POSTNIKOV DECOMPOSITION AND THE GROUP OF SELF-EQUIVALENCE OF A RATIONALIZED SPACE

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Abstract
Let $X$ be a simply connected rational CW complex of finite type. Write $X^{[n]}$ for the $n$th Postnikov section of $X$. Let $\mathcal{E}(X^{[n+1]})$ denote the group of homotopy self-equivalences of $X^{[n+1]}$. We use Sullivan models in rational homotopy theory to construct two short exact sequences:

$$\text{Hom}(\pi_{n+1}(X); H^{n+1}(X^{[n]})) \rightarrow \mathcal{E}(X^{[n+1]}) \rightarrow D_n^{n+1},$$

$$\text{Hom}(\pi_{n+1}(X); H^{n+1}(X^{[n]})) \rightarrow \mathcal{E}^*(X^{[n+1]}) \rightarrow G_n^{n+1},$$

where $D_n^{n+1}$ is a subgroup of $\text{aut}(\text{Hom}(\pi_q(X); \mathbb{Q})) \times \mathcal{E}(X^{[n]})$ which is defined in terms of the Whitehead exact sequence of $X$ and where $G_n^{n+1}$ is a certain subgroup of $\mathcal{E}^*(X^{[n]}).$ Here $\mathcal{E}^*(X^{[n]})$ is the subgroup of those elements inducing the identity on the homotopy groups. Moreover, we give an alternative proof of the Costoya–Viruel theorem [9]: Every finite group occurs as $\mathcal{E}(X)$ where $X$ is rational.

1. Introduction

Let $\mathcal{E}(X)$ denote the group of self homotopy equivalences of a simply connected CW-complex $X$ and let $\mathcal{E}_*^*(X)$ denote the subgroup represented by self-equivalences that induce the identity map on $\pi_*$ $(X).$ The subgroup $\mathcal{E}_*^*(X)$ is not in general trivial, for instance in [3] it is shown that $\mathcal{E}_*^*(S^2 \times S^n) \cong \pi_{n+2}(S^2) \oplus \mathbb{Z}_2$ where $n \geq 3.$

The study of the groups $\mathcal{E}(X)$ and $\mathcal{E}_*^*(X)$ by means of a cellular decomposition of $X$ is a difficult problem with a long history. See Rutter [11, Chapter 11] for a survey.

When $X$ is a simply connected rational CW complex of finite type, i.e., $\pi_n(X)$ is a vector space of finite dimension for every $n \geq 1,$ the group $\mathcal{E}(X)$ has emerged as a recent object of interest, for instance the realization problem, namely which group occurs as $\mathcal{E}(X)$? Arkowitz–Lupton [2] gave the first examples of finite groups occurring as $\mathcal{E}(X).$ Further examples were given by the author in [6, 7]. Costoya–Viruel [9] then proved the remarkable result that every finite group $G$ occurs as $G = \mathcal{E}(X)$ for some elliptic rational space $X.$ All these works have been accomplished.

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using Sullivan models in rational homotopy theory. A similar approach of studying the group $E(X)$ based on Anick’s DG Lie algebra models [1] over a certain subring of $\mathbb{Q}$ has been developed by Benkhalifa–Smith [8].

The aim of this paper is to study the effect of rational cell-attachment on the group of self-equivalences using the Postnikov tower. More precisely let $X$ be a simply connected rational CW complex of finite type. Write $X[\cdot]$ for the $n$th Postnikov section of $X$ [4, page 237]. We consider the situation in which

$$X[\cdot] \cup_{\alpha} \left( \bigcup_{i \in I} e^q_i \right), \quad \text{where } I \text{ is finite and } q > n,$$

(1)

is the space obtained by attaching rational cells to $X[\cdot]$ by a map $\alpha: \bigvee_{i \in I} S^q_{\mathbb{Q}} \to X[\cdot]$ where $S^q_{\mathbb{Q}}$ is the rational sphere of dimension $q$ [10, page 102]. Let $Z^q$ be the pullback of the map $X[\cdot] \to K(\pi_q(X); q + 1)$ over the path fibration:

$$Z^q \to PK(\pi_q(X); q + 1) \to X[\cdot] \to K(\pi_q(X); q + 1).$$

Recall that if $q = n + 1$, then the space $Z^q$ coincides with $(n + 1)$th Postnikov section $X[\cdot + 1]$ of $X$.

We prove:

**Theorem 1.1.** There exist two short exact sequences of groups:

$$\text{Hom}(\pi_q(X); H^{n+1}(X[\cdot])) \to \mathcal{E}(Z^q) \to D^q_n,$$

$$\text{Hom}(\pi_q(X); H^{n+1}(X[\cdot])) \to \mathcal{E}(Z^q) \to G^q_n,$$

where $D^q_n$ (respect. $G^q_n$) is a certain subgroup of $\text{aut}(\text{Hom}(\pi_q(X); \mathbb{Q})) \times \mathcal{E}(X[\cdot])$ (respect. of $\mathcal{E}_q(X[\cdot])$) (see (14) and (15) for the definitions).

Consequently, we prove the following first result concerning the question of (in)finiteness of the group $\mathcal{E}(X[\cdot])$.

**Corollary 1.2.** Let $X$ be a simply connected rational CW complex of finite type. Assume that $\pi_i(X) \otimes \mathbb{Q} \neq 0$ for $i = n - 1, n, n + 1$.

If $\mathcal{E}(X[\cdot])$ is finite, then $\mathcal{E}(X[\cdot + 1])$ and $\mathcal{E}(X[\cdot - 1])$ are infinite.

Next we prove the following result showing the relationship between the finiteness of the group $\mathcal{E}(Z^{n+2})$ and the Postnikov invariant $[k^n]$ of the space $X$ [4, page 237].

**Corollary 1.3.** Let $X$ be a simply connected rational CW complex of finite type. Assume that $\mathcal{E}(Z^{n+2})$ is finite, then the space $X[\cdot + 1]$ has the homotopy type of $X[\cdot] \times K(\pi_{n+1}(X), n + 1)$.

In addition we give a second proof of Costoya–Viruel in [9] that every finite group occurs as $\mathcal{E}(X)$ for some elliptic rational space $X$.

We establish these results in an algebraic context using the notion of Sullivan models in rational homotopy theory [12]. Recall that if $X$ is a simply connected
rational CW complex of finite type, then there exists a free commutative cochain algebra \((\Lambda V, \partial)\) called the Sullivan model of \(X\), unique up to isomorphism, which determines completely the homotopy type of the space \(X\). Moreover, the Sullivan model recovers homotopy data via the identifications:

\[
\text{Hom}(\pi_*(X; \mathbb{Q}), \mathbb{Q}^*) \cong V^*, \quad H^*(X; \mathbb{Q}) \cong H^*(\Lambda V, \partial) \quad \text{and} \quad \mathcal{E}(X) \cong \frac{\text{aut}(\Lambda V, \partial)}{\simeq},
\]

where \(\text{aut}(\Lambda V, \partial) \cong \mathbb{Z}\) is the group of homotopy cochain self-equivalences of \((\Lambda V, \partial)\) modulo the relation of homotopy between free commutative cochain algebras (see \([12]\)).

We write

\[
\mathcal{E}(\Lambda V) = \text{aut}(\Lambda V, \partial)/\simeq,
\]

for this group. Similarly, we have \(\mathcal{E}_g(X) \cong \text{aut}_g(\Lambda V, \partial)/\simeq\). We denote the latter group by \(\mathcal{E}_g(\Lambda V)\).

