ON THE COHOMOLOGY OF ORIENTED GRASSMANN MANIFOLDS

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Abstract

This paper presents a new approach to studying the kernel of the additive homomorphism from $H^q(G_{n,k})$ to $H^{q+1}(G_{n,k})$ given by the cup-product with the first Stiefel–Whitney class of the canonical $k$-plane bundle over the Grassmann manifold $G_{n,k}$ of all $k$-dimensional vector subspaces in Euclidean $n$-space. This method enables us to improve the understanding of the $\mathbb{Z}_2$-cohomology of the “oriented” Grassmann manifold $\tilde{G}_{n,k}$ of oriented $k$-dimensional vector subspaces in Euclidean $n$-space. In particular, we derive new information on the characteristic rank of the canonical oriented $k$-plane bundle over $\tilde{G}_{n,k}$ and the $\mathbb{Z}_2$-cup-length of $\tilde{G}_{n,k}$. Our results on the cup-length for three infinite families of the manifolds $\tilde{G}_{n,\alpha}$ confirm the corresponding claims of Fukaya’s conjecture from 2008.

1. Introduction

The $\mathbb{Z}_2$-cohomology algebra of the “unoriented” Grassmann manifold $G_{n,k}$ ($k \leq n-k$) of $k$-dimensional vector subspaces in $\mathbb{R}^n$ has a simple description in terms of generators and relations [3]: we can write

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, \ldots, w_k]/I_{n,k},$$

where $\text{dim}(w_i) = i$ and the ideal $I_{n,k}$ is generated by the $k$ homogeneous components of $(1 + w_1 + \cdots + w_k)^{-1}$ in dimensions $n - k + 1, \ldots, n$. If $\gamma_{n,k}$ (briefly $\gamma$) denotes the canonical $k$-plane bundle over $G_{n,k}$, then the indeterminate $w_i$ is a representative of the $i$th Stiefel–Whitney class $w_i(\gamma)$ in the quotient algebra $H^*(G_{n,k})$. For $w_i(\gamma)$, we shall also use $w_i$ as an abbreviation. Note that all cohomology in this paper will be taken with coefficients in $\mathbb{Z}_2$. Also note that $w_i = w_i(\gamma)$ should not be confused with the Stiefel–Whitney classes of the manifold, namely $w_i(G_{n,k}) = w_i(TG_{n,k})$, the Stiefel–Whitney classes of its tangent bundle.

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Besides $\gamma_{n,k}$ and its $(n - k)$-dimensional orthogonal complement $\gamma^\perp_{n,k}$ (briefly $\gamma^\perp$), over $G_{n,k}$, we have a nontrivial line bundle $\xi = \det(\gamma) = \det(\gamma^\perp)$. Then the “oriented” Grassmann manifold $\tilde{G}_{n,k}$ of oriented $k$-dimensional vector subspaces in $\mathbb{R}^n$ which, of course, is a double cover of $G_{n,k}$, can be interpreted as the 0-sphere bundle of $\xi$ ([9, Corollary 12.3] or [12, Theorem 5.7.11]), to which one has an exact sequence of Gysin type,

$$\overline{\psi} : H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{\nu^*} H^j(\tilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1}.$$

(2)

We write here and elsewhere $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ for the homomorphism given by the cup-product with the Stiefel–Whitney class $w_1(\xi) = w_1 = w_1(\gamma^\perp)$, and $p : \tilde{G}_{n,k} \to G_{n,k}$ is the obvious covering projection. Note that $\tilde{G}_{n,k}$ is always orientable as a manifold, whereas $G_{n,k}$ is an orientable manifold if and only if $n$ is even.

It is known [14] that $\text{Im}(p^* : H^*(G_{n,k}) \to H^*(\tilde{G}_{n,k}))$ is a self-annihilating subspace of $H^*(\tilde{G}_{n,k})$ of half the dimension. Very little is known about the algebra $H^*(\tilde{G}_{n,k})$, apart from the cases of $(n - 1)$-dimensional spheres $\tilde{G}_{n,1} \cong S^{n-1}$ and complex quadrics $\tilde{G}_{n,2}$. This is due to the fact that it is difficult to obtain information on cohomology classes that generate

$$\text{Im}(\psi) \cong H^*(\tilde{G}_{n,k})/\text{Ker}(\psi) = H^*(\tilde{G}_{n,k})/\text{Im}(p^*).$$

(3)

A reason for this is that, in general, it is hard to calculate explicitly in $H^*(G_{n,k})$ and determine the kernel of $w_1$; of course, by (2), the latter vector space is the same as $\text{Im}(\psi)$.

Over $\tilde{G}_{n,k}$ we have the canonical oriented $k$-plane bundle $\tilde{\gamma}_{n,k}$ (briefly $\tilde{\gamma}$), which is isomorphic to $p^*(\gamma)$. As a consequence, $p^*(w_i) = \tilde{w}_i$ for all $i$, where $\tilde{w}_i$ is an abbreviation, used throughout the paper, for the Stiefel–Whitney class $w_i(\tilde{\gamma}_{n,k})$; note $\tilde{w}_1 = 0$. The subspace $C(j; n, k) := \text{Im}(p^*)$ of the $\mathbb{Z}_2$-vector space $H^j(\tilde{G}_{n,k})$ is the characteristic subspace (all its elements can be expressed in the Stiefel–Whitney characteristic classes of $\tilde{\gamma}_{n,k}$; that is why we call it by this name). If we denote $\dim(C(j; n, k))$ by $\chi_j(\tilde{G}_{n,k})$ and $\dim(\text{Im}(\psi))$ by $\alpha_j(\tilde{G}_{n,k})$, then (see (3))

$$\chi_j(\tilde{G}_{n,k}) + \alpha_j(\tilde{G}_{n,k}) = b_j(\tilde{G}_{n,k}),$$

the right-hand side being the $j$th $\mathbb{Z}_2$-Betti number of $\tilde{G}_{n,k}$.

Recall [10, 8] that, for a real vector bundle $\alpha$ over a path-connected CW-complex $X$, its characteristic rank, $\text{charrank}(\alpha)$, is defined to be the greatest integer $q$, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X)$, $0 \leq j \leq q$, is a polynomial in the Stiefel–Whitney classes $w_i(\alpha) \in H^i(X)$. In particular (see [7]), if $TM$ is the tangent bundle of a smooth closed connected manifold $M$, then $\text{charrank}(TM)$ is the characteristic rank of $M$, denoted $\text{charrank}(M)$.

Now the greatest integer $q$ such that

$$\alpha_0(\tilde{G}_{n,k}) = \alpha_1(\tilde{G}_{n,k}) = \cdots = \alpha_q(\tilde{G}_{n,k}) = 0$$

is nothing but the characteristic rank of $\tilde{\gamma}_{n,k}$, briefly $\text{charrank}(\tilde{\gamma}_{n,k})$.

