R \eta g), \\
F^R(g, n)_R^{n-8i-2} &= \mathbb{Z}/2 = (B^i R \eta^2 g), & F^R(g, n)_R^{n-8i-4} &= \hat{\mathbb{Z}}_2 = (B^i R \xi g), \\
F^R(g, n)_R^{n-8i-k} &= 0 \text{ for } k = 3, 5, 6, 7.
\end{align*}
\]

We note that \( F^C(g, n) \) and \( F^R(g, n) \) are Bott exact for all \( n \). In general, a free
2-adic CR-module on a finite set of generators may be constructed as a direct sum of the corresponding monogenic free 2-adic CR-modules. To test for this freeness, we may use:

**Lemma 2.5.** For a Bott exact 2-adic CR-module $M$ (e.g. for some $M = K_{CR}^*(X; \mathbb{Z}_2)$), if $M_c$ is a free module over $K = \mathbb{Z}_2[B, B^{-1}]$ on the generators $\{a_i\}, \Pi \{b_j\}, \Pi \{b^*_j\}$ for finite sets of elements $\{a_i\}$ in $M^*_R$ and $\{b_j\}, \{b^*_j\}$ in $M^*_C$, then $M$ is a free 2-adic CR-module on the generators $\{a_i\}$ and $\{b^*_j\}$.

**Proof.** The canonical map to $M$ from the specified 2-adic CR-module is an isomorphism by Lemma 2.3.

To describe the multiplicative structure of $K_{CR}^*(X; \mathbb{Z}_2)$ for a space $X$, we introduce:

**Definition 2.6** (The 2-adic CR-algebras). By a 2-adic CR-algebra $A = \{A_C, A_R\}$, we mean a 2-adic CR-module with continuous bilinear multiplications $A^*_L \times A^*_L \to A^*_{L+n}$ and elements $1 \in A^*_L$ for $m, n \in \mathbb{Z}$ and $L = C, R$ such that:

(i) the multiplication in $A^*_C$ and $A^*_R$ is graded commutative and associative with identity 1;

(ii) $B(zw) = (Bz)w = z(Bw)$ and $(zw)^* = z^*w^*$ for $z \in A^*_L$ and $w \in A^*_C$;

(iii) $B_R(xy) = (B_Rx)y = x(B_Ry)$, $\eta(xy) = (\eta x)y = x(\eta y)$, and $\xi(xy) = (\xi x)y = x(\xi y)$ for $x \in A^*_R$ and $y \in A^*_R$;

(iv) $c1 = 1$ and $c(xy) = (cx)(cy)$ for $x \in A^*_R$ and $y \in A^*_R$;

(v) $r((cz)x) = (r(cz))x$ for $x \in A^*_R$ and $z \in A^*_C$.

Equivalently, a 2-adic CR-algebra $A$ consists of a 2-adic CR-module with a commutative associative multiplicative action $A \hat{\otimes}_{CR} A \to A$ with identity $\xi \to A$ for $\xi = FR(1, 0)$ $\cong K_{CR}^*(pt; \mathbb{Z}_2)$, where $\hat{\otimes}_{CR}$ is the (symmetric monoidal) complete tensor product for 2-adic CR-modules [11, 2.6].

**Definition 2.7** (Augmentations and nilpotency). For a 2-adic CR-algebra $A$, an augmentation is a map $A \to \xi$ of 2-adic CR-algebras which is left inverse to the identity $\xi \to A$. When $A$ is augmented, we let $\hat{A}(m)$ denote the augmentation ideal, and for $m \geq 1$ we let $A(m) = \hat{A}(m)$ denote the $m$-th power of $A$ given by the image of the $m$-fold product $\hat{A} \hat{\otimes}_{CR} \cdots \hat{\otimes}_{CR} \hat{A} \to \hat{A}$. Thus, $\hat{A}(m)_C$ is the image of the $m$-fold product $A^*_C \hat{\otimes}_{CR} \cdots \hat{\otimes}_{CR} A^*_C \to A^*_C$, while $A(m)_R$ is the image of the $m$-fold product $A^*_R \hat{\otimes}_{CR} \cdots \hat{\otimes}_{CR} A^*_R \to A^*_R$ plus the realization of $A(m)_C$. The indecomposables of $A$ are given by the 2-adic CR-module $\hat{Q}A = \hat{A}/\hat{A}(2)$. We call $A$ nilpotent when $\hat{A}(m) = 0$ for sufficiently large $m$ and call $A$ pro-nilpotent when $\cap_m \hat{A}(m) = 0$ or equivalently when $A \cong \lim_m A/\hat{A}(m)$. For a space $X$, the cohomology $K_{CR}^*(X; \mathbb{Z}_2)$ has a canonical augmentation $K_{CR}^*(X; \mathbb{Z}_2) \to \xi$ induced by the basepoint $pt \subset X$ with the usual augmentation ideal $\hat{K}_{CR}^*(X; \mathbb{Z}_2) = \{\hat{K}^*(X; \mathbb{Z}_2), \hat{KO}^*(X; \mathbb{Z}_2)\}$. Moreover, when $X$ is connected, the cohomology $K_{CR}^*(X; \mathbb{Z}_2)$ is pro-nilpotent since it is the inverse limit of the cohomologies $K_{CR}^*(X_\alpha; \mathbb{Z}_2)$ for the finite connected subspaces $X_\alpha \subset X$, where each $K_{CR}^*(X_\alpha; \mathbb{Z}_2)$ is nilpotent.
3. The 2-adic $\phi CR$-algebras

To capture some additional features of the 2-adic $CR$-algebras $K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ for spaces $X$, we now introduce the 2-adic $\phi CR$-algebras. These structures are often surprisingly rigid and will allow us to construct convenient bases for $K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ in some important general cases, for instance, when $X$ is a simply-connected compact Lie group.

**Definition 3.1** (The 2-adic $\phi CR$-algebras). By a 2-adic $\phi CR$-algebra $A$, we mean a 2-adic $CR$-algebra with continuous functions $\phi: A^0_C \to A^0_R$ and $\phi: A^{-1}_C \to A^0_R$ such that:

- (i) $c\phi a = a^*a$ and $c\phi x = B^{-1}x^*x$ for $a \in A^0_C$ and $x \in A^{-1}_C$;
- (ii) $\phi(a + b) = \phi a + \phi b + r(a^*b)$ and $\phi(x + y) = \phi x + \phi y + rB^{-1}(x^*y)$ for $a, b \in A^0_C$ and $x, y \in A^{-1}_C$;
- (iii) $\phi(ab) = (\phi a)(\phi b)$, $\phi(ax) = (\phi a)(\phi x)$, and $\phi B^{-1}(xy) = (\phi x)(\phi y)$ for $a, b \in A^0_C$ and $x, y \in A^{-1}_C$;
- (iv) $\phi(1) = 1$, $\phi(ka) = k^2\phi a$, $\phi(a^*) = \phi a$, $\phi(kx) = k^2\phi x$, and $\phi(x^*) = -\phi x$ for $a \in A^0_C$, $x \in A^{-1}_C$, and $k \in \hat{\mathbb{Z}}_2$.

For convenience, we extend the operation $\phi$ periodically to give $\phi: A^0_C \to A^0_R$ and $\phi: A^{-1}_C \to A^0_R$ with $\phi w = \phi B^i w$ for all $i$ and elements $w$. For a space $X$, the cohomology $K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ has a natural 2-adic $\phi CR$-algebra structure with $\phi: K^*(X; \hat{\mathbb{Z}}_2) \to KO^0(X; \hat{\mathbb{Z}}_2)$ as in [11, Section 3]. In particular, $\xi \cong K^*_{CR}(pt; \hat{\mathbb{Z}}_2)$ is a 2-adic $\phi CR$-algebra with $\phi(k1) = k^21$ for $k \in \hat{\mathbb{Z}}_2$. For a 2-adic $\phi CR$-algebra $A$, an augmentation is a map $A \to \xi$ of 2-adic $\phi CR$-algebras which is left inverse to the identity, and we retain the other notation and terminology of Definition 2.7. Thus, for a space $X$, the $\phi CR$-algebra $K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ has a canonical augmentation and is pro-nilpotent whenever $X$ is connected. To capture some other needed features, we introduce:

**Definition 3.2** (The special 2-adic $\phi CR$-algebras). A 2-adic $\phi CR$-algebra $A$ is called special when:

- (i) $A$ is augmented and pro-nilpotent;
- (ii) $z^2 = 0$ for $z \in A_0^C$ with $n$ odd;
- (iii) $y^2 = 0$ for $y \in A_R^0$ with $n \equiv 1, -3 \mod 8$;
- (iv) $c\phi x = 0$ for $x \in A_R^n$ with $n \equiv -1, -5 \mod 8$.

For a connected space $X$, the cohomology $K^*_{CR}(X; \hat{\mathbb{Z}}_2)$ is a special 2-adic $\phi CR$-algebra by [11, Section 3].

**Definition 3.3** (Simple systems of generators). Let $A$ be a special 2-adic $\phi CR$-algebra. By a simple system of generators of odd degree for $A$, we mean finite ordered sets of odd-degree elements $\{x_i\}_i$ in $A_R$ and $\{z_j\}_j$ in $A_C$ such that $A_C$ is an exterior algebra over $\tilde{K}^* = \hat{\mathbb{Z}}_2[B, B^{-1}]$ on the generators $\{cx_i\}_i \cup \{z_j\}_j \cup \{z_j^*\}_j$. 
Such a simple system determines associated products

\[ x_{i_1} \cdots x_{i_m}(\phi z_{j_1}) \cdots (\phi z_{j_n}) \in A_R, \]

\[ (cx_{i_1}) \cdots (cx_{i_m})(c\phi z_{j_1}) \cdots (c\phi z_{j_n})w_{k_1} \cdots w_{k_q} \in A_C, \]

where: \( i_1 < \cdots < i_m \) with \( m \geq 0 \); \( j_1 < \cdots < j_n \) with \( n \geq 0 \); \( k_1 < \cdots < k_q \) with \( q \geq 1 \); each \( w_{k_i} \) is \( z_{k_i} \) or \( z_{k_i}^* \) with \( w_{k_q} = z_{k_q} \); and \( \{k_1, \ldots, k_q\} \) is disjoint from \( \{j_1, \ldots, j_n\} \) in each complex product.

**Proposition 3.4.** If \( A \) is a Bott exact special 2-adic \( \phi CR \)-algebra with a simple system of generators of odd degree, then \( A \) is a free 2-adic CR-module on the associated products.

**Proof.** This follows by Lemma 2.5. \( \square \)

When the cohomology \( K^*_CR(X; \hat{Z}_2) \) of a connected space \( X \) has a simple system of generators of odd degree, this result will determine the 2-adic CR-algebra structure of the cohomology, provided that we can compute the squares of the real simple generators of degree \( \equiv -1, -5 \mod 8 \), since the squares of the other simple generators and of their \( \phi \)’s must vanish. For a simply-connected compact Lie group \( G \), we shall see that the cohomology \( K^*_CR(G; \hat{Z}_2) \) must always have a simple system of generators of odd degree by Theorem 10.3 below.

### 4. The universal 2-adic \( \phi CR \)-algebra functor \( \hat{L} \)

We must now go beyond simple systems of generators and develop functorial descriptions of cohomologies \( K^*_CR(X; \hat{Z}_2) \) using universal special 2-adic \( \phi CR \)-algebras. Our results will apply, for instance, when \( X \) is a suitable infinite loop space (Theorem 6.7) or a simply-connected compact Lie group (Theorem 10.3). We start by introducing the algebraic modules that will generate our universal algebras.

**Definition 4.1** (The 2-adic \( \Delta \)-modules). By a 2-adic \( \Delta \)-module \( N = \{N_C, N_R, N_H\} \), we mean a triad of 2-profinite abelian groups \( N_C, N_R, \) and \( N_H \) with continuous additive operations

\[ t: N_C \cong N_C, \quad c: N_R \to N_C, \quad r: N_C \to N_R, \]

\[ c': N_H \to N_C, \quad q: N_C \to N_H \]

satisfying the relations

\[ t^2 = 1, \quad cr = 1 + t, \quad rc = 2, \quad tc = c, \quad rt = r, \]

\[ c'q = 1 + t, \quad qd' = 2, \quad tc' = c', \quad qt = q \]

as in [10, 4.5]. For \( z \in N_C \), the element \( tz \) is sometimes written as \( z^* \) or \( \psi^{-1}z \). For a 2-adic CR-module \( N \) and integer \( n \), we obtain a 2-adic \( \Delta \)-module \( \Delta^n N = \{N_C^n, N_R^n, N_R^{n-4}\} \) with \( c' = B^{-2}c: N_R^{n-4} \to N_C^n \) and \( q = rB^2: N_C^n \to N_R^{n-4} \). In particular, we obtain a 2-adic \( \Delta \)-module \( K^*_\Delta(X; \hat{Z}_2) = \Delta^n K^*_CR(X; \hat{Z}_2) \) for a space \( X \).
We say that a 2-adic $\Delta$-module $N$ is torsion-free when $N_C$, $N_R$, and $N_H$ are torsion-free, and we say that $N$ is exact when the sequence

$$\cdots \rightarrow N_C \xrightarrow{(r,q)} N_R \oplus N_H \xrightarrow{c-c'} N_C \xrightarrow{1-t} N_C \xrightarrow{(r,q)} N_R \oplus N_H \rightarrow \cdots$$

is exact (see $[10, 4.5]$). It is straightforward to show:

**Lemma 4.2.** A 2-adic $\Delta$-module $N = \{N_C, N_R, N_H\}$ is torsion-free and exact if and only if:

(i) $c: N_R \rightarrow N_C$ and $c': N_H \rightarrow N_C$ are monic;

(ii) $N_C$ is torsion-free with $\ker(1 + t) = \im(1 - t)$ for $t: N_C \rightarrow N_C$;

(iii) $cN_R + c'N_H = \ker(1 - t)$ and $cN_R \cap c'N_H = \im(1 + t)$.

The 2-adic $\Delta$-module

$$K_*^\Delta(X; 2\mathbb{Z}) = \{K^{-1}(X; 2\mathbb{Z}_2), KO^{-1}(X; 2\mathbb{Z}_2), KO^{-5}(X; 2\mathbb{Z}_2)\}$$

of a space $X$ has additional operations $\theta$ which we now include in:

**Definition 4.3** (The 2-adic $\theta\Delta$-modules). By a 2-adic $\theta\Delta$-module $M = \{M_C, M_R, M_H\}$, we mean a 2-adic $\Delta$-module with continuous additive operations $\theta: M_C \rightarrow M_C, \theta: M_R \rightarrow M_R$, and $\theta: M_H \rightarrow M_H$ satisfying the following relations for elements $z \in M_C, x \in M_R$, and $y \in M_H$:

$$\theta cx = c \theta x, \quad \theta c'y = c \theta y, \quad \theta tz = t \theta z, \quad \theta qz = \theta rz, \quad \theta rz = \theta rz.$$

In general, $\theta rz$ may differ from $r \theta z$, and we let $\tilde{\theta}: M_C \rightarrow M_R$ be the difference operation with $\tilde{\theta} z = \theta rz - r \theta z$ for $z \in M_C$. Using the above relations, we easily deduce:

$$\tilde{\theta} cx = 0, \quad \tilde{\theta} c' x = 0, \quad \tilde{\theta} tz = \tilde{\theta} z, \quad 2 \tilde{\theta} z = 0,$$

$$c \tilde{\theta} z = 0, \quad \tilde{\theta} rz = 0.$$

For a space $X$, the cohomology $K_*^\Delta(X; 2\mathbb{Z}_2)$ has a natural 2-adic $\theta\Delta$-module structure by $[11$, Section 3$]$ with the operations

$$\theta = -\lambda^2: K^{-1}(X; 2\mathbb{Z}_2) \rightarrow K^{-1}(X; 2\mathbb{Z}_2),$$

$$\theta = -\lambda^2: KO^{-1}(X; 2\mathbb{Z}_2) \rightarrow KO^{-1}(X; 2\mathbb{Z}_2),$$

$$\theta = -\lambda^2: KO^{-5}(X; 2\mathbb{Z}_2) \rightarrow KO^{-1}(X; 2\mathbb{Z}_2).$$

Moreover, this structure interacts with the 2-adic $\phi CR$-algebra structure of $K_*^{CR}(X; 2\mathbb{Z}_2)$ in several ways.