The exact sequences in Theorem 1.1 are then the translation of the exact sequences given, in (11), for \(\mathcal{E}(\Lambda V)\) and \(\mathcal{E}_g(\Lambda V)\). We end this work by giving an example showing that there exists a free commutative cochain algebra \((\Lambda V, \partial)\), which is not of finite type, such that \(\mathcal{E}(\Lambda V) \cong \mathbb{Z}\).

2. Homotopy self-equivalences of cochain morphisms

2.1. Notation and fundamental results

All vector spaces, algebras, tensor products, etc. are defined over \(\mathbb{Q}\) and this ground field will be in general suppressed from the notation.

A commutative cochain algebra is a (positive) graded differential vector space equipped with a graded linear map \(\partial^*\) (called the differential) \((\partial^n: A^n \rightarrow A^{n+1}\), such that \(\partial^2 = 0\)) together with linear maps \(A^* \otimes A^* \rightarrow A^*\), denoted by \(a \otimes b \rightarrow a.b\) and \(i: \mathbb{Q} \rightarrow A^*\) which satisfy the associativity, commutativity (in the graded sense), and unit conditions and the following relations:

\[
A^i.A^j \subset A^{i+j} \quad \text{and} \quad d(a.b) = d(a).b + (-1)^{|a|}a.d(b) , \quad a, b \in A.
\]

A commutative cochain algebra \((A^*, d)\) is said to be 1-connected if \(H^0(A) = \mathbb{Q}\) and \(H^1(A) = 0\), and of finite type if each vector space \(H^n(A)\) is finite dimensional.

**Definition 2.1.** Let \(V\) be a (positive) graded vector space and let \(T(V)\) be the free tensor algebra over \(V\). Put \(T(V) = \mathbb{Q} \oplus T^{\geq 1}(V)\) and define \(\Lambda V\), called the free commutative algebra, to be \(T(V)\) divided by the two-sided ideal generated by elements of the form \((a.b).c - a.(b.c)\) and \(a.b - (-1)^{|a||b|}b.a\) where \(a, b, c \in T(V)\).

The free commutative algebra \(\Lambda V\) can also be described as follows: \(\Lambda V = E(V^{\text{odd}}) \otimes S(V^{\text{even}})\), where \(E(V^{\text{odd}})\) is the exterior algebra on the oddly graded part of \(V\) and \(S(V^{\text{even}})\) is the symmetric algebra on the evenly graded part of \(V\). Moreover, if \(\partial\) is a differential on \(\Lambda V\), then \((\Lambda V, \partial)\) is called a free commutative cochain algebra (fcca for short).

Let us recall the following results (see \([10, 12]\)) which are fundamental in rational homotopy theory.
Theorem 2.2. For every 1-connected commutative cochain algebra \((A^*, d)\), there exists a quasi-isomorphism \((\Lambda V, \partial) \to (A^*, d)\) such that \(\partial\) is decomposable, i.e., \(\text{Im} \partial \subset \Lambda^+ V \Lambda^+ V\). The fcca \((\Lambda V, \partial)\) is called a minimal Sullivan model of \((A^*, d)\), it is unique up to isomorphism. Moreover, two 1-connected commutative cochain algebras are quasi-isomorphic if and only if their minimal Sullivan models are isomorphic.

Theorem 2.3. Let \(X\) be a simply connected CW-complex having rational homology of finite type. Let \(A_{PL}(X)\) be the simplicial cochain algebra associated with \(X\) (see [10, 12]). The minimal Sullivan model \((\Lambda V, \partial)\) of \(A_{PL}(X)\) is called the minimal Sullivan model of \(X\). Recall that \(H^*(\Lambda V, \partial) \cong H^* (\pi_\ast (X); \mathbb{Q})\), as graded algebras, and \(V^n \cong \text{Hom}_{\mathbb{Z}} (\pi_n (X); \mathbb{Q})\) for every \(n \geq 2\). In addition there exists a bijection between the set of rational homotopy types of simply connected CW-complexes having rational homology of finite type and the set of isomorphism classes of fccas.

2.2. Notion of homotopy for free commutative cochain algebras

Definition 2.4. Let \((\Lambda (V), \partial)\) be a 1-connected fcca. Define the vector spaces \(V\) and \(\hat{V}\) by \(V^n = V^{n+1}\) and \((\hat{V})^n = V^n\). We then define the fcca \((\Lambda (V, V), \hat{V}), D\) with the differential \(D\) is given by

\[ D(v) = \partial(v), \quad D(\hat{v}) = 0, \quad D(\overline{v}) = \hat{v}. \]

We define a derivation \(S\) of degree -1 of the fcca \((\Lambda (V, V, \hat{V}), D)\) by putting \(S(v) = v\), \(S(\hat{v}) = 0\) and \(S(\overline{v}) = 0\) for \(v \in V\) and \(\theta = D \circ S + S \circ D\).

A homotopy between two cochain morphisms \(\alpha, \alpha' : (\Lambda (V), \partial) \to (\Lambda (V), \partial)\) is a cochain morphism

\[ F : (\Lambda (V, V), \hat{V}), D) \to (\Lambda (V), \partial), \]

such as \(F(v) = \alpha(v)\) and \(F \circ e^\theta(v) = \alpha'(v)\), where

\[ e^\theta(v) = v + \hat{v} + \sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n (v), \quad v \in V \quad \text{and} \quad \theta = D \circ S + S \circ D. \]

Thereafter we will need the following lemma:

Lemma 2.5. Let \(q > n\) and let \(V = V^q \oplus V^{\leq n}\) and \(\alpha, \alpha' : (\Lambda (V), \partial) \to (\Lambda (V), \partial)\) be two cochain morphisms satisfying:

\[ \alpha(v) = v + z, \quad \alpha(v) = v + z' \quad \text{on} \quad V^q \quad \text{and} \quad \alpha = \alpha' = \text{id} \quad \text{on} \quad V^{\leq n}. \]

Assume that \(z - z' = \partial(u)\), where \(u \in \Lambda (V)\). Then \(\alpha\) and \(\alpha'\) are homotopic.

Proof. Define \(F\) by setting

\[ F(v) = v + y, \quad F(\hat{v}) = z' - z \quad \text{and} \quad F(\overline{v}) = z \quad \text{for} \quad v \in V^q, \]

\[ F(v) = v, \quad F(\hat{v}) = 0 \quad \text{and} \quad F(\overline{v}) = 0 \quad \text{for} \quad v \in V^{\leq n} \]

then \(F\) is the needed homotopy.

\[ \square \]
2.3. The graded linear map $b^*$

**Definition 2.6.** Let $(\Lambda(V^q \oplus V^{\leq n}), \partial)$ be a 1-connected fcca where $q > n$. We define the linear map $b^q : V^q \rightarrow H^{q+1}(\Lambda V^{\leq n})$ by setting

$$b^q(v) = [\partial(v)], \quad v \in V^q.$$  

(2)

Here $[\partial(v)]$ denotes the cohomology class of $\partial(v) \in (\Lambda V^{\leq n})_{q+1}$.

For every 1-connected cdga $(\Lambda(V^q \oplus V^{\leq n}), \partial)$, the linear map $b^q$ is natural. Namely, if $[\alpha] \in \mathcal{E}(\Lambda(V^q \oplus V^{\leq n}))$, then the following diagram commutes:

$$
\begin{array}{ccc}
V^q & \xrightarrow{\alpha^q} & V^q \\
\downarrow{b^q} & & \downarrow{b^q} \\
H^{q+1}(\Lambda V^{\leq n}) & \xrightarrow{H^{q+1}(\alpha^{\leq n})} & H^{q+1}(\Lambda V^{\leq n}),
\end{array}
$$

(3)

where $\tilde{\alpha} : V^* \rightarrow V^*$ is the graded homomorphism induced by $\alpha$ on the indecomposables and where $\alpha^{\leq n} : (\Lambda V^{\leq n}, \partial) \rightarrow (\Lambda V^{\leq n}, \partial)$ is the restriction of $\alpha$.

