KEI MODULES AND UNORIENTED LINK INVARIANTS

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(communicated by Ronald Brown)

Abstract

We define invariants of unoriented knots and links by enhancing the integral kei counting invariant \( \Phi^Z_X(K) \) for a finite kei \( X \) using representations of the kei algebra, \( \mathbb{Z}_K[X] \), a quotient of the quandle algebra \( \mathbb{Z}[X] \) defined by Andruskiewitsch and Graña. We give an example that demonstrates that the enhanced invariant is stronger than the unenhanced kei counting invariant. As an application, we use a quandle module over the Takasaki kei on \( \mathbb{Z}_3 \) which is not a \( \mathbb{Z}_K[X] \)-module to detect the non-invertibility of a virtual knot.

1. Introduction

In [10], Mituhisa Takasaki introduced an algebraic structure known as kei (or \( \圭 \) in the original kanji). In [7] this same structure was reintroduced under the name involutory quandle, a special case of a more general algebraic structure related to oriented knots known as quandles. These algebraic structures can be understood as arising from the unoriented and oriented Reidemeister moves respectively via a certain labeling scheme, encoding knot structures in algebra.

In [1], an associative algebra \( \mathbb{Z}[X] \) was defined for every finite quandle \( X \); in [3] a geometric interpretation of \( \mathbb{Z}[X] \) was given, with generators representing coefficients of “beads” indexed by quandle labelings of arcs, with relations defined from the Reidemeister moves. Representations of \( \mathbb{Z}[X] \), known as quandle modules, were used to define new invariants of oriented knots and links in [3]. In [8] a modification of \( \mathbb{Z}[X] \) for finite racks (a generalization of quandles to the case of blackboard-framed isotopy) was used to define invariants of framed and unframed oriented knots and links.

In this paper we define a modification of the quandle algebra we call the kei algebra \( \mathbb{Z}_K[X] \) and use it to extend the invariants defined in [8] to unoriented knots and links. The paper is organized as follows. In section 2 we review the basics of kei and the kei counting invariant. In section 3 we define the kei algebra and kei modules. In section 4 we define the kei module enhanced counting invariant. As an application, we use a module over \( \mathbb{Z}[X] \) for a kei \( X \) which is not a \( \mathbb{Z}_K[X] \)-module to detect the non-invertibility of a virtual knot.
non-invertibility of a virtual knot. In section 5 we collect a few questions for future research.

2. Kei

Kei or involutory quandles were introduced by Mituhisa Takasaki in 1945 [10] and later reintroduced independently by David Joyce and S.V. Matveev in the early 1980s [7, 9].

Definition 2.1. A kei or involutory quandle is a set $X$ with a binary operation $\triangleright$ satisfying for all $x, y, z \in X$

(i) $x \triangleright x = x$,
(ii) $(x \triangleright y) \triangleright y = x$, and
(iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Example 2.2. Let $X$ be any abelian group regarded as a $\mathbb{Z}$-module. Then $X$ is a kei under the operation

$$x \triangleright y = 2y - x.$$ 

Such a kei is known as a Takasaki kei. If $X \cong \mathbb{Z}_n$ then $X$ is often denoted as $R_n$ in the knot theory literature, known as the dihedral quandle on $n$ elements. $R_n$ can also be understood as the set of reflections of a regular $n$-gon.

Example 2.3. Let $X$ be any module over $\mathbb{Z}[t]/(t^2 - 1)$. Then $X$ is a kei known as an Alexander kei under the operation

$$x \triangleright y = tx + (1 - t)y.$$ 

A Takasaki kei is an Alexander kei with $t = -1$.

Example 2.4. Let $L$ be an unoriented link diagram and let $A = \{a_1, \ldots, a_n\}$ be a set of generators corresponding bijectively with the set of arcs of $L$. The Fundamental Kei of $L$, $FK(L)$, is defined in the following way. First, let $W(L)$ be the set of kei words in $A$, defined recursively by the rules

- $a \in A \Rightarrow a \in W(L)$ and
- $x, y \in W(L) \Rightarrow x \triangleright y \in W(L)$.