**Lemma 4.4.** For a space $X$, we have:

(i) $\eta \tilde{\theta} z = \tilde{\theta} z$ for $z \in K^{-1}(X; 2\mathbb{Z}_2)$;

(ii) $x^2 = \eta \theta x$ for $x \in KO^{-1}(X; 2\mathbb{Z}_2)$;

(iii) $y^2 = B_R q \theta y$ for $y \in KO^{-5}(X; 2\mathbb{Z}_2)$.

**Proof.** This follows from $[11$, Section 3$]$.  \square
We shall take account of these relations in our universal algebras. For a 2-adic \( \theta \Delta \)-module \( M \) and a special 2-adic \( \phi CR \)-algebra \( A \), an admissible map \( \alpha: M \to A \) consists of a 2-adic \( \Delta \)-module map \( \alpha: M \to \Delta^{-1} \tilde{A} \) such that:

(i) \( \eta \phi \alpha z = \alpha \bar{\phi} z \) in \( A^{-1}_C \) for each \( z \in M_C \);
(ii) \( (\alpha x)^2 = \eta \theta x \) in \( A^{-2}_R \) for each \( x \in M_R \);
(iii) \( (\alpha y)^2 = B_R \eta \theta y \) in \( A^{-10}_R \) for each \( y \in M_H \).

We say that a special 2-adic \( \phi CR \)-algebra \( A \) with an admissible map \( \alpha: M \to A \) is universal if, for each special 2-adic \( \phi CR \)-algebra \( B \) with admissible map \( g: M \to B \), there exists a unique \( \phi CR \)-algebra map \( \bar{g}: A \to B \) such that \( \bar{g} \alpha = g \).

**Lemma 4.5.** For each 2-adic \( \theta \Delta \)-module \( M \), there exists a universal special 2-adic \( \phi CR \)-algebra \( \hat{LM} \) with admissible map \( \alpha: M \to \hat{LM} \).

This will be proved later in Section 11. By universality, \( \hat{LM} \) is unique up to isomorphism and is natural in \( M \), so that we have a functor \( \hat{L} \) from the category of 2-adic \( \theta \Delta \)-modules to the category of special 2-adic \( \phi CR \)-algebras. We believe that the \( \phi CR \)-algebra \( \hat{LM} \) can be given canonical operations \( \theta \) satisfying all the formulae of [11, Section 3] and that this provides a strengthened version of \( \hat{L} \) that is right adjoint to \( \Delta^{-1} \tilde{()}. \) However, for simplicity, we rely on the present basic functor \( \hat{L} \).

**Lemma 4.6.** For a 2-adic \( \theta \Delta \)-module \( M \), the canonical map \( \hat{\Lambda} M_C \to (\hat{LM})_C \) is an algebra isomorphism.

This will be proved later in Section 11. We must impose extra conditions on \( M \) to ensure that \( \hat{LM} \) is Bott exact and hence topologically relevant.

**Definition 4.7** (The robust 2-adic \( \theta \Delta \)-modules). We say that a 2-adic \( \theta \Delta \)-module \( M \) is profinite when it is the inverse limit of an inverse system of finite 2-adic \( \theta \Delta \)-modules, and we let \( M/\bar{\phi} \) denote the 2-adic \( \Delta \)-module \( \{ M_C, M_R/\bar{\phi}M_C, M_H \} \). We call \( M \) robust when:

(i) \( M \) is profinite;
(ii) \( M/\bar{\phi} \) is torsion-free and exact;
(iii) \( \ker \bar{\phi} = cM_R + c'M_H + 2M_C \).

When \( M \) is obtained from \( K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) \) for a space \( X \), the profiniteness condition will usually hold automatically since \( K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) = \lim_{\alpha,i} K_{\Delta}^{-1}(X_{\alpha}; \hat{\mathbb{Z}}_2)/2^i \) for the system of finite subcomplexes \( X_\alpha \subset X \) and \( i \geq 1 \). The following key lemma will be proved later in Section 12.

**Lemma 4.8.** If \( M \) is a robust 2-adic \( \theta \Delta \)-module, then the special 2-adic \( \phi CR \)-algebra \( \hat{LM} \) is Bott exact; in fact, \( \hat{LM} \) is the inverse limit of an inverse system of finitely generated free 2-adic CR-modules.

This leads to a crucial comparison theorem.
Theorem 4.9. For a connected space $X$ and a robust $2$-adic $\theta\Delta$-module $M$, suppose that $g: M \to \hat{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ is a $2$-adic $\theta\Delta$-module map that induces an isomorphism $\hat{\Lambda}M_C \cong K^*(X; \hat{\mathbb{Z}}_2)$. Then $g$ induces an isomorphism $\hat{\Lambda}M \cong K_{CR}(X; \hat{\mathbb{Z}}_2)$ of special $2$-adic $\phi_{CR}$-algebras.

Proof. Since $g$ gives an admissible $M \to K_{CR}(X; \hat{\mathbb{Z}}_2)$ by Lemma 4.4, the result follows by Lemmas 2.3, 4.6, and 4.8. \qed

When $M$ is finitely generated in this theorem, we may easily choose a simple system of odd-degree generators (see Definition 3.3) for $K_{CR}(X; \hat{\mathbb{Z}}_2)$ from $M_C$, $M_R$, and $M_H$. However, the present description of $K_{CR}(X; \hat{\mathbb{Z}}_2)$ as $\hat{\Lambda}M$ is more natural and includes the full multiplicative structure. To check whether such a description is possible for a given space $X$, we may use:

Remark 4.10 (Determination of $M$ from $K_{CR}(X; \hat{\mathbb{Z}}_2)$). For a connected space $X$, we may take the indecomposables $\hat{QK}_{CR}^*(X; \hat{\mathbb{Z}}_2)$ as in Definition 2.7 with the operations $\theta$ of Definition 4.3 to produce a $2$-adic $\theta\Delta$-module $\hat{QK}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) = \{\hat{Q}K^{-1}(X; \hat{\mathbb{Z}}_2), \hat{Q}K^{-1}(X; \hat{\mathbb{Z}}_2), \hat{Q}K^{-5}(X; \hat{\mathbb{Z}}_2)\}$ together with a natural quotient map $\hat{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) \to \hat{QK}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$. Now by Lemma 4.11 below, whenever Theorem 4.9 applies to $X$, there is a canonical isomorphism $M \cong \hat{QK}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ and the map $g: M \to \hat{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ in the theorem corresponds to a splitting of $\hat{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) \to \hat{QK}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$. When $X$ is an $H$-space, we may often obtain the required splitting by mapping $\hat{QK}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ to the primitives in $\hat{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$. For instance, this applies when $X$ is a suitable infinite loop space or simply-connected compact Lie group (see Theorems 6.7 and 10.3). Finally, we note that the $2$-adic $\theta\Delta$-module $\hat{QK}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ will automatically be robust by Proposition 3.4 whenever $K_{CR}(X; \hat{\mathbb{Z}}_2)$ has a simple system of odd-degree generators with no real generators of degree $\equiv 1, -3 \mod 8$. We have used:

Lemma 4.11. For a $\theta\Delta$-module $M$, the canonical map $M \to \Delta^{-1} \hat{Q}LM$ is an isomorphism.

This will be proved later in Section 11.

5. Stable $2$-adic Adams operations and $K/2_+\text{-local spectra}$

We now bring stable Adams operations into our united $2$-adic $K$-cohomology theory and use this theory to classify the needed $K/2_+\text{-local spectra}$. We first recall some terminology from [8, 2.6].

Definition 5.1 (The stable $2$-adic Adams modules). By a finite stable $2$-adic Adams module $A$, we mean a finite abelian $2$-group with automorphisms $\psi^k: A \cong A$ for the odd $k \in \mathbb{Z}$ such that:

(i) $\psi^1 = 1$ and $\psi^j \psi^k = \psi^{jk}$ for the odd $j, k \in \mathbb{Z}$;

(ii) when $n$ is sufficiently large, the condition $j \equiv k \mod 2^n$ implies $\psi^j = \psi^k$. 

By a stable 2-adic Adams module $A$, we mean the topological inverse limit of an inverse system of finite stable 2-adic Adams modules. Such an $A$ has an underlying 2-profinite abelian structure with continuous automorphisms $\psi^k : A \cong A$ for the odd $k \in \mathbb{Z}$ (and in fact for $k \in \hat{\mathbb{Z}}^\times$). We note that the operations $\psi^{-1}$ and $\psi^3$ on $A$ determine all of the other stable Adams operations $\psi^k$ as in [5, 6.4]. Our main examples of stable 2-adic Adams modules are the cohomologies $K^n(X; \hat{\mathbb{Z}}_2)$ and $KO^n(X; \hat{\mathbb{Z}}_2)$ for a spectrum or space $X$ and integer $n$ with the usual stable Adams operations $\psi^k$. We let $\hat{A}$ denote the abelian category of stable 2-adic Adams modules, and for $i \in \mathbb{Z}$, we let $\hat{S}^i : \hat{A} \to \hat{A}$ be the functor with $\hat{S}^i A$ equal to $A$ as a group but with $\psi^k$ on $\hat{S}^i A$ equal to $k^i \psi^k$ on $A$ for the odd $k \in \mathbb{Z}$. We note that $\hat{S}^i A = A$ in $\hat{A}$ for all $i$ when $2A = 0$.

**Definition 5.2** (The stable 2-adic Adams CR-modules). By a stable 2-adic Adams CR-module $M$, we mean a 2-adic CR-module consisting of stable 2-adic Adams modules $\{M^i_C, M^i_R\}$ such that the operations $B : \hat{S}M^i_C \cong M^{i-2}_C$, $t : M^i_C \cong M^i_C$, $B_R : \hat{S}^4M^i_R \cong M^i_R$, $\eta : M^i_R \to M^{i-1}_R$, $c : M^i_R \to M^i_C$, and $r : M^i_C \to M^i_R$ are all maps in $\hat{A}$, where $\psi^{-1} = t$ in $M^i_C$ and $\psi^{-1} = 1$ in $M^i_R$. For a spectrum or space $X$, the united 2-adic $K$-cohomology

$$K^n_{CR}(X; \hat{\mathbb{Z}}_2) = \{K^n(X; \hat{\mathbb{Z}}_2), KO^n(X; \hat{\mathbb{Z}}_2)\}$$

has a natural stable 2-adic Adams CR-module structure with the usual operations.

**Definition 5.3** (The stable 2-adic Adams $\Delta$-modules). By a stable 2-adic Adams $\Delta$-module $N$, we mean a 2-adic $\Delta$-module consisting of stable 2-adic Adams modules $\{N_C, N_R, N_H\}$ such that the operations $t : N_C \cong N_C$, $c : N_R \to N_C$, $r : N_C \to N_R$, $c' : N_H \to N_C$, and $q : N_C \to N_H$ are all maps in $\hat{A}$, where $\psi^{-1} = t$ in $N_C$ and $\psi^{-1} = 1$ in both $N_R$ and $N_H$. For a stable 2-adic Adams CR-module $M$ and integer $n$, we obtain a stable 2-adic Adams $\Delta$-module

$$\Delta^n M = \{M^n_C, M^n_R, \hat{S}^{-2}M^n_R^{-4}\}$$

as in Definition 4.1. Thus, for a spectrum or space $X$ and integer $n$, we now obtain a stable 2-adic Adams $\Delta$-module

$$K^n_\Delta(X; \hat{\mathbb{Z}}_2) = \Delta^nK^n_{CR}(X; \hat{\mathbb{Z}}_2) = \{K^n(X; \hat{\mathbb{Z}}_2), KO^n(X; \hat{\mathbb{Z}}_2), \hat{S}^{-2}KO^n(X; \hat{\mathbb{Z}}_2)\}.$$ 

To give another example, we say that a 2-profinite abelian group $G$ with involution $t : G \cong G$ is positively torsion-free when $G$ is torsion-free with $\ker(1 + t) = \im(1 - t)$. By [5, Proposition 3.8], this is equivalent to saying that $G$ factors as a (possibly infinite) product of $\hat{\mathbb{Z}}_2$’s with $t = 1$ and $\hat{\mathbb{Z}}_2 \oplus t\hat{\mathbb{Z}}_2$’s. For a positively torsion-free stable 2-adic Adams module $A$, we may use the operation $\psi^{-1} : A \cong A$ to construct a torsion-free exact stable 2-adic Adams $\Delta$-module $\{A, A^+, A_+\}$ with $A^+ = \ker(1 - \psi^{-1})$, $A_+ = \coker(1 - \psi^{-1})$, $t = \psi^{-1}$, $c = 1$, $r = 1 + \psi^{-1}$, $c' = 1 + \psi^{-1}$, and $q = 1$.

We let $\hat{\mathcal{ACR}}$ (resp. $\hat{\mathcal{A}}\Delta$) denote the abelian category of stable 2-adic Adams CR-modules (resp. $\Delta$-modules), and we note that the functor $\Delta^n : \hat{\mathcal{ACR}} \to \hat{\mathcal{A}}\Delta$ for $n \in \mathbb{Z}$ has a left adjoint $CR^n : \hat{\mathcal{A}}\Delta \to \hat{\mathcal{ACR}}$ with $CR^n(N)^C_C = N_C$, with $CR^n(N)^C_{C-1} = 0$. 
and with

$$CR^n(N)_{R}^{n-i} = \begin{cases} 
N_R & \text{for } i = 0 \\
N_R/r & \text{for } i = 1 \\
SN_C/c' & \text{for } i = 2 \\
0 & \text{for } i = 3, 7 \\
S^2N_H & \text{for } i = 4 \\
S^2N_H/q & \text{for } i = 5 \\
\tilde{S}^3NC/c & \text{for } i = 6 
\end{cases}$$

as in [10, 4.10]. We easily see that $CR^n(N)$ is Bott exact whenever $N$ is torsion-free and exact. Our next lemma will often allow us to work in the simpler category $\tilde{A}\Delta$ instead of $A\Delta$.

**Lemma 5.4.** For $n \in \mathbb{Z}$, the adjoint functors $CR^n: \tilde{A}\Delta \rightarrow \tilde{ACR}$ and $\Delta^n: \tilde{ACR} \rightarrow \tilde{A}\Delta$ restrict to equivalences between the full subcategories of all torsion-free exact $N \in \mathcal{A}\Delta$ and all Bott exact $M \in \tilde{ACR}$ with $M^n_0$ positively torsion-free and $M_0^{n-1} = 0$.

**Proof.** For $M \in \tilde{ACR}$ as above, we see that $\Delta^n M$ is a torsion-free exact $\Delta$-module by [10, 4.4 and 4.7] with an adjunction isomorphism $CR^n\Delta^n M \rightarrow M$ by Lemma 2.3. The corresponding result for $N \in \tilde{A}\Delta$ is obvious. \hfill $\square$

When $E$ is a spectrum with $K^n(E; \mathbb{Z}/2)$ positively torsion-free and $K^{n-1}(E; \mathbb{Z}/2) = 0$ for some $n$, we now have $K^n_{CR}(E; \mathbb{Z}/2) \cong CR^n(N)$ in $\tilde{ACR}$ for the torsion-free exact module $N = \Delta^n K^n_{CR}(E; \mathbb{Z}/2)$ in $\tilde{A}\Delta$, and we have the following existence theorem for such spectra in the stable homotopy category.

**Theorem 5.5.** For each torsion-free exact $N \in \tilde{A}\Delta$ and $n \in \mathbb{Z}$, there exists a $K/2_*$-local spectrum $\mathcal{E}^n N$ with $K^n_{CR}(\mathcal{E}^n N; \mathbb{Z}/2) \cong CR^n(N)$ in $\tilde{ACR}$. Moreover, $\mathcal{E}^n N$ is unique up to (noncanonical) equivalence.