2.4. The groups $D^q_n$

**Definition 2.7.** Given a 1-connected fcca $(\Lambda(V^q \oplus V^{\leq n}), \partial)$ where $q \geq n$, let $D^q_n$ be the subset of $\text{aut}(V^q) \times \mathcal{E}(\Lambda V^{\leq n})$ consisting of the couples $(\xi, [\alpha^{\leq n}])$ making the following diagram commute:

$$
\begin{array}{ccc}
V^q & \xrightarrow{\xi^q} & V^q \\
\downarrow{b^q} & & \downarrow{b^q} \\
H^{q+1}(\Lambda V^{\leq n}) & \xrightarrow{H^{q+1}(\alpha^{\leq n})} & H^{q+1}(\Lambda V^{\leq n}).
\end{array}
$$

(4)

Clearly, $D^q_n$ is a subgroup of $\text{aut}(V^q) \times \mathcal{E}(\Lambda V^{\leq n})$.

**Proposition 2.8.** The map $g : \mathcal{E}(\Lambda(V^q \oplus V^{\leq n})) \rightarrow D^q_n$ given by

$$g([\alpha]) = ([\tilde{\alpha}^q], [\alpha^{\leq n}])$$

is a surjective homomorphism of groups.

**Proof.** First it is well-known [10, Proposition 12.8] that if two cochain morphisms are homotopic, then they induce the same graded linear maps on the indecomposables, i.e., $\tilde{\alpha} = \tilde{\alpha}'$, moreover, $\alpha^{\leq n}, \alpha'^{\leq n}$ are homotopic and by using the diagram (3) we deduce that the map $g$ is well-defined.

Next let $(\xi, [\alpha^{\leq n}]) \in D^q_n$. Recall that, in the diagram (4), we have:

$$H^{q+1}(\alpha^{\leq n}) \circ b^q(v) = \alpha^{\leq n} \circ \partial(v) + \text{Im} \partial^{\leq n},$$

$$b^q \circ \xi^q(v) = \partial \circ \xi^q(v) + \text{Im} \partial^{\leq n},$$

(5)

where $\partial^{\leq n} : (\Lambda V^{\leq n})^q \rightarrow (\Lambda V^{\leq n})^{q+1}$.

Since by Definition 2.7 this diagram commutes, the element $(\alpha^{\leq n} \circ \partial - \partial \circ \xi^q)(v) \in \text{Im} \partial^{\leq n}$. As a consequence there exists $u_v \in (\Lambda V^{\leq n})^q$ such that

$$\alpha^{\leq n} \circ \partial - \partial \circ \xi^q)(v) = \partial^{\leq n}(u_v).$$

(6)
Thus we define $\alpha: (\Lambda(V^q \oplus V^\leq n), \partial) \to (\Lambda(V^q \oplus V^\leq n), \partial)$ by setting

$$\alpha(v) = \xi^q(v) + u_v, \quad \text{and} \quad \alpha = \alpha^\leq n \text{ on } V^\leq n.$$  

As $\partial(v) \in (\Lambda V^\leq n)^q$ then, by (6), we get

$$\partial \circ \alpha(v) = \partial(\xi^q(v)) + \partial^\leq n(u_v) = \alpha^\leq n \circ \partial(v) = \alpha \circ \partial(v).$$

So $\alpha$ is a cochain morphism. Now due to the fact that $u_v \in (\Lambda V^\leq n)^q$ and $q > n$, the linear map $\tilde{\alpha}^q: V^q \to V^q$ coincides with $\xi^q$.

Then it is well-known (see [10]) that any cochain morphism between two 1-connected fccas inducing a graded isomorphism on the indecomposables is a homotopy equivalence. Consequently, $[\alpha] \in \mathcal{E}(\Lambda V, \partial)$. Therefore $g$ is surjective.

Finally, the following relations:

$$g([\alpha][\alpha']) = [\alpha \circ \alpha'] = (\alpha \circ \alpha^q, [\alpha^\leq n \circ \alpha'^\leq n])$$

$$= (\tilde{\alpha}^q, [\alpha^\leq n]) \circ (\alpha^q, [\alpha'^\leq n]) = g([\alpha]) \circ g([\alpha'])$$

assure that $g$ is a homomorphism of groups. \hfill \square

### 2.5. Characterization of $\ker g$

Next by definition we have

$$\ker g = \{ [\alpha] \in \mathcal{E}(\Lambda(V)) \mid \tilde{\alpha}^q = id_{V^q}, \quad [\alpha^\leq n] = [id_{\Lambda(V^\leq n)}] \},$$

therefore for every $[\alpha] \in \ker g$ we have

$$\alpha(v) = v + z, \quad z \in \Lambda^q(V^\leq n),$$

$$\alpha^\leq n \simeq id_{\Lambda(V^\leq n)}. \quad (7)$$

So define

$$\theta_\alpha: V^q \to \Lambda^q(V^\leq n) \quad \text{by} \quad \theta_\alpha(v) = \alpha(v) - v. \quad (8)$$

Notice that the relations (7) and (8) imply that

$$\theta_{\alpha' \circ \alpha} = \theta_{\alpha'} + \theta_\alpha. \quad (9)$$

**Lemma 2.9.** Let $[\alpha] \in \ker g$. Then there exists $[\beta] \in \ker g$ satisfying:

1. $\theta_\beta(v)$ is a cocycle in $\Lambda^q(V^\leq n)$ for every $v \in V^q$
2. $\beta^\leq n = \alpha^\leq n$
3. $[\beta] = [\alpha]$

**Proof.** Since $[\alpha^\leq n] = [id_{\Lambda(V^\leq n)}]$ there is a homotopy $F: (\Lambda(V^\leq n), \partial) \to (\Lambda(V^\leq n), \partial)$, such that $F(v) = v$ and $F \circ e^\theta(v) = \alpha^\leq n(v)$. Therefore for $v \in V^q$ the element $F\left( \sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n(v) \right)$ is a well-defined element in $\Lambda^q(V^\leq n)$. Thus we define $\beta$ by setting

$$\beta(v) = \begin{cases} 
    v, & \text{for } v \in V^q; \\
    \alpha(v) - F\left( \sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n(v) \right), & \text{for } v \in V^\leq n.
\end{cases}$$
Thus $\beta$ satisfies (1). For (2) and (3), we define $G: (\Lambda(V, V, \widetilde{V}), D) \rightarrow (\Lambda(V), \partial)$ by setting $G = F$ on $(\Lambda(V^{\le n}, V^{\le n}, \widetilde{V}^{\le n}), D)$ while, for $v \in V^q$, we set $G(v) = \beta(v)$ and $G(\widetilde{v}) = G(\overline{v}) = 0$. First, it is easy to check that $G$ is a cochain morphism. Next, for $v \in V^q$ we have

$$G \circ e^\theta(v) = G\left(v + \widetilde{v} + \sum_{n \geq 1} \frac{1}{n!}(S \circ \partial)^n(v)\right) = G(v) + G\left(\sum_{n \geq 1} \frac{1}{n!}(S \circ \partial)^n(v)\right)$$

$$= \beta(v) + F\left(\sum_{n \geq 1} \frac{1}{n!}(S \circ \partial)^n(v)\right) = \alpha(v).$$

Therefore $\beta \simeq \alpha$ and the lemma is proved. \hfill \square

Thus Lemma 2.9 and the relation (8) allow us to define a map $\Phi: \ker g \rightarrow \text{Hom}(V^q, H^q(\Lambda(V^{\le n})))$ by setting $\Phi([\beta])(v) = \{\theta_\beta(v)\}$ for $v \in V^q$ where $[\beta]$ is chosen as in Lemma 2.9.