Then the free kei on $A$ is the set of equivalence classes of kei words in $A$ under the equivalence relation generated by relations of the forms

- $x \triangleright x \sim x$,
- $(x \triangleright y) \triangleright y \sim x$, and
- $(x \triangleright y) \triangleright z \sim (x \triangleright z) \triangleright (y \triangleright z)$

for all $x, y, z \in W(L)$. The free kei is a kei under the operation $[x] \triangleright [y] = [x \triangleright y]$. Now, at each crossing in $L$, we have a crossing relation given by $z = x \triangleright y$ where $y$ is the
overcrossing arc and $x$ and $z$ are the undercrossing arcs. That is, we have

\[
\begin{array}{c}
 x \triangleright y \\
\downarrow \\
 y \\
\uparrow \\
x
\end{array}
\]

Then the *fundamental kei* of $L$, $FK(L)$, is the set of equivalence classes of free kei elements modulo the crossing relations of $L$, or equivalently $FK(L)$ is the set of equivalence classes of kei words in $A$ modulo the equivalence relation determined by the crossing relations together with the free kei relations.

It is convenient to describe a finite kei $X = \{x_1, \ldots, x_n\}$ with a matrix encoding the operation table of $X$, i.e. a matrix $M_X$ whose $(i, j)$ entry is $k$ where $x_k = x_i \triangleright x_j$. For example, the Takasaki kei on $\mathbb{Z}_3$ has matrix

\[
M_X = \begin{bmatrix}
 1 & 3 & 2 \\
 3 & 2 & 1 \\
 2 & 1 & 3
\end{bmatrix}
\]

where we set $x_1 = 1$, $x_2 = 2$, and $x_3 = 3 = 0$.

As with groups and other algebraic structures, we have the following standard notions:

**Definition 2.5.** Let $X$ and $Y$ be kei.

- A map $f: X \to Y$ is a *kei homomorphism* if for all $x, x' \in X$ we have $f(x \triangleright x') = f(x) \triangleright f(x')$.
- A subset $Y \subseteq X$ which is itself a kei under the kei operation $\triangleright$ of $X$ is a *subkei* of $X$. It is easy to check that $Y \subseteq X$ is a subkei if and only if $Y$ is closed under $\triangleright$.

For defining invariants of unoriented links, we have the following well-known result:

**Theorem 2.6.** If $L$ and $L'$ are ambient isotopic unoriented links, then there is an isomorphism of kei $\phi: FK(L) \to FK(L')$. For any finite kei $X$, the sets of homomorphisms $\text{Hom}(FK(L), X)$ and $\text{Hom}(FK(L'), X)$ are finite and there is an induced bijection $\phi_* : \text{Hom}(FK(L), X) \to \text{Hom}(FK(L'), X)$. In particular, the cardinality of the set of rack homomorphisms $\Phi^Z_X(L) = |\text{Hom}(FK(L), X)|$ is a non-negative integer-valued invariant of unoriented links known as the integral kei counting invariant.

A kei homomorphism $f: FK(L) \to X$ can be represented as a labeling of the arcs of $L$ with elements of $X$ satisfying the crossing relations at every crossing—such a labeling defines a unique homomorphism, and every $f \in \text{Hom}(FK(L), X)$ can be so represented.

**Example 2.7.** We can use the kei counting invariant to see that the trefoil knot 3$_1$ is nontrivially knotted. Let $X$ be the Takasaki kei on $\mathbb{Z}_3$; we have $x \triangleright y = 2y - x =
The crossing relations in $3_1$ give us the system of linear equations

$$
\begin{align*}
    z &= 2x + 2y \\
    y &= 2z + 2x \\
    x &= 2x + 2y
\end{align*}
$$

and the solution space is two-dimensional, giving us a total of $\Phi_X^Z(3_1) = 9$ solutions. Since $\Phi_X^Z(\text{Unknot}) = 3$, the integral kei counting invariant detects the knottedness of the trefoil.

**Remark 2.8.** Replacing the second kei axiom with the alternative axiom

(iii') There exists a second operation $\triangleright^{-1}$ satisfying

$$(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$$

for all $x, y \in X$

yields an algebraic object known as a *quandle*, which is the oriented analog of kei. Labeling oriented links according to the signed crossing conditions

![Diagram](https://via.placeholder.com/150)

defines homomorphisms from the fundamental quandle of the link $L$ into $X$; the *integral quandle counting invariant* $\Phi_X^Z(L)$ is then an invariant of oriented links.