**Proof.** This follows by Lemma 5.4 and [10, Theorem 5.3]. \hfill $\square$

The spectrum $\mathcal{E}^n N$ in the theorem will be endowed with an isomorphism $K^n_{CR}(\mathcal{E}^n N; \mathbb{Z}/2) \cong CR^n(N)$ in $\tilde{ACR}$. Thus, for an arbitrary spectrum $E$, a map $g: E \rightarrow \mathcal{E}^n N$ induces a map $g^*: CR^n(N) \rightarrow K^n_{CR}(E; \mathbb{Z}/2)$ in $\tilde{ACR}$. Each algebraic map of this sort must come from a topological map by:

**Theorem 5.6.** For a torsion-free exact $N \in \tilde{A}\Delta$, $n \in \mathbb{Z}$, and an arbitrary spectrum $E$, if $\gamma: CR^n(N) \rightarrow K^n_{CR}(E; \mathbb{Z}/2)$ is a map in $\tilde{ACR}$, then there exists a map of spectra $g: E \rightarrow \mathcal{E}^n N$ with $g^* = \gamma$.

**Proof.** Let $\tau_2 E$ denote the 2-torsion part of $E$ given by the homotopy fiber of its localization away from 2. By Pontrjagin duality [10, Theorem 3.1], the map $\gamma$ corresponds to an $ACR$-module map $K^n_{CR}(\tau_2 E) \rightarrow K^n_{CR}(\tau_2 \mathcal{E}^n N)$ in the sense of [5], where $K^n_{CR}(\tau_2 \mathcal{E}^n N)$ is CR-exact with $K^n_{CR}(\tau_2 \mathcal{E}^n N)$ divisible. This $ACR$-module map prolongs canonically to an $ACRT$-module map $K^n_{CRT}(\tau_2 E) \rightarrow K^n_{CRT}(\tau_2 \mathcal{E}^n N)$ by [5, Theorem 7.14], and the results of [5, 9.8 and 7.11] now show that this prolonged
algebraic map must come from a topological map \( \tau_2 E \to \tau_2 \mathcal{E}^n N \), which gives the desired \( g: \mathcal{E} \to \mathcal{E}^n N \).

The map \( g \) in this theorem is generally not unique (see [10, 5.4]).

6. On the united 2-adic \( K \)-cohomologies of infinite loop spaces

In preparation for our work on \( K/2 \)-localizations of spaces, we functorially determine the united 2-adic \( K \)-cohomologies of the needed infinite loop spaces (see Theorem 6.7). We must first introduce:

**Definition 6.1** (The 2-adic Adams \( \Delta \)-modules). By a 2-adic Adams \( \Delta \)-module \( M \), we mean a 2-adic \( \theta \Delta \)-module (see Definition 4.3) consisting of stable 2-adic Adams modules \( \{ M_C, M_R, M_H \} \) such that the operations \( t: M_C \cong M_C, \ c: M_R \to M_C, \ r: M_C \to M_R, \ e': M_H \to M_C, \ q: M_C \to M_H, \ \theta: M_C \to M_C, \ \theta: M_R \to M_R, \) and \( \theta: M_H \to M_R \) are all maps in \( \mathcal{A} \), where \( \psi^{-1} = t \) in \( M_C \) and \( \psi^{-1} = 1 \) in both \( M_R \) and \( M_H \). We let \( \mathcal{M}\Delta \) denote the abelian category of 2-adic Adams \( \Delta \)-modules. We say that \( M \) is \( \theta \)-nilpotent when it has \( \theta^i = 0 \) for sufficiently large \( i \), and we say that \( M \) is \( \theta \)-pro-nilpotent when it is the inverse limit of an inverse system of \( \theta \)-nilpotent 2-adic Adams \( \Delta \)-modules. Thus, \( M \) is \( \theta \)-pro-nilpotent if and only if \( M \cong \lim_i M/\theta^i \) where \( M/\theta^i \) is the quotient module of \( M \) in \( \mathcal{M}\Delta \) with

\[
\begin{align*}
(M/\theta^i)_C &= M_C/\theta^i M_C, \\
(M/\theta^i)_R &= M_R/(\theta^i M_R + \theta^i M_H + r \theta^i M_C), \\
(M/\theta^i)_H &= M_H/q \theta^i M_C
\end{align*}
\]

for \( i \geq 1 \). More simply, \( M \) is \( \theta \)-pro-nilpotent if and only if \( \cap_i \theta^i M_C = 0 \) and \( \cap_i \theta^i M_R = 0 \). It is not hard to show that whenever \( M \) is \( \theta \)-pro-nilpotent, \( M \) must be profinite (i.e. \( M \) must be the inverse limit of an inverse system of finite 2-adic Adams \( \Delta \)-modules). For a space \( X \), the cohomology

\[
\overline{K}^{-1}_\Delta(X; \hat{Z}_2) = \{ \overline{K}^{-1}(X; \hat{Z}_2), \overline{KO}^{-1}(X; \hat{Z}_2), S^{-2} \overline{KO}^{-5}(X; \hat{Z}_2) \}
\]

has a natural 2-adic Adams \( \Delta \)-module structure by [11, Section 3], and we find:

**Lemma 6.2.** If \( X \) is a connected space with \( H^1(X; \hat{Z}_2) = 0 \), then the 2-adic Adams \( \Delta \)-module \( \overline{K}^{-1}_\Delta(X; \hat{Z}_2) \) is \( \theta \)-pro-nilpotent.

**Proof.** The condition \( \cap_i \theta^i \overline{K}^0(\Sigma X; \hat{Z}_2) = 0 \) holds by [6, 5.4 and 5.5] since \( H^2(\Sigma X; \hat{Z}_2) = 0 \), and a similar proof shows \( \cap_i \theta^i \overline{KO}^0(\Sigma X; \hat{Z}_2) = 0 \) since \( H^1(\Sigma X; \hat{Z}_2) = 0 \). This proof uses the fact that the \( \lambda \)-ideal \( \overline{KO}^0 \) is \( \gamma \)-nilpotent for a connected finite CW complex \( Y \) by [10, Theorem 6.7] and the fact that the real line bundles over \( Y \) are classified by \( H^1(Y; \mathbb{Z}/2) \). \( \square \)

**Definition 6.3** (The functor \( \tilde{F} \)). We shall construct a functor \( \tilde{F}: \hat{\mathcal{A}}\Delta \to \hat{\mathcal{M}}\Delta \) where \( \hat{\mathcal{A}} \Delta \) is the abelian category of stable 2-adic Adams \( \Delta \)-modules and \( \hat{\mathcal{M}} \Delta \)
We finally define operations $\hat{\iota}$ for $\tilde{\iota}$.

**Proof.**

Let $\rho N$ be the stable 2-adic Adams $\Delta$-module with operations given by

$$
\rho N = \{N_C, N_{RH} \oplus N_{C\phi}, N_{C+}\}
$$

be the map in $\hat{\iota}$ for $\tilde{\iota}$, $c(x,w) = \bar{c}x$, and $qz = (\bar{r}z, [z])$, $c'[z] = (1+t)z$, and $qz = [z]$. We then obtain a stable 2-adic Adams $\Delta$-module

$$
\tilde{\hat{\iota}} = N \times \rho N \times \rho N \times \cdots
$$

with components

$$
\tilde{\hat{\iota}}_N = N_C \times N_C \times N_C \times \cdots,
$$

$$
\tilde{\hat{\iota}}_R N = N_R \times N_{RH} \times N_{C\phi} \times N_{C\phi} \times \cdots,
$$

$$
\tilde{\hat{\iota}}_H N = N_H \times N_{C+} \times N_{C+} \times \cdots.
$$

We finally define operations $\theta: \tilde{\hat{\iota}}_C N \rightarrow \tilde{\hat{\iota}}_C N$, $\theta: \tilde{\hat{\iota}}_R N \rightarrow \tilde{\hat{\iota}}_R N$, and $\theta: \tilde{\hat{\iota}}_H N \rightarrow \tilde{\hat{\iota}}_R N$ respectively by the formulae

$$
\theta(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots),
$$

$$
\theta(x_1, x_2, x_3, z_3, \ldots) = (0, [x_1], 0, x_2, 0, x_3, 0, \ldots),
$$

$$
\theta(y_1, z_2, z_3, \ldots) = (0, [y_1], 0, \bar{r}z_2, 0, \bar{r}z_3, 0, \ldots).
$$

This gives a natural 2-adic Adams $\Delta$-module $\tilde{\hat{\iota}} N$ and hence a functor $\tilde{\hat{\iota}}: \hat{\Delta} \rightarrow \hat{\Delta}$. We let $\iota: N \rightarrow \tilde{\hat{\iota}} N$ be the map in $\hat{\Delta}$ with $\iota_{\theta}(z) = (z, 0, 0, \ldots)$, $\iota_{\theta}(x) = (x, 0, 0, \ldots)$, and $\iota_{\theta}(y) = (y, 0, 0, \ldots)$, and we show:

**Theorem 6.4.** For a stable 2-adic Adams $\Delta$-module $N \in \hat{\Delta}$, the 2-adic Adams $\Delta$-module $\tilde{\hat{\iota}} N \in \hat{\Delta}$ is $\theta$-pro-nilpotent and the map $\iota: N \rightarrow \tilde{\hat{\iota}} N$ has the universal property that, for each $\theta$-pro-nilpotent $M \in \hat{\Delta}$ and map $f: N \rightarrow M$ in $\hat{\Delta}$, there exists a unique map $f: \tilde{\hat{\iota}} N \rightarrow M$ in $\hat{\Delta}$ with $\tilde{\hat{\iota}} f = f$.

**Proof.**

$\tilde{\hat{\iota}} N$ is $\theta$-pro-nilpotent since it is the inverse limit of its quotient modules

$$
\tilde{\hat{\iota}} N / \theta^{i+1} \cong N \times \rho N \times \cdots \times \rho N.
$$

For $i \geq 1$, we define a map $f^{(i)}: \rho N \rightarrow M$ in $\hat{\Delta}$ by

$$
\begin{align*}
\hat{\iota}^{(i)} C &= \theta^i \hat{\iota} C: N_C \rightarrow M_C, \\
\hat{\iota}^{(i)} R &= (\theta^i \hat{\iota} R, \theta^i \hat{\iota} H) + \theta^i \hat{\iota} C: N_{RH} \oplus N_{C\phi} \rightarrow M_R, \\
\hat{\iota}^{(i)} H &= \theta^i \hat{\iota} H: N_{C+} \rightarrow M_H.
\end{align*}
$$

We then define $f: \tilde{\hat{\iota}} N \rightarrow M$ as the inverse limit of the maps

$$
\begin{align*}
f + f^{(1)} + \cdots + f^{(i)}: N \times \rho N \times \cdots \times \rho N \rightarrow M / \theta^{i+1}
\end{align*}
$$

in $\hat{\Delta}$, and we check that $\hat{\iota} f = f$. The uniqueness condition for $\hat{\iota} f$ follows since the
2-adic Adams \( \Delta \)-modules \( \tilde{F}N/\theta^{i+1} = N \times \rho N \times \cdots \times \rho N \) are generated by \( \ell N \). \( \square \)

To show the robustness (see Definition 4.7) of \( \tilde{F}N \) for suitable \( N \), we need:

**Definition 6.5** (The functor \( \bar{\rho}: \bar{\Delta} \rightarrow \bar{\Delta} \)). For \( N \in \bar{\Delta} \), we let \( \bar{\rho}N = \{ N_C, N_{RH}, N_{C^+} \} \) be the stable 2-adic Adams \( \Delta \)-module with operations given by \( tz = tz, \ cx = \bar{c}x, \ rz = \bar{r}z, \ c'[z] = (1 + t)z \), and \( qz = [z] \). Thus, \( \bar{\rho}N \) is the quotient of \( \rho N = \{ N_C, N_{RH} + N_{C^+}, N_{C^+} \} \) by \( N_{C^+} \). If \( N \) is torsion-free and exact, then \( \bar{\rho}N \) is also torsion-free and exact by Lemma 4.2 since it is isomorphic to the module \( \{ N_C, N_R + N_H, N_R \cap N_H \} \) with \( c \) and \( c' \) treated as inclusions.

**Lemma 6.6.** If \( N \in \bar{\Delta} \) is torsion-free and exact, then \( \tilde{F}N \in \bar{M} \Delta \) is robust.

**Proof.** We check that \( \bar{\phi}: \tilde{F}C N \rightarrow \tilde{F}R N \) is given by

\[
\bar{\phi}(z_1, z_2, z_3, \ldots) = (0, 0, [z_1], 0, [z_2], 0, \ldots)
\]

for \( z_i \in N_C \) and \( [z_i] \in N_{C^+} \). Thus, \( \ker \bar{\phi} = c\tilde{F}R N + \tilde{c}\tilde{F}H N + 2\tilde{F}C N \) and \( \tilde{F}N/\bar{\phi} \cong N \times \tilde{\rho}N \times \tilde{\rho}N \times \cdots \). Hence, \( \tilde{F}N/\bar{\phi} \) is torsion-free and exact by Definition 6.5 as required. \( \square \)

Our main result in this section is:

**Theorem 6.7.** If \( E \) is a 0-connected spectrum with \( H^1(E; \tilde{\mathbb{Z}}_2) = 0 = H^2(E; \tilde{\mathbb{Z}}_2) \), with \( K^0(E; \tilde{\mathbb{Z}}_2) = 0 \), and with \( K^{-1}(E; \tilde{\mathbb{Z}}_2) \) positively torsion-free (5.3), then there is a natural isomorphism \( \tilde{L}FK_{\Delta}^{-1}(E; \tilde{\mathbb{Z}}_2) \cong K_{CR}^*(\Omega^\infty E; \tilde{\mathbb{Z}}_2) \).

**Proof.** Since \( K_{\Delta}^{-1}(\Omega^\infty E; \tilde{\mathbb{Z}}_2) \) is \( \theta \)-pro-nilpotent by Lemma 6.2, the infinite suspension map \( \sigma: K_{\Delta}^{-1}(E; \tilde{\mathbb{Z}}_2) \rightarrow K_{\Delta}^{-1}(\Omega^\infty E; \tilde{\mathbb{Z}}_2) \) induces a map \( \bar{\sigma}: \tilde{F}K_{\Delta}^{-1}(E; \tilde{\mathbb{Z}}_2) \rightarrow \tilde{K}_{\Delta}^{-1}(\Omega^\infty E; \tilde{\mathbb{Z}}_2) \) in \( \bar{M} \Delta \), where \( \tilde{F}K_{\Delta}^{-1}(E; \tilde{\mathbb{Z}}_2) \) is robust by Lemmas 5.4 and 6.6. Thus \( \bar{\sigma} \) induces an isomorphism \( \tilde{L}FK_{\Delta}^{-1}(E; \tilde{\mathbb{Z}}_2) \cong K_{CR}^*(\Omega^\infty E; \tilde{\mathbb{Z}}_2) \) by Theorem 4.9, since it induces an isomorphism of the complex components by [6, Theorem 8.3]. \( \square \)

## 7. Strong 2-adic Adams \( \Delta \)-modules

Our main results on \( K/2 \)-localizations in Section 8 will involve a space \( X \) with \( K_{CR}^*(X; \tilde{\mathbb{Z}}_2) \cong LM \) for a 2-adic Adams \( \Delta \)-module \( M \) that is strong in the sense that it is robust, \( \psi^3 \)-splittable, and regular. In this section, we provide the required algebraic definitions and explanations of these notions. We first recall:

**Definition 7.1** (The robust modules). We say that a 2-adic Adams \( \Delta \)-module \( M \) is robust when it is robust in the sense of Definition 4.7, ignoring stable Adams operations. When \( M \) is robust, the underlying 2-adic \( \Delta \)-module \( M/\bar{\phi} \) satisfies the conditions of Lemma 4.2 and may be factored as a (possibly infinite) product of
monogenic free 2-adic $\Delta$-modules

\[
\begin{align*}
F^C(z) &= \{ \hat{z} + t\hat{z}, \hat{z}, \hat{z} \} = \{ \langle z \rangle + \langle tz \rangle, \langle rz \rangle, \langle qz \rangle \}, \\
F^R(x) &= \{ \hat{z}, \hat{z}, \hat{z} \} = \{ \langle cx \rangle, \langle x \rangle, \langle qcx \rangle \}, \\
F^H(y) &= \{ \hat{z}, \hat{z}, \hat{z} \} = \{ \langle c', y \rangle, \langle rc' \rangle, \langle y \rangle \}
\end{align*}
\]

by an argument using the factorization of positively torsion-free groups in Definition 5.3. We let $\operatorname{gen}_C M$, $\operatorname{gen}_R M$, and $\operatorname{gen}_H M$ respectively denote the number of complex, real, and quaternionic monogenic free factors of $M/\hat{\phi}$. These numbers do not depend on the factorization since they equal the dimensions of the respective $Z/2$-vector spaces $(M_{C(0)})^\#$, $(M_{R}/(\hat{\phi}M + rM))^\#$, and $(M_{H}/qM)^\#$, where $(-)^\#$ is the Pontrjagin duality functor from 2-profinite abelian groups to discrete 2-torsion abelian groups. Using the factorization of $M/\hat{\phi}$, we find that

\[
\operatorname{gen}_C M = 2 \operatorname{gen}_C M + \operatorname{gen}_R M + \operatorname{gen}_H M
\]

where $\operatorname{gen}_M$ denotes the number of $\hat{Z}_2$ factors in the 2-profinite abelian group $M_C$.

