ON GROEBNER BASES AND IMMERSIONS OF GRASSMANN MANIFOLDS $G_{2,n}$

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Abstract

Mod 2 cohomology of the Grassmann manifold $G_{2,n}$ is a polynomial algebra modulo a certain well-known ideal. A Groebner basis for this ideal is obtained. Using this basis, some new immersion results for Grassmannians $G_{2,n}$ are established.

1. Introduction

Mod 2 cohomology of Grassmann manifolds $G_{k,n} = O(n+k)/O(n) \times O(k)$ has a rather simple description. It is the polynomial algebra on the Stiefel-Whitney classes $w_1, w_2, \ldots, w_k$ of the canonical vector bundle $\gamma_k$ over $G_{k,n}$ modulo the ideal $I_{k,n}$ generated by the dual classes $\overline{w}_{n+1}, \overline{w}_{n+2}, \ldots, \overline{w}_{n+k}$. Alas, from this description it is not at all easy to establish whether a certain cohomology class is zero or not. In [6], Monks found Groebner bases for the ideal $I_{2,n}$ in the cases $n = 2^s - 3$ and $n = 2^s - 4$. Using these bases, some new results concerning the mod 2 cohomology of $G_{2,2^s-3}$ and $G_{2,2^s-4}$ were established in that paper. Also, the author used the method of modified Postnikov towers and gave an immersion result for the spaces $G_{2,2^s-3}$ into $\mathbb{R}^d$. In [9], Shimkus improved this immersion result by the same method.

Motivated by these results, we have found a reduced Groebner basis for the ideal $I_{2,n}$ for all $n$. This result is stated in Theorem 2.7. In Corollary 2.8 we present a convenient vector space bases for $H^*(G_{2,n}; \mathbb{Z}_2)$.

Using these bases and modified Postnikov towers, in Theorem 3.11 we generalize the immersion result established in [9] and prove that $G_{2,n}$ immerses into $\mathbb{R}^{4n-5}$ where $n$ is any odd integer $\geq 7$. Our result improves upon the previously known best result (obtained by Cohen in [2]) whenever $\alpha(n) = \alpha(2n) < 5$ (where $\alpha(n)$ denotes the number of ones in the binary expansion of $n$).

The lower bounds for the immersion dimension of $G_{2,n}$ (which is defined by $\text{imm}(G_{2,n}) := \min\{d \mid G_{2,n} \text{ immerses into } \mathbb{R}^d\}$) were established by Oproiu ([8]) using the method of the Stiefel-Whitney classes. For example, he has shown that $\text{imm}(G_{2,2s-1+1}) \geq 2^{s+1} - 2$. Our result states that $\text{imm}(G_{2,2s-1+1}) \leq 2^{s+1} - 1$ for $s \geq 4$, so it only remains to check whether $G_{2,2s-1+1}$ can be immersed into $\mathbb{R}^{2^{s+1}-2}$. One more example where the lower bound from [8] almost reaches the upper bound obtained in Theorem 3.11 is $G_{2,7}$; $22 \leq \text{imm}(G_{2,7}) \leq 23$.

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In addition to these main results, in Theorem 3.1 we use Groebner bases to give a
simple proof of the previous result of Oproiu concerning lower bounds for $\text{imm}(G_{2,n})$.

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2. Groebner bases

For positive integer $b$ and arbitrary integer $a$, the binomial coefficient is defined by
$(a \choose b) := \frac{a(a-1)\cdots(a-b+1)}{b!}$. Also, $(a \choose 0) := 1$. If $b$ is a negative integer, we define $(a \choose b)$ to be
equal to zero. Then it is easy to see that the well-known formula

\begin{equation}
\frac{a}{b} = \frac{a-1}{b} + \frac{a-1}{b-1}
\end{equation}

is valid for all $a, b \in \mathbb{Z}$.

From formula (1) we deduce directly that $(a \choose b) + (a-1 \choose b-1) \equiv (a \choose b) \pmod{2}$, $a, b \in \mathbb{Z}$,
or equivalently $(a \choose b-1) \equiv (a \choose b) \pmod{2}$, $a, b \in \mathbb{Z}$.

Henceforth, all binomial coefficients are considered mod 2.

Let $G_{k,n}$ be the Grassmann manifold of unoriented $k$-dimensional vector subspaces
in $\mathbb{R}^{n+k}$. It is known that the cohomology algebra $H^*(G_{k,n}; \mathbb{Z}_2)$ is isomorphic to the
quotient $\mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{k,n}$ of the polynomial algebra $\mathbb{Z}_2[w_1, w_2, \ldots, w_k]$ by the
ideal $I_{k,n}$ generated by polynomials $\overline{w}_{n+1}, \overline{w}_{n+2}, \ldots, \overline{w}_{n+k}$, which are obtained from the
equation

$$(1 + w_1 + w_2 + \cdots + w_k)(1 + \overline{w}_1 + \overline{w}_2 + \cdots) = 1.$$ 

For $k = 2$ (which is the case from now on), we have

\begin{align*}
1 + \overline{w}_1 + \overline{w}_2 + \cdots &= \frac{1}{1 + w_1 + w_2} \\
&= \sum_{t \geq 0} (w_1 + w_2)^t = \sum_{t \geq 0} \sum_{a+b=t} \binom{a+b}{a} w_1^a w_2^b \\
&= \sum_{a, b \geq 0} \binom{a+b}{a} w_1^a w_2^b.
\end{align*}

By identifying the homogeneous parts of (cohomological) degree $r \geq 0$, we obtain

$$\overline{w}_r = \sum_{a+2b=r} \binom{a+b}{a} w_1^a w_2^b.$$ 

It is understood that $a$ and $b$ are nonnegative integers.
We use the grelex ordering on the monomials in $\mathbb{Z}_2[w_1, w_2]$ with $w_1 > w_2$. That is, $w_i^a w_j^b \prec w_i^c w_j^d$ if either $a + b < c + d$ or else $a + b = c + d$ and $a \leq c$. Of course, we will write $w_i^a w_j^b \prec w_i^c w_j^d$ when $w_i^a w_j^b \not\leq w_i^c w_j^d$ and $w_i^a w_j^b \not\prec w_i^c w_j^d$.

We shall prove that, with respect to this ordering, the reduced Groebner basis for the ideal $I_{2,n} := (\overline{w}_{n+2})$ is of the form $G = \{g_0, g_1, \ldots, g_{n+1}\}$ where $\text{LT}(g_m) = w_1^{n+1-m} w_2^m$, $0 \leq m \leq n + 1$. From this it follows immediately that a vector space basis for $H^*(G_{2,n}; \mathbb{Z}_2)$ is the set of all monomials $w_i^a w_j^b$ such that $a + b \leq n$.