**Proposition 2.10.** The map $\Phi: \ker g \rightarrow \text{Hom}(V^q, H^q(\Lambda(V^{\le n})))$ is an isomorphism.

**Proof.** First we prove that $\Phi$ is well-defined. Suppose $[\beta] = [\beta']$ satisfy the conclusion of Lemma 2.9. Since both maps then restrict to the identity on $\Lambda(V^{\le n})$, the homotopy $F: (\Lambda(V, V, \widetilde{V}), D) \rightarrow (\Lambda(V), \partial)$ between them can be chosen so that

$$F(\widetilde{V}^{\le n}) = F(\overline{V}^{\le n}) = 0. \quad (10)$$

Given $v \in V^q$, according to (8) we then have

$$\theta_{\beta'}(v) - \theta_\beta(v) = \beta(v) - \beta'(v) = F \circ e^\theta(v) - F(v)$$

$$= F(v') + F\left(\sum_{n \geq 1} \frac{1}{n!}(S \circ \partial)^n(v)\right)$$

$$= F(D(sv)) + F\left(\sum_{n \geq 1} \frac{1}{n!}(S \circ \partial)^n(v)\right)$$

$$= \partial(F(sv)) + F\left(\sum_{n \geq 1} \frac{1}{n!}(S \circ \partial)^n(v)\right) = \partial(F(sv)).$$
Thus \( \theta_{\beta'}(v) - \theta_{\beta}(v) \) is a coboundary. Notice that the relation (10) implies that
\[
F\left( \sum_{n \geq 1} \frac{1}{n!} (S \circ \partial)^n (v) \right) = 0.
\]

For the injectivity, assume that \( \Phi([\beta])(v) = \Phi([\beta'])(v) \) in \( H^{q+1}(\Lambda(V^{\leq n})) \), then \( \theta_{\beta'}(v) - \theta_{\beta}(v) = \beta(v) - \beta'(v) \) is a coboundary and Lemma 2.5 implies that \([\beta] = [\beta']\).

For the surjectivity, given a homomorphism \( \chi \in \text{Hom}(V^q, \widetilde{H}(\Lambda(V^{\leq n}))) \), write \( \chi(v) = \{ \chi(v) \} \), where \( \chi(v) \) is a cocycle. We define \( \beta : (\Lambda(V), \partial) \to (\Lambda(V), \partial) \) by
\[
\beta(v) = v + \widetilde{\chi}(v) \quad \text{for} \quad v \in V^q \quad \text{and} \quad \beta = \text{id} \quad \text{on} \quad V^{\leq n}.
\]

Then \( \beta \) is a cochain morphism with \( \Phi([\beta]) = \chi \).

Finally, given \( \beta, \beta' \in \ker g \) as in Lemma 2.9. So \( \beta(v) = v + \theta_{\beta}(v) \) and \( \beta'(v) = v + \theta_{\beta'}(v) \) for \( v \in V^q \). Therefore, by (9) we get
\[
\beta' \circ \beta(v) = v + \theta_{\beta'}(v) + \theta_{\beta}(v) = v + \theta_{\beta' \circ \beta}(v).
\]

Consequently, \( \Phi([\beta'], [\beta]) = \Phi([\beta' \circ \beta]) = \theta_{\beta' \circ \beta} = \theta_{\beta'} + \theta_{\beta} = \Phi([\beta']) + \Phi([\beta]) \). Thus \( \Phi \) is a homomorphism of groups.

Summarizing, we have proven:

**Theorem 2.11.** Let \( q > n \) and let \( (\Lambda(V^q \oplus V^{\leq n}), \partial) \) be a 1-connected fcca. Then there exists a short exact sequence of groups
\[
\text{Hom}(V^q, H^q(\Lambda(V^{\leq n}))) \to \mathcal{E}(\Lambda(V^q \oplus V^{\leq n})) \xrightarrow{\Psi} D^q_n.
\]

We now focus on the subgroup \( \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \) of \( \mathcal{E}(\Lambda(V^q \oplus V^{\leq n})) \) of the elements inducing the identity on the graded vector space of indecomposables. Let us define \( \mathcal{G}_d^q \) as the subgroup of \( \mathcal{E}_d(\Lambda(V^{\leq n})) \) of those elements \([\alpha] \) satisfying \( H^{q+1}(\alpha) \circ b^q = b^q \) where \( b^q : V^q \to H^{q+1}(\Lambda(V^{\leq n})) \) is as in (2).

**Theorem 2.12.** Let \( q > n \) and let \( (\Lambda(V^q \oplus V^{\leq n}), \partial) \) be a 1-connected fcca. Then there exists a short exact sequence of groups
\[
\text{Hom}(V^q, H^q(\Lambda(V^{\leq n}))) \to \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \to \mathcal{G}_d^q.
\]

**Proof.** First let \([\alpha] \in \ker g\). From the relation (7) we deduce that \( \tilde{\alpha} = \text{id}_{V^q} \) and \( \alpha^{\leq n} \simeq \text{id}_{\Lambda(V^{\leq n})} \), therefore \( \alpha^{\leq n} \) induces the identity on the indecomposables. So \( \tilde{\alpha} = \text{id}_{V} \). It follows that \( \ker g \subseteq \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \).

Next from (11) we obtain the short exact sequence
\[
\text{Hom}(V^q, H^q(\Lambda(V^{\leq n}))) \to \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \to g\left( \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \right),
\]
where
\[
g\left( \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \right) = \left\{ \Psi([\alpha]) = (\tilde{\alpha}, [\alpha^{\leq n}]) \mid [\alpha] \in \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \right\}.
\]

As \([\alpha] \in \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n}))\), the graded automorphism \( \tilde{\alpha} \in \text{aut}(V^q \oplus V^{\leq n}) \) is the identity which, in turn, implies
\[
g\left( \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \right) = \left\{ (\text{id}_{V^q}, [\alpha^{\leq n}]) \mid [\alpha^{\leq n}] \in \mathcal{E}_d(\Lambda(V^{\leq n})) \right\}.
\]

As \([\alpha] \in \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n}))\), the pair \( (\text{id}_{V^q}, [\alpha^{\leq n}]) \) makes the diagram (3) commute. As a result we can identify \( g\left( \mathcal{E}_d(\Lambda(V^q \oplus V^{\leq n})) \right) \) with the subgroup \( \mathcal{G}_d^q \). \( \square\)
Corollary 2.13. Let $q > n$ and let $(\Lambda(V^q \oplus V^\leq n), \partial)$ be a 1-connected fcca. If $E_\ast(\Lambda(V^\leq n))$ is trivial, then
\[
\text{Hom}(V^q, H^q(\Lambda(V^\leq n))) \cong E_\ast(\Lambda(V^q \oplus V^\leq n)).
\]

Corollary 2.14. Let $q > 2n + 1$ and let $(\Lambda(V^q \oplus V^\leq 2n+1), \partial)$ be an $n$-connected fcca. Then
\[
\text{Hom}(V^q, H^q(\Lambda(V^\leq 2n+1))) \cong E_\ast(\Lambda(V^q \oplus V^\leq 2n+1)).
\]

Proof. As $(\Lambda(V^q \oplus V^\leq 2n+1), \partial)$ is $n$-connected, then $V^1 = \cdots = V^n = 0$. So, for degree reasons, the group $E_\ast(\Lambda(V^\leq 2n+1))$ is trivial and we then apply Corollary 2.13. \qed

3. Topological applications

All the CW-complexes which we consider in this section are simply connected having rational homology of finite type.