### 3. Kei algebras and modules

Let $X$ be a finite kei. We would like to define an associative algebra on $X$ generated by “beads” such that secondary labelings of $X$-labeled link diagrams by beads are preserved by Reidemeister moves. Specifically, at a crossing in a link diagram with arcs labeled $x, y$ and $x \triangleright y$, we define the following relationship between the beads $a$, $b$, and $c$:

$$c = t_{x,y}a + s_{x,y}b.$$
The *kei algebra* of $X$, $\mathbb{Z}[K[X]]$, is the quotient of the polynomial algebra $\mathbb{Z}[t_{x,y}, s_{x,y}]$ by the ideal $I$ required to obtain invariance under unoriented Reidemeister moves.

First, we note that the bead relationship above also requires that

$$a = t_{x>y, y}c + s_{x>y, y}b;$$

together these imply

$$a = t_{x,y}t_{x>y,y}a + (t_{x,y}s_{x>y,y} + s_{x,y})b,$$

which yields

$$t_{x,y}t_{x>y,y} = 1 \quad \text{and} \quad t_{x,y}s_{x>y,y} + s_{x,y} = 0. \quad (1)$$

From the Reidemeister I move, we must have $t_{x,x} + s_{x,x} = 1$:

$$a = t_{x,x}a + s_{x,x}a \quad (2)$$

The Reidemeister II move yields conditions equivalent to equation (1):

$$c = t_{x,y}a + s_{x,y}b$$

$$a = t_{x>y,y}c + s_{x>y,y}b$$

$$\Rightarrow a = t_{x>y,y}t_{x,y}a + (t_{x>y,y}s_{x,y} + s_{x>y,y})b$$
The Reidemeister III move yields the defining equations for the original rack algebra \( \mathbb{Z}[X] \) from [1], comparing coefficients on bead \( e \):

\[
\begin{align*}
t_{xy,z}t_{x,y} &= t_{xz,yz}t_{x,z}, & t_{xy,z}s_{x,y} &= s_{xz,yz}t_{y,z} \\
\text{and } & & s_{xy,z} &= s_{xz,yz}s_{y,z} + t_{xz,yz}s_{x,z}.
\end{align*}
\]

### Definition 3.1

Let \( X \) be a finite kei. The **kei algebra** \( \mathbb{Z}_K[X] \) of \( X \) is the quotient of the polynomial algebra \( \mathbb{Z}[t_{x,y}, s_{x,y}] \) for all \( x, y \in X \) by the ideal \( I \) generated by all elements of the forms:

- \( t_{x,y}s_{xy,y} + s_{x,y} \)
- \( t_{x,x} + s_{x,x} - 1 \)
- \( t_{xy,z}s_{xy,y} - s_{xz,yz}t_{y,z} \)
- \( t_{xy,y}t_{xy,y} - t_{xz,yz}t_{x,z} \)
- \( s_{xy,y} - s_{xz,yz}s_{y,z} - t_{xz,yz}s_{x,z} \)

for all \( x, y, z \in X \). A \( \mathbb{Z}_K[X] \)-**module** or just an \( X \)-module is a representation of \( \mathbb{Z}_K[X] \), i.e., an abelian group \( A \) with a family of automorphisms \( t_{x,y} : A \to A \) and endomorphisms \( s_{x,y} : A \to A \) such that each of the bulleted maps listed above are zero.

We can now define the kei algebra of a finite kei \( X \).

### Example 3.2

Let \( X \) be a kei. Any ring \( R \) becomes a \( \mathbb{Z}_K[X] \)-module by choosing invertible elements \( t_{x,y} \) and elements \( s_{x,y} \) for \( x, y \in X \) satisfying the conditions (1), (2), and (3). In particular, if \( X = \{x_1, x_2, \ldots, x_n\} \) is a finite kei, we can specify a \( \mathbb{Z}_K[X] \)-module structure on \( R \) with an \( n \times 2n \) block matrix

\[
M_R = \begin{bmatrix}
  t_{1,1} & t_{1,2} & \cdots & t_{1,n} & s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\
  t_{2,1} & t_{2,2} & \cdots & t_{2,n} & s_{2,1} & s_{2,2} & \cdots & s_{2,n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  t_{n,1} & t_{n,2} & \cdots & t_{n,n} & s_{n,1} & s_{n,2} & \cdots & s_{n,n}
\end{bmatrix}
\]

such that the entries satisfy the conditions (1), (2), and (3) above.