**Definition 7.2** (The $\psi^3$-splittable modules). For a 2-adic Adams $\Delta$-module $M \in \mathcal{M}\Delta$, we consider the stable 2-adic Adams $\Delta$-module $M = M/\hat{\phi} \in \mathcal{A}\Delta$, and we say that $M$ is $\psi^3$-splittable when the quotient map $M \to \hat{M}$ has a right inverse $s: \hat{M} \to M$ in $\mathcal{A}\Delta$. We call such a map $s$ a $\psi^3$-splitting of $M$, and we note that it corresponds to a left inverse $s': \mathcal{A}_R/rM \to \hat{\phi}M$ of the canonical map $\hat{\phi}M \to \mathcal{A}_R/rM$ in the category $\mathcal{A}$ of stable 2-adic Adams modules, or equivalently in the category of profinite $Z/2$-modules with automorphisms $\psi^3$. We deduce that $M$ is automatically $\psi^3$-splittable in some important cases:

**Lemma 7.3.** If $M$ is a robust 2-adic Adams $\Delta$-module with $\operatorname{gen}_C M = 0$ or $\operatorname{gen}_R M = 0$, then $M$ is $\psi^3$-splittable.

**Proof.** Since $M_C$ is positively torsion-free, the map $cr = 1 + t: M_{C+} \to M_C$ is monic, and hence $c: rM_C \to M_C$ is also monic. Thus, $\hat{\phi}M_C \cap rM_C = 0$ and there is a short exact sequence

\[
0 \to \hat{\phi}M_C \to M_R/rM_C \to M_{R}/(\hat{\phi}M_C + rM_C) \to 0
\]

in $\mathcal{A}$. Since $\operatorname{gen}_C M = 0$ or $\operatorname{gen}_R M = 0$, this has $\hat{\phi}M_C = 0$ or $M_{R}/(\hat{\phi}M_C + rM_C) = 0$, and hence the map $\hat{\phi}M_C \to M_R/rM_C$ has an obvious left inverse in $\mathcal{A}$. \hfill \Box

We shall use the $\psi^3$-splitability condition to give:

**Definition 7.4** (The $\theta$-resolutions of modules). Let $M \in \mathcal{M}\Delta$ be a 2-adic Adams $\Delta$-module that is $\theta$-pro-nilpotent, robust, and $\psi^3$-splittable. These conditions will hold when $M$ is strong (see Definition 7.11). For a $\psi^3$-splitting $s: \hat{M} \to M$ in $\mathcal{A}\Delta$, we shall construct an associated $\theta$-resolution

\[
0 \to \hat{\phi}M \xrightarrow{\hat{d}} \hat{F} \hat{M} \xrightarrow{\hat{s}} M \to 0
\]

of $M$ in $\mathcal{M}\Delta$, with $\hat{\phi}M = \{ \hat{M}_C, \hat{M}_{RH}, \hat{M}_{C+} \}$ as in Definition 6.5, where $\hat{s}: \hat{F} \hat{M} \to$
$M$ is induced by $s$ via Theorem 6.4. To specify $d$, we use the commutative square
\[
\begin{array}{ccc}
\rho \tilde{M} & \xrightarrow{\theta} & \tilde{M} \\
\downarrow & & \downarrow \\
\rho \tilde{M} & \xrightarrow{s^{(1)}} & M
\end{array}
\]
in $\tilde{\Delta}$ with $\rho \tilde{M} = \{\tilde{M}_C, M_{RH} \oplus M_{C_\phi}, \tilde{M}_{C_+}\}$ as in Definition 6.3, where $s^{(1)}$ is given by the proof of Theorem 6.4, where $\theta = \{\theta, (\theta, \theta), q\theta\}$, and where $\sigma = \{1, (1, \theta_\phi), 1\}$, using the map $\theta_\phi: M_{RH} \to \tilde{M}_{C_\phi} = M_{C_\phi}$ given by the composition of the sequence
\[
\tilde{M}_{RH} \xrightarrow{\rho} M_{RH} \xrightarrow{(\rho, \theta)} M_R \cong \tilde{M}_R \oplus M_{C_\phi} \xrightarrow{\text{proj}} M_{C_\phi}
\]
in which the isomorphism is the inverse of $(s, \tilde{\phi}): M_R \oplus M_{C_\phi} \cong M_R$. The commutative square now gives a map
\[
d = (\theta, -\sigma, 0, 0, \ldots): \rho \tilde{M} \to \tilde{F}\tilde{M}
\]
in $\tilde{\Delta}$ with $s\tilde{d} = 0$, and this induces the required map $\tilde{d}: \tilde{F}\rho \tilde{M} \to \tilde{F}\tilde{M}$ in $\tilde{\Delta}$ with $\tilde{s}\tilde{d} = 0$.

Lemma 7.5. If $M \in \tilde{\Delta}$ is $\theta$-pro-nilpotent and robust with a $\psi$-splitting $s: \tilde{M} \to M$, then the $\theta$-resolution $0 \to \tilde{F}\rho \tilde{M} \xrightarrow{\tilde{d}} \tilde{F}\tilde{M} \xrightarrow{\tilde{s}} M \to 0$ is exact in $\tilde{\Delta}$.

Proof. We easily check that $0 \to \tilde{M}(\tilde{F}\rho \tilde{M})_C \to \tilde{\phi}(\tilde{F}\tilde{M})_C \to \tilde{\phi}M_C \to 0$ is exact and that $\tilde{s}/\tilde{\phi}: \tilde{F}M/\tilde{\phi} \to M/\tilde{\phi}$ is onto. Hence, it suffices to show that the map $\tilde{F}\rho \tilde{M}/\tilde{\phi} \to \ker(\tilde{s}/\tilde{\phi})$ is an isomorphism. This follows by [10, Lemma 4.8] since the map $(\tilde{F}\rho \tilde{M}/\tilde{\phi})_C \to \ker(\tilde{s}/\tilde{\phi})_C$ is clearly an isomorphism and since the 2-adic $\Delta$-modules $\tilde{F}\rho \tilde{M}/\tilde{\phi}$ and $\ker(\tilde{s}/\tilde{\phi})$ are exact by Lemma 6.6 and by the short exact sequence rule of [10, 4.5].

To formulate our regularity condition for $M$, we use:

Definition 7.6 (The 2-adic Adams modules). These are the unstable versions of the stable 2-adic Adams modules and were previously discussed in [8, 28]. By a finite 2-adic Adams module $A$, we mean a finite abelian 2-group with endomorphisms $\psi^k: A \to A$ for $k \in \mathbb{Z}$ such that:

(i) $\psi^1 = 1$ and $\psi^j \psi^k = \psi^{jk}$ for $j, k \in \mathbb{Z}$;

(ii) when $n$ is sufficiently large, the condition $j \equiv k \mod 2^n$ implies $\psi^j = \psi^k$.

By a 2-adic Adams module $A$, we mean the topological inverse limit of an inverse system of finite 2-adic Adams modules. Such an $A$ has an underlying 2-profinite abelian group with continuous endomorphisms $\psi^k: A \to A$ for $k \in \mathbb{Z}$ (and in fact for $k \in \hat{\mathbb{Z}}_2$). For a space $X$, the cohomology $K^3(X; \hat{\mathbb{Z}}_2)$ is a 2-adic Adams module with the usual Adams operations $\psi^k$ for $k \in \hat{\mathbb{Z}}_2$ as in [6, Example 5.2]. We note that the operations $\psi^2$ and $\psi^k$, for $k$ odd, in $K^3(X; \hat{\mathbb{Z}}_2)$ correspond via Bott periodicity to $\theta$ and to $k^{-1}\psi^k$ in $K^{-1}(X; \hat{\mathbb{Z}}_2)$. In general, for a $\theta$-pro-nilpotent 2-adic Adams $\Delta$-module $M$, we obtain a 2-adic Adams module $M_C$ having the same group as $M_C$ but having $\psi^0 = 0$ and having $\psi^{k^{2^i}}$ equal to $k^{-1}\psi^k\theta^i$ on $M_C$ for $k$ odd and $i \geq 0$. 
Definition 7.7 (The linear and strictly nonlinear modules). As in [8, Section 4] and [7, Section 2], a 2-adic Adams module $H$ is called linear when it has $\psi^k = k$ for all $k \in \mathbb{Z}$, and $H$ is called quasilinear when $2H \subset \psi^2 H$. Each 2-adic Adams module $A$ has a largest linear quotient module

$$\text{Lin} A = A/((\psi^2 - 2)A + (\psi^{-1} + 1)A + (\psi^3 - 3)A)$$

and also has a largest quasilinear submodule $A_{ql} \subset A$ by Lemma 13.1 below. A 2-adic Adams module $A$ is called strictly nonlinear when $A_{ql} = 0$. This implies that $A$ is torsion-free with $\cap_i (\psi^2)^i A = 0$, and $A$ will be strictly nonlinear by Remark 13.2 and [7, 2.5] whenever it is torsion-free with $(\psi^2)^i A \subset 2^{i+1} A$ for some $i \geq 1$.

Definition 7.8 (The regular modules). As in [8, 4.4], we say that a 2-adic Adams module $A$ is regular when the kernel of $A \to \text{Lin} A$ is strictly nonlinear. This implies that $\cap_i (\psi^2)^i A = 0$, and $A$ will be regular whenever it is an extension of a strictly nonlinear submodule by a linear quotient module. We also say that a 2-adic Adams $\Delta$-module $M$ is regular when it is $\theta$-pro-nilpotent with $M^C$ regular as a 2-adic Adams module. For a connected space $X$ with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, the 2-adic Adams $\Delta$-module $\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ is always $\theta$-pro-nilpotent by Lemma 6.2, and hence $\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular if and only if $\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular as a 2-adic Adams module. The following two lemmas will often guarantee regularity for our modules.

Lemma 7.9. Let $X$ be a connected space with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, with $H^m(X; \hat{\mathbb{Z}}_2) = 0$ for sufficiently large $m$, and with $\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ torsion-free. Then $\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular with $\psi^2: \tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2) \to \tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ monic, and hence $\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular with $\theta: \tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2) \to \tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ monic.

Lemma 7.10. For a regular 2-adic Adams module $A$, each submodule is regular, and each torsion-free quotient module is regular when $A$ is finitely generated over $\hat{\mathbb{Z}}_2$.

The proofs are in Section 13. Combining the preceding definitions, we finally introduce:

Definition 7.11 (The strong modules). We say that a 2-adic Adams $\Delta$-module $M \in \mathcal{M}\Delta$ is strong when:

(i) $M$ is robust;

(ii) $M$ is $\psi^3$-splittable;

(iii) $M$ is regular.

Such an $M$ is automatically $\theta$-pro-nilpotent (and hence profinite) since it is regular.

8. On the $K/2_\ast$-localizations of our spaces

We recall that the $K/2_\ast$-localizations of spaces or spectra are the same as the $K^\ast(-; \hat{\mathbb{Z}}_2)$-localizations since the $K/2_\ast$-equivalences are the same as the $K^\ast(-; \hat{\mathbb{Z}}_2)$-equivalences. In this section, we give our main result (Theorem 8.6) on the $K/2_\ast$-localization of a connected space $X$ with $K^\ast_{CR}(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a strong 2-adic Adams $\Delta$-module $M$. We first consider:
Definition 8.1 (Building blocks for $K/2_\ast$-localizations). For a torsion-free exact stable 2-adic Adams $\Delta$-module $N \in \hat{\Delta}$, we let $\mathcal{E}N$ denote the $K/2_\ast$-local spectrum $\mathcal{E}^{-1}N$ of Theorem 5.5 with an isomorphism $K_{CR}^\ast(\mathcal{E}N; \hat{\mathbb{Z}}_2) \cong CR^{-1}N$ in the category $\hat{\mathcal{ACR}}$ of stable 2-adic Adams $CR$-modules. As in $[8, 3.5]$, we let $\hat{\mathcal{E}}N \to \mathcal{E}N \to \hat{P}^2\mathcal{E}N$ denote the Postnikov fiber sequence of spectra with $\pi_i\mathcal{E}N \cong \pi_i\hat{\mathcal{E}}N$ for $i > 2$, with $\pi_i\mathcal{E}N = 0$ for $i < 2$, and with $\pi_2\mathcal{E}N \cong \hat{\mathcal{E}}_2\mathcal{E}N$, where $\hat{\mathcal{E}}_2\mathcal{E}N \subset \pi_2\mathcal{E}N$ denotes the Ext-2-completion of the torsion subgroup of $\pi_2\mathcal{E}N$. We now obtain a simply-connected infinite loop space $\Omega^\infty\mathcal{E}N$ which is $K/2_\ast$-local by $[8, \text{Theorem 3.8}]$. These $\Omega^\infty\mathcal{E}N$, with their companions $\Omega^\infty\hat{\mathcal{E}}\hat{\rho}\mathcal{N}$, will serve as our building blocks for $K/2_\ast$-localizations of spaces, where $\hat{\rho}\mathcal{N}$ denotes the torsion-free exact stable 2-adic Adams $\Delta$-module $\rho\mathcal{N} = \{N_C, N_R + N_H, N_R \cap N_H\}$ of Definition 6.5.

Definition 8.2 (Strict homomorphisms and isomorphisms). For a 2-adic Adams $\Delta$-module $M \in \hat{\mathcal{M}}\Delta$ and a connected space $X$, a strict homomorphism (resp. strict isomorphism) $\hat{\mathcal{L}}M \to K_{CR}(X; \hat{\mathbb{Z}}_2)$ is a homomorphism (resp. isomorphism) of special 2-adic $\phi CR$-algebras induced by a map $M \to K^{-1}_\Delta(X; \hat{\mathbb{Z}}_2)$ of 2-adic Adams $\Delta$-modules. For instance, there is a strict isomorphism

$$\hat{\mathcal{L}}\mathcal{F}N \cong K_{CR}^\ast(\Omega^\infty\mathcal{E}N; \hat{\mathbb{Z}}_2)$$

for each torsion-free exact stable 2-adic Adams $\Delta$-module $N \in \hat{\Delta}$ by Theorem 6.7, and we have:

Lemma 8.3. For a torsion-free exact module $N \in \hat{\Delta}$ and a connected space $X$ with $H^1(X; \hat{\mathbb{Z}}_2) = 0 = H^2(X; \hat{\mathbb{Z}}_2)$, each strict homomorphism $\hat{\mathcal{L}}\mathcal{F}N \to K_{CR}^\ast(X; \hat{\mathbb{Z}}_2)$ is induced by a (possibly non-unique) map $X \to \Omega^\infty \mathcal{E}N$.

Proof. A strict homomorphism $\hat{\mathcal{L}}\mathcal{F}N \to K_{CR}^\ast(X; \hat{\mathbb{Z}}_2)$ corresponds successively to: a map $\hat{\mathcal{L}}\mathcal{F}N \to K^{-1}_\Delta(X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$, a map $N \to K^{-1}_\Delta(X; \hat{\mathbb{Z}}_2)$ in $\hat{\Delta}$, and a map $CR^{-1}N \to K_{CR}^\ast(\Omega^\infty X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{ACR}}$. By Theorem 5.6, this last map is induced by a map $\Omega^\infty X \to \mathcal{E}N$, which lifts uniquely to a map $\Sigma^\infty X \to \mathcal{E}N$, and we can easily check that the adjoint map $X \to \Omega^\infty \mathcal{E}N$ induces the original strict homomorphism.