Let $(\Lambda(V), \partial)$ be a 1-connected fcca. Recall that in [5] it is shown that with $(\Lambda(V), \partial)$ we can associate the following long exact sequence
\[
\cdots \to V^n \xrightarrow{b^n} H^{n+1}(\Lambda(V^{\leq n-1})) \to H^{n+1}(\Lambda(V)) \to V^{n+1} \xrightarrow{b^{n+1}} \cdots,
\]

called the Whitehead exact sequence of $(\Lambda(V), \partial)$. Recall that $b^\ast$ is the graded linear map defined in (2).

Now let $X$ be a simply connected rational CW-complex of finite type and let $X^{[n]}$ be the $n$th Postnikov section of $X$. For $q > n$, as in (1), let
\[
X^{[n]} \cup_\alpha \left( \bigcup_{i \in I} e_i^q \right), \quad \text{where } I \text{ is finite and } q > n,
\]
be the space obtained by attaching rational cells to $X^{[n]}$ by a map $\alpha : \bigvee_{i \in I} S^q_\mathbb{Q} \to X^{[n]}$ where $S^q_\mathbb{Q}$ is the rational sphere of dimension $q$. If $(\Lambda(V), \partial)$ is the Sullivan model of $X$, then it is well-known that $(\Lambda(V^\leq n), \partial)$ is the Sullivan model of $X^{[n]}$ while $(\Lambda(V^q \oplus V^\leq n), \partial)$ is the Sullivan model of the pullback $Z^q$ of the map $X^{[n]} \to K(\pi_q(X); q+1)$, whose homotopy class is the cohomology class in $H^{q+1}(X^{[n]}; \pi_q(X)) = [X^{[n]}; K(\pi_q(X); q+1)]$ given algebraically by
\[
b^q \in \text{Hom} \left( V^q; H^{q+1}(\Lambda(V^\leq n)) \right) \cong H^{q+1}(X^{[n]}; \pi_q(X)),
\]
over the path fibration:
\[
\begin{array}{ccc}
Z^q & \to & PK(\pi_q(X); q+1) \\
\downarrow & & \downarrow \\
X^{[n]} & \to & K(\pi_q(X); q+1).
\end{array}
\]

Observe that the fibration long exact sequence implies that $\pi_q(X) \cong \pi_q(Z^q)$. Hence by the virtues of the properties of the Sullivan model, the Whitehead exact sequence of $(\Lambda(V^q \oplus V^\leq n), \partial)$ yields the following exact sequence
\[
\cdots \xrightarrow{h^n} \text{Hom}(\pi_q(X), \mathbb{Q}) \xrightarrow{b^n} H^{q+1}(X^{[n]}) \to H^{q+1}(Z^q) \xrightarrow{h^{q+1}} \text{Hom}(\pi_{q+1}(X), \mathbb{Q}) \xrightarrow{b^{q+1}} \cdots,
\]

where \(h^r\) is the dual of the Hurewicz homomorphism.

Let \(D^q_n\) be the subgroup of \(\text{aut}(\text{Hom}(\pi_q(X), \mathbb{Q}) \times \mathcal{E}(X^{[n]})\) of those pairs \((\xi, [f])\) making the following diagram commute:

\[
\begin{array}{ccc}
\text{Hom}(\pi_q(X), \mathbb{Q}) & \xrightarrow{\xi} & \text{Hom}(\pi_q(X), \mathbb{Q}) \\
\downarrow{b^n} & & \downarrow{b^n} \\
H^{q+1}(X^{[n]}) & \xrightarrow{H^{q+1}(f)} & H^{q+1}(X^{[n]})
\end{array}
\]

and let

\[
G^q_n = \{ [f] \in \mathcal{E}_n(X^{[n]}) \mid \text{such that } H^{q+1}(f) \circ b^n = b^q \}.
\]

Clearly \(G^q_n\) is a subgroup of \(\mathcal{E}_n(X^{[n]})\). From Theorems 2.11 and 2.12 we deduce the following topological results:

**Theorem 3.1.** Let \(X\) be a CW-complex. Then for every \(n\) and for every \(q > n\) there exist two short exact sequence of groups:

\[
\begin{align*}
\text{Hom}(\pi_q(X); H^q(X^{[n]})) & \to \mathcal{E}(Z^q) \to D^q_n, \\
\text{Hom}(\pi_q(X); H^q(X^{[n]})) & \to \mathcal{E}_q(Z^q) \to G^q_n.
\end{align*}
\]

Moreover, if \(\mathcal{E}_n(X^{[n]})\) is a trivial group, then \(\text{Hom}(\pi_q(X); H^q(X^{[n]})) \cong \mathcal{E}_n(Z^q)\). Here \(Z^q\) is the space given in (12).

**Proof.** The two sequences (16) follow from Theorems 2.11 and 2.12. Notice that Sullivan theory implies the following identifications:

\[
\mathcal{E}(Z^q) \cong \mathcal{E}(\Lambda(V^q \oplus V \leq n)), \quad \mathcal{E}_n(Z^q) \cong \mathcal{E}_n(\Lambda(V^q \oplus V \leq n)),
\]

\[
D^q_n \cong D^q_n, \quad G^q_n \cong G^q_n.
\]

Finally, the last assertion follows by applying Corollary 2.13.

**Remark 3.2.** According to Theorem 3.1, if we take \(X^{[n]} = K(\pi, n)\), where \(\pi\) is a vector space of finite dimension, then we get

\[
\mathcal{E}_n(Z^q) \cong \text{Hom}(\pi_q(X); H^q(K(\pi, n))).
\]

Indeed, we know that the Sullivan model of \(K(\pi, n)\) is \((\Lambda(V^n), 0)\) where \(V^n \cong H^n(K(\pi, n), \mathbb{Q})\). Therefore the group \(\mathcal{E}_n(K(\pi, n)) \cong \mathcal{E}_n(\Lambda(V^n))\) is trivial.

Let \(X\) be a simply connected rational CW-complex of finite type. As the space \(Z^{n+1}\) coincides with \(X^{[n+1]}\), Theorem 3.1 implies:

**Corollary 3.3.** Let \(X\) be a simply connected rational CW-complex of finite type. The following two short sequences are exact:

\[
\begin{align*}
\text{Hom}(\pi_q(X); H^q(X^{[n]})) & \to \mathcal{E}(X^{[n+1]}) \to D^q_{n-1}, \\
\text{Hom}(\pi_q(X); H^q(X^{[n]})) & \to \mathcal{E}_q(X^{[n+1]}) \to G^q_{n-1}.
\end{align*}
\]

Moreover, if \(\mathcal{E}_n(X^{[n-1]})\) is finite, then \(\text{Hom}(\pi_q(X); H^q(X^{[n]})) \cong \mathcal{E}_n(X^{[n]})\).
Now let us consider the dual of the Hurewicz homomorphism $h^*$ given in the long exact sequence (13) and let $E^{(q+1)}_{q+1}(X^{[n]})$ denote the subgroup of $E_q(X^{[n]})$ consisting of the self-homotopy equivalences $[f]$ such that $H^{q+1}(f) : H^{q+1}(X^{[n]}) \to H^{q+1}(X^{[n]})$ is the identity.