### Remark 3.3

The **quandle algebra** defined in [1] is the quotient of the polynomial algebra \( \mathbb{Z}[t_{x,y}, s_{x,y}] \) by the ideal generated by the relations coming from the Reidemeister I and III moves, i.e.,

- \( t_{x,x} + s_{x,x} - 1 \)
• \( t_{x'y',z} t_{x,y} - t_{x'z',y'z} t_{x,z} \),
• \( t_{x'y',z} s_{x,y} - s_{x'z',y'z} t_{y,z} \),
• \( s_{x'y',z} - s_{x'z',y'z} s_{y,z} - t_{x'z',y'z} s_{x,z} \),

with the type II move condition handled by the bead labeling rule below.

\[
\begin{align*}
&x \triangleright y \\
&\begin{array}{c}
  \circ \\
  y
\end{array}
&x \triangleright^{-1} y \\
&\begin{array}{c}
  \circ \\
  y
\end{array}
\end{align*}
\]

\[
c = t_{x,y} a + s_{x,y} b
\]

The kei algebra \( \mathbb{Z}_K[X] \) is a quotient of the quandle algebra by the additional relations

\[
t_{x,y} s_{x'z',y'z} + s_{x,y} \quad \text{and} \quad 1 - t_{x,y} t_{x'z',y'z}.
\]

**Example 3.4.** For a specific instance of the type of kei module defined in example 3.2, let \( X \) be the 3-element Takasaki kei with kei matrix

\[
M_X = \begin{bmatrix}
1 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 & 3
\end{bmatrix}
\]

and let the ring \( R = \mathbb{Z}_5 \). Our python computations indicate that there are 48 \( \mathbb{Z}_K[X] \)-module structures on \( R \), including for instance

\[
M_R = \begin{bmatrix}
4 & 1 & 3 & 2 & 4 & 1 \\
3 & 4 & 2 & 3 & 2 & 3 \\
2 & 1 & 4 & 3 & 1 & 2
\end{bmatrix}
\]

**Remark 3.5.** For a given kei \( X \), the set of \( \mathbb{Z}_K[X] \)-modules over a given ring \( R \) is a subset of the set of \( \mathbb{Z}[X] \)-modules, and can be a proper subset depending on \( R \), since a \( \mathbb{Z}[X] \)-module satisfies the conditions in equations (1) and (3) but not necessarily those of equation (2). For instance, our python computations reveal a total of 32 \( \mathbb{Z}[X] \)-modules on the kei \( X \) and ring \( R \) in example 3.4 which are not \( \mathbb{Z}_K[X] \)-modules, including for instance

\[
M_R = \begin{bmatrix}
2 & 1 & 2 & 4 & 2 & 3 \\
1 & 2 & 2 & 4 & 3 & 2 \\
4 & 4 & 2 & 4 & 4 & 4
\end{bmatrix}
\]

The invariants defined in the next section associated with such modules are invariants of oriented links but not invariants of unoriented links.

**Example 3.6.** Another important example of a \( \mathbb{Z}_K[X] \) module is the fundamental \( \mathbb{Z}_K[X] \)-module of an \( X \)-labeled link. Let \( L \) be an unoriented link with a labeling \( f : FK(L) \to X \) by a kei \( X \). On each arc of \( L \), we place a bead; the set of crossing relations then determines a presentation for a \( \mathbb{Z}_K[X] \)-module, denoted \( \mathbb{Z}_f[X] \), which
we can represent concretely with a coefficient matrix of the resulting homogeneous system of linear equations. For instance, let $X$ be the kei with matrix

$$M_X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix};$$

then the (4,2)-torus link with the $X$-labeling below has fundamental $\mathbb{Z}_k[X]$-module presented by the matrix $M_{Z_5[X]}$:

$$M_{Z_5[X]} = \begin{bmatrix} t_{13} & s_{13} & -1 & 0 \\ 0 & t_{32} & s_{32} & -1 \\ -1 & 0 & t_{23} & s_{23} \\ s_{31} & -1 & 0 & t_{31} \end{bmatrix}.$$

4. Kei module enhancements of the counting invariant

We can now define invariants of unoriented knots and links using kei modules. The idea is to use the set of homomorphisms $g: \mathbb{Z}_f[X] \to R$ from the fundamental kei module of an $X$-labeled diagram $L$ to the kei module $R$ as a signature for each kei homomorphism $f: FK(L) \to X$.