Of course, we have

$$\alpha_1 + \text{charrank}(\tilde{\gamma}_{n,k})(\tilde{G}_{n,k}) \neq 0,$$

since $1 + \text{charrank}(\tilde{\gamma}_{n,k})$ is the least degree, in which an “anomalous” (not expressible
exclusively in the Stiefel–Whitney classes of $\tilde{\gamma}_{n,k}$ generator of $H^*(\tilde{G}_{n,k})$ appears. Note that (see (2)) $\text{charrank}(\tilde{\gamma}_{n,k}) \geq j$ (for some $j$) if and only if $p^*: H^j(G_{n,k}) \rightarrow H^j(\tilde{G}_{n,k})$ is surjective or, equivalently, $w_1: H^j(G_{n,k}) \rightarrow H^{j+1}(G_{n,k})$ is injective, for all non-negative integers $i \leq j$.

When we know that some cohomology group of $\tilde{G}_{n,k}$, in a degree not exceeding half of $\dim(\tilde{G}_{n,k})$, does not vanish, we can use it to obtain an upper bound for $\text{charrank}(\tilde{\gamma}_{n,k})$. More precisely, due to the fact that the subspace $\text{Im}(p^*) \subset H^*(\tilde{G}_{n,k})$ is self-annihilating, one can easily adjust the proof of [2, Theorem 2.1] to verify that if $H^j(\tilde{G}_{n,k}) \neq 0$ for some $t \leq \frac{1}{2}k(n-k)$, then we have

$$\text{charrank}(\tilde{\gamma}_{n,k}) \leq k(n-k) - t - 1.$$ 

Under certain conditions, the characteristic rank of a vector bundle over a smooth closed connected manifold $M$ and the $\mathbb{Z}_2$-cup-length, denoted by $\text{cup}(M)$, are nicely related, as shown in the following generalization of [7, Theorem 1.1] which, in particular, will be used (in Section 3) for deriving upper bounds or exact values for the cup-length of $\tilde{G}_{n,k}$.

**Theorem 1.1** (Naolekar and Thakur [10]). Let $M$ be a connected closed smooth $d$-dimensional manifold. Let $\alpha$ be a vector bundle over $M$ satisfying the following: there exists $j$, $j \leq \text{charrank}(\alpha)$, such that every monomial $w_{i_1}(\alpha) \cdots w_{i_t}(\alpha)$, $0 \leq i_t \leq j$, in dimension $d$ vanishes. Then

$$\text{cup}(M) \leq 1 + \frac{d - j - 1}{r_M},$$

where $r_M$ is the smallest positive integer such that $\tilde{H}^{r_M}(M) \neq 0$.

In addition, for any $j \leq \text{charrank}(\tilde{\gamma}_{n,k})$, one sees that both the $w_1$-homomorphisms in the Gysin sequence (2) are injective and the homomorphism $p^*: H^j(G_{n,k}) \rightarrow H^j(\tilde{G}_{n,k})$ is surjective, thus we have

$$H^j(\tilde{G}_{n,k}) \cong H^j(G_{n,k})/\text{Im}(w_1: H^{j-1}(G_{n,k}) \rightarrow H^j(G_{n,k})).$$

Of course, now $\dim(\text{Im}(w_1: H^{j-1}(G_{n,k}) \rightarrow H^j(G_{n,k}))) = b_{j-1}(G_{n,k})$. Consequently, if $j \leq \text{charrank}(\tilde{\gamma}_{n,k})$, then we have for the Betti number $b_j(G_{n,k})$ that

$$b_j(\tilde{G}_{n,k}) = b_j(G_{n,k}) - b_{j-1}(G_{n,k}).$$

The difference of the $\mathbb{Z}_2$-Betti numbers of the Grassmann manifold $G_{n,k}$ on the right-hand side is readily calculable from the Poincaré polynomial, and is nothing but the number of linearly independent semi-invariants of degree $k$ and weight $j$ of a binary form of degree $n-k$, provided $j \leq \frac{k(n-k)}{2}$ (note that the latter number equals $\frac{1}{2}\dim(G_{n,k})$), by a theorem of Cayley and Sylvester (see [11, Satz 2.21]). This interesting interpretation of the Betti numbers $b_j(\tilde{G}_{n,k})$ for

$$j \leq \text{min}\{\text{charrank}(\tilde{\gamma}_{n,k}), \frac{k(n-k)}{2}\}$$

seems to have remained unnoticed thus far.

Theorem 2.1 in [8], on lower bounds or exact values for $\text{charrank}(\tilde{\gamma}_{n,k})$ ($3 \leq k \leq n-k$), gives information on the structure of the $\mathbb{Z}_2$-cohomology of the manifold $\tilde{G}_{n,k}$.
In the present paper, we add further results. As compared to [8], we present a different approach to studying the kernel of $w_1$. Some of the new results on the characteristic rank presented here imply new exact values of the $\mathbb{Z}_2$-cup-length of $\tilde{G}_{n,k}$. In particular, our results on the cup-length of three infinite families of the manifolds $\tilde{G}_{n,3}$ in Theorem 3.6(2) confirm the corresponding claims of Fukaya’s conjecture [4, p. 196].

2. An approach to studying the kernel of $w_1$

The aim of this section is to develop tools for studying the kernel of the homomorphism $w_1 : H^j(G_{n,k}) \to H^{j+1}(G_{n,k})$.

A key rôle will be played by the fact that, for the $\mathbb{Z}_2$-vector space $H^j(G_{n,k})$ ($k \leq n - k$), the set

$$\{w_{1}^{a_{1}} \cdots w_{k}^{a_{k}}; \sum_{i=1}^{k} i a_{i} = j, \sum_{i=1}^{k} a_{i} \leq n - k\}$$

(4)

is an additive basis. This follows from [9, Corollary 6.7]; another proof can be found in [5]. We shall refer to the basis (4) as “standard basis” in this paper. We say that an element, $w_{1}^{a_{1}} \cdots w_{k}^{a_{k}} \in H^j(G_{n,k})$, of the standard basis is regular (with respect to the homomorphism $w_1 : H^j(G_{n,k}) \to H^{j+1}(G_{n,k})$) if its $w_1$-image is an element of the standard basis for $H^{j+1}(G_{n,k})$, that is, if $\sum_{i=1}^{k} a_{i} < n - k$. An element of the standard basis that is not regular is said to be singular.

Of course,

$$\dim(\text{Im}(w_1 : H^j(G_{n,k}) \to H^{j+1}(G_{n,k})))$$

is greater than or equal to the number of regular elements of the standard basis for $H^j(G_{n,k})$ and

$$\dim(\text{Ker}(w_1 : H^j(G_{n,k}) \to H^{j+1}(G_{n,k}))) = \alpha_j(\tilde{G}_{n,k})$$

does not exceed the number of singular elements of the standard basis for $H^j(G_{n,k})$. The latter inequality can be concretized. Indeed, let $p(\{1, 2, \ldots, k-1\}, x)$ denote the number of partitions of a non-negative integer $x$ into parts, each taken from the set $\{1, 2, \ldots, k-1\}$; in particular, if $x \leq k-1$, then $p(\{1, 2, \ldots, k-1\}, x) = p(x)$ is the total number of partitions of $x$.