Let us now define the polynomials $g_m (0 \leq m \leq n + 1)$.

**Definition 2.1.** For $0 \leq m \leq n + 1$, let

$$g_m := \sum_{a + 2b = n + 1 + m} \binom{a + b - m}{a} w_1^a w_2^b.$$

As before, it is understood that $a, b \geq 0$. Note that the (cohomological) degree of the polynomial $g_m$ is $n + 1 + m$.

By comparing with the above formula for $\overline{w}_r$, it is obvious that $g_0 = \overline{w}_{n+1}$. Also,

$$w_2 \overline{w}_n = \sum_{a + 2b = n} \binom{a + b}{a} w_1^a w_2^{b+1} = \sum_{a + 2b = n + 2} \binom{a + b - 1}{a} w_1^a w_2^b = g_1.$$

The change of variable $b \mapsto b - 1$ does not affect the requirement that $b \geq 0$ since for $b = 0$ the binomial coefficient $\binom{a + b - 1}{a} = \binom{n + 1}{a}$ is equal to 0.

From the defining formula, one can see that $b$ must be such that $m \leq b \leq \frac{n + 1 + m}{2}$. Namely, $a + b - m$ cannot be negative since $a + b - m < 0$ implies $a + 2b < 2(a + b) < 2m \leq n + 1 + m$, contradicting the requirement that $a + 2b = n + 1 + m$. Now, $a + b - m$ must be $\geq a$ in order for $\binom{a + b - m}{a}$ to be nonzero, and we conclude that $b \geq m$. The second inequality comes from the condition $a + 2b = n + 1 + m$. Therefore, we have

$$g_m = \sum_{b = m}^{\frac{n + 1 + m}{2}} \binom{n + 1 - b}{b - m} w_1^{n + 1 + m - 2b} w_2^b. \quad (2)$$

It is obvious that the summand obtained for $b = m$ provides the leading term $\text{LT}(g_m) = w_1^{n + 1 - m} w_2^m$.

In order to show that $G = \{g_0, g_1, \ldots, g_{n+1}\}$ is a Groebner basis for $I_{2,n}$, we define the ideal $I_G := (G) = (g_0, g_1, \ldots, g_{n+1})$ in $\mathbb{Z}_2[w_1, w_2]$. As we have already noticed, $\overline{w}_{n+1} = g_0 \in I_G$, $\overline{w}_{n+2} = w_1 \overline{w}_{n+1} + w_2 \overline{w}_n = w_1 g_0 + g_1 \in I_G$, so $I_{2,n} \subseteq I_G$.

It remains to prove that $I_G \subseteq I_{2,n}$ and that $G$ is a Groebner basis. It turns out that the following proposition plays the crucial role in proving these facts.

**Proposition 2.2.** For each $m \in \{0, 1, \ldots, n - 1\}$, $w_2 g_m + w_1 g_{m+1} = g_{m+2}$. Also, we have that $w_2 g_n + w_1 g_{n+1} = 0$. 

Proof. We calculate
\[ w_2 g_m + w_1 g_{m+1} \]
\[ = \sum_{a+2b=n+1+m} \left( \frac{a+b-m}{a} \right) w_1^a w_2^{b+1} + \sum_{a+2b=n+m+2} \left( \frac{a+b-m-1}{a} \right) w_1^{a+1} w_2^b \]
\[ = \sum_{a+2b=n+m+3} \left( \frac{a+b-m-1}{a} \right) w_1^a w_2^b + \sum_{a+2b=n+m+3} \left( \frac{a+b-m-2}{a-1} \right) w_1^a w_2^b \]
\[ = \sum_{a+2b=n+m+3} \left( \frac{a+b-m-2}{a} \right) w_1^a w_2^b = g_{m+2}. \]

We note that, for the similar reasons as above, the change of variable \( b \mapsto b - 1 \) (\( a \mapsto a - 1 \)) does not affect the requirement that \( b \geq 0 \) (\( a \geq 0 \)).

The second statement is a consequence of the equalities \( g_n = \text{LT}(g_n) = w_1 w_2^n \) and \( g_{n+1} = \text{LT}(g_{n+1}) = w_2^{n+1} \) which are easily seen from (2).

\[ \square \]

**Corollary 2.3.** \( I_G \subseteq I_{2,n} \).

Proof. We already know that \( g_0 = \mathfrak{w}_{n+1} \in I_{2,n} \) and \( g_1 = w_1 \mathfrak{w}_{n+1} + \mathfrak{w}_{n+2} \in I_{2,n} \).

Proposition 2.2 applies, and by induction on \( m \) we have that \( g_m \in I_{2,n} \) (\( 0 \leq m \leq n+1 \)). The corollary follows.

Therefore, \( G \) is a basis for \( I_{2,n} \), and we wish to prove that it is a Groebner basis. We recall that (for a fixed monomial ordering) the \( S \)-polynomial of polynomials \( f, g \in \mathbb{Z}_2[x_1, x_2, \ldots, x_k] \) is given by (we work with mod 2 coefficients)
\[ S(f, g) = \frac{L}{\text{LT}(f)} \cdot f + \frac{L}{\text{LT}(g)} \cdot g, \]
where \( L = \text{lcm}(\text{LT}(f), \text{LT}(g)) \) denotes the least common multiple of \( \text{LT}(f) \) and \( \text{LT}(g) \).

If \( 0 \leq m < m+s \leq n+1 \), we see that
\[ \text{lcm}(\text{LT}(g_m), \text{LT}(g_{m+s})) = \text{lcm}(w_1^{n+1-m} w_2^m, w_1^{n+1-m-s} w_2^{m+s}) = w_1^{n+1-m} w_2^{m+s}, \]
and so we have
\[ S(g_m, g_{m+s}) = w_2^5 g_m + w_1^5 g_{m+s}. \]  

(3)

We are going to prove that \( G \) satisfies a sufficient condition (see [1]) for being a Groebner basis. In order to do that, we recall the following definition and theorem ([1, p. 219]). We formulate them for the field \( R = \mathbb{Z}_2 \). It is assumed that we have an ordering \( \preceq \) on the monomials in \( \mathbb{Z}_2[x_1, x_2, \ldots, x_k] \).

**Definition 2.4.** Let \( F \) be a finite subset of \( \mathbb{Z}_2[x_1, x_2, \ldots, x_k] \), \( f \in \mathbb{Z}_2[x_1, x_2, \ldots, x_k] \) a nonzero polynomial and \( t \) a fixed monomial. If \( f \) can be written as a finite sum of the form \( \sum_i m_i f_i \), where \( f_i \in F \) and \( m_i \in \mathbb{Z}_2[x_1, x_2, \ldots, x_k] \) are nonzero monomials such that \( \text{LT}(m_i f_i) \preceq t \) for all \( i \), we say that \( \sum_i m_i f_i \) is a \( t \)-representation of \( f \) with respect to \( F \).
Theorem 2.5. Let \( F \) be a finite subset of \( \mathbb{Z}_2[x_1, x_2, \ldots, x_k] \), \( 0 \notin F \). If for all \( f_1, f_2 \in F \), \( S(f_1, f_2) \) either equals zero or has a \( t \)-representation with respect to \( F \) for some monomial \( t < \text{lcm}(\text{LT}(f_1), \text{LT}(f_2)) \), then \( F \) is a Groebner basis.