**Definition 4.1.** Let $L$ be an unoriented knot or link, $X$ a finite kei, and $R$ a finite $\mathbb{Z}_K[X]$-module. The **kei module enhanced multiset** invariant of $L$ associated to $X$ and $R$ is the multiset of cardinalities of the sets of $\mathbb{Z}_K[X]$-module homomorphisms, i.e.,

$$\Phi_{X,R}(L) = \left\{ |\text{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R)| : f \in \text{Hom}(FK(L), X) \right\}.$$  

Taking the generating function of this multiset gives us a polynomial-form invariant for easy comparison: the **kei module enhanced invariant** of $L$ with respect to $X$ and $R$ is

$$\Phi_{X,R}(L) = \sum_{f \in \text{Hom}(FK(L), X)} u^{|\text{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R)|}.$$  

By construction, we have the following:

**Theorem 4.2.** If $L$ and $L'$ are ambient isotopic unoriented links, $X$ is a finite kei, and $R$ is a $\mathbb{Z}_K[X]$-module, then $\Phi_{X,R}^K(L) = \Phi_{X,R}^K(L')$ and $\Phi_{X,R}(L) = \Phi_{X,R}(L').$  

**Remark 4.3.** If $R$ is not finite, we can replace the cardinality $|\text{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R)|$ with the rank of the $\mathbb{Z}_K[X]$-module $\text{Hom}_{\mathbb{Z}_K[X]}(\mathbb{Z}_f[X], R).$

To compute $\Phi_{X,R}^K$, for each kei labeling $f: FK(L) \to X$ of $L$ by $X$, we first obtain the matrix for $\mathbb{Z}_f[X]$, replace each $t_{x,y}$ and $s_{x,y}$ with its value in $R$, and solve the resulting system of equations to obtain the contributions to $\Phi_{X,R}^K$ for $f$.

**Example 4.4.** Let $L$ be the figure eight knot 4_3 and let $X$ and $R$ be the kei and kei module on $\mathbb{Z}_5$ from example 3.4. The set of $X$-labelings of $L$ includes only constant
labelings, i.e. every arc is labeled with a 1, 2, or 3. For example, the constant labeling with every arc labeled 1 yields the listed \( \mathbb{Z}_f[X] \)-presentation matrix:

\[
M_{\mathbb{Z}_f[X]} = \begin{bmatrix}
t_{11} & -1 & s_{11} & 0 \\
s_{11} & t_{11} & 1 & -1 \\
-1 & 0 & t_{11} & s_{11} \\
s_{11} & t_{11} & 0 & -1
\end{bmatrix}
\]

Replacing the \( t_{xy} \) and \( s_{x,y} \) with their values in \( R \) and row-reducing over \( \mathbb{Z}_5 \), we obtain

\[
\begin{bmatrix}
4 & 4 & 2 & 0 \\
0 & 2 & 4 & 4 \\
4 & 0 & 4 & 2 \\
2 & 4 & 0 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 3 & 0 \\
0 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and this \( X \)-labeling contributes a \( u^{25} \) to the invariant \( \Phi^K_{X,R}(4_1) \). Summing these contributions over the complete set of \( X \)-labelings gives us \( \Phi^K_{X,R}(4_1) = 3u^{25} \). Comparing this to the unknot, which has \( \Phi^K_{X,R}(\text{Unknot}) = 3u^{5} \), we see that \( \Phi^K_{X,R} \) distinguishes the unoriented figure eight from the unoriented unknot despite the two having equal kei counting invariant values. In particular, since \( \Phi^Z_X(k) \) is obtained from \( \Phi^K_{X,R} \) by evaluating at \( u = 1 \), \( \Phi^K_{X,R} \) is a strictly stronger invariant than \( \Phi^Z_X(k) \).