**Proposition 2.1.** For the Grassmann manifold $G_{n,k}$ ($2 \leq k \leq n - k$), we have the following:

(a) If $1 \leq x \leq n - k$, then all the elements in the standard basis for $H^{n-k-x}(G_{n,k})$ are regular, thus we have $\alpha_{n-k-x}(\tilde{G}_{n,k}) = 0$.

(b) If $x \geq 0$, then precisely $p(\{1, 2, \ldots, k-1\}, x)$ elements of the standard basis for $H^{n-k+x}(G_{n,k})$ are singular; thus $\alpha_{n-k+x}(\tilde{G}_{n,k}) \leq p(\{1, 2, \ldots, k-1\}, x)$.

**Proof.** Part (a). Let $w_{1}^{a_{1}} \cdots w_{k}^{a_{k}}$ be an element of the standard basis in $H^{n-k-x}(G_{n,k})$. We have

$$\sum_{i=1}^{k} a_{i} \leq \sum_{i=1}^{k} i a_{i} = n - k - x,$$

thus the equality $\sum_{i=1}^{k} a_{i} = n - k$ is impossible; in other words, each basis element
is now regular.

Part (b). If \( w_1^{a_1} \cdot \ldots \cdot w_k^{a_k} \) is a singular element of the standard basis in \( H^{n-k+x}(G_{n,k}) \), then clearly \( a_1 + a_2 + \ldots + a_k = n - k \), whence

\[ a_2 + 2a_3 + \ldots + (k-1)a_k = x. \]

Since \( a_1 \) is uniquely determined by the equation \( a_1 = n - k - a_2 - \ldots - a_k \), the number of singular elements of the standard basis is equal to \( p(\{1, 2, \ldots, k-1\}, x) \). \( \square \)

Remark 2.2. Of course, the vanishing of \( \alpha_{n-k-x}(\overline{G}_{n,k}) \) for all \( x \) in Proposition 2.1(a) is equivalent to the known inequality \( [8, (2.5)] \)

\[ \text{charrank}(\overline{\gamma}_{n,k}) \geq n - k - 1. \]

The next lemma is elementary and stated without proof.

Lemma 2.3. Let \( \vec{a}_1, \ldots, \vec{a}_m, \vec{b}_1, \ldots, \vec{b}_n \) be linearly independent vectors in a vector space \( V \) over a field \( K \). If \( \vec{c}_1, \ldots, \vec{c}_s \) (\( s \leq n \)) are linearly independent vectors in the linear span \( [\vec{b}_1, \ldots, \vec{b}_n] \subset V \), then also the vectors \( \vec{a}_1, \ldots, \vec{a}_m, \vec{c}_1, \ldots, \vec{c}_s \) are linearly independent.

In \( \mathbb{Z}_2[w_1, \ldots, w_k] \), let \( \overline{w}_i(w_1, \ldots, w_k) \) (briefly \( \overline{w}_i \)) denote the homogeneous component of \((1 + w_1 + \ldots + w_k)^{-1} = 1 + w_1 + \ldots + w_k + (w_1 + \ldots + w_k)^2 + \ldots \) in dimension \( i \). Of course, in particular, \( \overline{w}_i \) with \( i = n - k + 1, n - k + 2, \ldots, n \) are the generators of the ideal \( I_{n,k} \); see (1). In addition, let \( g_i(w_2, \ldots, w_k) \) (briefly just \( g_i \)) denote the reduction of \( \overline{w}_i(w_1, \ldots, w_k) \) modulo \( w_1 \). We note that \( \overline{w}_i \) is a representative of the Stiefel–Whitney class \( w_i(\gamma^+) \in H^i(G_{n,k}) \); we shall also use \( \overline{w}_i \) as an abbreviation for \( w_i(\gamma^+) \).

If \( i = n - k + 1, n - k + 2, \ldots, n \), then the polynomials \( g_i(w_2, \ldots, w_k) \) are representatives of some multiples (by an abuse of notation, also denoted by \( g_i(w_2, \ldots, w_k) \), briefly \( g_i \)) of the first Stiefel–Whitney class \( w_1 \) in the quotient algebra \( H^*(G_{n,k}) \). The singular elements of the standard basis in \( H^{n-k+x}(G_{n,k}) \) (\( x \geq 0 \)), when multiplied by \( w_1 \), do not produce elements of the standard basis in \( H^{n-k+x+1}(G_{n,k}) \). Combined with Lemma 2.3 (where we take those elements of the standard basis divisible by \( w_1 \) in the rôle of the vectors \( \vec{a}_i \) and the others in the rôle of the vectors \( \vec{b}_j \)), this explains why the following proposition focuses on elements of the form \( w_2^{c_2} \cdot \ldots \cdot w_k^{c_k} g_{n-k+1+i} \in H^{n-k+x+1}(G_{n,k}) \) (\( i = 0, 1, \ldots, k-1 \)).

Proposition 2.4. For a non-negative integer \( x \), we associate with \( H^{n-k+x+1}(G_{n,k}) \) \( (2 \leq k \leq n - k) \) the set

\[ N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{ w_2^{b_2} \cdot \ldots \cdot w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \ldots + kb_k = x - i \}. \]

(1) The cardinality of \( N_x(G_{n,k}) \) is equal to \( p(\{1, 2, \ldots, k-1\}, x) \), which is the same (by Proposition 2.1) as the number of singular elements in the standard basis for \( H^{n-k+x}(G_{n,k}) \).

(2) If \( x \leq n - k - 1 \), then each element of \( N_x(G_{n,k}) \) consists exclusively of monomials \( w_2^{c_2} \cdot \ldots \cdot w_k^{c_k} \) such that \( c_2 + \ldots + c_k \leq n - k \), thus of elements not divisible by \( w_1 \) and belonging to the standard basis of \( H^{n-k+x+1}(G_{n,k}) \).
If $x \leq n - k - 1$ and there are $t$ linearly independent (over $\mathbb{Z}_2$) elements in the set $N_x(G_{n,k})$, then

$$\alpha_{n-k+x}(\widetilde{G}_{n,k}) \leq p(\{1,2,\ldots,k-1\},x) - t.$$ 

In particular, if $x \leq n - k - 1$ and the set $N_x(G_{n,k})$ is linearly independent, then

$$w_1: H^{n-k+x}(G_{n,k}) \to H^{n-k+x+1}(G_{n,k})$$ is a monomorphism and $\alpha_{n-k+x}(\widetilde{G}_{n,k})$ vanishes.

