TOPOLOGICAL HOCHSCHILD HOMOLOGY OF $K/p$

AS A $K^\wedge_p$ MODULE

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Abstract

For commutative ring spectra $R$, one can construct a Thom spectrum for spaces over $BGL_1 R$. This specialises to the classical Thom spectra for spherical fibrations in the case of the sphere spectrum. The construction is useful in detecting $A_{\infty}$-structures: a loop space (up to homotopy) over $BGL_1 R$ yields an $A_{\infty}$-ring structure on the Thom spectrum. The topological Hochschild homology of these $A_{\infty}$-ring spectra may be expressed as Thom spectra.

This paper uses the identification of topological Hochschild homology of Thom spectra to make computations. Specifically, we take $R$ to be the $p$-adic $K$-theory spectrum and consider a certain map from $S^1$ to $BGL_1 R$, so that the Thom spectrum is equivalent to the mod $p$ $K$-theory spectrum. We make computations at odd primes.

1. Introduction

The goal of this paper is to use generalised Thom spectra to calculate the topological Hochschild homology of $K/p$ in the category of modules over $K^\wedge_p$.

Let $R$ be a ring spectrum and $GL_1 R$ its space of units. It is the $H$-space of homotopy automorphisms of $R$ as an $R$-module. An $R$-twisting of a space $X$ is a continuous map $\zeta$ from $X$ to $BGL_1 R$. Associated to $\zeta$, one can define the Thom spectrum of $\zeta$, $X^\zeta$ (see [2]). This notion specialises for $R = S^0$ to the Thom spectrum of a spherical fibration. The homotopy groups of $X^\zeta$ is the group of twisted $R$ homology classes with respect to the twisting $\zeta$.

Suppose that $R$ is an $E_{\infty}$-ring spectrum. Then its space of units is an infinite loop space. Given a map $f: BG \to B^2GL_1 R$, let $\zeta \simeq \Omega f: G \to BGL_1 R$. Then the Thom spectrum $G^\zeta$ admits an $A_{\infty}$ $R$-algebra structure.

1.1. $K/p$ as a module over $K^\wedge_p$

Suppose that $R = K^\wedge_p$, the spectrum of $p$-adic $K$-theory. Let $G$ be the group $S^1$. A twisting on $S^1$ is a map $\zeta: S^1 \to BGL_1 K^\wedge_p$. This is classified by the group
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$\pi_1(BGL_1K_p^\wedge) \cong \pi_0(GL_1K_p^\wedge) \cong \mathbb{Z}_p^\times$. If we choose $\zeta = 1 - p \in \mathbb{Z}_p^\times$, then the Thom spectrum $(S^1)^\zeta \simeq K/p$, the mod $p$ $K$-theory spectrum. Moreover, the twisting $\zeta$ can be realised as a loop map, and so, for every way of writing $\zeta \simeq \Omega f$ we get an $A_\infty$-ring structure on $K/p$ as an $K_r$-module.

1.2. Topological Hochschild homology of Thom spectra

Given a map $f$ from $X$ to $B^2GL_1R$, let $G \simeq \Omega X$ and $\zeta \simeq \Omega f : G \simeq \Omega X \to BGL_1R$. In this case, the Thom spectrum $G^\zeta$ has an $A_\infty$-ring structure. We write $\eta^* f$ for the composite

$$LX \longrightarrow LB^2GL_1R \xrightarrow{\cong} B^2GL_1R \times BGL_1R \xrightarrow{\eta \times \text{id}} BGL_1R \times BGL_1R \longrightarrow BGL_1R,$$

where $\eta: \Sigma R \to R$ is induced from $S^1 \xrightarrow{\eta} S^0$ via $S^1 \wedge R \to S^0 \wedge R \simeq R$. In the above situation, $THH^R(G^\zeta) \simeq LX^{\ast f}$. The case $R = S^0$ was proved in [5]. The same argument applies for general $R$ [3].

Using this identification of $THH$ as a Thom spectrum, we compute the topological Hochschild homology of $K/p$. For odd primes $p$,

$$\pi_*(THH^{K_p^\wedge}(K/p)) = \begin{cases} \left(\mathbb{Z}/(p^\infty)\right)^i & \text{if } \ast = 2k, \\ 0 & \text{if } \ast = 2k + 1, \end{cases}$$

where $i$ is an integer between 1 and $p - 1$ depending on the choice of $f$ with $\zeta \simeq \Omega f$.

Similar results were obtained before by Angeltveit in [1]. He used the Bökstedt spectral sequence (see [6, Chapter IX]).

We can also form mod $p$ $K$-theory as a Thom spectrum by starting with $X = S^3$, $R = K_p^\wedge$ and $\zeta = p \in \pi_3(BGL_1K_p^\wedge) = \pi_2(GL_1K_p^\wedge) = \mathbb{Z}_p$. Again, this $\zeta$ can be realised as a loop map and we can compute $THH$ of these $A_\infty$-ring structures in an analogous way. This gives the same results.