We need the following lemma.

Lemma 2.6. For \( 0 \leq m < m + s \leq n + 1 \), \( S(g_m, g_{m+s}) = \sum_{i=0}^{s-1} w_1^i w_2^{s-1-i} g_{m+2+i} \).

It is understood that for \( m + s = n + 1 \), the last summand in this sum (for \( i = s - 1 \)) is zero.

Proof. We proceed by induction on \( s \). For \( s = 1 \), we obtain

\[ S(g_m, g_{m+1}) = w_2 g_m + w_1 g_{m+1} = g_{m+2} = \sum_{i=0}^{0} w_1^i w_2^{0} g_{m+2+i}, \]

using (3) and Proposition 2.2. For the inductive step, we have

\[
S(g_m, g_{m+s}) = w_2^s g_m + w_1^s g_{m+s} \\
= w_2^s g_m + w_2 w_1^{s-1} g_{m+s-1} + w_2 w_1^{s-1} g_{m+s-1} + w_2^s g_{m+s} \\
= w_2 S(g_m, g_{m+s-1}) + w_1^{s-1} g_{m+s+1} \\
= w_1^{s-1} g_{m+s+1} + \sum_{i=0}^{s-2} w_1^i w_2^{s-1-i} g_{m+2+i} \\
= \sum_{i=0}^{s-1} w_1^i w_2^{s-1-i} g_{m+2+i},
\]

again by (3), Proposition 2.2 and the induction hypothesis. It is clear that if \( m + s = n + 1 \) then the summand \( w_1^{s-1} g_{m+s+1} \) does not appear in the sum (Proposition 2.2) and so \( 0 \leq i \leq s - 2 \) in this case.

Theorem 2.7. Let \( n \geq 2 \). Then \( G = \{g_0, g_1, \ldots, g_{n+1}\} \) defined above is the reduced Groebner basis for the ideal \( I_{2,n} \) in \( \mathbb{Z}_2[w_1, w_2] \) with respect to the grlex ordering \( \preceq \).

Proof. We have already shown that \( G \) is a basis for \( I_{2,n} \). We wish to apply Theorem 2.5. Let \( g_m \) and \( g_{m+s} \) \( (0 \leq m < m + s \leq n + 1) \) be two arbitrary elements of \( G \). If \( m = n \), then \( m + s \) must be \( n + 1 \) and, using (3) and Proposition 2.2, one obtains \( S(g_m, g_{m+s}) = S(g_n, g_{n+1}) = w_2 g_n + w_1 g_{n+1} = 0 \). If \( m \leq n - 1 \), then according to Lemma 2.6,

\[
S(g_m, g_{m+s}) = \sum_{i=0}^{s-1} w_1^i w_2^{s-1-i} g_{m+2+i}.
\]

Define \( t = t(m, s) := w_1^{n-1-m} w_2^{m+s+1} \). First of all, observe that

\[ t < w_1^{n+1-m} w_2^{m+s} = \text{lcm}(\text{LT}(g_m), \text{LT}(g_{m+s})). \]
Let normal bundle \( M^s \) we recall the theorem of Hirsch (\([g]\) and \( \gamma \) in our later calculations. As we have already noticed, by formula (2), we have proved the following corollary.

Theorem 2.5 applies, and we conclude that \( G \) is a Groebner basis for \( I_{2,n} \).

To see that it is the reduced one, we observe that \( \{\text{LT}(g) \mid g \in G\} \) is the set of all monomials \( w_1^n w_2^b \) such that \( a + b = n + 1 \). Also, by looking at formula (2), we see that all other terms appearing in \( g_m \) have the sum of the exponents < \( n + 1 \), and so they cannot be divisible by any of the leading terms in \( G \).

Since \( G \) is a Groebner basis for \( I_{2,n} \), a vector space basis for \( \mathbb{Z}_2[w_1, w_2]/I_{2,n} \) could be formed by taking all the monomials in \( \mathbb{Z}_2[w_1, w_2] \) (more precisely, their classes) which are not divisible by any of \( \text{LT}(g_0), \text{LT}(g_1), \ldots, \text{LT}(g_{n+1}) \). As we have noticed in the proof of Theorem 2.7, the set \( \{\text{LT}(g) \mid g \in G\} \) is the set of all monomials \( w_1^n w_2^b \) such that \( a + b = n + 1 \). From this it is obvious that \( w_1^n w_2^b \) is not divisible by any of the leading terms \( \text{LT}(g_m) \) if and only if \( a + b \leq n \). By collecting all these facts, we have proved the following corollary.

**Corollary 2.8.** Let \( n \geq 2 \). If \( w_i \) is the \( i \)-th Stiefel-Whitney class of the canonical vector bundle \( \gamma_2 \) over \( G_{2,n} \), then the set \( \{w_1^n w_2^b \mid a + b \leq n\} \) is a vector space basis for \( H^*(G_{2,n}; \mathbb{Z}_2) \).

Let us now determine a few elements of the Groebner basis \( G \) which will be used in our later calculations. As we have already noticed, by formula (2), \( g_{n+1} = w_2^{n+1} \) and \( g_n = w_1 w_2^n \). Using this and Proposition 2.2, we obtain \( w_2 g_{n-1} = w_1 g_n + g_{n+1} = w_1^n w_2^{n+1} = w_1^n w_2^{n-1} + w_2^n \), and so we deduce that \( g_{n-1} = w_1^n w_2^{n+1} + w_2^n \).

Continuing in the same manner, one gets

\[
\begin{align*}
g_{n-2} &= w_1^n w_2^{n-2}; \\
g_{n-3} &= w_1^n w_2^{n-3} + w_2^n w_2^{n-2} + w_2^{n-1}; \\
g_{n-4} &= w_1^n w_2^{n-4} + w_1 w_2^{n-2}; \\
g_{n-5} &= w_1^n w_2^{n-5} + w_1^n w_2^{n-4} + w_2^{n-2}.
\end{align*}
\]

### 3. Immersions

In order to construct the immersions of Grassmannians \( G_{2,n} \) into Euclidean spaces, we recall the theorem of Hirsch ([4]) which states that a smooth compact \( m \)-manifold \( M^m \) immerses in \( \mathbb{R}^{m+l} \) if and only if the classifying map \( f_\nu : M^m \rightarrow BO \) of the stable normal bundle \( \nu \) of \( M^m \) lifts up to \( BO(l) \).

\[
\begin{array}{c}
\text{BO}(l) \\
\downarrow p \\
\text{M}^m \\
\downarrow f_\nu \\
\text{BO}
\end{array}
\]
Let \( \text{imm}(M^m) \) denote the least integer \( d \) such that \( M^m \) immerses into \( \mathbb{R}^d \). By Hirsch’s theorem, if \( w_k(\nu) \neq 0 \) then \( \text{imm}(M^m) \geq m + k \).