Proof. Part (1). The set $\{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}$ consists of $p(\{2,\ldots,k\},x-i)$ elements. Thus the cardinality of $N_x(G_{n,k})$ is equal to

$$\sum_{i=0}^{k-1} p(\{2,\ldots,k\},x-i). \quad (5)$$

If $S$ is a set of positive integers and $p(S,j)$ is the number of partitions of $j$ whose parts are from $S$, then (see [1, Theorem 1.1] if needed) we have, for $|q| < 1$,

$$\sum_{i \geq 0} p(S,i)q^i = \prod_{i \in S} (1 - q^i)^{-1}.$$ 

Since

$$(1 + q + q^2 + \cdots + q^{k-1}) \prod_{i=2}^{k} (1 - q^i)^{-1} = \prod_{i=1}^{k-1} (1 - q^i)^{-1},$$

the cardinality of $N_x(G_{n,k})$ (the sum in (5)) is $p(\{1,2,\ldots,k-1\},x)$.

Part (2). Since each element of $N_x(G_{n,k})$ is some $w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}$ such that $2b_2 + 3b_3 + \cdots + kb_k = x - i$, it consists of monomials of the form

$$w_2^{b_2} \cdots w_k^{b_k} w_2^{c_2} \cdots w_k^{c_k} = w_2^{b_2+c_2} \cdots w_k^{b_k+c_k},$$

where $\sum_{i=2}^{k} (b_i + c_i) = n - k + x + 1$. Thus, if $x \leq n - k - 1$, then

$$\sum_{i=2}^{k} (b_i + c_i) \leq \frac{n - k + x + 1}{2} \leq n - k.$$

Part (3). This is obviously implied by Lemma 2.3 and the first two parts of this proposition. \hfill \square

3. Results on the characteristic rank and cup-length

In this section, the tools developed in Section 2 yield new bounds or exact results on the characteristic rank of $\tilde{\gamma}_{n,k}$ (for odd $n$, also on the characteristic rank of $\tilde{G}_{n,k}$). These lead to obtaining infinitely many new exact values of the cup-length of $\tilde{G}_{n,3}$, regarded as likely in Fukaya’s conjecture [4, p. 196].

Theorem 3.1. For the oriented Grassmann manifold $\tilde{G}_{n,k}$ ($4 \leq 2k \leq n$), with the unique integer $t$ such that $2^{t-1} < n \leq 2^t$, we have the following:
Thus, of course, if \( n \) is odd, then \( \text{charrank}(\overline{\gamma}_{n,2}) \geq n - 2 \), and if \( n \) is even, then \( \text{charrank}(\overline{\gamma}_{n,2}) \geq n - 3 \).

If \( s = 1 \) or \( s = 2 \), \( r \) is a non-negative integer, and \( 2^{t-1} + \lfloor \frac{s-1}{2} \rfloor < n < 2^{t} - s - 2 \), then \( \text{charrank}(\overline{\gamma}_{n+r,3+r}) \geq n + s - 2 \). If \( 3 \leq s \leq 6 \) and \( 2^{t-1} + \lfloor \frac{s-1}{2} \rfloor < n < 2^{t} - s - 2 \), then \( \text{charrank}(\overline{\gamma}_{n,3}) \geq n + s - 2 \).

In addition, if \( n \) is odd, then the replacement of the canonical bundle \( \overline{\gamma}_{n,k} \) by the corresponding manifold \( \overline{G}_{n,k} \) gives the corresponding result on \( \text{charrank}(\overline{G}_{n,k}) \).

Proof. It is known ([8, Theorem 2.1] and the final part of its proof) that if \( n \) is odd, then \( \text{charrank}(\overline{\gamma}_{n,k}) = \text{charrank}(\overline{G}_{n,k}) \). Thus it suffices to prove Parts (1) and (2).

Part (1). By Remark 2.2, \( \text{charrank}(\overline{\gamma}_{n,2}) \geq n - 3 \) for all \( n \). If \( n \) is odd, then \( N_0(G_{n,2}) \) (see Proposition 2.4) only contains \( g_{n-1} \). From \( (1 + w_2)^{-1} = 1 + w_2 + w_3^2 + w_3^3 + \cdots \), one sees that \( g_{n-1} \neq 0 \). Thus, by Proposition 2.4(3), \( \alpha_{n-2}(G_{n,2}) \) vanishes and we have

\[
\text{charrank}(\overline{\gamma}_{n,2}) \geq n - 2.
\]

Part (2) We shall repeatedly use the fact that, by [8, Lemma 2.3(i)],

\[
g_i(w_2, w_3) = 0 \text{ if and only if } i = 2^j - 3 \text{ for some } j \geq 2.
\]

The following lemma (when combined with (6)) will also be useful; cf. each of the four tables that occur in the proof of Theorem 3.1.

**Lemma 3.2.** For \( G_{n,3} \), let \( g_i \) denote the same polynomial in \( \mathbb{Z}_2[w_2, w_3] \) as in the rest of this paper.

(a) If \( m \neq 4 \) is such that \( g_{m-1} \neq 0 \) and \( g_m \neq 0 \), then \( w_2^2 g_{m-1} + w_3 g_m \neq 0 \).

(b) If \( m \neq 9 \) is such that \( g_{m-2} \neq 0 \) and \( g_{m+1} \neq 0 \), then \( w_2^3 g_{m-2} + w_3 g_{m+1} \neq 0 \).

Proof of the lemma. We know [6], for all \( j \geq 1 \), that

\[
g_j = \sum_{\frac{1}{2} \leq i \leq \frac{3}{2}} \binom{i}{3i - j} w_2^{3i-j} w_3^{2i}.
\]

Part (a). Of course, a necessary condition for \( w_2^2 g_{m-1} + w_3 g_m = 0 \) is that

\[
w_2^2 \mid g_m \text{ and } w_3 \mid g_{m-1}.
\]

Writing \( m = 6a + b \) (0 \( \leq b \leq 5 \)), from (7), one either directly sees that \( w_2^2 g_{m-1} + w_3 g_m \neq 0 \), or that the divisibility condition (8) is not satisfied.

Indeed, if \( b = 0 \), then \( g_m \), and if \( b = 1 \), then \( g_{m-1} \) is equal to \( w_3^{2a} + \cdots + w_2^3 \), thus the condition (8) fails for \( b = 0 \), 1. Similarly, if \( b = 2 \), then \( g_m \), and if \( b = 3 \), then \( g_{m-1} \) is equal to \( w_2 w_3^{2a} + \cdots + w_2^{3a+1} \), thus the condition (8) fails for \( b = 2 \), 3.