Definition 8.4 (The key construction). For a strong 2-adic Adams $\Delta$-module $M \in \hat{\mathcal{M}}\Delta$, we may take a $\theta$-resolution (see Definition 7.4)

$$0 \to \hat{\mathcal{F}}\hat{\rho}\hat{M} \xrightarrow{d} \hat{\mathcal{F}}\hat{M} \xrightarrow{\hat{s}} M \to 0$$

using the torsion-free exact module $\hat{M} = M/\hat{\phi} \in \hat{\Delta}$. We may then apply Lemma 8.3 to give a map $f : \Omega^\infty \hat{\mathcal{F}}\hat{M} \to \Omega^\infty \hat{\mathcal{E}}\hat{\rho}\hat{M}$ inducing the $K_{CR}^\ast(-; \hat{\mathbb{Z}}_2)$-homomorphism $f^* = Ld : \hat{\mathcal{L}}\hat{F}\hat{\rho}\hat{M} \to \hat{L}\hat{F}\hat{M}$. Any such $f$ will be called a companion map of $M$, and its homotopy fiber $\text{Fib} f$ will be $K/2_\ast$-local since $\Omega^\infty \hat{\mathcal{E}}\hat{M}$ and $\Omega^\infty \hat{\mathcal{E}}\hat{\rho}\hat{M}$ are. As in $[8, 4.6]$ and Definition 8.1, we let

$$\hat{\text{Fib}} f \to \text{Fib} f \to \hat{P}^2 \text{Fib} f$$

denote the Postnikov fiber sequence with $\pi_i\hat{\text{Fib}} f \cong \pi_i\text{Fib} f$ for $i > 2$, with $\pi_i\hat{\text{Fib}} f = 0$ for $i < 2$, and with $\pi_i\hat{\text{Fib}} f \cong \hat{\mathcal{E}}_2\pi_2\text{Fib} f$. We note that $\hat{P}^2 \text{Fib} f$ is an infinite loop
space which is $K/2_*$-local by [8, Theorem 3.8], and we conclude that $\overline{\text{Fib}}f$ is also $K/2_*$-local. Moreover, we have $K^*_{CR}(\overline{\text{Fib}}f; \overline{\mathbb{Z}_2}) \cong \hat{LM}$ by:

**Theorem 8.5.** For a strong 2-adic Adams $\Delta$-module $M \in \hat{M} \Delta$ and any companion map $f: \Omega^{\infty}\tilde{E}M \to \Omega^{\infty}\tilde{E}\rho M$, there is a strict isomorphism $\hat{LM} \cong K^*_{CR}(\overline{\text{Fib}}f; \overline{\mathbb{Z}_2})$.

Thus, $\hat{LM}$ is topologically realizable for each strong $M \in \hat{M} \Delta$. This theorem will be proved in Section 14 and leads immediately to our main result on $K/2_*$-localizations of spaces.

**Theorem 8.6.** If $X$ is a connected space with a strict isomorphism $\hat{LM} \cong K^*_{CR}(X; \overline{\mathbb{Z}_2})$ for a strong 2-adic Adams $\Delta$-module $M \in \hat{M} \Delta$, then there is an equivalence $X_{K/2} \simeq \overline{\text{Fib}}f$ for some companion map $f: \Omega^{\infty}\tilde{E}M \to \Omega^{\infty}\tilde{E}\rho M$ of $M$, where the equivalence induces the canonical isomorphism $K^*_{CR}(\overline{\text{Fib}}f; \overline{\mathbb{Z}_2}) \cong \hat{LM} \cong K^*_{CR}(X; \overline{\mathbb{Z}_2})$. Moreover, $H^1(X; \overline{\mathbb{Z}_2}) = 0 = H^2(X; \overline{\mathbb{Z}_2})$.

**Proof.** The last statement follows by [6, 5.4]. For the first, we take a $\theta$-resolution $0 \to \tilde{E}\rho M \to \tilde{E}M \to M \to 0$ of $M$ and apply Lemma 8.3 to give a map $h: X \to \Omega^{\infty}\tilde{E}M$ with $h^* = \tilde{L}s: \tilde{L}\hat{E}M \to \hat{LM}$. We then apply Lemma 8.3 again to give a map $k^*: \text{Cof} h \to \Omega^{\infty}\tilde{E}\rho M$ with

$$k^* = \tilde{L}d: \tilde{L}\hat{E}M \to K^*_{CR}(\text{Cof} h; \overline{\mathbb{Z}_2}) \subset \tilde{L}\hat{E}M.$$ 

Composing $k$ with the cofiber map, we obtain a companion map $f: \Omega^{\infty}\tilde{E}M \to \Omega^{\infty}\tilde{E}\rho M$ of $M$ such that $h$ lifts to a map $u: X \to \overline{\text{Fib}}f$ which is a $K/2_*$-equivalence by Theorem 8.5. Since $\overline{\text{Fib}}f$ is $K/2_*$-local, this gives the desired equivalence $X_{K/2} \simeq \overline{\text{Fib}}f$. 

In this theorem, $M$ is uniquely determined by the space $X$ since there is a canonical isomorphism $M \cong \check{Q}K^{-1}_{\Delta}X(\overline{\mathbb{Z}_2})$ in $\hat{M} \Delta$ by Remark 4.10 and [11, Section 3].

9. **On the $v_1$-periodic homotopy groups of our spaces**

The $p$-primary $v_1$-periodic homotopy groups $v_1^{-1}\pi_* X$ of a space $X$ at a prime $p$ were defined by Davis and Mahowald [15] and have been studied extensively (see [13]). In this section, we apply the preceding result (Theorem 8.6) on the $K/2_*$-localizations of our spaces to approach $v_1$-periodic homotopy groups at $p = 2$ using:

**Definition 9.1** (The functor $\Phi_1$). As in [4], [9], [16], and [18], there is a $v_1$-stabilization functor $\Phi_1$ from the homotopy category of spaces to that of spectra such that:

(i) for a space $X$, there is a natural isomorphism $v_1^{-1}\pi_* X \cong \pi_* \tau_2 \Phi_1 X$ where $\tau_2 \Phi_1 X$ is the 2-torsion part of $\Phi_1 X$ (given by the fiber of its localization away from 2);

(ii) $\Phi_1 X$ is $K/2_*$-local for each space $X$;
(iii) for a spectrum $E$, there is a natural equivalence $\Phi_1(\Omega^\infty E) \simeq E_{K/2}$;
(iv) $\Phi_1$ preserves fiber squares.

Various other properties of $\Phi_1$ are described in \cite{10, Section 2}, and the isomorphism $v_1^{-1}\pi_*X \cong \pi_*\tau_2\Phi_1X$ may be applied as in \cite[Theorem 3.2]{10} to show:

**Theorem 9.2.** For a space $X$, there is a natural long exact sequence

\[ \cdots \to KO^{n-3}(\Phi_1X; \hat{Z}_2) \xrightarrow{\psi^3-9} KO^{n-3}(\Phi_1X; \hat{Z}_2) \to (v_1^{-1}\pi_nX)^# \]
\[ \to KO^{n-2}(\Phi_1X; \hat{Z}_2) \xrightarrow{\psi^3-9} KO^{n-2}(\Phi_1X; \hat{Z}_2) \to \cdots \]

where $(-)^#$ is the Pontrjagin duality functor from discrete 2-torsion abelian groups to 2-profinite abelian groups.

This may be used to calculate $v_1^{-1}\pi_*X$ from $KO^*(\Phi_1X; \hat{Z}_2)$ up to extension. To approach $KO^*(\Phi_1X; \hat{Z}_2)$ or $K^*(\Phi_1X; \hat{Z}_2)$, we require:

**Definition 9.3** (The $K/2_+$-durable spaces). Following \cite[7.8]{8}, we say that a space $X$ is $K/2_+$-durable when the $K/2_+$-localization $X \to X_{K/2}$ induces an equivalence $\Phi_1X \simeq \Phi_1X_{K/2}$ (or equivalently induces an isomorphism $v_1^{-1}\pi_*X \cong v_1^{-1}\pi_*X_{K/2}$), and we recall that each connected $H$-space is $K/2_+$-durable. For such $X$, we may apply our key result on $K/2_+$-localizations (Theorem 8.6) to deduce:

**Theorem 9.4.** If $X$ is a connected $K/2_+$-durable space (e.g. $H$-space) with a strict isomorphism $\hat{LM} \cong K^*_{CR}(X; \hat{Z}_2)$ for a strong module $M \in \hat{M}\Delta$, then there is a (co)fiber sequence of spectra $\Phi_1X \to E\hat{M} \xrightarrow{\pi} E\hat{p}M$ such that $\epsilon^*: K^*_{CR}(E\hat{p}M; \hat{Z}_2) \to K^*_{CR}(E\hat{M}; \hat{Z}_2)$ is given by $CR^{-1}\epsilon: CR^{-1}\hat{p}M \to CR^{-1}\hat{M}$.

Here, the map $\theta: \hat{p}M \to \hat{M}$ is given by

\[ \theta = (\theta, \theta, \theta): \{\hat{M}_C, \hat{M}_R + \hat{M}_H, \hat{M}_R \cap \hat{M}_H\} \to \{\hat{M}_C, \hat{M}_R, \hat{M}_H\} \]

in $\hat{M}\Delta$. This theorem will be proved below and may be used to calculate $K^*(\Phi_1X; \hat{Z}_2)$ and $KO^*(\Phi_1X; \hat{Z}_2)$ since it immediately implies:

**Theorem 9.5.** For $X$ as in Theorem 9.4, there is a $K^*(-; \hat{Z}_2)$ cohomology exact sequence

\[ 0 \to K^{-2}(\Phi_1X; \hat{Z}_2) \to \hat{M}_C \xrightarrow{\theta} \hat{M}_C \to K^{-1}(\Phi_1X; \hat{Z}_2) \to 0, \]

and there is a $KO^*(-; \hat{Z}_2)$ cohomology exact sequence

\[ 0 \to KO^{-8}(\Phi_1X; \hat{Z}_2) \to \hat{M}_C/(\hat{M}_R + \hat{M}_H) \xrightarrow{\theta} \hat{M}_C/\hat{M}_R \to KO^{-7}(\Phi_1X; \hat{Z}_2) \to 0 \to \hat{M}_H/(\hat{M}_R \cap \hat{M}_H) \to KO^{-6}(\Phi_1X; \hat{Z}_2) \to \hat{M}_R \cap \hat{M}_H \xrightarrow{\theta} \hat{M}_H \to KO^{-5}(\Phi_1X; \hat{Z}_2) \to 0 \to 0 \to KO^{-4}(\Phi_1X; \hat{Z}_2) \to \hat{M}_C/(\hat{M}_R \cap \hat{M}_H) \xrightarrow{\theta} \hat{M}_C/\hat{M}_H \to KO^{-3}(\Phi_1X; \hat{Z}_2) \to (\hat{M}_R + \hat{M}_H)/(\hat{M}_R \cap \hat{M}_H) \xrightarrow{\theta} \hat{M}_R/(\hat{M}_R \cap \hat{M}_H) \to KO^{-2}(\Phi_1X; \hat{Z}_2) \to \hat{M}_R + \hat{M}_H \xrightarrow{\theta} \hat{M}_R \to KO^{-1}(\Phi_1X; \hat{Z}_2) \to 0. \]
In these sequences, θ may be replaced by $\chi^2 = -\theta$. Also, for $i, k \in \mathbb{Z}$ with $k$ odd, the Adams operation $\psi^k$ in $K^{2i-1}(\Phi X; \hat{Z}_2)$, $K^{2i-2}(\Phi X; \hat{Z}_2)$, or $K\Omega^{2i-2}(\Phi X; \hat{Z}_2)$ agrees with $k^{-1}\psi^k$ in the adjacent $\hat{M}$ terms.

Thus, for $X$ as in Theorem 9.4, we may essentially calculate $v_1^{-1} \pi_* X$ from $\hat{M}$ (up to extension problems) using Theorems 9.2 and 9.5. By [10, 7.6], this approach to $v_1^{-1} \pi_* X$ may be extended to various other important spaces $X$ using:

**Definition 9.6** (The $\hat{K}\Phi_1$-goodness condition). For a space $X$, we let $\Phi_1 : \hat{K}^{CR}_0(X; \hat{Z}_2) \to \hat{K}^{CR}_0(\Phi X; \hat{Z}_2)$ denote the $v_1$-stabilization homomorphism of [10, 7.1], and we recall that it induces a homomorphism $\Phi_1 : \hat{Q}K^0_\Delta(X; \hat{Z}_2)/\theta \to \hat{K}_\Delta(\Phi X; \hat{Z}_2)$ in $\hat{A}\hat{\Delta}$ for $n = -1, 0$ by [10, 7.4], where $\hat{Q}K^0_\Delta(X; \hat{Z}_2)/\theta$ is as in Remark 4.10 and Definition 6.1. Following [10, 7.5], we say that a space $X$ is $\hat{K}\Phi_1$-good when the complex $v_1$-stabilization homomorphism $\Phi_1 : \hat{Q}K^n(X; \hat{Z}_2)/\theta \to K^n(\Phi X; \hat{Z}_2)$ is an isomorphism for $n = -1, 0$. Our next theorem will provide initial examples of $\hat{K}\Phi_1$-good spaces from which other examples may be built.

**Theorem 9.7.** If $X$ is a connected $K/2_*$-durable space (e.g. $H$-space) with a strict isomorphism $\hat{L}M \cong \hat{K}^{CR}_0(X; \hat{Z}_2)$ for a strong module $M \in \hat{M}\Delta$ such that $\theta : \hat{M}_C \to \hat{M}_C$ is monic, then $X$ is $\hat{K}\Phi_1$-good with $K^0(\Phi X; \hat{Z}_2) = 0$, with $K^{-1}(\Phi X; \hat{Z}_2) = \hat{M}_C/\theta$, and with $K^{-1}(\Phi X; \hat{Z}_2) \cong \hat{M}/\theta$.

To prove Theorems 9.4 and 9.7, we first consider the spectrum $\hat{E}N$ for a torsion-free exact module $N \in \hat{A}\hat{\Delta}$ and note that $\Phi_1 \hat{\Omega}^\infty \hat{E}N \cong (\hat{E}N)_{K/2} \cong \hat{E}N$.

**Lemma 9.8.** The space $\hat{\Omega}^\infty \hat{E}N$ is $\hat{K}\Phi_1$-good, and the $v_1$-stabilization gives a natural isomorphism

$$\Phi_1 : \hat{Q}K^1_\Delta(\hat{\Omega}^\infty \hat{E}N; \hat{Z}_2)/\theta \cong \hat{K}^1_\Delta(\hat{E}N; \hat{Z}_2).$$

**Proof.** By [10, 7.1], the homomorphism $\Phi_1 : \hat{K}^{-1}_\Delta(\hat{\Omega}^\infty \hat{E}N; \hat{Z}_2) \to \hat{K}^{-1}_\Delta(\hat{E}N; \hat{Z}_2)$ is left inverse to the infinite suspension homomorphism, and the lemma now follows by Theorem 6.7 together with Lemma 4.11, and Definition 6.3.

**Proof of Theorem 9.4.** Applying the functor $\Phi_1$ to the fiber sequence of Theorem 8.6, we obtain a (co)fiber sequence of spectra

$$\Phi_1 X_{K/2} \longrightarrow \Phi_1 \hat{\Omega}^\infty \hat{E}\hat{M} \xrightarrow{\Phi_1f} \Phi_1 \hat{\Omega}^\infty \hat{E}\hat{p}\hat{M}$$

for some companion map $f$ of $M$. We then deduce that $\Phi_1f$ corresponds to a map $\hat{E}M \to \hat{E}\hat{p}\hat{M}$ having the desired properties by Lemmas 9.8 and 5.4.