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2. The Thom spectrum

The notion of a generalised Thom spectrum used here is discussed in detail in [2]. The construction resembles a twisted version of the group ring. Given an extension of a group $G$ by the units in a field $k$,

$$(\tau): 1 \to k^\times \to E \to G \to 1$$

the algebra $k^\tau[G] = \mathbb{Z}[E] \otimes_{\mathbb{Z}[k^\times]} k$ is a twisted group ring. If the extension $\tau$ is trivial, one gets the group ring $k[G]$. Imitating this definition of a twisted group ring for
spectra leads to the construction of the Thom spectrum. One replaces the field $k$ by an $E_\infty$-ring spectrum $R$, and the units $k^*$ by the space of units $GL_1R$ acting on $R$.

### 2.1. The space of units and the Thom spectrum

The space of units of a ring spectrum is a generalisation of the group of units of a commutative ring, the set of invertible elements under multiplication. It is defined to be the components of $\Omega^\infty R$ that lie over the units in $\pi_0(R)$. Following [2], we make the definition:

**Definition 2.1.** Let $R$ be an $E_\infty$-ring spectrum. Its space of units $GL_1R$ is defined to be the homotopy pullback

$$
\begin{array}{ccc}
GL_1R & \longrightarrow & \Omega^\infty(R) \\
\downarrow & & \downarrow \\
\pi_0(R)^\times & \longrightarrow & \pi_0(R). 
\end{array}
$$

It follows from the definition that the homotopy classes of maps from a space $X$ to $GL_1R$ are given by

$$[X, GL_1R] = R^0(X)^\times$$

the units of the cohomology ring $R^0(X) = [X, \Omega^\infty R]$.

From the pullback diagram one can read off the homotopy groups of $GL_1R$:

$$\pi_n(GL_1R) = \begin{cases} 
\pi_n(R) & \text{if } n > 0, \\
\pi_0(R)^\times & \text{if } n = 0. 
\end{cases}$$

We note that $GL_1R$ is an $H$-space for any ring spectrum $R$. If $R$ is $E_\infty$, then $GL_1R$ is an infinite loop space: there is a connective spectrum $gl_1R$ with $0^{th}$-space is $GL_1R$ (Theorem 3.2 in [2]).

We can view $\Omega^\infty R$ as the space of endomorphisms $End_R(R, R)$, in the topological category of $R$-modules, and $GL_1R = Aut_R(R, R) \subset End_R(R, R)$ as the subset of weak equivalences. Therefore, the units $GL_1R$ is the space of homotopy automorphisms of $R$ in the category of $R$-modules. In this way, the infinite loop space $GL_1R$ acts on the spectrum $R$ by weak equivalences, and $R$ is a module over the $E_\infty$ ring spectrum $\Sigma^\infty GL_1R$.

**Definition 2.2.** Given a map $\zeta: X \to BGL_1R$, let $P$ be the $GL_1R$ bundle classified by $\zeta$ described as the pullback

$$
\begin{array}{ccc}
P & \longrightarrow & EGL_1(R) \\
\downarrow & & \downarrow \\
X & \longrightarrow & BGL_1(R) \\
\zeta & & 
\end{array}
$$

and define the associated Thom spectrum to be

$$X^\zeta = \Sigma^\infty P_+ \wedge^L \Sigma^\infty GL_1(R)_+R.$$

In the above $\wedge^L$ denotes the derived smash product in the category of modules over the $E_\infty$-ring spectrum $\Sigma^\infty GL_1R$ as in [6]. We note from Section 7 of [2], that
the Thom spectrum functor commutes with homotopy colimits, and from Section 8.6 of [2] that it generalises the classical Thom spectrum of a spherical fibration.

The Thom spectrum of the map \(* \to BGL_1 R\) is weakly equivalent to \(R\), since the universal bundle associated to the inclusion of a point in \(BGL_1 R\) is isomorphic to \(GL_1 R\) and \(\Sigma^\infty GL_1 R_+ \wedge_{\Sigma^\infty GL_1 R_+} R \simeq R\).

Similarly, the Thom spectrum of a map \(X \to BGL_1 R\) which is null homotopic is weakly equivalent to \(R \wedge X_+\). Indeed, the universal bundle associated to the constant map is \(X \times GL_1 R\). Then the Thom spectrum is \(\Sigma^\infty(X \times GL_1 R)_+ \wedge_{\Sigma^\infty GL_1 R_+} R \simeq (\Sigma^\infty X_+ \wedge \Sigma^\infty GL_1 R_+) \wedge_{\Sigma^\infty GL_1 R_+} R \simeq R \wedge X_+\).