If \( b = 4 \), then we see that the condition (8) is fulfilled. But one calculates that \( w_2^2 g_{m-1} + w_3 g_m \) is equal to

\[
aw_2^2 w_3^{2a+1} + \cdots + \alpha w_2^{3a-4} w_3^5 + \beta w_2^{3a-1} w_3^3 + aw_2^{3a+2} w_3,
\]

where the third last and second last coefficients are abbreviated,

\[
\alpha = \binom{3a - 1}{5} + \binom{3a}{4}, \quad \beta = \binom{3a}{3} + \binom{3a + 1}{2}.
\]

Thus, of course, \( w_2^2 g_{m-1} + w_3 g_m \neq 0 \) if \( a \) is odd. For even \( a \), one readily verifies
that if \( a = 8l, 8l + 4 \), then \( \alpha = 1 \), and if \( a = 8l + 2, 8l + 6 \), then \( \beta = 1 \), thus again \( w_2^3g_{m-1} + w_3g_m \neq 0 \).

Finally, if \( b = 5 \), then \( g_{m-1} = \cdots + w_2^{3a+2} \), thus \( w_3 \uparrow g_{m-1} \).

**Part (b).** We proceed similarly as in the proof of Part (a). A necessary condition for \( w_2^3g_{m-2} + w_3g_{m+1} = 0 \) is that

\[
  w_2^3 \mid g_{m+1} \text{ and } w_3 \mid g_{m-2}.
\]  

(9)

Writing \( m = 6a + b \) \((0 \leq b \leq 5)\), one either directly sees, from (7), that \( w_2^3g_{m-2} + w_3g_{m+1} \neq 0 \), or that the divisibility condition (9) is not satisfied. Indeed, if \( b = 0 \), then \( g_{m-2} = g_6a_2 = g_6(a-1)+4 = w_2^{2a-1} + \cdots \), and if \( b = 4 \), then \( g_{m-2} = g_6a_2 = \cdots + w_2^{3a+1} \), thus \( w_3 \uparrow g_{m-2} \); the condition (9) fails for \( b = 0, 4 \). If \( b = 1 \), then \( g_{m+1} = g_6a_2 + w_2^{2a} + \cdots \), if \( b = 2 \), then \( g_{m+1} = g_6a_2 + w_2^{2a+1} + \cdots \), and if \( b = 5 \), then \( g_{m+1} = g_6(a+1) = w_2^{2a+2} + \cdots \), thus \( w_2 \uparrow g_{m+1} \); the condition (9) fails for \( b = 1, 2, 5 \). Finally, let us suppose that \( b = 3 \). Then \( g_{m+1} = g_6a_4 = (a+1)w_2^{2a} + w_2^{3a+2} \), thus \( w_2 \uparrow g_{m+1} \) if \( a \) is even. It remains to see what happens for odd \( a \). If \( a = 8l + 1 \) \((l \geq 1)\), then one calculates that \( w_2^{3a}g_{m-2} + w_3g_{m+1} = w_3^{2a+1} \), \( w_3g_{m+1} = \cdots + w_2^{3a-10}w_3^{9+\cdots} \neq 0 \), if \( a = 8l + 3 \), then one calculates that \( w_2^{3a}g_{m-2} + w_3g_{m+1} = \cdots + w_2^{3a-1}w_3^{9+\cdots} \neq 0 \), if \( a = 8l + 5 \), then we have \( w_2^{3a}g_{m-2} + w_3g_{m+1} = \cdots + w_2^{3a-7}w_3^{9+\cdots} \neq 0 \) and, finally, if \( a = 8l + 7 \), then we have \( w_2^{3a}g_{m-2} + w_3g_{m+1} = \cdots + w_2^{3a-1}w_3^{9+\cdots} \neq 0 \). This proves the lemma.

Now we are ready to verify the claims of Theorem 3.1(2).

**Case** \( s = 1 \). We have \( 2t-1 < n < 2t - 3 \) and assumptions of the theorem imply that \( n \geq 9 \). By [8, Theorem 2.1], we know that \( \text{charrank}(\overline{\gamma}_{n+r,3+r}) \geq n - 2 \). One readily calculates (see Proposition 2.4) that

\[
  N_2(G_{n,3}) = \{w_2g_{n-2}(w_2, w_3), g_n(w_2, w_3)\}.
\]

Since (see (6)) \( w_2g_{n-2} \neq 0 \), \( g_n \neq 0 \) and, since \( w_2g_{n-2} + g_n = w_3g_{n-3} \neq 0 \), the set \( N_2(G_{n,3}) \) is linearly independent. At the same time,

\[
  N_2(G_{n+r,3+r}) = \{w_2g_{n-2}(w_2, w_3, \ldots, w_{3+r}), g_n(w_2, w_3, \ldots, w_{3+r})\}.
\]

By iterating the obvious “inclusion” \( G_{n,k} \to G_{n+1,k+1} \) \((D \mapsto D \oplus \mathbb{R})\), we obtain an inclusion

\[
  j: G_{n,k} \to G_{n+r,k+r}
\]  

(10)

such that, for the pullbacks, we have \( j^*(\gamma) \cong \gamma \oplus r\varepsilon \) (here \( r\varepsilon \) is the trivial \( r \)-plane bundle) and \( j^*(\overline{\gamma}) \cong \overline{\gamma} \). Of course, for the induced cohomology homomorphism, we have that \( j^*(w_i) = w_i \) (with the right-hand side zero when \( k = 3 \) and \( i \geq 4 \)) and \( j^*(\overline{w_i}) = \overline{w_i} \). Thus, since the set \( N_2(G_{n,3}) = j^*(N_2(G_{n+r,3+r})) \) is linearly independent, \( N_2(G_{n+r,3+r}) \) has this property as well. Proposition 2.4(3) implies that \( \alpha_{n-1}(\overline{G}_{n+r,3+r}) = 0 \) and \( \text{charrank}(\overline{\gamma}_{n+r,3+r}) \geq n - 1 \).

**Case** \( s = 2 \). Now \( 2t-1 < n < 2t - 4 \) and assumptions of the theorem imply that \( n \geq 9 \). By the result for \( s = 1 \), we know that \( \text{charrank}(\overline{\gamma}_{n+r,3+r}) \geq n - 1 \). Since \( w_3g_{n-2} \neq 0, w_2g_{n-1} \neq 0 \), and \( w_2g_{n-2} + w_2g_{n-2} = g_{n+1} \neq 0 \), the set

\[
  N_3(G_{n,3}) = \{w_3g_{n-2}, w_2g_{n-1}\}
\]

is linearly independent. Similarly to the case of \( s = 1 \), one sees for \( r > 0 \) that the set \( N_3(G_{n+r,3+r}) = \{w_3g_{n-2}, w_2g_{n-1}, g_{n+1}\} \) is independent. Thus Proposition 2.4(3)
implies that we have $\alpha_n(\bar{G}_{n+r,3+r}) = 0$ and $\text{charrank}(\bar{\gamma}_{n+r,3+r}) \geq n$.