**Proof of Theorem 9.7.** The results on $K^*(\Phi X; \hat{Z}_2)$ and $K_\Delta^{-1}(\Phi X; \hat{Z}_2)$ follow from Theorem 9.5. Since $K^*(\Phi X; \hat{Z}_2) \cong \hat{A}\hat{M}_C$ by Lemma 4.6, we obtain isomorphisms $\hat{Q}K^0(X; \hat{Z}_2)/\theta = 0$ and $\hat{Q}K^{-1}(X; \hat{Z}_2)/\theta \cong \hat{M}_C/\theta$, and we deduce that $\Phi_1 : \hat{Q}K^n(X; \hat{Z}_2)/\theta \cong K^n(\Phi X; \hat{Z}_2)$ for $n = -1, 0$ by Lemma 9.8 and naturality.
10. Applications to simply-connected compact Lie groups

We now apply the preceding results to a simply-connected compact Lie group $G$. We first use the representation theory of $G$ to functorially determine the united 2-adic $K$-cohomology ring $K^*_{CR}(G; \mathbb{Z}_2) = \{K^*(G; \mathbb{Z}_2), KO^*(G; \mathbb{Z}_2)\}$ in Theorem 10.3. Then, with slight restrictions on the group, we use the representation theory of $G$ to give expressions for the $K/2$-localization $G_{K/2}$, for the $v_1$-stabilization $\Phi_1 G$, and for the cohomology $KO^*(\Phi_1 G; \mathbb{Z}_2)$, and we also show that $G$ is $\hat{K}\Phi_1$-good. Our results are summarized in Theorem 10.6 and permit calculations of the 2-primary $v_1$-periodic homotopy groups $\pi_{1,1} G$ using Theorem 9.2, as accomplished very successfully by Davis [14]. In this section, we assume some general familiarity with the representation rings of our Lie groups as described in [12, Sections II.6 and VI.4] and [14, Theorem 2.3].

**Definition 10.1** (The representation ring $R_\Delta G$). For a simply-connected compact Lie group $G$, we let $RG$ be the complex representation ring and let $R_H G, R_H G \subset RG$ be the real and quaternionic parts of $RG$ with the usual $\lambda$-ring structures on $RG$ and $R_H G \oplus R_H G$. We also let $t = \psi^{-1}: RG \cong RG$, $c: R_H G \subset RG$, $r: RG \rightarrow R_H G$, $c': R_H G \subset RG$, and $q: RG \rightarrow R_H G$ be the usual operations satisfying the $\Delta$-module relations of Definition 4.1. These structures are compatible in the expected ways and combine to give a $\Delta\lambda$-ring $R_\Delta G = \{RG, R_H G, R_H G\}$ in the sense of [10, 6.2]. We let $\hat{R}_\Delta G = \{\hat{R} G, \hat{R} H G, \hat{R} H G\}$ be the augmentation ideal of $R_\Delta G$ given by the kernel $RG$ of the complex augmentation $dim: RG \rightarrow \mathbb{Z}$, where $\hat{R} G = R_H G \cap \hat{R} G$ and $\hat{R} H G = R_H G \cap \hat{R} G$. We also let $QR_\Delta G = \{QR G, QR H G, QR H G\}$ be the indecomposables of $R_\Delta G$ given by

$$QR G = \hat{R} G/\hat{R} G^2,$$

$$QR H G = \hat{R} H G/((\hat{R} G)^2 + r(\hat{R} G)^2),$$

$$QR H G = \hat{R} H G/((\hat{R} G)^2 + q(\hat{R} G)^2).$$

It is straightforward to show that $\hat{R}_\Delta G$ and $QR_\Delta G$ inherit $\Delta\lambda$-ring structures (without identities) from $R_\Delta G$. Since $QR_\Delta G$ is a $\Delta\lambda$-ring with trivial multiplication, it is equipped with additive operations $t: QR G \cong QR G$, $c: QR H G \rightarrow QR G$, $r: QR G \rightarrow QR H G$, $c': QR H G \rightarrow QR G$, $q: QR G \rightarrow QR H G$, $\theta = -\lambda^2: QR G \rightarrow QR G$, $\theta = -\lambda^2: QR H G \rightarrow QR H G$, $\psi: QR G \rightarrow QR G$, $\psi^k: QR H G \rightarrow QR H G$, and $\psi^k: QR H G \rightarrow QR H G$ for the odd $k \in \mathbb{Z}$. We now let $QR H G = \{QR G, QR H G, QR H G\}$ be the 2-adic completion of $QR_\Delta G$ with the induced additive operations on the components $QR G = \hat{Z}_2 \otimes QR G$, $QR H G = \hat{Z}_2 \otimes QR H G$, and $QR H G = \hat{Z}_2 \otimes QR H G$.

**Lemma 10.2.** For a simply-connected compact Lie group $G$, $QR_\Delta G$ is a robust 2-adic Adams $\Delta$-module.

This will be proved below. To determine the cohomology ring $K^*_{CR}(G; \mathbb{Z}_2) = \{K^*(G; \mathbb{Z}_2), KO^*(G; \mathbb{Z}_2)\}$ from the representation theory of $G$, we now let $\beta: QR_\Delta G \rightarrow K_{\Delta}^{-1}(G; \mathbb{Z}_2)$ be the 2-adic Adams $\Delta$-module homomorphism induced by the composition of the canonical homomorphisms $\hat{R} G \rightarrow \bar{K}_{\Delta}^\alpha(BG; \mathbb{Z}_2) \rightarrow \bar{K}_{\Delta}^{-1}(G; \mathbb{Z}_2)$. 


Theorem 10.3. For a simply-connected compact Lie group $G$, there is a natural strict isomorphism \( \hat{\beta}: \hat{L}(QR\Delta G) \cong K^*_{CR}(G; \hat{\mathbb{Z}}_2) \).

Proof. This follows by Lemma 10.2 and Theorem 4.9 since $\hat{\beta}: \hat{QR}G \to K^{-1}(G; \hat{\mathbb{Z}}_2)$ induces an isomorphism $\hat{\Lambda}(\hat{QR}G) \cong K^*(G; \hat{\mathbb{Z}}_2)$ by [17].

We note that $K^*_{CR}(G; \hat{\mathbb{Z}}_2)$ has a simple system of generators (see Definition 3.3) consisting of the $\beta \hat{z}_i \in K^{-1}(G; \hat{\mathbb{Z}}_2)$, the $\beta \hat{x}_a \in KO^{-1}(G; \hat{\mathbb{Z}}_2)$, and the $\beta y_\beta \in KO^{-5}(G; \hat{\mathbb{Z}}_2)$ obtained from the analysis of $\hat{QR}\Delta G$ below in Remark 10.7. Thus, by Proposition 3.4, $K^*_{CR}(G; \hat{\mathbb{Z}}_2)$ is a free 2-adic CR-module on the associated products. However, our description of $K^*_{CR}(G; \hat{\mathbb{Z}}_2)$ as $\hat{L}(\hat{QR}\Delta G)$ is more natural and includes the full multiplicative structure. Moreover, it will let us apply our main results to $G$.

Lemma 10.4. For a simply-connected compact Lie group $G$, the 2-adic Adams $\Delta$-module $\hat{QR}\Delta G$ is regular with $\theta: \hat{QR}G \to \hat{QR}G$ monic.

Proof. This follows by Lemmas 7.9 and 7.10 since $\beta: \hat{QR}G \to \hat{K}^{-1}(G; \hat{\mathbb{Z}}_2)$ is monic by Theorem 10.3.

Thus, $\hat{QR}\Delta G$ is strong (robust, $\psi^3$-splittable, and regular) if and only if it is $\psi^3$-splittable, and this is usually the case by:

Lemma 10.5. For a simply-connected compact simple Lie group $G$, the 2-adic Adams $\Delta$-module $\hat{QR}\Delta G$ is $\psi^3$-splittable (and hence strong) if and only if $G$ is not $E_6$ or $\text{Spin}(4k + 2)$ with $k$ not a 2-power.

This will be proved below using work of Davis [14]. For a simply-connected compact Lie group $G$, we now let $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ briefly denote the associated stable 2-adic Adams $\Delta$-module $\hat{Q}_\Delta RG = (\hat{Q}_\Delta RG)/\hat{\phi}$. This agrees with the notation of [10, 9.2] and [14], since our $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_H, \hat{Q}_H\}$ is the 2-adic completion of their $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$. Our main results now give the following omnibus theorem, whose four parts may be expanded in the obvious ways to match the cited theorems.

Theorem 10.6. Let $G$ be a simply-connected compact Lie group such that the 2-adic Adams $\Delta$-module $\hat{Q}_\Delta RG$ is $\psi^3$-splittable (see Lemma 10.5), and let $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ be the associated stable 2-adic Adams $\Delta$-module. Then:

(i) the $K/2$-localization $G_{K/2}$ is the homotopy fiber of a map $\Omega^\infty \hat{E}Q_\Delta \to \Omega^\infty \hat{E} \hat{Q}_\Delta$ with low dimensional modifications as in Theorem 8.6;

(ii) the 2-adic $v_1$-stabilization $\Phi_1G$ is the homotopy fiber of a map of spectra $\hat{E}Q_\Delta \to \hat{E} \hat{Q}_\Delta$ as in Theorem 9.4;

(iii) there is an exact sequence

$$0 \to KO^{-8}(\Phi_1G; \hat{\mathbb{Z}}_2) \to \hat{Q}/(\hat{Q}_R + \hat{Q}_H) \xrightarrow{\theta} \hat{Q}/\hat{Q}_R \to \cdots$$

continuing as in Theorem 9.5;

(iv) $G$ is $K\Phi_1$-good at the prime 2 as in Theorem 9.7.
The exact sequence in (iii) permits calculations of the 2-primary \( v_1 \)-periodic homotopy groups \( \pi_* G \) using Theorem 9.2 as accomplished by Davis [14]. This exact sequence was previously obtained in [10, Theorem 9.3] using indirect algebraic methods under the hard-to-verify condition that \( G \) was \( \hat{\Phi}_1 \)-good. It is now obtained using the \( KO^*(\cdot; \mathbb{Z}_2) \) cohomology exact sequence of the (co)fiber sequence in (ii) under an accessible algebraic condition that implies the \( \hat{\Phi}_1 \)-goodness of \( G \) by (iv).

We devote the rest of the section to proving Lemmas 10.2 and 10.5 using:

**Remark 10.7** (Generators for representation rings). For a simply-connected compact Lie group \( G \), standard results summarized in [14, Theorem 2.3] show that \( RG \) is a finitely generated polynomial ring \( \mathbb{Z}[z_\gamma, z_\gamma^*, x_\alpha, y_\beta]_{\gamma, \alpha, \beta} \) on certain basic complex representations \( z_\gamma \) together with their conjugates \( z_\gamma^* = t z_\gamma \), certain basic real representations \( x_\alpha \), and certain basic quaternionic representations \( y_\beta \). Moreover, in terms of these generators, the \( \mathbb{Z}/2 \)-graded ring \( \{ RG, RH/G \} \) is characterized by the fact that its quotient \( \{ R_R G/rRG, R_H G/qRG \} \) is a \( \mathbb{Z}/2 \)-graded polynomial algebra \( \mathbb{Z}/2[x_\alpha, \phi z_\gamma, y_\beta]_{\alpha, \gamma, \beta} \) on the real generators \( x_\alpha \) and \( \phi z_\gamma \) (with \( c \phi z_\gamma = z_\gamma^* z_\gamma \)) and the quaternionic generators \( y_\beta \). Consequently, the indecomposables \( QR_\Delta G = \{ QRG, QR_R G, QR_H G \} \) may be expressed as

\[
\begin{align*}
QRG &= \mathbb{Z}\{z_\gamma, z_\gamma^*, c_\alpha x_\alpha, c' \gamma y_\beta\}_{\gamma, \alpha, \beta}, \\
QR_R G &= \mathbb{Z}\{r z_\gamma, \bar{x}_\alpha, r' c' \gamma y_\beta\}_{\gamma, \alpha, \beta} \oplus \mathbb{Z}/2\{\bar{\phi} z_\gamma\}_{\gamma}, \\
QR_H G &= \mathbb{Z}\{q z_\gamma, qc x_\alpha, \bar{y}_\beta\}_{\gamma, \alpha, \beta}
\end{align*}
\]

where \( \bar{w} \) denotes \( w - \dim w \) for \( w \in RG \). Thus, the 2-adic indecomposables \( \hat{QR} \Delta G = \{ \hat{QRG}, \hat{QR_R G}, \hat{QR_H G} \} \) may be expressed similarly using \( \hat{\mathbb{Z}}_2 \) in place of \( \mathbb{Z} \), and the stable 2-adic indecomposables \( \hat{Q} = \{ \hat{QR}, \hat{QR_R}, \hat{QR_H} \} \) may be expressed as

\[
\begin{align*}
\hat{Q} &= \hat{\mathbb{Z}}_2\{\hat{z}_\gamma, \hat{z}_\gamma^*, c_\alpha x_\alpha, c' \gamma y_\beta\}_{\gamma, \alpha, \beta}, \\
\hat{Q}_R &= \hat{\mathbb{Z}}_2\{r z_\gamma, \bar{x}_\alpha, r' c' \gamma y_\beta\}_{\gamma, \alpha, \beta}, \\
\hat{Q}_H &= \hat{\mathbb{Z}}_2\{q z_\gamma, qc x_\alpha, \bar{y}_\beta\}_{\gamma, \alpha, \beta}
\end{align*}
\]

**Proof of Lemma 10.2.** Since \( QR_\Delta G \) is a \( \Delta \lambda \)-ring with trivial multiplication, it is straightforward to check all of the required relations for operations (see Definitions 4.3 and 6.1) in particular, we deduce \( \theta \theta r = \theta r \theta \) from the relations \( \lambda^4 r = \lambda^4 + \phi \lambda^2, \lambda^4 - \lambda^2 \lambda^2, \phi = \lambda^2 r - r \lambda^2, 2 \phi = 0, \) and \( \theta = -\lambda^2 \), which hold generally in \( \Delta \lambda \)-rings with trivial multiplication [10, 6.2]. We next observe that \( QRG, \hat{QR}_R G, \) and \( QR_H G \) are stable 2-adic Adams modules by [6, 6.2], since \( QR_G \) and \( QR_R G \oplus QR_H G \) are \( \gamma \)-nilpotent and finitely generated abelian (because they have trivial multiplications and have finite generating sets of elements \( \bar{w} \) for representations \( w \)). Thus, \( QR_\Delta G \) is a 2-adic Adams \( \Delta \)-module, and it must be robust by the analysis of Remark 10.7. \( \square \)

To check the \( \psi^3 \)-splittability of \( \hat{QR} \Delta G \), we let \( hG = \text{ker}(1 - t)/\text{im}(1 + t) \) be the augmented algebra over \( \mathbb{Z}/2 \) obtained from \( RG \) using the involution \( t = \psi^{-1} : RG \cong \)
This is a polynomial algebra $hG \cong \mathbb{Z}/2[\tilde{x}_\alpha, \tilde{z}_\gamma \tilde{z}_\gamma, c'y_{\beta}]_{\alpha, \gamma, \beta}$ which is $\mathbb{Z}/2$-graded, since there is an isomorphism
\[
c + c' : R|G/rRG \oplus R|H/G/qRG \cong hG,
\]
and we let $QhG \cong \mathbb{Z}/2[\tilde{x}_\alpha, \tilde{z}_\gamma \tilde{z}_\gamma]_{\alpha, \gamma}$ denote the real (degree 0) indecomposables.

We define a homomorphism $s : QRG \to QhG$ by $s[u] = [u^* u]$ for $u \in \hat{R}G$ and note that $sQRG = \mathbb{Z}/2[\tilde{z}_\gamma \tilde{z}_\gamma]_{\gamma}$. We view $s$ as a homomorphism of $\psi^3$-modules (abelian groups with endomorphisms $\psi^3$) as in [14, 2.4].

**Lemma 10.8.** For a simply-connected compact Lie group $G$, $QRG$ is $\psi^3$-splittable if and only if the $\psi^3$-submodule $sQRG \subset QhG$ is a direct summand.

**Proof.** By Definition 7.2 and the proof of Lemma 7.3, $QRG$ is $\psi^3$-splittable if and only if the $\psi^3$-submodule $\phi QRG \subset QRG$ (or equivalently $\phi QRG \subset QRG/rQRG$) is a direct summand. The lemma now follows since $\phi QRG$ corresponds to $sQRG$ under the isomorphism $c : QRG/rQRG \cong QhG$. \hfill \Box

**Proof of Lemma 10.5.** By Lemma 10.8 and Davis [14, Theorem 1.3], the following conditions are successively equivalent: $QRG$ is $\psi^3$-splittable; the $\psi^3$-submodule $sQRG \subset QhG$ is a direct summand; $G$ satisfies the Technical Condition of [14, Definition 2.4]; $G$ is not $E_6$ or $Spin(4k + 2)$ with $k$ not a 2-power. \hfill \Box

### 11. Proofs of basic lemmas for $\hat{L}$

We shall prove Lemmas 4.5, 4.6, and 4.11 showing the basic properties of the functor $\hat{L} : \theta \Delta \text{Mod} \to \phi CR\hat{\Delta}\text{Alg}$, where $\theta \Delta \text{Mod}$ is the category of 2-adic $\theta \Delta$-modules and $\phi CR\hat{\Delta}\text{Alg}$ is that of special 2-adic $\phi CR$-algebras (see Definitions 4.3 and 3.2). We first introduce an intermediate category of modules.