As in Corollary 2.8, let \( w_i \) be the \( i \)-th Stiefel-Whitney class of the canonical vector bundle \( \gamma_2 \) over \( G_{2,n} (n \geq 2) \). It is well known (see [8, p. 179]) that, if \( 2^s \) is the least power of 2 exceeding \( n \), i.e., \( 2^{s-1} \leq n < 2^s \), then for the total Stiefel-Whitney class \( w(\nu) \) of the stable normal bundle \( \nu \) of \( G_{2,n} \), one has

\[
    w(\nu) = (1 + w_1^2)(1 + w_1 + w_2)^{2^{s+1} - 2 - n}.
\]  

(4)

For \( n = 2^s - 2 \), from formula (2) we have that

\[
    g_0 = \sum_{b=0}^{2^{s-1}-1} \left( \begin{array}{l} 2^s - 1 - b \\ b \end{array} \right) w_1^{2^s - 1 - 2b} w_2^b = w_1^{2^s-1}
\]

since the binomial coefficient \( \left( \begin{array}{l} 2^s - 1 - b \\ b \end{array} \right) \) is odd only for \( b = 0 \) (by Lucas formula). This means that \( w_1^{2^s-1} = 0 \) in \( H^*(G_{2,2^s-2}; \mathbb{Z}_2) \). But then \( w_1^{2^s-1} = 0 \) in \( H^*(G_{2,n}; \mathbb{Z}_2) \) for all \( n \leq 2^s - 2 \) since the inclusion \( i: G_{2,n} \rightarrow G_{2,2^s-2} \) is obviously covered by a map of canonical bundles \( \gamma_2 \).

If \( 2^s-1 \leq n \leq 2^s - 2 \), then by formula (4) we have

\[
    w(\nu) = (1 + w_1^2)(1 + w_1 + w_2)^{2^{s-2} - n} = (1 + w_1^2)(1 + w_1^2 + w_2^2)(1 + w_1 + w_2)^{2^{s-2} - n}.
\]

Now, \( w_2^2 = 0 \) because it is a class of degree \( 2^{s+1} > 2^{s+1} - 4 \geq 2n = \dim(G_{2,n}) \). Also, by the previous discussion \( w_1^2 = 0 \) and (4) simplifies to

\[
    w(\nu) = (1 + w_1^2)(1 + w_1 + w_2)^{2^{s-2} - n}.
\]  

(5)

If \( n = 2^s - 1 \), then from (4) we obtain

\[
    w(\nu) = (1 + w_1^2)(1 + w_1 + w_2)^{2^{s-1}}
\]

\[
    = (1 + w_1^2)^2 \sum_{i=0}^{2^s-1} \left( \begin{array}{l} 2^s - 1 \\ i \end{array} \right) (1 + w_1)^i w_2^{2^s-1-i}
\]

\[
    = \sum_{i=0}^{2^s-1} (1 + w_1)^{i+2} w_2^{2^s-1-i}
\]

\[
    = \sum_{i=0}^{2^s-1} \sum_{j=0}^{i+2} \left( \begin{array}{l} i + 2 \\ j \end{array} \right) w_1^i w_2^{2^s-1-i}.
\]  

(6)

We now recall a theorem of Oproiu ([8]), and we prove it using the Groebner basis from Theorem 2.7.

**Theorem 3.1** (Oproiu [8]). For \( 2 \leq 2^{s-1} \leq n < 2^s \), we have:

(a) \( \text{imm}(G_{2,n}) \geq 2^{s+1} - 2 \).

(b) \( \text{imm}(G_{2,2^s-1}) \geq 3 \cdot 2^s - 2 \).

**Proof.** (a) The top class in the expression (5) is \( w_{2^{s+1}-2-n} = w_1^2 w_2^{2^s-2-n} \), and since the sum of the exponents \( 2 + 2^s - 2 - n = 2^s - n \leq 2^{s-1} \leq n \), by Corollary 2.8
we have that $w_{2s+1-2-2n}(\nu) \neq 0$ and we conclude that
\[ \text{imm}(G_{2,n}) \geq \dim(G_{2,n}) + 2^{s+1} - 2 - 2n = 2^{s+1} - 2. \]

(b) From the equality (6) we calculate
\[ w_{2^s}(\nu) = \sum_{i=2^{s-1}-1}^{2^s-1} \left( \frac{i+2}{2^s-2s} \right) w_1^{2i+2-2^s} w_2^{2^s-i} \]
\[ = \sum_{l=0}^{2^s-1} \left( \frac{2^s + 1 - l}{2^s - 2l} \right) w_1^{2^s-2l} w_2^l = \sum_{l=0}^{2^s-1} \left( \frac{2^s + 1 - l}{l + 1} \right) w_1^{2^s-2l} w_2^l \]
\[ = \left( \frac{2^s + 1}{1} \right) w_1^{2^s} + \left( \frac{2^s}{2} \right) w_1^{2^s-2} w_2 + \left( \frac{2^s - 1}{3} \right) w_1^{2^s-4} w_2^2 + \cdots \]
\[ = w_1^{2^s} + w_1^{2^s-4} w_2^2 + \cdots, \]
where the unwritten monomials (if there are any) have the sum of the exponents \( \leq 2^s - 3 = n - 2 \). Note that, since $2^{s-1} \geq 2$, three written summands must appear in the sum.

On the other hand, from the equality (2) we see that the first element of the Groebner basis in this case is
\[ g_0 = \sum_{b=0}^{2^s-1} \left( \frac{2^s - b}{b} \right) w_1^{2s-2b} w_2^b = w_1^{2^s} + (2^s - 1) w_1^{2^s-2} w_2 + \left( \frac{2^s - 2}{2} \right) w_1^{2^s-4} w_2^2 + \cdots \]
\[ = w_1^{2^s} + w_1^{2^s-2} w_2 + w_1^{2^s-4} w_2^2 + \cdots. \]
Again, the unwritten monomials have the sum of the exponents \( \leq n - 2 \), and three written ones must be here.