Proposition 2.3. Suppose that \(\zeta\) is a map from \(X \simeq \Sigma Y\) to \(BGL_1 R\). Then, the Thom spectrum \(X^\zeta\) is equivalent to the homotopy colimit of \((R \leftarrow R \wedge Y_+ \to R)\) where one of the maps is the projection \(p_Y\) and the other is \(u_{\zeta}\).

Proof. The space \(X\) is the homotopy colimit of \(* \leftarrow Y \to *\), and this gives a homotopy pushout square of Thom spectra

\[
\begin{array}{ccc}
Y^\zeta & \to & *^\zeta \\
\downarrow & & \downarrow \\
*^\zeta & \to & (\Sigma Y)^\zeta.
\end{array}
\]

The Thom spectrum \(*^\zeta\) is weakly equivalent to \(R\) and \(Y^\zeta \simeq R \wedge Y_+\), so the homotopy pushout can be written as

\[
\begin{array}{ccc}
R \wedge Y_+ & \to & R \\
\downarrow & & \downarrow \\
R & \to & (\Sigma Y)^\zeta.
\end{array}
\]

From this, one obtains a Mayer Vietoris sequence for calculating the homotopy groups

\[
\ldots \to \pi_*(R \wedge Y_+) \to \pi_*(R) \oplus \pi_*(R) \to \pi_*((\Sigma Y)^\zeta) \ldots .
\]

To compute the maps in this sequence, one must examine the \(GL_1 R\)-bundle over \(X \simeq \Sigma Y\). This restricts to trivial bundles over the two copies of the cone of \(Y\) inside \(X\) and on their intersection \(Y\), the bundles are identified via the map \(\tilde{\zeta} : Y \to GL_1 R\).

In the long exact sequence, there are two maps \(R_*(Y_+) \to \pi_*(R)\). One of these maps is given by the map from \(Y\) to a point \((p_Y)\) and the other is the map \(u_{\zeta}\) defined in the preceding paragraph.

Remark 2.4. The proposition describes the homotopy groups of the Thom spectrum as twisted \(R\)-homology groups. An \(R\)-twisting on a space \(X\) can be defined as a 1-cocycle in the sheaf (of groupoids) \(- \{\text{units in } R^0(X)\}\). The groupoid of units in \(R^0\) is classified by the units \(GL_1 R\), and therefore, 1-cocycles on \(X\) are equivalent to \([X, BGL_1 R]\). Therefore, a twisting is given by a continuous map \(\zeta\) from \(X\) to \(BGL_1 R\).
For $X = \bigcup U_i$ a 1-cocycle defines units over $U_i \cap U_j$ satisfying a cocycle condition on further intersections. A twisted $R$ homology class is an element in each $R_*(U_i)$, two of which are identified using the values of the 1-cocycle on the intersections. The abelian group of these classes is defined to be the twisted $R$-homology of $X$ with respect to the twisting $\zeta$. This is isomorphic to the homotopy groups of the Thom spectrum $X^\zeta$. The proposition above verifies this in the case $X = \Sigma Y$, where $X$ is the union of two contractible open sets.

2.2. Computations of some Thom spectra

**Proposition 2.5.** Suppose that $\zeta: S^1 \to BGL_1 K^\wedge_p$ represents

$$1 - p \in \pi_1(BGL_1(K^\wedge_p)) = \pi_0(GL_1(K^\wedge_p)) = \mathbb{Z}_p.$$  

Then, $(S^1)^\zeta \simeq K/p$.

**Proof.** By Proposition 2.3 with $Y = S^0$, the Thom spectrum is a homotopy pushout

$$K^\wedge_p \vee K^\wedge_p \to K^\wedge_p \to (S^1)^\zeta.$$  

Therefore, there is a cofibre sequence

$$K^\wedge_p \vee K^\wedge_p \to (S^1)^\zeta.$$  

Proposition 2.3 also identifies the left map in the sequence in suitable coordinates, to be given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 - p \end{pmatrix}.$$  

Therefore, the cofibre sequence can be rewritten as

$$K^\wedge_p \overset{p}{\to} K^\wedge_p \to (S^1)^\zeta,$$

so that $(S^1)^\zeta \simeq K^\wedge_p/p \simeq K/p$. \hfill \Box

**Remark 2.6.** Consider the map $\zeta: S^1 \to BGL_1((S^0)^\wedge_p)$ given by $(1 - p)$ as in the previous proposition. Then, $(S^1)^\zeta \simeq (S^0)^\wedge_p/p \simeq M_p$ is the mod $p$ Moore spectrum. In fact, for any $\zeta: S^1 \to BGL_1 R$, $(S^1)^\zeta \simeq cofibre(1 - \zeta: R \to R)$. This follows from the argument above.