**Corollary 3.4.** Let $X$ be a simply connected rational CW-complex and let $Z^q$ be the space given in (12). Assume that $h^q$ is nil and $h^{q+1}$ is injective. There exist two short exact sequence of groups:

$$
\text{Hom}(\pi_q(X); H^q(X^{[n]})) \to E(Z^q) \to E^{(q+1)}_{q+1}(X^{[n]}),
$$

$$
\text{Hom}(\pi_q(X); H^q(X^{[n]})) \to E_q(Z^q) \to E^{(q+1)}_{q+1}(X^{[n]}).
$$

**Proof.** First notice that if $h^q$ is nil and $h^{q+1}$ is injective, then according to the long exact sequence (13) the map $b^q : \text{Hom}(\pi_q(X), \mathbb{Q}) \to H^{q+1}_+(X^{[n]})$ is an isomorphism. Then for every $[f] \in E(X^{[n]})$ the pair $((b^q)^{-1} \circ H^{q+1}_+(f) \circ b^q, [f])$ makes the diagram (14) commute. Therefore we get a map $E(X^{[n]}) \to D^q_n$ defined by $[f] \mapsto ((b^q)^{-1} \circ H^{q+1}_+(f) \circ b^q, [f])$ and it is easy to see that it is an isomorphism of groups.

Likewise by (15) we can say that the group $G^q_n$ coincides with the subgroup $E^{(q+1)}_{q+1}(X^{[n]})$. Thus the sequences (16) imply the sequences (17). □

**Corollary 3.5.** Let $X$ be simply connected rational CW-complex of finite order. Assume that $\pi_i(X) \neq 0$ for $i = n - 1, n, n + 1$. If $E(X^{[n]})$ is finite, then $E(X^{[n+1]})$ and $E(X^{[n+1]})$ are infinite.

**Proof.** Working algebraically, we assume given a feca $(\Lambda(V), \partial)$ with $V^i \neq 0$ for $i = n - 1, n, n + 1$ and suppose that $E(\Lambda(V_{\leq n+1}))$ is finite. We prove $E(\Lambda(V_{\leq n}))$ is infinite. The result then follows from the properties of the Sullivan model.

Since $E(\Lambda(V_{\leq n+1}))$ is finite, applying Theorem 2.11 gives that $H^{n+1}_+(\Lambda(V_{\leq n-1})) = 0$. This implies the the linear map $b^n : V^n \to H^{n+1}_+(\Lambda(V_{\leq n-1})) = 0$ vanishes. Now by taking $\alpha = \text{id} : (\Lambda(V_{\leq n-1}), \partial) \to (\Lambda(V_{\leq n-1}), \partial)$ and $\xi^a \in \text{aut}(V^n), a \in \mathbb{Q}$, such that $\xi^a(v) = av$ for $v \in V^n$, the following diagram commutes obviously:

$$
\begin{array}{ccc}
V^n & \xymatrix{ \ar[r]^-{\xi^a} & } & V^n \\
\downarrow^{b^n=0} & & \downarrow^{b^n=0} \\
H^{n+1}_+(\Lambda(V_{\leq n-1})) & \xymatrix{ \ar[r]^-{H^{n+1}_+(\alpha) = \text{id}} & } & H^{n+1}_+(\Lambda(V_{\leq n-1})).
\end{array}
$$

Therefore there exist an infinity of pairs $(\xi^a, [\text{id}]) \in D^q_{n-1}$, so the group $D^q_{n-1}$ is infinite, it follows that $E(\Lambda(V_{\leq n}))$ is infinite by Theorem 2.11. Finally, if $E(\Lambda(V_{\leq n-1}))$ is finite, then by the above argument the group $E(\Lambda(V_{\leq n}))$ must be infinite. Contradiction. □

**Corollary 3.6.** Let $X$ be a simply connected rational CW complex of finite type. Assume that $E(Z^{n+2})$ is finite, then the space $X^{[n+1]}$ has the homotopy type of $X^{[n]} \times K(\pi_{n+1}(X), n + 1)$.

**Proof.** As $E(Z^{n+2})$ is finite, Theorem 3.1 implies that

$$
\text{Hom}(\pi_q(X); H^q(X^{[n]})) = H^{n+2}(X^{[n]}, \pi_{n+1}(X)) = 0.
$$
This implies that the Postnikov invariant \([k^n] \in H^{n+2}(X[n], \pi_{n+1}(X))\) is nil. As a result the space \(X^{[n+1]}\) has the homotopy type of \(X[n] \times K(\pi_{n+1}(X), n + 2)\).

**Remark 3.7.** Let \(X\) be as in Corollary 3.6. Then if the Postnikov invariant \([k^{n+1}]\) is not nil, then \(\mathcal{E}(Z^{n+2})\) is infinite.

### 3.1. Realization problem

Recall that the realizability problem for groups deals with the following question: Given a group \(G\), is there a space \(X\) such that \(G = \mathcal{E}(X)\)? A complete answer to the realizability problem for finite groups is given by Costoya–Viruel [9]. Here, in this section, we give an alternative proof based on the main result of this work.

Let \(G = \{g_1, g_2, \ldots, g_n\}\) be a group of order \(n\). By Cayley’s theorem, there is a monomorphism \(G \rightarrow S_n\) given by \(g_k \mapsto \sigma_s: g_k \mapsto g_sg_k, 1 \leq k \leq n\). For \(2 \leq s \leq n\), write \(\sigma_s = \begin{pmatrix} 1 & 2 & \cdots & n \\ s & \sigma_s(2) & \cdots & \sigma_s(n) \end{pmatrix}\) and let

\[
\sigma_2 = \left(1, 2\right) \left(\sigma_1(2) \cdots, \sigma_2^{\alpha_2}(2)\right) (i_1 \sigma_1(i_1) \cdots \sigma_2^{\alpha_1}(i_1)) \cdots (i_k \sigma_1(i_k) \cdots \sigma_2^{\alpha_k}(i_k))
\]  (18)

be the decomposition of \(\sigma_2\) as a product of cycles. Notice that the monomorphism \(G \rightarrow S_n\) implies that

\[
\sigma_s(i)(j) = \sigma_s \circ \sigma_i(j), \quad 1 \leq i, j, s \leq n.
\]  (19)

For a group \(G\), we define \((\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n}), \partial\), where \(|x_1| = 8, |x_2| = 10, |w_j| = 40\), by

\[
\partial(x_1) = \partial(x_2) = \partial(w_j) = 0, \quad \partial(y_1) = x_3^3 x_2, \quad \partial(y_2) = x_1^2 x_2^2, \quad \partial(y_3) = x_1 x_2^3,
\]

\[
\partial(z_1) = w_1^3 + w_1 w_2 x_2^4 + \sum_{\tau = 1}^{k} w_1 w_{\tau, i} x_2^4 + u + x_1^{15}, \quad u = y_1 y_2 x_2^4 x_2^2 - y_1 y_3 x_1^3 x_2 + y_2 y_3 x_1^6,
\]

\[
\partial(z_j) = w_j^3 + w_j w_{\sigma(j)}(2) x_2^4 + \sum_{\tau = 1}^{k} w_j w_{\sigma_s(i_s j)}(i_s) x_2^4 + u + x_1^{15}, \quad 2 \leq j \leq n.
\]  (20)

Thus applying Theorem 2.11 to the fcca \((\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n}), \partial\) we get the following exact sequence

\[
H^{19}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \ldots, w_n)) \rightarrow \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n})) \rightarrow D^{40}_{19}.
\]

**Lemma 3.8.** \(H^{19}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \ldots, w_n)) = 0\).

**Proof.** First \(L^{19}\) is spanned by

\[
y_1 x_1^7 x_2^3, \quad y_1 x_2^7, \quad y_2 x_1^3 x_2, \quad y_2 x_1^7, \quad y_2 x_2^3, \quad y_2 x_2^7, \quad y_3 x_1^3 x_2, \quad y_3 x_1^7, \quad y_3 x_2^3, \quad y_3 x_2^7, \quad y_1 w_1 x_1^3 x_2, \quad y_2 w_2 x_1^3 x_2, \quad y_3 w_3 x_1^3 x_2.
\]