By adding these two equalities together, using the fact that $g_0 = 0$ in $H^*(G_{2,n}; \mathbb{Z}_2)$ we obtain
\[ w_{2^s}(\nu) = w_1^{2^s-2} w_2 + \cdots. \]

The sum of the exponents in the monomial $w_1^{2^s-2} w_2$ is $2^s - 1 = n$, and in the remaining monomials (if there are any) this sum is \( \leq n - 2 \), so none of these monomials is divisible by any of the leading terms $\text{LT}(g_m)$. This means that we have obtained the remainder of dividing $w_{2^s}(\nu)$ by $G$. Since $w_1^{2^s-2} w_2$ must appear in this remainder, we conclude that $w_{2^s}(\nu) \neq 0$. Finally, this implies that
\[ \text{imm}(G_{2,2^s-1}) \geq \dim(G_{2,2^s-1}) + 2^s = 3 \cdot 2^s - 2 \]
and we are done. \( \square \)

Example 3.2. If $n = 2^{s-1} > 2$, then $\text{imm}(G_{2,2^{s-1}}) \geq 2^{s+1} - 2 = 2 \cdot \dim(G_{2,2^{s-1}}) - 2$. By the result of Massey [5, Theorem V], if $M^m$ is orientable, $m > 4$ and $w_2(\nu) \cdot w_{m-2}(\nu) = 0$, then $M^m$ immerses into $\mathbb{R}^{2m-2}$. Now, $G_{2,2^{s-1}}$ is orientable (Grassmannian $G_{k,n}$ is orientable if and only if $n + k$ is even; see [8, p. 179]), and from formula (5) we have
\[ w(\nu) = (1 + w_1^2)(1 + w_1 + w_2)^{2^{s-1}-2}, \]
so
\[ w_2(\nu) = \left( 1 + \binom{2^{s-1} - 2}{2} \right) w_1^2 + (2^{s-1} - 2)w_2 = 0 \]
since \(2^{s-1} > 2\). This implies that \(G_{2,2^{s-1}}\) immerses into \(\mathbb{R}^{2^{s+1} - 2}\), i.e., \(\text{imm}(G_{2,2^{s-1}}) \leq 2^{s+1} - 2\), so for \(2^{s-1} > 2\), we actually have the equality
\[ \text{imm}(G_{2,2^{s-1}}) = 2^{s+1} - 2. \]

Also, we note that for \(G_{2,3}\), Oproiu’s Theorem 3.1(b) gives \(\text{imm}(G_{2,3}) \geq 10\), and by Cohen’s theorem ([2]), \(\text{imm}(G_{2,3}) \leq 10\), so \(\text{imm}(G_{2,3}) = 10\). For \(G_{2,5}\), the results of Oproiu ([8]) and Monks ([6]) provide inequalities \(14 \leq \text{imm}(G_{2,5}) \leq 17\).

We now turn to the proof of the immersion result.

**Lemma 3.3.** Let \(n\) be an odd integer \(\geq 5\). For the stable normal bundle \(\nu\) of \(G_{2,n}\) we have:

(a) \(w_i(\nu) = 0\) for \(i \geq 2n - 5\);

(b) \(w_1(\nu) = w_1\);

(c) \(w_2(\nu) = w_2\) if \(n \equiv 3 \pmod{4}\); \(w_2(\nu) = w_1^2 + w_2\) if \(n \equiv 1 \pmod{4}\).

**Proof.** As above, let \(s\) be the integer such that \(2^{s-1} \leq n < 2^s\). Since \(n\) is odd, we have that \(n \geq 2^{s+1} + 1\). This implies \(4n \geq 2^{s+1} + 4\), i.e., \(2^{s+1} - 2 - 2n < 2n - 5\).

If \(n \neq 2^s - 1\), then from formula (5) we see that the top class in the expression for \(w(\nu)\), namely \(w_1^2w_2^{2^s - 2 - n}\), is of degree \(2^{s+1} - 2 - 2n\), and by the previous inequality, we deduce that \(w_i(\nu) = 0\) for \(i \geq 2n - 5\).

For \(n = 2^s - 1\), Oproiu shows [8, p. 182] that the top nonzero class in the expression (4) is of degree \(2^s\) and, since \(n \geq 7\) in this case, we conclude that \(2^s = n + 1 < 2n - 5\) obtaining (a).

From formula (4) we read off
\[ w_1(\nu) = (2^{s+1} - 2 - n)w_1; \]
\[ w_2(\nu) = \left( 1 + \binom{2^{s+1} - 2 - n}{2} \right) w_1^2 + (2^{s+1} - 2 - n)w_2 \]
and obtain (b) and (c).

A few more lemmas will be useful.

**Lemma 3.4.** In \(H^*(G_{2,n}; \mathbb{Z}_2)\), for all nonnegative integers \(a\) and \(b\), the following relations hold:

(a) \(Sq^1(w_1^aw_2^b) = (a + b)w_1^{a+1}w_2^b\);

(b) \(Sq^2(w_1^aw_2^b) = bw_1^{a+1}w_2^b + (a + b)w_1^{a+2}w_2^b\).

**Proof.** Since \(Sq^2(w_1^a) = \binom{a}{2}w_1^{a+1}\), the formulas are true for \(b = 0\). We proceed by induction on \(b\).

(a) By the Wu formula, \(Sq^1w_2 = w_1w_2\). Using the Cartan formula and the induction hypothesis, we have...
By Corollary 2.8, the set \( H \) is an odd integer

Proof. >

(b) For the induction step we use again formulas of Cartan and Wu, the statement (a) and the fact that \( Sq^2 w_2 = w_2^2 \). We calculate

\[
Sq^2(w_1^a w_2^b) = Sq^2(w_2 w_1^a w_2^{b-1}) \\
= w_1 w_2^a w_2^{b-1} + (a + b - 1) w_2^a w_2^{b-1} \\
= (a + b) w_1^a w_2^b.
\]

and the proof is complete.

\[ \square \]

**Lemma 3.5.** The map \( (Sq^2 + w_2(\nu)) : H^{2n-5}(G_{2n};\mathbb{Z}_2) \to H^{2n-3}(G_{2n};\mathbb{Z}_2) \), where \( n \) is an odd integer \( \geq 5 \), is determined by the equalities

\[
(Sq^2 + w_2(\nu))(w_1^5 w_2^{n-5}) = (Sq^2 + w_2(\nu))(w_1^3 w_2^{n-4}) = (Sq^2 + w_2(\nu))(w_1 w_2^{n-3}) \\
= w_1 w_2^{n-2}.
\]

**Proof.** By Corollary 2.8, the set \( \{w_1^5 w_2^{n-5}, w_1^3 w_2^{n-4}, w_1 w_2^{n-3}\} \) is a vector space basis for \( H^{2n-3}(G_{2n};\mathbb{Z}_2) \).