**Example 4.5.** Our python computations\(^1\) yield the listed values for \( \Phi^K_{X,R} \) with \( X \) the 3-element Takasaki kei and the randomly selected \( \mathbb{Z}_K[X] \)-module over \( \mathbb{Z}_7 \) below for the prime knots with up to eight crossings and prime links with up to seven crossings as listed in the knot atlas [2]:

<table>
<thead>
<tr>
<th>( \Phi^K_{X,M}(L) )</th>
<th>( L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3u^7 )</td>
<td>unknot, 4_1, 5_1, 6_2, 6_3, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_8, 8_9, 8_{12}, 8_{13}, 8_{14}, 8_{17}, L2a1, L4a1, L6a2, L6a4, L6n1, L7a2, L7a3, L7a4, L7a7, L7n1, L7n2</td>
</tr>
<tr>
<td>( 3u^7 + 6u^{49} )</td>
<td>3_1, 6_1, 7_4, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}, 8_{21}, L6a1, L6a3, L6a5, L7a1, L7a5</td>
</tr>
<tr>
<td>( 3u^7 + 24u^{49} )</td>
<td>8_{18}</td>
</tr>
<tr>
<td>( 3u^{19} )</td>
<td>5_2, 7_1, 8_{16}, L7a6</td>
</tr>
<tr>
<td>( 9u^{19} )</td>
<td>7_7, 8_{5}</td>
</tr>
</tbody>
</table>

**Remark 4.6.** As with most enhancements of quandle-related counting invariants, \( \Phi^K_{X,M} \) is well-defined for unoriented virtual links as well as classical links.

In our final example, we use a quandle module which is not a kei module to detect the non-invertibility of a virtual knot.

Example 4.7. Let $X$ be the kei from example 3.4 and $M$ the quandle module from remark 3.5. Since $M$ is not a kei module, $\Phi^M_X$ is an invariant of oriented knots and links, but not unoriented knots and links. Thus, we can potentially use $\Phi^M_X$ to compare the two orientations of a non-invertible knot. In particular, consider the virtual knot numbered 4.97 in the Knot Atlas [2]; it is the closure of the virtual braid below. Let us denote 4.97 with the upward orientation by 4.97" and 4.97 with the downward orientation as 4.97#. The only labelings of 4.97 by $X$ are constant labelings, of which there are three for both orientations, the unenhanced integral kei counting invariant $\Phi^Z_X(4.97") = 3 = \Phi^Z_X(4.97#)$, and $\Phi^Z_X$ does not distinguish 4.97" from 4.97#. However, the constant labeling with every arc labeled with a 1 yields the listed fundamental kei module presentation matrices. Replacing $t_{1,1}$ and $s_{1,1}$ with their values from $M$ yields the listed matrices, which we row-reduce over $\mathbb{Z}_5$ to obtain the cardinalities of the solution spaces which form the signature of the constant labeling by the element $1 \in X$.

Since $t_{1,1} = t_{2,2} = t_{3,3} = 2$ and $s_{1,1} = s_{2,2} = s_{3,3} = 4$, we get the same signatures for all three labelings for each knot, respectively $u^{25}$ and $u^3$, and thus we have

$$\Phi^M_X(4.97") = 3u^{25} \neq 3u^3 = \Phi^M_X(4.97#)$$

and for non-kei module quandle modules $M$ over a finite kei, $X$, the quandle module enhanced counting invariant $\Phi^M_X$ is capable of detecting invertibility of virtual (and hence classical) knots.

5. Questions

In this section we collect a few open questions for future research.

In our computations we have only considered the simplest type of $\mathbb{Z}_K[X]$ modules, namely $\mathbb{Z}_k[X]$-module structures on $\mathbb{Z}_n$ with the action of $t_{x,y}$ and $s_{x,y}$ given by
multiplication by fixed elements of $\mathbb{Z}_K[X]$. Expanding to other abelian groups with other automorphisms $t_{x,y}: X \to X$ and endomorphisms $s_{x,y}: X \to X$ should give interesting results. We are particularly interested in the case of non-commuting $t_{x,y}$ and $s_{x,y}$ values.

We have generalized the rack module bead counting invariant from [8], but several other oriented link invariants using the quandle algebra were defined in [3]; these invariants should have generalizations to the unoriented case using the kei algebra.

References


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