Next, if

\[
\phi = A_1 y_1 x_1^7 x_2^3 + A_2 y_1 x_2^7 + A_3 y_2 x_1^3 x_2 + A_4 y_2 x_2^3 x_2 + A_5 y_3 x_1^3 x_2 + A_6 y_3 x_2^3 x_2
\]

\[
+ \left(\sum_{j=1}^{n} A_j^{(j)} y_1 w_j x_1^3 x_2^3\right) + \left(\sum_{j=1}^{n} A_j^{(j)} y_2 w_j x_1^3 x_2^3\right) + \left(\sum_{j=1}^{n} A_j^{(j)} y_3 w_j x_1^3 x_2^3\right),
\]

then

\[
\phi \in H^{19}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \ldots, w_n)) \rightarrow \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n})) \rightarrow D^{40}_{19}.
\]
it follows that
\[
\partial(\phi) = (A_1 + A_4 + A_5)x_1^{10}x_2^4 + (A_2 + A_3 + A_6)x_1^5x_2^8 + \left(\sum_{j=1}^{n} A_j^{(j)} + A_8^{(j)} + A_9^{(j)}\right)w_jx_1^5x_2^4.
\]

So the space of 119-cocycles is spanned by \(y_3x_1^9x_2 - y_1x_1^7x_2^3, y_2x_1^2x_2^8 - y_1x_1^7x_2^3, y_1x_1^2x_2^2 - y_3x_1^4x_2^5, y_2x_1^2x_2^5 - y_3x_1^4x_2^5, y_3w_jx_1^4x_2 - y_1w_jx_1^2x_2^3, y_2w_jx_1^3x_2^2 - y_1w_jx_1^2x_2^3.

But we have
\[
\begin{align*}
\partial(y_3y_1x_2^6) &= y_1x_1^7x_2^3 - y_3x_1^9x_2, \quad \partial(y_2y_1x_1^5x_2^2) = y_2x_1^8x_2^2 - y_1x_1^7x_2^3, \\
\partial(y_3y_1x_1^4x_2^7) &= y_1x_1^7x_2^7 - y_3x_1^4x_2^5, \quad \partial(y_2y_3x_1^2x_2^3) = y_2x_1^3x_2^6 - y_3x_1^4x_2^5, \\
\partial(y_1y_2w_jx_2^3) &= y_2w_jx_1^3x_2^2 - y_1w_jx_1^4x_2^3, \quad \partial(y_1y_3w_jx_1) = y_3w_jx_1^4x_2 - y_1w_jx_1^4x_2^3
\end{align*}
\]
and the lemma is proved. \(\square\)

Let \(g_s \in G\). For every \(j \leq n\), define
\[
\xi_s(z_j) = z_{s,(j)}, \quad \alpha_s(w_j) = w_{s,(j)}, \quad \alpha_s = \text{id} \text{ on } x_1, x_2, y_1, y_2, y_3.
\]

Clearly, \([\alpha_s] \in \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \ldots, w_n))\) and \(b^{119} \circ \xi_s = H^{120}(\alpha_s) \circ b^{119}\). So we get a map \(\Omega: G \rightarrow \mathcal{D}_{40}^{119}\) defined by \(\Omega(g_s) = (\xi_s, [\alpha_s])\).

**Proposition 3.9.** \(\Omega\) is an isomorphism of groups.

**Proof.** Let \((\xi, [\alpha]) \in \mathcal{D}_{40}^{119}\). For degree reasons we have
\[
\begin{align*}
\alpha(x_1) &= \beta_1x_1, \quad \alpha(x_2) = \beta_2x_2, \quad \alpha(y_1) = p_1y_1, \quad \alpha(y_2) = p_2y_2, \quad \alpha(y_3) = p_3y_3, \\
\alpha(w_j) &= a_jw_1 + \cdots + a_jw_n + \gamma_jx_1^5 + \gamma_jx_2^4, \quad \xi(z_j) = \lambda_jz_1 + \cdots + \lambda_jz_n.
\end{align*}
\]

As \(\alpha(\partial(y_1)) = \partial(\alpha(y_1))\) it follows that
\[
p_1 = \beta_1^3\beta_2, \quad p_2 = \beta_1^2\beta_2^2, \quad p_3 = c_1 = \beta_1\beta_2^3.
\]

Thus
\[
\begin{align*}
\alpha(\partial(z_j)) &= \alpha(w_j^3) + \alpha(w_jw_{\sigma_j}(z)x_2^4) + \alpha\left(\sum_{\tau=1}^{k} w_{j\tau}w_{\sigma_j, (i\tau)}x_2^4\right) + \alpha(u) + \alpha(x_1^{15}), \\
\partial(\xi(z_j)) &= \lambda_j(w_1^3 + w_1w_2x_2^4 + \sum_{\tau=1}^{k} w_{1\tau}w_{\sigma_j, (i\tau)}x_2^4 + u + x_1^{15}) + \cdots \\
&\quad + \lambda_j(w_n^3 + w_nw_{\sigma_j}(z)x_2^4 + \sum_{\tau=1}^{k} w_{n\tau}w_{\sigma_j, (i\tau)}x_2^4 + v^2 + u + x_1^{15}).
\end{align*}
\]

As \((\xi, [\alpha]) \in \mathcal{D}_{40}^{119}\) and due to (5) there exists \(\varphi_j \in \Lambda^{119}(x_1, x_2, y_1, y_2, y_3, w_1, \ldots, w_n)\) such that
\[
\partial(\xi(z_j)) - \alpha(\partial(z_j)) = \partial(\varphi_j), \quad \forall j \leq n.
\]

By expanding \(\alpha(w_j^3) = (a_jw_1 + \cdots + a_{jn}w_n + \gamma_jx_1^5 + \gamma_jx_2^4)^3\) the monomials \(a_{jn}^3, a_{jn}w_s^2w_t, \gamma_jx_1^5w_s, \gamma_jx_2^4x_1^2\) where \(1 \leq s \neq t \leq n\) appear. As \(\alpha\) is a homotopy
equivalence it induces an isomorphism on the indecomposables. So \(a_{j1}, \ldots, a_{jn}\) cannot be all nil. Equating the coefficients in (24) and (21) leads to

\[
a_{jt}^2 a_{jt} = 0, \quad \gamma_{j1} = \gamma_{j2} = 0, \quad 1 \leq s \neq t \leq n, \quad 1 \leq j \leq n. \tag{25}
\]

As a result if \(a_{1s} \neq 0\), then \(a_{1j} = 0\) for every \(1 \leq s \neq j \leq n\). It follows that \(\alpha(w_1) = a_{1s}w_s\) and

\[
\alpha(\partial(z_1)) = a_{1s}^3 w_s^3 + a_{1s} w_s \alpha(w_2 x_2^4) + \sum_{\tau=1}^{k} a_{1s} w_s \alpha(w_\sigma x_2^4) + \alpha(u) + \alpha(x_1^{15}),
\]

\[
\partial(\xi(z_1)) = \lambda_{s1} (w_s^2 + w_s w_{\sigma s}(2)x_2^4 + \sum_{\tau=1}^{k} w_s w_{\sigma s}(i) x_2^4 + u + x_1^{15}) + \cdots. \tag{26}
\]