**Definition 11.1** (The 2-adic $\eta$-modules). By a 2-adic $\eta$-module $N = \{N_C, N_R, N_H, N_S\}$, we mean a 2-adic $\Delta$-module $\{N_C, N_R, N_H\}$, with operations $t, c, r, c'$, and $q$ as in Definition 4.1, together with a 2-profinite abelian group $N_S$ and continuous additive operations $\phi : N_C \to N_R$, $\eta : N_R \to N_S$, $\lambda$ and $\lambda'$ such that $\eta x = [x]$ for $x \in N_R$, and $y \in N_H$:

\[
\begin{align*}
\phi c x &= 0, & \phi c' y &= 0, & \phi t z &= \phi z, & 2\tilde{\phi} z &= 0, & c\phi z &= 0, \\
(\tilde{\phi} z)^{[2]} &= 0, & 2q x &= 0, & \eta r z &= 0, & (qz)^{[2]} &= (rz)^{[2]} = \eta\tilde{z}.
\end{align*}
\]

We let $\eta \Delta \hat{\text{Mod}}$ denote the category of 2-adic $\eta$-modules.

**Remark 11.2** (A functorial interpretation of admissible maps). Let $J : \theta \Delta \text{Mod} \to \eta \Delta \text{Mod}$ be the functor carrying a 2-adic $\theta \Delta$-module $M$ to the 2-adic $\eta$-module $JM = \{M_C, M_R, M_H, M_R/rM_C\}$ having the original operations $t, c, r, c'$, and $q$ together with operations $\eta : M_R \to M_R/rM_C$, $\lambda$ and $\lambda'$ such that $\eta x = [x]$, $x^{[2]} = [\theta x]$, and $y^{[2]} = [\theta y]$ for $x \in M_R$ and $y \in M_H$. Let $I : \phi CR\hat{\Delta}\text{Alg} \to \eta \Delta \text{Mod}$ be the functor carrying a special 2-adic $\phi CR$-algebra $A$ to the 2-adic $\eta$-module $IA = \{A_C, A_R, A_H, A_R^2\}$ having the
operations $t$, $c$, $r$, $c'$, and $q$ of $\Delta^{-1} \tilde{A}$ (see Definition 4.1) together with operations $\tilde{\phi}: \tilde{A}^1_C \to \tilde{A}^{-1}_R$, $\eta: \tilde{A}_R^{-1} \to \tilde{A}_R^{-2}$, $(\eta^2): \tilde{A}_R^{-1} \to \tilde{A}_R^{-2}$, and $(\eta^2): \tilde{A}_R^{-5} \to \tilde{A}_R^{-2}$ given by $\tilde{\phi}z = \eta \phi z$, $\eta x = \eta x$, $x^{[2]} = x^2$, and $y^{[2]} = B_R^{-1} y^2$ for $z \in \tilde{A}_C^{-1}$, $x \in \tilde{A}_R^{-1}$, and $y \in \tilde{A}_R^{-5}$. We now easily see:

**Lemma 11.3.** For $M \in \theta \Delta \hat{\text{Mod}}$ and $A \in \phi \text{CR} \hat{\text{Alg}}$, an admissible map $f: M \to A$ is equivalent to a map $f: JM \to IA$ in $\eta \Delta \text{Mod}$.

To construct the functor $\hat{L}$, we need:

**Lemma 11.4.** The functor $I: \phi \text{CR} \hat{\text{Alg}} \to \eta \Delta \text{Mod}$ has a left adjoint $\hat{V}: \eta \Delta \text{Mod} \to \phi \text{CR} \hat{\text{Alg}}$.

**Proof.** This follows by the Special Adjoint Functor Theorem (see [19]) since $I$ preserves small limits and since $\phi \text{CR} \hat{\text{Alg}}$ has a small cogenerating set by Lemma 11.5 below.

A special 2-adic $\phi \text{CR}$-algebra $A$ will be called **finite** when the groups $\tilde{A}_m^n$ and $\tilde{A}_R^n$ are finite for all $m$.

**Lemma 11.5.** Each special 2-adic $\phi \text{CR}$-algebra $A$ is the inverse limit of its finite quotients in $\phi \text{CR} \hat{\text{Alg}}$.

**Proof.** This is similar to the corresponding result for topological rings in [22, 5.1.2]. For a 2-adic CR-submodule $G \subset A$ with $A/G$ finite, we must obtain a special 2-adic $\phi \text{CR}$-ideal $H$ of $A$ with $H \subset G$ and $A/H$ finite. We first obtain an ideal $M$ of $A_R$ (closed under $B_R$, $B_R^{-1}$, $\eta$, and $\xi$) with $M \subset G_R$ and $A_R/M$ finite as in [22]. We next obtain an ideal $N$ of $A_C$ (closed under $B$, $B^{-1}$, and $t$) with $N \subset G_C \cap t^{-1}N \cap \phi^{-1}M_0$ and $A_C/N$ finite as in [22]. The desired ideal $H$ is now given by $H_C = N$ and $H_R = M \cap c^{-1}N$.

**Proof of Lemma 4.5.** Using Lemmas 11.3 and 11.4, we obtain the desired universal algebra $LM$ from the functor $\hat{L} = \hat{V}J: \theta \Delta \text{Mod} \to \phi \text{CR} \text{Mod}$.

A 2-adic $\eta \Delta$-module $N$ is called **sharp** when $\eta: N_R/rN_C \to N_S$ is an isomorphism, and we may now derive the properties of $\hat{L}$ from the corresponding properties of $V$ on such sharp modules.

**Lemma 11.6.** For a sharp 2-adic $\eta \Delta$-module $N$, the canonical map $\hat{A}N_C \to (\hat{V}N)_C$ is an algebra isomorphism.

**Proof.** Let $W: \phi \text{CR} \hat{\text{Alg}} \to C \hat{\text{Alg}}$ be the forgetful functor carrying each $A \in \phi \text{CR} \hat{\text{Alg}}$ to its complex part $A_C \subset C \hat{\text{Alg}}$ where $C \hat{\text{Alg}}$ is the category of special 2-adic $C$-algebras, which are defined similarly to special 2-adic $\phi \text{CR}$-algebras (see Definition 3.2) but using only complex terms and their operations. The functor $W$ has a right adjoint $H: C \hat{\text{Alg}} \to \phi \text{CR} \hat{\text{Alg}}$ where $(HX)_C = X$ and $(HX)_R = \{ x \in X | t x = x \}$ with $c = 1$, $r = 1 + t$, $\eta = 0$, $\phi z = z^*z$ for $z \in X_0$, and $\phi w = B^{-1}w*w$ for $w \in X^{-1}$. For each $N \in \eta \Delta \text{Mod}$ and each $X \in C \hat{\text{Alg}}$, a map $N \to IHX$ in $\eta \Delta \text{Mod}$ corresponds to a map $N_C \to X^{-1}$ respecting $t$, which in turn corresponds to a map $\hat{A}N_C \to X$ in $C \hat{\text{Alg}}$. Hence, since $W\hat{V}$ is left adjoint to $IH$, the canonical map $\hat{A}N_C \to W\hat{V}N$ is an isomorphism.
Proof of Lemma 4.6. For a 2-adic $\theta\Delta$-module $M$, the canonical map $\hat{\Lambda}M_C \to (\hat{L}M)_C$ is an isomorphism by Lemma 11.6 and by the above proof of Lemma 4.5.

Let $\hat{Q} : \phi\mathcal{CR}\hat{A}\text{alg} \to \phi\mathcal{CR}\mathcal{M}\text{od}$ be the functor carrying each $A \in \phi\mathcal{CR}\hat{A}\text{alg}$ to its indecomposables $\hat{Q}A \in \phi\mathcal{CR}\mathcal{M}\text{od}$ where $\phi\mathcal{CR}\mathcal{M}\text{od}$ is the category of special 2-adic $\phi\mathcal{CR}$-modules, which may be defined as the augmentation ideals of the special 2-adic $\phi\mathcal{CR}$-algebras having trivial multiplication.

Lemma 11.7. For a sharp 2-adic $\eta\Delta$-module $N$, the canonical map $\{N_C, N_R, N_H\} \to \Delta^{-1}QVN$ is an isomorphism.

Proof. The functor $\hat{Q}$ has a right adjoint $E : \phi\mathcal{CR}\mathcal{M}\text{od} \to \phi\mathcal{CR}\hat{A}\text{alg}$ where $EX = \mathcal{E} \oplus X$. Since $\hat{Q}V : \eta\Delta\text{Mod} \to \phi\mathcal{CR}\mathcal{M}\text{od}$ is left adjoint to $IE$, a detailed analysis shows that $\hat{Q}VN$ is a special 2-adic $\phi\mathcal{CR}$-module with $(\hat{Q}VN)^{-1}_C = N_C$, $(\hat{Q}VN)^{1}_R = N_R$, and $(\hat{Q}VN)^{0}_R = N_H$.

Proof of Lemma 4.11. For a 2-adic $\theta\Delta$-module $M$, the canonical map $M \to \Delta^{-1}QLM$ is an isomorphism by Lemma 11.7 and the above proof of Lemma 4.5.

12. Proof of the Bott exactness lemma for $\hat{L}$

We must now prove Lemma 4.8 showing the Bott exactness of $\hat{L}M$ for a robust 2-adic $\theta\Delta$-module $M$. This lemma will follow easily from the corresponding result for $\eta\Delta$-modules (Lemma 12.1), whose proof will extend through most of this section. We say that a 2-adic $\eta\Delta$-module $N$ is profinitely sharp when it is the inverse limit of an inverse system of finite sharp 2-adic $\eta\Delta$-modules. This obviously implies that $N$ is sharp. We call $N$ robust when:

$(i)$ $N$ is profinitely sharp;

$(ii)$ the 2-adic $\Delta$-module $\{N_C, N_R/\phi N_C, N_H\}$ is torsion-free and exact;

$(iii)$ $\text{ker}\phi = cN_R + c'N_H + 2N_C$.

Lemma 12.1. If $N$ is a robust 2-adic $\eta\Delta$-module, then the special 2-adic $\phi\mathcal{CR}$-algebra $VN$ is Bott exact; in fact, $VN$ is the inverse limit of an inverse system of finitely generated free 2-adic $\mathcal{CR}$-modules.

This will be proved at the end of the section.

Proof of Lemma 4.8. For a robust 2-adic $\theta\Delta$-module $M$, the 2-adic $\eta\Delta$-module $JM$ is also robust, and hence $\hat{L}M$ has the required properties by Lemma 12.1 and the proof of Lemma 4.5 in Section 11.

Before proving Lemma 12.1, we must analyze the robust 2-adic $\eta\Delta$-modules, and we start with:

Definition 12.2 (The complex 2-adic $\eta\Delta$-modules). The functor $(-)_C : \eta\Delta\text{Mod} \to \hat{A}\text{b}$ from the 2-adic $\eta\Delta$-modules to the profinite abelian groups has a left adjoint $C : \hat{A}\text{b} \to \eta\Delta\text{Mod}$ with $C(G)_C = G \oplus G = G \oplus tG$, $C(G)_R = G \oplus G/2 = rG \oplus \hat{G}$, $C(G)_H = G = qG$, and $C(G)_S = G/2 = (\hat{G})^{[2]}$ for $G \in \hat{A}\text{b}$. A 2-adic $\eta\Delta$-module
Lemma 12.3. If $\tilde{N} \subset N$ is an inclusion of robust 2-adic $\eta\Delta$-modules such that $N_C/\tilde{N}_C$ is torsion-free and $\tilde{N}_C^- = N_C^-$, then each map $\tilde{N} \to C(G)$ for $G \in \tilde{Ab}$ may be extended to a map $N \to C(G)$ of 2-adic $\eta\Delta$-modules.

Proof. For a given map $F(\tilde{f}, \tilde{g}) : N \to C(G)$, we first extend $\tilde{g} : \tilde{N}_S \to G/2$ to a map $g : N_S \to G/2$. Since $\tilde{N}_C/\tilde{N}_C^+ \cong N_C^+$, $N_C/\tilde{N}_C^+ \cong N_C^-$, and $\tilde{N}_C^- = N_C^-$, we see that $N_C$ is a pushout of the inclusions $N_C^+ \to \tilde{N}_C^+ \to \tilde{N}_C$. Thus, the maps $\tilde{g} : N_C^+ \to G/2$ and $[\tilde{f}] : N_C \to G/2$ induce a map $f' : N_C \to G/2$, and we obtain a commutative diagram

\[
\begin{array}{ccc}
N_C^+ & \xrightarrow{f} & G \\
\downarrow \pi & & \downarrow 1 \\
N_C & \xrightarrow{g} & G/2
\end{array}
\]

where $N_C^+ = \{z \in N_C | tz = z\}$ and $\pi$ is the composition of $(c, c') : N_R/\bar{\delta}N_C \coprod_{N_C} N_H \cong N_C^+$ and $()^2 : N_R/\bar{\delta}N_C \coprod_{N_C} N_H \to N_S$. Letting $N_C^- = \{z \in N_C | tz = -z\}$, we now have:

**Lemma 12.4.** For a robust 2-adic $\eta\Delta$-module $N$, there exists a decomposition $N \cong C(G) \oplus P$ where $G$ is torsion-free and $P$ is robust with $t = 1$ on $P_C$.

Proof. By the factorization of positively torsion-free groups in Definition 5.3, there exists a decomposition $N_C \cong (G \oplus tG) \oplus H$ with $t = 1$ on $H$, and we let $i : C(G) \to$
Proof of Lemma 12.1

Definition 12.5 (The $t$-trivial 2-adic $\eta\Delta$-modules). A 2-adic $\eta\Delta$-module $N$ will be called $t$-trivial when $t = 1$ on $N_C$. When $N$ is $t$-trivial and robust, it must have $\delta = 0$: $N_C \to N_R$ since $N_C = cN_R + c'N_H$ by the exactness of $\{N_C, N_R/\delta N_C, N_H\}$. Moreover, it must also have $(rN_C)^{[2]} = 0$, $(qN_C)^{[2]} = 0$, and $c + c': N_R/rN_C \oplus N_H/qN_C \cong N_C/2$ by [10, Lemma 4.7]. Hence, the operations $(\cdot)^{[2]}: N_R \to N_S$ and $(\cdot)^{[2]}: N_H \to N_S$ induce operations $\bar{\alpha}: N_R/rN_C \to N_R/rN_C$ and $\tilde{\alpha}: N_H/rN_C \to N_R/rN_C$, where the $\bar{\alpha}$-module $N_R/rN_C$ is profinite since $N$ is profinitely sharp. In this way, a $t$-trivial robust 2-adic $\eta\Delta$-module $N$ corresponds to a torsion-free group $G \in \hat{A}b$ together with a decomposition $(G/2)_R \oplus (G/2)_H = G/2$ equipped with operations $\theta: (G/2)_R \to (G/2)_R$ and $\tilde{\theta}: (G/2)_H \to (G/2)_R$ such that the $\bar{\alpha}$-module $(G/2)_R$ is profinite. We say that a 2-adic $\eta\Delta$-module $N$ is of finite type when $N_C$, $N_R$, $N_H$, and $N_S$ are finitely generated over $\mathbb{Z}_2$, and we now easily deduce:

Lemma 12.6. A $t$-trivial robust 2-adic $\eta\Delta$-module may be expressed as the inverse limit of an inverse system of $t$-trivial robust quotient modules of finite type.

A similar result obviously holds for the robust 2-adic $\eta\Delta$-modules $C(G)$ with $G$ torsion-free, and the following lemma will now let us restrict our study of $\hat{V}$ to the robust modules of finite type.

Lemma 12.7. If a 2-adic $\eta\Delta$-module $N$ is the inverse limit of an inverse system $\{N_\alpha\}_\alpha$ of quotient modules, then $\hat{V}N \cong \text{lim}_\alpha \hat{V}N_\alpha$.