Now, if \( n \equiv 3 \pmod{4} \), using Lemma 3.3, Lemma 3.4 and Groebner basis from Theorem 2.7, we calculate

\[
(Sq^2 + w_2(\nu))(w_1^5 w_2^{n-5}) = Sq^2(w_1^5 w_2^{n-5}) + w_2 w_1^5 w_2^{n-5} \\
= (n - 5) w_1^5 w_2^{n-5} + \binom{n}{2} w_1^7 w_2^{n-5} + w_1^5 w_2^{n-4} \\
= w_1^7 w_2^{n-5} + w_1^5 w_2^{n-4} \\
= w_1(g_{n-5} + w_1^4 w_2^{n-4} + w_2^{n-2}) + w_1^5 w_2^{n-4} = w_1 w_2^{n-2};
\]

\[
(Sq^2 + w_2(\nu))(w_1^3 w_2^{n-4}) = Sq^2(w_1^3 w_2^{n-4}) + w_2 w_1^3 w_2^{n-4} \\
= (n - 4) w_1^3 w_2^{n-4} + \binom{n-1}{2} w_1^5 w_2^{n-4} + w_1^3 w_2^{n-3} \\
= w_1^5 w_2^{n-4} = g_{n-4} + w_1 w_2^{n-2} = w_1 w_2^{n-2};
\]

\[
(Sq^2 + w_2(\nu))(w_1 w_2^{n-3}) = Sq^2(w_1 w_2^{n-3}) + w_2 w_1 w_2^{n-3} \\
= (n - 3) w_1 w_2^{n-2} + \binom{n-2}{2} w_1^3 w_2^{n-3} + w_1 w_2^{n-2} \\
= w_1 w_2^{n-2}.
\]
Similarly, if \( n \equiv 1 \pmod{4} \), we have
\[
(Sq^2 + w_2(\nu))(w_1^5w_2^{n-5}) = Sq^2(w_1^5w_2^{n-5}) + w_1^2w_1^5w_2^{n-5} + w_2w_1^5w_2^{n-5}
\]
\[
= (n - 5)w_1^5w_2^{n-5} + \binom{n}{2}w_1^7w_2^{n-5} + w_1^7w_2^{n-5} + w_1^5w_2^{n-4}
\]
\[
= w_1^5w_2^{n-5} + w_1^5w_2^{n-4}
\]
\[
= w_1(g_{n-5} + w_4w_2^{n-4} + w_2^{n-2}) + w_1^5w_2^{n-4} = w_1w_2^{n-2};
\]
\[
(Sq^2 + w_2(\nu))(w_1^3w_2^{n-4}) = Sq^2(w_1^3w_2^{n-4}) + w_1^2w_1^3w_2^{n-4} + w_2w_1^3w_2^{n-4}
\]
\[
= (n - 4)w_1^3w_2^{n-3} + \binom{n-1}{2}w_1^5w_2^{n-4} + w_1^5w_2^{n-4} + w_1^3w_2^{n-3}
\]
\[
= w_1^5w_2^{n-4} = g_{n-4} + w_1w_2^{n-2} = w_1w_2^{n-2};
\]
\[
(Sq^2 + w_2(\nu))(w_1^w_2^{n-3}) = Sq^2(w_1^w_2^{n-3}) + w_1^2w_1^w_2^{n-3} + w_2w_1^w_2^{n-3}
\]
\[
= (n - 3)w_1w_2^{n-2} + \binom{n-2}{2}w_1^3w_2^{n-3} + w_1^3w_2^{n-3} + w_1w_2^{n-2}
\]
\[
= w_1w_2^{n-2},
\]
and the proof of the lemma is complete. \(\square\)

**Lemma 3.6.** The map \( Sq^1 : H^{2n-2}(G_{2,n};\mathbb{Z}_2) \to H^{2n-1}(G_{2,n};\mathbb{Z}_2) \), where \( n \) is an odd integer \( \geq 5 \), is trivial.

**Proof.** The set \( \{w_1^2w_2^{n-2}, w_2^{n-1}\} \) is a vector space basis for \( H^{2n-2}(G_{2,n};\mathbb{Z}_2) \) (Corollary 2.8). Using Lemma 3.4, we obtain
\[
Sq^1(w_1^2w_2^{n-2}) = nw_1^2w_2^{n-2} = w_1^3w_2^{n-2} = g_{n-2} = 0;
\]
\[
Sq^1(w_2^{n-1}) = (n - 1)w_1w_2^{n-1} = 0,
\]
which proves the lemma. \(\square\)

**Lemma 3.7.** The map \( (Sq^2 + w_3(\nu)) : H^{2n-3}(G_{2,n};\mathbb{Z}_2) \to H^{2n-1}(G_{2,n};\mathbb{Z}_2) \), where \( n \) is an odd integer \( \geq 5 \), is determined by the equalities:
\[
(Sq^2 + w_2(\nu))(w_1^w_2^{n-3}) = w_1w_2^{n-1} \neq 0;
\]
\[
(Sq^2 + w_2(\nu))(w_1^w_2^{n-2}) = 0.
\]

**Proof.** Again by Corollary 2.8, the classes \( w_1^3w_2^{n-3} \) and \( w_1w_2^{n-2} \) form a vector space basis for \( H^{2n-3}(G_{2,n};\mathbb{Z}_2) \), and the class \( w_1w_2^{n-1} \) is nontrivial in \( H^{2n-1}(G_{2,n};\mathbb{Z}_2) \cong \mathbb{Z}_2 \).

By Lemma 3.3 and Lemma 3.4, for \( n = 3 \pmod{4} \) we have
\[
(Sq^2 + w_2(\nu))(w_1^w_2^{n-3}) = Sq^2(w_1^w_2^{n-3}) + w_2w_1^w_2^{n-3}
\]
\[
= (n - 3)w_1^w_2^{n-3} + \binom{n}{2}w_1^w_2^{n-3} + w_1^w_2^{n-2}
\]
\[
= w_1^w_2^{n-3} + w_1^w_2^{n-2}
\]
\[
= w_1(g_{n-3} + w_2^{n-2} + w_2^{n-1}) + w_1^w_2^{n-2}
\]
\[
= w_1w_2^{n-1};
\]
If $H$ is a vector space basis for $H^{n-2}(G_2;\mathbb{Z}_2)$, which was to be proved.