Case $s = 3$. We have $2^t-1 + 1 < n < 2^t - 5$; assumptions of the theorem imply that $n \geq 10$. By the result for $s = 2$, we know that $\text{charrank}(\bar{\gamma}_{n,3}) \geq n$. One verifies that $N_4(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2^2 g_{n-2}$, $w_3 g_{n-1}$, and $w_2^3 g_n$; they are linearly independent, as the following table shows.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_1 w_2^2 g_{n-2} + h_2 w_3 g_{n-1} + h_3 w_2 g_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$g_{n+2} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$w_2 w_3 g_{n-3} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$w_2^3 g_{n-2} + w_3 g_{n-1} \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$w_2^3 g_{n-4} \neq 0$</td>
</tr>
</tbody>
</table>

By Proposition 2.4(3), now $\alpha_{n+1}(\bar{G}_{n,3}) = 0$ and $\text{charrank}(\bar{\gamma}_{n,3}) \geq n + 1$.

Case $s = 4$. Now $2^t-1 + 1 < n < 2^t - 6$ and, by assumptions of the theorem, we have $n \geq 18$. By the result for $s = 3$, we know that $\text{charrank}(\bar{\gamma}_{n,3}) \geq n + 1$. We see that $N_5(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2 w_3 g_{n-2}$, $w_2^3 g_{n-1}$, and $w_3 g_n$; they are linearly independent, as the following table shows.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_1 w_2 w_3 g_{n-2} + h_2 w_2^2 g_{n-1} + h_3 w_3 g_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$w_2^3 g_{n-1} + w_3 g_n \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$w_2^3 g_{n-3} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$w_2 g_{n+1} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$g_{n+3} \neq 0$</td>
</tr>
</tbody>
</table>

So we have proved that $\alpha_{n+2}(\bar{G}_{n,3}) = 0$, and Proposition 2.4(3) implies that now $\text{charrank}(\bar{\gamma}_{n,3}) \geq n + 2$.

Case $s = 5$. We have $2^t-1 + 2 < n < 2^t - 7$ and assumptions of the theorem imply that $n \geq 19$. By the result for $s = 4$, we know that $\text{charrank}(\bar{\gamma}_{n,3}) \geq n + 2$. One calculates that $N_6(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2^3 g_{n-2}$, $w_2^3 g_{n-3}$, $w_2 w_3 g_{n-1}$, and $w_2^3 g_n$. The following table shows that they are linearly independent.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_4$</th>
<th>$h_1 w_2^3 g_{n-2} + h_2 w_2^2 g_{n-2} + h_3 w_2 w_3 g_{n-1} + h_4 w_2^3 g_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$w_2 g_{n+2} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$g_{n+4} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$w_2^2 w_3 g_{n-3} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$w_3 g_{n+1} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$w_2 (w_2^2 g_{n-2} + w_3 g_{n-1}) \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$(w_2^3 + w_2^2) g_{n-2} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$w_2^3 g_n + w_3 g_{n+1} \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$w_2 w_3^2 g_{n-4} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$w_3 (w_2^2 g_{n-3} + w_3 g_{n-2}) \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$w_2^3 g_{n-2} + w_3 g_{n+1} \neq 0$, Lemma 3.2(b)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$w_2^3 g_{n-5} \neq 0$</td>
</tr>
</tbody>
</table>
So we have proved that now $\alpha_{n+3}(\tilde{G}_{n,3}) = 0$. By Proposition 2.4(3), we have

$$\text{charrank}(\tilde{G}_{n,3}) \geq n + 3.$$ 

*Case $s = 6$. We have $2^{t-1} + 2 < n < 2^t - 8$ and assumptions of the theorem imply that $n \geq 19$. By the result for $s = 5$, we know that $\text{charrank}(\tilde{G}_{n,3}) \geq n + 3$. One calculates that $N_7(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2^3w_3g_{n-2}$, $w_2^3g_{n-1}$, $w_3^2g_{n-1}$, and $w_2w_3g_n$. The following table shows that they are linearly independent.

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_4$</th>
<th>$h_1w_2^2w_3g_{n-2} + h_2w_2^3g_{n-1} + h_3w_3^2g_{n-1} + h_4w_2w_3g_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$w_3g_{n+2} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$w_2(w_2^2g_{n-1} + w_3g_n) \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$w_2w_3^2g_{n-3} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$(w_3^2 + w_3^3)g_{n-1} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$w_3(w_2^2g_{n-2} + w_3g_{n-1}) \neq 0$, Lemma 3.2(a)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$w_2^3g_{n+1} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$w_2^3g_{n-1} + w_3g_{n+2} \neq 0$, Lemma 3.2(b)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$w_3^2g_{n-4} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$w_2g_{n+3} \neq 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$w_2^2g_{n+1} + w_3^2g_{n-1} = g_{n+5} \neq 0$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$w_2^2g_{n+1} + w_3g_{n+2} \neq 0$, Lemma 3.2(a)</td>
</tr>
</tbody>
</table>

So we have proved that now $\alpha_{n+4}(\tilde{G}_{n,3}) = 0$. By Proposition 2.4(3), we have

$$\text{charrank}(\tilde{G}_{n,3}) \geq n + 4.$$ 

The proof of Theorem 3.1 is finished. □

**Conjecture 3.3.** We conjecture that Theorem 3.1(2) holds true for all $s \geq 1$ such that $2^{t-1} + \lfloor \frac{s-1}{2} \rfloor < n < 2^t - s - 2$, and not just when $s \leq 6$.

In the proof of Theorem 3.1(2) for $s = 1$ or $s = 2$, we have extended a specific lower bound for the characteristic rank of $\tilde{G}_{n,3}$ to a lower bound for the characteristic rank of $\tilde{G}_{n+r,3+r}$ ($r \geq 0$). The following theorem brings an additional piece of information on the homomorphism $w_1$ and offers further possibilities for extensions of results on the characteristic rank of the vector bundle $\tilde{G}_{n,k}$.

**Proposition 3.4.** For the Grassmann manifold $G_{n,k}$ ($1 \leq k \leq n - k$) and any positive integer $l$ not exceeding $n - 1$, we have the following:

1. If the homomorphism $w_1: H^l(G_{n,k}) \rightarrow H^{l+1}(G_{n,k})$ is injective, then also $w_1: H^l(G_{n+1,k+1}) \rightarrow H^{l+1}(G_{n+1,k+1})$ is injective.
2. An obvious consequence of (1) is that if $\text{charrank}(\tilde{G}_{n,k}) \geq l$ then, for any nonnegative integer $r$, we have $\text{charrank}(\tilde{G}_{n+r,k+r}) \geq l$. 

Proof. Let \( j : G_{n,k} \to G_{n+1,k+1} \) denote the inclusion described in (10). The diagram

\[
\begin{array}{ccc}
H^l(G_{n+1,k+1}) & \xrightarrow{w_1} & H^{l+1}(G_{n+1,k+1}) \\
\downarrow j^* & & \downarrow j^* \\
H^l(G_{n,k}) & \xrightarrow{w_1} & H^{l+1}(G_{n,k})
\end{array}
\]

obviously commutes. Let us suppose that the lower homomorphism \( w_1 \) is injective; we should prove that the upper homomorphism \( w_1 \) is injective as well.