Proof. For a finite special 2-adic $\phi CR$-algebra $F$, there is a canonical isomorphism $\text{Hom}(\text{lim}_\alpha \hat{V}N_\alpha, F) \cong \text{Hom}(\hat{V}N, F)$. Hence the map $\hat{V}N \to \text{lim}_\alpha \hat{V}N_\alpha$ is an isomorphism by Lemma 11.5.

Proof of Lemma 12.1. It now suffices to show that $\hat{V}N$ is a free 2-adic $CR$-module when $N = C(G) \oplus P$ for a finitely generated free $\mathbb{Z}_2$-module $G$ and a $t$-trivial robust 2-adic $\eta\Delta$-module $P$ of finite type. By Definition 7.1, we may choose finite ordered sets of elements $\{z_k\}_k$ in $G$, $\{x_i\}_i$ in $P_R$, and $\{y_j\}_j$ in $P_H$ such that $G$ is a free $\mathbb{Z}_2$-module on $\{z_k\}_k$ and $\{P_R, P_H\}$ is a free 2-adic $\Delta$-module on $\{x_i\}_i$ and $\{y_j\}_j$. Since $P_R$ is a free $\mathbb{Z}/2$-module on the generators $\{x_i\}_i$, there are expressions $x_i^{[2]} = r_i$ and $y_j^{[2]} = s_j$ for each $i$ and $j$ where the $r_i$ and $s_j$ are $\mathbb{Z}/2$-linear combinations of these generators. We may now obtain $\hat{V}N$ as the free augmented 2-adic $CR$-algebra on the generators $x_i \in (\hat{V}N)_R^{-1}$, $y_j \in (\hat{V}N)_R^{-5}$, $z_k \in (\hat{V}N)_C^{-1}$, and $\phi z_k \in (\hat{V}N)_H^{-1}$ subject to the relations $x_i^2 = r_i$, $y_j^2 = B_R s_j$, $z_k^2 = 0$, $z_k^2 z_k = B c \phi z_k$, and $(\phi z_k)^2 = 0$ for each $i$, $j$, and $k$. It follows by a straightforward analysis that $\hat{V}N$ is a free 2-adic $CR$-module on the associated products (see Definition 3.3) of $\{x_i\}_i$, $\{y_j\}_j$, and $\{z_k\}_k$. 

\[\square\]
13. Proofs for regular modules

We first show that our strict nonlinearity condition (see Definition 7.7) for 2-adic Adams modules agrees with that of [7, 2.4], and we then prove Lemmas 7.9 and 7.10 for regular modules. For a 2-adic Adams module $A$, we let $T \subset A$ be given by the pullback square

$$
\begin{array}{c}
\begin{array}{c}
T \subset A \\
\downarrow \subset \\
A \\
\downarrow \\
A / \psi^2 A
\end{array}
\end{array}
\end{array}$$

where $(A / \psi^2 A) \setminus 2$ is the kernel of 2: $A / \psi^2 A \to A / \psi^2 A$. Since the square is also a pushout, $A$ is quasilinear if and only if $TA = A$. Now let $T^\infty A$ be the intersection of the submodules $T^i A \subset A$ for $i > 0$.

Lemma 13.1. $T^\infty A$ is the largest quasilinear submodule of $A$, and hence $A_{ql} = T^\infty A$.

Proof. Using the inverse limit of the pullback squares for $T^i A$ with $i > 0$, we find that $T^\infty A$ contains each quasilinear submodule of $A$ and that $T(T^\infty A) = T^\infty A$.

Remark 13.2 (Strict nonlinearity conditions). Our definition of strict nonlinearity in Section 7 is equivalent to our earlier definition in [7, 2.3 and 2.4]. In fact, for a 2-adic Adams module $A$, the largest quasilinear submodule $A_{ql}$ remains unchanged in the earlier category of 2-adic $\psi^2$-modules, since it is still given by $T^\infty A$. To prove Lemma 7.10, we need:

Lemma 13.3. For a strictly nonlinear 2-adic Adams module $A$, each submodule is strictly nonlinear. Moreover, when $A$ is finitely generated over $\hat{\mathbb{Z}}_2$, each torsion-free quotient module is strictly nonlinear.

Proof. The first statement is clear, and we shall prove the second by working in the earlier category $\mathcal{N}$ of 2-adic $\psi^2$-modules that are $\psi^2$-pro-nilpotent. Let $0 \to \hat{A} \to A \to A \to 0$ be a short exact sequence in $\mathcal{N}$ with $A$ strictly nonlinear and finitely generated over $\hat{\mathbb{Z}}_2$ and with $\hat{A}$ torsion-free. To show that $\hat{A}$ is strictly nonlinear, it suffices to show that $\text{Hom}_{\mathcal{N}}(H, \hat{A}) = 0$ for each torsion-free quasilinear $H \in \mathcal{N}$ that is finitely generated over $\hat{\mathbb{Z}}_2$. Since $\hat{A}$ is torsion-free, it now suffices to show that $\text{Hom}_{\mathcal{N}}(H, \hat{A})$ is finite for such $H$. Hence, since $\text{Hom}_{\mathcal{N}}(H, A) = 0$ by strict nonlinearity, it suffices to show that $\text{Ext}_{\mathcal{N}}^1(H, \hat{A})$ is finite for such $H$. This finiteness follows using the exact sequence

$$
0 \to \text{Hom}_{\mathcal{N}}(H, \hat{A}) \to \text{Hom}_{\hat{\mathcal{A}}_\text{pro}}(H, \hat{A}) \to \text{Hom}_{\hat{\mathcal{A}}_\text{pro}}(H, \hat{A}) \to \text{Ext}_{\mathcal{N}}^1(H, \hat{A}) \to 0
$$

with $\text{Hom}_{\mathcal{N}}(H, \hat{A}) = 0$ by strict nonlinearity, where $\hat{\mathcal{A}}_\text{pro}$ is the category of 2-profinite abelian groups.

Proof of Lemma 7.10. This result follows easily from Definition 7.8 and Lemma 13.3.
14. Proof of the realizability theorem for $\hat{\mathcal{M}}$

We shall prove Theorem 8.5, giving a strict isomorphism $\hat{\mathcal{M}} \cong K^*(C_\mathcal{M}(\text{Fib} \hat{f}; \mathbb{Z}_2))$ for a companion map $f: \Omega^\infty\mathcal{M} \to \Omega^\infty\mathcal{M} \hat{f}$ of a strong 2-adic Adams $\Delta$-module $M$. For this, it will suffice by Theorem 4.9 to obtain an isomorphism $\hat{\mathcal{M}} \cong K^*(\text{Fib} f; \mathbb{Z}_2)$ of the complex components. We do this by adapting our proof of the corresponding odd primary result (Theorem 4.7) in [8]. First, to determine the 2-adic $K$-cohomology of the loops on $\Omega^\infty\mathcal{M}$ or $\Omega^\infty\mathcal{M} \hat{f}$, we may replace Theorem 11.2 of [8] by the following two theorems.

**Theorem 14.1.** If $X = \Omega^\infty E$ for a 1-connected spectrum $E$ with $H^2(E; \mathbb{Z}_2) = 0$, with $K^0(E; \mathbb{Z}_2) = 0$, and with $K^1(E; \mathbb{Z}_2)$ torsion-free, then $K^1(\Omega X; \mathbb{Z}_2) = 0$ and $K^0(\Omega X; \mathbb{Z}_2)$ is torsion-free.

**Proof.** This follows from [6, Theorem 8.3].

Using notation and terminology of [7] for a 1-connected space $X$, we obtain an augmented 2-adic $\psi^2$-module $QK^1(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}_2)$ representing the Atiyah-Hirzebruch map $K^1(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}_2)$, and we have:

**Theorem 14.2.** If $X$ is a 1-connected $\mathcal{H}$-space with $K^1(\Omega X; \mathbb{Z}_2) = 0$ and $K^0(\Omega X; \mathbb{Z}_2)$ torsion-free, then $\sigma: U(QK^1(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}_2)) \cong K^0(\Omega X; \mathbb{Z}_2)$.

**Proof.** This follows from [7, Theorem 10.2].

When $X$ is $\Omega^\infty\mathcal{M}$ or $\Omega^\infty\mathcal{M} \hat{f}$, we shall determine $H^3(X; \mathbb{Z}_2)$ from the united 2-adic $K$-cohomology of $X$. For any 1-connected space $X$, we let $\alpha_R: KO^{-1}(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}_2)$ be the homomorphism induced by the Postnikov section $KO\mathbb{Z}_2 \to$...
Lemma 14.3. If \( X \) is a 1-connected space with \( H^2(X; \mathbb{Z}_2) = 0 \), then \( \alpha_R: \tilde{QKO}^{-1}(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}_2)\) factors through \( \tilde{QKO}^{-1}(X; \mathbb{Z}_2) \) and vanishes on the following subgroups: \( \tilde{QK}^{-1}(X; \mathbb{Z}_2) \), \( (\theta - 2)\tilde{KO}^{-1}(X; \mathbb{Z}_2) \), \( (\theta - rB^{-2}c)\tilde{KO}^{-5}(X; \mathbb{Z}_2) \), and \( (\psi^3 - 9)\tilde{KO}^{-1}(X; \mathbb{Z}_2) \).

Proof. The map \( \alpha_R \) factors through \( \tilde{QKO}^{-1}(X; \mathbb{Z}_2) \) by a suspension argument using the isomorphism \( H^3(X; \mathbb{Z}_2) \cong H^2(\Omega X; \mathbb{Z}_2) \). Since \( X \) is 1-connected with \( H^2(X; \mathbb{Z}_2) = 0 \), there is a natural isomorphism \( H^3(X; \mathbb{Z}_2) \cong (\pi_2(\tau_2 X))^\# \) by [8, Lemma 11.4]. Thus, it suffices by naturality to prove the desired vanishing results when \( X \) is \( S^2 \cup_{2k} e^3 \) for \( k \geq 1 \), and these results now follow from the elementary case \( X = S^3 \) since the collapsing map \( S^2 \cup_{2k} e^3 \to S^3 \) induces epimorphisms of the cohomologies \( \tilde{K}^{-1}(-; \mathbb{Z}_2) \), \( \tilde{KO}^{-1}(-; \mathbb{Z}_2) \), and \( \tilde{KO}^{-5}(-; \mathbb{Z}_2) \).

For a 1-connected space \( X \) with \( H^2(X; \mathbb{Z}_2) = 0 \), the above \( \alpha_R \) now induces a homomorphism \( \tilde{\alpha}_R: \text{Lin}^\Delta \tilde{QK}^{-1}(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z}_2) \) where \( \tilde{QK}^{-1}(X; \mathbb{Z}_2) \) is the 2-adic Adams \( \Delta \)-module of indecomposables given by Remark 4.10 and Definition 6.1, and where \( \text{Lin}^\Delta \) carries a 2-adic Adams \( \Delta \)-module \( M \) to the group

\[
\text{Lin}^\Delta M = M_R/\langle \tilde{\phi}M_C + (\theta - 2)M_R + (\theta - rC_1)M_H + (\psi^3 - 9)M_R \rangle.
\]

To determine \( H^3(X; \mathbb{Z}_2) \) when \( X \) is \( \Omega^\infty \hat{E}M \) or \( \Omega^\infty \hat{E}pM \), we may replace Proposition 11.3 of [8] by:

Proposition 14.4. If \( N \) is a torsion-free exact stable 2-adic Adams \( \Delta \)-module, then

\[
\tilde{\alpha}_R: \text{Lin}^\Delta \tilde{QK}^{-1}(\Omega^\infty \hat{E}N; \mathbb{Z}_2) \cong H^3(\Omega^\infty \hat{E}N; \mathbb{Z}_2).
\]

Proof. Since there is a stable isomorphism \( \tilde{\alpha}_R: KO^{-1}(\hat{E}N; \mathbb{Z}_2)/(\psi^3 - 9) \cong H^3(\hat{E}N; \mathbb{Z}_2) \) by [10, Theorem 3.2] and [8, Lemma 11.4], the proposition follows using Theorem 6.7 and Lemma 4.11.

For any \( \theta \)-pro-nilpotent 2-adic Adams \( \Delta \)-module \( M \), we obtain a homomorphism \( r: M^C \to \text{Lin}^\Delta M \) of 2-adic Adams modules with \( M^C \) as in Definition 7.6 and \( \text{Lin}^\Delta M \) linear. Such a homomorphism is called properly torsion-free [7, 4.5] when its source is torsion-free and its kernel is strictly nonlinear (see Definition 7.7). We shall need:

Lemma 14.5. If \( M \) is a strong 2-adic Adams \( \Delta \)-module, then \( r: M^C \to \text{Lin}^\Delta M \) is properly torsion-free.

Proof. Since \( M \) is strong, \( M^C \) is torsion-free and \( \ker(M^C \to \text{Lin}^\Delta M) \) is strictly nonlinear. Using the maps \( r: \text{Lin} M^C \to \text{Lin}^\Delta M \) and \( c: \text{Lin}^\Delta M \to \text{Lin} M^C \) with \( cr = 2 \), we see that \( 2\ker(M^C \to \text{Lin}^\Delta M) \) is contained in \( \ker(M^C \to \text{Lin} M) \). Thus \( \ker(M^C \to \text{Lin}^\Delta M) \) is strictly nonlinear by Lemma 13.3.
As in [8, Section 11], for a strong 2-adic Adams $\Delta$-module $M$ and a companion map $f$, we obtain a ladder of $p$-complete fiber sequences

$$
\text{Fib} f \longrightarrow X \overset{f}{\longrightarrow} Y
$$

such that:

(i) $X$ and $Y$ satisfy the hypotheses of Theorems 14.1 and 14.2;
(ii) the vertical maps from $X$ and $Y$ are $K^*(\cdot; \hat{\mathbb{Z}}_2)$-equivalences;
(iii) $H^3(Y; \hat{\mathbb{Z}}_2) = 0$ and the sequence $H^3(\Omega^\infty \hat{\mathcal{F}}M; \hat{\mathbb{Z}}_2) \rightarrow H^3(\Omega^\infty \hat{\mathcal{E}}\hat{\rho}M; \hat{\mathbb{Z}}_2) \rightarrow H^3(X; \hat{\mathbb{Z}}_2) \rightarrow 0$ is exact.

**Lemma 14.6.** There is a canonical isomorphism $H^3(X; \hat{\mathbb{Z}}_2) \cong \text{Lin}^\Delta M$.

**Proof.** Since $f^*: K^*_{CR}(\Omega^\infty \hat{\mathcal{E}}\hat{\rho}M; \hat{\mathbb{Z}}_2) \rightarrow K^*_{CR}(\Omega^\infty \hat{\mathcal{E}}M; \hat{\mathbb{Z}}_2)$ is equivalent to $\hat{d}: \hat{L}\hat{F}\hat{\rho}M \rightarrow \hat{L}\hat{F}M$ for the $\theta$-resolution map $\hat{d}$, the homomorphism $f^*: H^3(\Omega^\infty \hat{\mathcal{E}}\hat{\rho}M; \hat{\mathbb{Z}}_2) \rightarrow H^3(\Omega^\infty \hat{\mathcal{E}}M; \hat{\mathbb{Z}}_2)$ is equivalent to $\text{Lin}^\Delta \hat{d}: \text{Lin}^\Delta \hat{F}\hat{\rho}M \rightarrow \text{Lin}^\Delta \hat{F}M$ by Proposition 14.4. Hence, there is an isomorphism of cokernels $H^3(X; \hat{\mathbb{Z}}_2) \cong \text{Lin}^\Delta M$. \qed

**Proof of Theorem 8.5.** The proof of Theorem 4.7 in [8] is now easily adapted to give Theorem 8.5. In more detail, Propositions 11.5 and 11.6 of [8] remain valid in our setting using Lemmas 14.5 and 14.6 together with the short exact sequence

$$0 \longrightarrow (\hat{F}M_C \downarrow 0) \longrightarrow (\hat{F}M_C \downarrow \text{Lin}^\Delta M) \longrightarrow (M_C \downarrow \text{Lin}^\Delta M) \longrightarrow 0$$
induced by the $\theta$-resolution. Propositions 11.7 and 11.8 likewise remain valid, and thus $\Lambda M_C \cong K^*(\text{Fib} f; \hat{\mathbb{Z}}_2)$, so that Theorem 8.5 follows by Theorem 4.9. \qed

**References**


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