Likewise, for $n \equiv 1 \pmod{4}$, we obtain

\[
(Sq^2 + w_2(\nu))(w_1^{n-2}) = Sq^2(w_1^{n-2}) + w_2 w_1^{n-2} = (n-2)w_1 w_2^{n-1} + \left(\frac{n-1}{2}\right)w_1^3 w_2^{n-2} + w_1 w_2^{n-1}
\]

which was to be proved. \hfill \Box

**Lemma 3.8.** The map $Sq^1 : H^{2n-3}(G_2;\mathbb{Z}_2) \to H^{2n-2}(G_2;\mathbb{Z}_2)$, where $n$ is an odd integer $\geq 5$, is given by the equalities:

\[
Sq^1(w_1^{n-3}) = w_1^2 w_2^{n-2} + w_2^{n-1};
\]

\[
Sq^1(w_1 w_2^{n-2}) = 0.
\]

**Proof.** As we have noticed in the proof of the previous lemma, the set

\[
\{w_1^3 w_2^{n-3}, w_1 w_2^{n-2}\}
\]

is a vector space basis for $H^{2n-3}(G_2;\mathbb{Z}_2)$. So, we calculate

\[
Sq^1(w_1^{n-3}) = nw_1^4 w_2^{n-3} = w_1^4 w_2^{n-3} = g_{n-3} + w_1^2 w_2^{n-2} + w_2^{n-1} = w_1^2 w_2^{n-2} + w_2^{n-1},
\]

\[
Sq^1(w_1 w_2^{n-2}) = (n-1)w_1^2 w_2^{n-2} = 0
\]

by Lemma 3.4.

**Lemma 3.9.** If $n$ is an odd integer $\geq 5$, then in $H^*(G_2;\mathbb{Z}_2)$ we have

\[
(Sq^2 + w_1(\nu)^2 + w_2(\nu)) Sq^1(w_1^{n-4} + w_1 w_2^{n-3}) = w_1^2 w_2^{n-2}.
\]

**Proof.** By Lemma 3.4(a),

\[
Sq^1(w_1^{n-4} + w_1 w_2^{n-3}) = (n-1)w_1^4 w_2^{n-4} + (n-2)w_1^2 w_2^{n-3} = w_1^2 w_2^{n-3}.
\]

If $n \equiv 3 \pmod{4}$, by Lemma 3.3 and Lemma 3.4(b), one obtains

\[
(Sq^2 + w_1(\nu)^2 + w_2(\nu))(w_1^{n-3}) = Sq^2(w_1^{n-3}) + w_1^2 w_1^2 w_2^{n-3} + 2w_1 w_2^{n-3}
\]

\[
= (n-3)w_1^2 w_2^{n-2} + \left(\frac{n-1}{2}\right)w_1^4 w_2^{n-3} + w_1^2 w_2^{n-3}
\]

\[+ w_1^2 w_2^{n-2} = w_1^2 w_2^{n-3}.\]
If \( n \equiv 1 \pmod{4} \), again by Lemma 3.3 and Lemma 3.4(b), we have
\[
(Sq^2 + w_1(\nu)^2 + w_2(\nu))(w_1^2 w_2^{n-3}) = Sq^2(w_1^2 w_2^{n-3}) + w_2 w_1^2 w_2^{n-3} = (n - 3)w_1^2 w_2^{n-2} + \binom{n - 1}{2} w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = w_1^2 w_2^{n-2},
\]
and we are done.

**Lemma 3.10.** If \( n \) is an odd integer \( \geq 5 \), then in \( H^*(G_{2,n};\mathbb{Z}_2) \) the following equality holds:
\[
(Sq^2 + w_2(\nu))(w_1^2 w_2^{n-3}) = w_2^{n-1}.
\]

**Proof.** As before, we use Lemma 3.3, Lemma 3.4 and Groebner basis from Theorem 2.7.

If \( n \equiv 3 \pmod{4} \),
\[
(Sq^2 + w_2(\nu))(w_1^2 w_2^{n-3}) = Sq^2(w_1^2 w_2^{n-3}) + w_2 w_1^2 w_2^{n-3} = (n - 3)w_1^2 w_2^{n-2} + \binom{n - 1}{2} w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = g_{n-3} + w_2^{n-1} = w_2^{n-1}.
\]

If \( n \equiv 1 \pmod{4} \),
\[
(Sq^2 + w_2(\nu))(w_1^2 w_2^{n-3}) = Sq^2(w_1^2 w_2^{n-3}) + w_2 w_1^2 w_2^{n-3} = (n - 3)w_1^2 w_2^{n-2} + \binom{n - 1}{2} w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = g_{n-3} + w_2^{n-1} = w_2^{n-1},
\]
completing the proof.

We are now ready to prove our immersion result.

**Theorem 3.11.** If \( n \) is an odd integer \( \geq 7 \), then \( G_{2,n} \) immerses into \( \mathbb{R}^{4n-5} \).

**Proof.** Let \( f_\nu : G_{2,n} \to BO \) be the classifying map for the stable normal bundle \( \nu \) of \( G_{2,n} \). We want to show that \( f_\nu \) can be lifted up to \( BO(2n-5) \). We will use the 2n-MPT for the fibration \( p : BO(2n-5) \to BO \) which can be constructed by the method of Gitler and Mahowald ([3]) using the result of Nussbaum ([7]) who proved that their method is applicable to the fibrations \( p : BO(l) \to BO \) when \( l \) is odd. The tower is presented in Figure 1 (\( K_m \) stands for the Eilenberg-MacLane space \( K(\mathbb{Z}_2, m) \)).

The relations that produce the \( k \)-invariants are
\[
k_1^2 : (Sq^2 + w_2)w_{2n-4} = 0,
\]
\[
k_2^2 : (Sq^2 + w_1^2 + w_2)S_1^1 w_{2n-4} + S_1^1 w_{2n-2} = 0,
\]
\[
k_3^2 : \begin{cases} (Sq^4 + w_4)w_{2n-4} + w_2 w_{2n-2} = 0, & n \equiv 3 \pmod{4} \\ (Sq^4 + w_4)w_{2n-4} + S_1^2 w_{2n-2} = 0, & n \equiv 1 \pmod{4} \end{cases},
\]
\[
k_1^3 : (Sq^2 + w_2)k_1^2 + Sq^4 k_2^2 = 0.
\]
Figure 1: $2n$-MPT for $p : BO(2n - 5) \to BO$

Since $\dim(G_{2,n}) = 2n$, $f_\nu$ lifts up to $BO(2n - 5)$ if and only if it lifts up to $E_3$.

By Lemma 3.3(a), $f^*_\nu(w_{2n-4}) = w_{2n-4}(\nu) = 0$, $f^*_\nu(w_{2n-2}) = w_{2n-2}(\nu) = 0$, so $f_\nu$ can be lifted up to $E_1$, i.e., there is a map $g_1 : G_{2,n} \to E_1$ such that $g_1 \circ g_1 = f_\nu$.