The standard basis vectors (see (4)) in \( H^l(G_{n+1,k+1}) \) are

\[
w_{1}^{a_1}(\gamma_{n+1,k+1}) \cdots w_{k}^{a_k}(\gamma_{n+1,k+1}) w_{k+1}^{a_{k+1}}(\gamma_{n+1,k+1}),
\]

such that \( a_1 + 2a_2 + \cdots + ka_k + (k+1)a_{k+1} = l \) and \( a_1 + a_2 + \cdots + a_k + a_{k+1} \leq n - k \). The images, under \( w_1 : H^l(G_{n+1,k+1}) \to H^{l+1}(G_{n+1,k+1}) \), of those vectors (11) having \( a_{k+1} = 0 \) are linearly independent. Indeed, these images are

\[
w_{1}^{1+a_1}(\gamma_{n+1,k+1}) \cdots w_{k}^{a_k}(\gamma_{n+1,k+1});
\]

the vectors \( j^*(w_{1}^{1+a_1}(\gamma_{n+1,k+1}) \cdots w_{k}^{a_k}(\gamma_{n+1,k+1})) = w_{1}^{1+a_1}(\gamma_{n,k}) \cdots w_{k}^{a_k}(\gamma_{n,k}) \) (being images of the standard basis vectors \( w_{1}^{a_1}(\gamma_{n,k}) \cdots w_{k}^{a_k}(\gamma_{n,k}) \in H^l(G_{n,k}) \)) under the injective linear map \( w_1 : H^l(G_{n,k}) \to H^l(G_{n,k}) \) are linearly independent. Thus also \( w_{1}^{1+a_1}(\gamma_{n+1,k+1}) \cdots w_{k}^{a_k}(\gamma_{n+1,k+1}) \) are linearly independent. In addition, the images under \( w_1 : H^l(G_{n+1,k+1}) \to H^{l+1}(G_{n+1,k+1}) \) of those vectors (11) having \( a_{k+1} \geq 1 \) are also linearly independent, because all the standard basis vectors (11) having \( a_{k+1} \geq 1 \) are regular. Indeed, we have for any of these standard basis vectors in \( H^l(G_{n+1,k+1}) \) that

\[
(a_1 + a_2 + \cdots + a_k + a_{k+1}) + ka_k + (k+1)a_{k+1} = l,
\]

thus

\[
a_1 + a_2 + \cdots + a_k + a_{k+1} \leq l - ka_k + a_{k+1} \leq l - k \leq n - k - 1.
\]

Finally, the \( w_1 \)-images of all the standard basis vectors (11) of \( H^l(G_{n+1,k+1}) \) are linearly independent. Indeed, let us suppose that a linear combination of all these images vanishes, that is,

\[
\sum_{a_{k+1} \geq 1} \alpha_{(a_1, \ldots, a_k, 0)} w_{1}^{1+a_1} w_{2}^{a_2} \cdots w_{k}^{a_k} w_{k+1}^{a_{k+1}} = 0.
\]

When mapped by \( j^* : H^{l+1}(G_{n+1,k+1}) \to H^{l+1}(G_{n,k}) \), this gives that

\[
\sum_{a_{k+1} \geq 1} \alpha_{(a_1, \ldots, a_k, 0)} w_{1}^{1+a_1}(\gamma_{n,k}) w_{2}^{a_2}(\gamma_{n,k}) \cdots w_{k}^{a_k}(\gamma_{n,k}) = 0,
\]

implying that all the coefficients \( \alpha_{(a_1, \ldots, a_k, 0)} \) vanish, since \( w_{1}^{1+a_1}(\gamma_{n,k}) \cdots w_{k}^{a_k}(\gamma_{n,k}) \) are linearly independent. So the left-hand side of (12) is reduced to a linear combination of vectors already known to be linearly independent, we have

\[
\sum_{a_{k+1} \geq 1} \alpha_{(a_1, \ldots, a_k, a_{k+1})} w_{1}^{1+a_1}(\gamma_{n+1,k+1}) w_{2}^{a_2}(\gamma_{n+1,k+1}) \cdots w_{k+1}^{a_{k+1}}(\gamma_{n+1,k+1}) = 0,
\]

thus also all the coefficients \( \alpha_{(a_1, \ldots, a_k, a_{k+1})} \) \( (a_{k+1} \geq 1) \) must vanish. This finishes the proof of Proposition 3.4. \( \square \)
Remark 3.5. The assumption \( l \leq n - 1 \) in Proposition 3.4(1) is the best possible, in the sense that the claim is false, in general, for \( l = n \). Indeed, \( w_1 : H^7(G_{7,2}) \rightarrow H^8(G_{7,2}) \) is readily seen to be a monomorphism (apply Proposition 2.4(3); the set \( N_2(G_{7,2}) = \{ w_2 g_6 \} \) is linearly independent), but the homomorphism \( w_1 : H^7(G_{8,3}) \rightarrow H^8(G_{8,3}) \) is not injective (by a calculation in the cohomology algebra \( H^*(G_{8,3}) \), (1), or consulting Stong’s result on the height of \( w_1 \) in [13], one sees that the kernel of this homomorphism contains \( w_1^7 \neq 0 \)).

Theorem 3.1 enables us to derive, among others, new exact results on the characteristic rank and \( \mathbb{Z}_2 \)-cup-length of three infinite families of the manifolds \( \tilde{G}_{n,3} \).

Theorem 3.6. For the oriented Grassmann manifolds \( \tilde{G}_{n,k} \) (\( 4 \leq 2k \leq n \)) we have the following:

(1) If \( n \) is odd, then

\[
\text{charrank}(\tilde{\gamma}_{n,2}) = n - 2, \quad \text{cup}(\tilde{G}_{n,2}) = \frac{n - 1}{2},
\]

and if \( n \) is even, then

\[
\text{charrank}(\tilde{\gamma}_{n,2}) = n - 3, \quad \text{cup}(\tilde{G}_{n,2}) = \frac{n}{2}.
\]

(2) If \( q \geq 4 \), then

\[
\text{charrank}(\tilde{\gamma}_{2q - 1 + 1,3}) = 2q - 1 + 1, \quad \text{cup}(\tilde{G}_{2q - 1 + 1,3}) = 2q - 1 - 3,
\]

\[
\text{charrank}(\tilde{\gamma}_{10,3}) = 11, \quad \text{cup}(\tilde{G}_{10,3}) = 5,
\]

and, if \( q \geq 5 \), then

\[
\text{charrank}(\tilde{\gamma}_{2q - 1 + 2,3}) = 2q - 1 + 4, \quad \text{cup}(\tilde{G}_{2q - 1 + 2,3}) = 2q - 1 - 3,
\]

\[
\text{charrank}(\tilde{\gamma}_{2q - 1 + 3,3}) = 2q - 1 + 7, \quad \text{cup}(\tilde{G}_{2q - 1 + 3,3}) = 2q - 1 - 3.
\]

Remark 3.7. The results on the cup-length in Theorem 3.6(2) confirm the corresponding claims of Fukaya’s conjecture [4, p. 196]; another claim contained in this conjecture was proved in [8].