Notice the relations (23) imply that \(\alpha(u) = \beta_1^9 \beta_2^5 (u)\) and \(\alpha(x_1^{15}) = \beta_1^{15}\). Consequently, equating the coefficients in (26) and using (22), (23), for every \(\tau \leq k\) and for every \(2 \leq j \leq n\) we get

\[
a_{1s}^3 = a_{1s} a_{2\sigma s(2)} \beta_2^4 = a_{1s} a_{i\sigma s(i)} \beta_2^4 = \beta_1^9 \beta_2^5 = \beta_1^{15} = \lambda_{s1}, \quad \lambda_{sj} = 0 \tag{27}
\]

it follows that \(a_{2\sigma s(2)} \neq 0\) and \(a_{i\sigma s(i)} \neq 0\) for every \(\tau \leq k\). So

\[
\alpha(w_2) = a_{2\sigma s(2)} w_2, \quad \alpha(w_{i\sigma}) = a_{i\sigma s(i)} w_{i\sigma}.
\]

Likewise we have

\[
\alpha(\partial(z_2)) = a_{2\sigma s(2)}^3 w_{\sigma s(2)}^3 + a_{2\sigma s(2)} w_{\sigma s(2)} \alpha(w_{\sigma s(2)} x_2^4)
\]

\[
+ \sum_{\tau=1}^{k} a_{2\sigma s(2)} w_{\sigma s(2)} \alpha(w_{\sigma s(2)} x_2^4) + \alpha(u) + \alpha(x_1^{15}),
\]

\[
\partial(\xi(z_2)) = \lambda_{\sigma s(2)} (w_{\sigma s(2)}^3 + w_{\sigma s(2)} w_{\sigma s(2)} x_2^4
\]

\[
+ \sum_{\tau=1}^{k} w_{\sigma s(2)} w_{\sigma s(2)} x_2^4 + u + x_1^{15}) + \cdots. \tag{28}
\]

Due to (19) we can write:

\[
w_{\sigma s(2)}(2) = w_{\sigma s(2)}, \quad w_{\sigma s(2)}(i\sigma) = w_{\sigma s(i\sigma)}, \quad \tau \leq k.
\]

Taking in consideration (25) and comparing the coefficients in (28) we get

\[
a_{2\sigma s(2)}^3 = a_{2\sigma s(2)} a_{\sigma s(2)} \beta_2^4 = a_{2\sigma s(2)} a_{\sigma s(i)} \beta_2^4 = \beta_1^9 \beta_2^5 = \beta_1^{15} = \lambda_{\sigma s(2)},
\]

\[
\lambda_{\sigma s(j)} = 0, \quad 1 \leq j \neq 2 \leq n, \quad \forall \tau \leq k. \tag{29}
\]

Now set \(\sigma_{r(p)}^2(2) = r_p\). By iterating the above process we get

\[
a_{r(p)s(r_p)}^3 = a_{r(p)s(r_p)} a_{s(r_p), \sigma s(r_p)} \beta_2^4 = a_{r(p)s(r_p)} a_{s(r_p), \sigma s(r_p)} a_{s(r_p), \sigma s(r_p)} \beta_2^4
\]

\[
= \beta_1^9 \beta_2^5 = \beta_1^{15} = \lambda_{s(r_p)} r_p, \quad \lambda_{s(r_p)}(j) = 0, \quad 1 \leq j \neq r_p \leq n. \tag{30}
\]

Now comparing (27), (29) and (30), for all \(1 \leq \tau \leq k\) we get

\[
a_{1s} = a_{2\sigma s(2)} = a_{r(p)s(r_p)} = \beta_2 = \beta_1 = \lambda_{s1} = \lambda_{s2}, \quad a_{i\sigma s(i\sigma)} = a_{i\sigma s(i\sigma)}, \quad \lambda_{s(r_p)}(r_p) = 1,
\]

\[
a_{i\sigma s(i\sigma)} = a_{i\sigma s(i\sigma)}, \quad a_{s(i\sigma), i\sigma s(i\sigma)} = a_{s(i\sigma), i\sigma s(i\sigma)} = 1.
\]
Hence, for every \( \tau \leq k \) and \( p \leq \kappa_2 \),
\[
\alpha(w_{\sigma_2^p(2)}) = w_{\sigma_s(\sigma_2^p(i_2))}, \quad \alpha(w_{\sigma_2^p(i_2)}) = w_{\sigma_s(\sigma_2^p(i_2))}.
\]
Finally, due to (18) we can say that every \( 1 \leq j \leq n \) occurs as \( \sigma_2^p(2) \) or \( \sigma_2^p(i_2) \), where \( \tau \leq k \), for a certain \( p \). Therefore from (31) we deduce that \( \alpha(w_j) = \alpha(w_{\sigma_s(j)}) \).

Summarizing, we have proven that if \((\xi, [\alpha]) \in D_{40}^{119}\), then there exists a permutation \( \sigma_s = \begin{pmatrix} 1 & 2 & \cdots & n \\ s & \sigma_s(2) & \cdots & \sigma_s(n) \end{pmatrix} \) such that
\[
\xi(z_j) = z_{\sigma_s(j)}, \quad \alpha(w_j) = w_{\sigma(j)}, \quad \alpha = \text{id}, \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3.
\]
This allows us to define \( \Omega : D_{40}^{119} \to G \) by setting \( \Omega'((\xi, [\alpha])) = g_s \), where the element \( g_s \) corresponds to the permutation \( \sigma_s \) via Cayley theorem. Clearly \( \Omega' \) is the inverse of \( \Omega \) and as \( g_s g_{s'} \) correspond to the permutation \( \sigma_s \circ \sigma_{s'} \) it is easy to see that \( \Omega \) is an isomorphism.

Applying Theorem 2.11, Lemma 3.8 and Proposition 3.9 we deduce that
\[
\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n})) \cong G
\]
and by the Sullivan model there exists a simply connected CW-complex \( X \) such that \( \mathcal{E}(X) \cong G \).

Remark 3.10. The fcca \((\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n}), \partial)\) given in (20), which is not elliptic, is a little modification of the elliptic fcca used by Costoya–Viruel in [9] to show their main result.

Using the arguments of the proof of Proposition 3.9 and omitting the details we can prove that:

Example 3.11. If we define \((\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_n, w_n\}_{n \in \mathbb{Z}}), \partial)\), where \( |x_1| = 8, |x_2| = 10, |w_n| = 40 \) for all \( n \in \mathbb{Z} \), by
\[
\partial(x_1) = \partial(x_2) = \partial(w_n) = 0, \quad \partial(y_1) = x_1^3 x_2, \quad \partial(y_2) = x_1^2 x_2^2, \quad \partial(y_3) = x_1 x_2^3,
\]
\[
\partial(z_n) = w_n^3 + w_n w_{n+1} x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},
\]
then
\[
\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_n, w_n\}_{n \in \mathbb{Z}})) \cong \mathbb{Z}.
\]
Indeed, let \( m \in \mathbb{Z} \). For every \( n \in \mathbb{Z} \), define
\[
\xi_m(z_n) = z_{n+m}, \quad \alpha_m(w_n) = w_{n+m}, \quad \alpha_m = \text{id} \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3,
\]
so that we get a homomorphism:
\[
\Omega : \mathbb{Z} \to D_{40}^{119}, \quad \Omega(m) = (\xi_m, [\alpha_m]).
\]
Now if \((\xi, [\alpha]) \in D_{40}^{119}\), then there is a unique \( m \in \mathbb{Z} \) such that
\[
\xi(z_n) = z_{n+m}, \quad \alpha(w_n) = w_{n+m}, \quad \alpha = \text{id} \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3.
\]
So \( \Omega \) is an isomorphism. Finally, (32) follows by applying Theorem 2.11 and Lemma 3.8.
References


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