In order to make the next step (to lift $f_\nu$ up to $E_2$), we need to modify (if necessary) the lifting $g_1$ to a lifting $g$ such that $g^*(k^2_1) = g^*(k^2_2) = g^*(k^2_3) = 0$. By choosing a map $\alpha \times \beta : G_{2,n} \to K_{2n-5} \times K_{2n-3} = \Omega(K_{2n-4} \times K_{2n-2})$ (i.e., classes $\alpha \in H^{2n-5}(G_{2,n}; \mathbb{Z}_2)$ and $\beta \in H^{2n-3}(G_{2,n}; \mathbb{Z}_2)$), we get another lifting $g : G_{2,n} \to E_1$ as the composition

$$G_{2,n} \xrightarrow{\Delta} G_{2,n} \times G_{2,n} \xrightarrow{(\alpha \times \beta) \times g_1} K_{2n-5} \times K_{2n-3} \times E_1 \xrightarrow{\mu} E_1,$$

where $\Delta$ is the diagonal mapping and $\mu : \Omega(K_{2n-4} \times K_{2n-2}) \times E_1 \to E_1$ is the action of the fibre in the principal fibration $q_1 : E_1 \to BO$. So, we are looking for classes $\alpha$ and $\beta$ such that $g^*(k^2_1) = g^*(k^2_2) = g^*(k^2_3) = 0$. By looking at the relations that produce the $k$-invariants $k^2_1$, $k^2_2$ and $k^2_3$, we conclude that the following equalities hold (see [3, p. 95]):

$$g^*(k^2_1) = g^*_1(k^2_1) + (Sq^2 + w_2(\nu))(\alpha);$$
$$g^*(k^2_2) = g^*_1(k^2_2) + (Sq^2 + w_1(\nu)^2 + w_2(\nu))Sq^1\alpha + Sq^1\beta;$$
$$g^*(k^2_3) = \begin{cases} g^*_1(k^2_3) + (Sq^4 + w_4(\nu))(\alpha) + w_2 \cdot \beta, & n \equiv 3 \pmod{4} \\ g^*_1(k^2_3) + (Sq^4 + w_4(\nu))(\alpha) + Sq^2\beta, & n \equiv 1 \pmod{4}. \end{cases}$$

First, we need to prove that the class $g^*_1(k^2_3)$ is in the image of the map $(Sq^2 + w_2(\nu)) : H^{2n-3}(G_{2,n}; \mathbb{Z}_2) \to H^{2n-3}(G_{2,n}; \mathbb{Z}_2)$. The $k$-invariant $k^3_1$ is produced by the relation $(Sq^2 + w_2)k^3_1 + Sq^2 k^3_2 = 0$ which holds in $H^*(E_1; \mathbb{Z}_2)$. Applying $g_1^*$, we get

$$(Sq^2 + w_2(\nu))g^*_1(k^2_3) = Sq^1 g^*_1(k^2_2).$$

But, by Lemma 3.6, $Sq^1 g^*_1(k^2_2) = 0$. Hence, $g^*_1(k^2_3)$ is in the kernel of the map $(Sq^2 + w_2(\nu)) : H^{2n-3}(G_{2,n}; \mathbb{Z}_2) \to H^{2n-3}(G_{2,n}; \mathbb{Z}_2)$.
w_2(\nu)) : H^{2n-3}(G_{2,n}; \mathbb{Z}_2) \rightarrow H^{2n-1}(G_{2,n}; \mathbb{Z}_2), and according to Lemmas 3.5 and 3.7, this kernel coincides with the image of the map (Sq^2 + w_2(\nu)) : H^{2n-5}(G_{2,n}; \mathbb{Z}_2) \rightarrow H^{2n-3}(G_{2,n}; \mathbb{Z}_2). Thus, we can find a class \alpha \in H^{2n-3}(G_{2,n}; \mathbb{Z}_2) such that g^* (k_2^3) = 0.

Since H^{2n-2}(G_{2,n}; \mathbb{Z}_2) is generated by the classes w_1^2 w_2^{n-2} and w_2^{n-1} (Corollary 2.8), by Lemma 3.8 and Lemma 3.9 we see that we can choose a class \beta \in H^{2n-3}(G_{2,n}; \mathbb{Z}_2) and modify \alpha (by adding, if necessary, the class w_1^2 w_2^{n-2} + w_1 w_2^{n-3}) to obtain g such that g^* (k_2^3) = 0. Since w_1^2 w_2^{n-2} + w_1 w_2^{n-3} is in the kernel of the map (Sq^2 + w_2(\nu)) : H^{2n-5}(G_{2,n}; \mathbb{Z}_2) \rightarrow H^{2n-3}(G_{2,n}; \mathbb{Z}_2) (Lemma 3.5), adding this class to the previous \alpha will not spoil the equality g^* (k_2^3) = 0.

Finally, observe the class \beta' := w_1 w_2^{n-2} \in H^{2n-3}(G_{2,n}; \mathbb{Z}_2). According to Corollary 2.8, w_2 \cdot \beta' = w_1 w_2^{n-1} \neq 0 and if \eta \equiv 1 (\text{mod } 4), by Lemma 3.4,

\[ Sq^2 \beta' = (n - 2)w_1 w_2^{n-1} + \left( \frac{n - 1}{2} \right) w_1^3 w_2^{n-2} = w_1 w_2^{n-1} \neq 0. \]

Since \beta' is in the kernel of the map Sq^1 : H^{2n-3}(G_{2,n}; \mathbb{Z}_2) \rightarrow H^{2n-2}(G_{2,n}; \mathbb{Z}_2) (Lemma 3.8), we can add this class to the previous \beta (if necessary) and obtain a lifting \eta such that g^* (k_2^3) = g^* (k_2^3) = g^* (k_2^3) = 0.

Therefore, we can lift \eta up to E_2, i.e., there is a map \eta_1 : G_{2,n} \rightarrow E_2 such that q_1 \circ q_2 \circ h_1 = q_1 \circ g = f_\nu.

We need to make one more step: to prove that the lifting \eta_1 can be modified to a lifting \eta which lifts up to E_3, i.e., such that h^* (k_2^3) = 0. Arguing as before, we see that it suffices to find classes \alpha \in H^{2n-4}(G_{2,n}; \mathbb{Z}_2) and \beta \in H^{2n-3}(G_{2,n}; \mathbb{Z}_2) such that (Sq^2 + w_2(\nu))(\alpha) + Sq^1 \beta = h^* (k_2^3) \in H^{2n-4}(G_{2,n}; \mathbb{Z}_2). But, since w_1^2 w_2^{n-2} and w_2^{n-1} generate H^{2n-3}(G_{2,n}; \mathbb{Z}_2), according to Lemma 3.8 and Lemma 3.10, such classes \alpha and \beta exist (that is, the indeterminacy of k_2^3 is all of H^{2n-2}(G_{2,n}; \mathbb{Z}_2)). This completes the proof of the theorem. 

\[ \square \]

References


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