Proof. Part (1). Let us first suppose that \( n \) is odd. It is clear (for instance, from (1)) that \( w_{\frac{n-3}{2}} \in H^{n-3}(G_{n,2}) \) is not a multiple of \( w_1 \), thus we have \( w_{\frac{n-3}{2}} \neq 0 \) and \( \text{cup}(\tilde{G}_{n,2}) \geq \frac{n-1}{2} \). We know, from Theorem 3.1, that \( \text{charrank}(\tilde{\gamma}_{n,2}) \geq n - 2 \). Thus Theorem 1.1 implies that \( \text{cup}(\tilde{G}_{n,2}) \leq \frac{n - 1}{2} \), and we see that \( \text{cup}(\tilde{G}_{n,2}) = \frac{n - 1}{2} \), as claimed. At the same time, this shows that \( \text{charrank}(\tilde{\gamma}_{n,2}) \leq n - 2 \); \( \text{charrank}(\tilde{\gamma}_{n,2}) \geq n - 1 \) would imply a false inequality, \( \text{cup}(\tilde{G}_{n,2}) \leq \frac{n - 2}{2} \), and so \( \text{charrank}(\tilde{\gamma}_{n,2}) = n - 2 \). [To see that \( \text{charrank}(\tilde{\gamma}_{n,2}) \leq n - 2 \), it also suffices to compare the Betti numbers \( b_{n-1}(G_{n,2}) = \frac{n-1}{2} \) and \( b_n(G_{n,2}) = \frac{n-3}{2} \), readily calculated from the Poincaré polynomial.]

Now let us suppose that \( n \) is even. First, we note that \( \tilde{G}_{4,2} \cong S^2 \times S^2 \); clearly \( \chi_2(\tilde{G}_{4,2}) = 1 = \alpha_2(\tilde{G}_{4,2}) \), \( \text{charrank}(\tilde{\gamma}_{4,2}) = 1 \), and \( \text{cup}(\tilde{G}_{4,2}) = 2 \), as claimed. So we may suppose that \( n \geq 6 \). Then \( w_{\frac{n-3}{2}} \in H^{n-2}(G_{n,2}) \) cannot be a multiple of \( w_1 \), thus
we have $w_2^{n-2} \neq 0$ and cup($\tilde{G}_{n,2}$) $\geq \frac{n}{2}$. We know, from Theorem 3.1, that
\[ \text{charrank}(\tilde{\gamma}_{n,2}) \geq n - 3; \]

Theorem 1.1 gives cup($\tilde{G}_{n,2}$) $= \frac{n}{2}$. We know, from Theorem 3.1, that
\[ \text{charrank}(\tilde{\gamma}_{n,2}) \geq n - 3. \]

Admitting that charrank($\tilde{\gamma}_{n,2}$) $\geq n - 2$ implies a false inequality, cup($\tilde{G}_{n,2}$) $\leq \frac{n-1}{2}$.

[An alternative: since $b_{2n-2}(G_{n,2}) = \frac{n}{2}$ and $b_{2n-3}(G_{n,2}) = \frac{n-3}{2}$, the homomorphism $w_1: H^{n-2}(G_{n,2}) \to H^{n-1}(G_{n,2})$ is not injective, and we conclude that charrank($\tilde{\gamma}_{n,2}$) $\leq n - 3.$] Thus charrank($\tilde{\gamma}_{n,2}$) $= n - 3$, as claimed.

Part (2). We first note that, for any non-negative integer $x$, one has an obvious “inclusion” $\tilde{j}: \tilde{G}_{2n-1,3} \to \tilde{G}_{2n-1+x,3}$, such that $j^*(\tilde{\gamma}_{2n-1+x,3}) \equiv \tilde{\gamma}_{2n-1,3}$. Thus, in cohomology, $j^*(w_2^{2n-1-4}(\tilde{\gamma}_{2n-1+x,3})) = w_2^{2n-1-4}(\tilde{\gamma}_{2n-1,3})$. It was proved in [7, p. 77] that the latter cohomology class does not vanish. As a consequence, we have that
\[ \text{cup}(\tilde{G}_{2n-1+x,3}) \geq 2^{q-1} - 3. \] (13)

For $\tilde{G}_{2n-1+1,3}$ ($q \geq 4$), Theorem 3.1(2) with $s = 2$ implies that
\[ \text{charrank}(\tilde{\gamma}_{2n-1+1,3}) \geq 2^{q-1} + 1. \]

Then, from Theorem 1.1, we obtain that cup($\tilde{G}_{2n-1+1,3}$) $\leq 2^{q-1} - 3$, thus we have (see (13)) cup($\tilde{G}_{2n-1+1,3}$) $= 2^{q-1} - 3$ and charrank($\tilde{\gamma}_{2n-1+1,3}$) $= 2^{q-1} + 1$.

For $\tilde{G}_{2n-1+2,3}$ with $q = 4$, that is, for $\tilde{G}_{10,3}$, Theorem 3.1(2) with $s = 3$ applies and gives that charrank($\tilde{\gamma}_{10,3}$) $= 11$. Thus from Theorem 1.1, we obtain that cup($\tilde{G}_{10,3}$) $\leq 5$ which, when combined with (13), implies that cup($\tilde{G}_{10,3}$) $= 5$ and charrank($\tilde{\gamma}_{10,3}$) $= 11$. Let us continue with $\tilde{G}_{2n-1+2,3}$, $q \geq 5$. Then Theorem 3.1(2) with $s = 4$ implies that charrank($\tilde{\gamma}_{2n-1+2,3}$) $\geq 2^{q-1} + 4$. From Theorem 1.1, we see that cup($\tilde{G}_{2n-1+2,3}$) $\leq 2^{q-1} - 3$; this, jointly with (13), yields
\[ \text{cup}(\tilde{G}_{2n-1+2,3}) = 2^{q-1} - 3 \text{ and charrank}(\tilde{\gamma}_{2n-1+2,3}) = 2^{q-1} + 4, \]
as claimed.

For $\tilde{G}_{2n-1+3,3}$ with $q \geq 5$, we apply Theorem 3.1(2) with $s = 6$ and see that charrank($\tilde{\gamma}_{2n-1+3,3}$) $\geq 2^{q-1} + 7$. Theorem 1.1 implies that cup($\tilde{G}_{2n-1+3,3}$) $\leq 2^{q-1} - 3$ which, when combined with (13), shows that
\[ \text{cup}(\tilde{G}_{2n-1+3,3}) = 2^{q-1} - 3 \text{ and charrank}(\tilde{\gamma}_{2n-1+3,3}) = 2^{q-1} + 7, \]
indeed. The proof of Theorem 3.6 is finished.

\[ \square \]

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