**Proposition 2.7.** Let $\zeta: S^3 \to BGL_1 K^\wedge_p$ represent the element $p$ of

$$[S^3, BGL_1(K^\wedge_p)] = \pi_3(BGL_1(K^\wedge_p)) = \pi_2(GL_1(K^\wedge_p)) = \pi_2(K^\wedge_p) \cong \mathbb{Z}_p.$$  

Then $(S^3)^\zeta \simeq K/p$.

**Proof.** The space $S^3$ is homotopy equivalent to the suspension of $S^2$. Proposition 2.3
implies the homotopy pushout
\[ K^\wedge_p \wedge S^2 \xrightarrow{\iota} K^\wedge_p \]
\[ \xrightarrow{\delta} K^\wedge_p \xrightarrow{} (S^3)^\zeta \]
and the associated Mayer Vietoris cofibre sequence
\[ K^\wedge_p \wedge (S^2) \vee K^\wedge_p \to K^\wedge_p \vee K^\wedge_p \to (S^3)^\zeta. \]
In suitable coordinates, the map in the Mayer Vietoris sequence is given by the matrix
\[
\begin{pmatrix}
1 & 0 \\
1 & p
\end{pmatrix}
\]
and the sequence can be rewritten as
\[ \Sigma^2 K^\wedge_p \xrightarrow{p} K^\wedge_p \xrightarrow{} (S^3)^\zeta. \]
By Bott periodicity \( \Sigma^2 K^\wedge_p \simeq K^\wedge_p \) so that \( (S^3)^\zeta \simeq K^\wedge_p / p \), as claimed.

### 2.3. Ring structures

Suppose \( R \) is an \( E_\infty \)-ring spectrum so that \( GL_1 R \) is an infinite loop space. Given \( f: X \to B^2 GL_1 R \), and \( \zeta: G \simeq \Omega X \xrightarrow{} BGL_1 R \), the Thom spectrum \( G^\wedge \) has an \( A_\infty \)-ring structure. This follows from [3] where it is proved that the Thom spectrum functor is symmetric monoidal (Proposition 4.10) and loop maps rectify to monoids over an appropriate model of \( BGL_1 R \) (Appendix A). This raises the question when a map
\[ \zeta: G \to BGL_1 R \]
from a monoid \( G \) is homotopy equivalent to a loop map, i.e., \( \zeta \simeq \Omega f \) for
\[ f: BG \to B^2 GL_1 R. \]
We have the standard maps
\[ \Sigma G \xrightarrow{\sigma} BG, \quad \Sigma GL_1 R \xrightarrow{\sigma} BGL_1 R, \]
so the question is if
\[ \sigma \circ \Sigma \zeta: \Sigma G \to B^2 GL_1 R \]
extends over \( BG \):
\[ \begin{tikzcd}
\Sigma G \arrow{r}{\Sigma \zeta} \arrow{d}{\sigma} & \Sigma BGL_1(R) \arrow{d}{\sigma} \\
BG \arrow{r}{f} & B^2 GL_1(R).
\end{tikzcd} \]

**Proposition 2.8.** Let \( G = S^1 \), \( R = K^\wedge_p \) and \( \zeta = 1 - p \) as in Proposition 2.5, then \( (S^1)^\zeta \simeq K/p \) has an \( A_\infty \)-ring structure.
Proof. The classifying space of $S^1$ is $CP^\infty$ so, in this case, the diagram above is

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\Sigma 1-p} & \Sigma BGL_1(K_p) \\
\downarrow & & \downarrow \\
CP^\infty & \xrightarrow{f} & B^2 GL_1(K_p) \\
\end{array}
$$

The space $CP^\infty$ has a CW structure made of even dimensional cells so that all the cells are attached along odd dimensional spheres. The spectrum $K_p^\wedge$ has non trivial homotopy groups only in even dimensions and hence, so does $B^2 GL_1 K_p^\wedge$. Thus, all the obstructions to extending the map $\Sigma 1 - p$ must vanish, which implies that there is an $A_\infty$-ring structure on the Thom spectrum $K/p$.

**Proposition 2.9.** Suppose that $G = S^3$, $R = K_p^\wedge$, and $\zeta = p$ as in Proposition 2.7, then the Thom spectrum has an $A_\infty$-ring structure.

Proof. The classifying space of $S^3$ is the infinite quarternionic projective space $HP^\infty$, and $\Sigma S^3 = S^4 \to BS^3 = HP^\infty$ is obtained by attaching even cells along maps of odd dimensional spheres. Therefore the extension problem can always be solved.

3. Topological Hochschild homology of Thom spectra

In the last section, we observed that the Thom spectrum of a loop map carries an induced $A_\infty$ structure. In this setting, there is a convenient description of the topological Hochschild homology as a Thom spectrum along the ideas of [5, 3] and [11]. In the following $G$ will be a group, $X$ a space, and $G$ homotopy equivalent to $\Omega X$ as $A_\infty$-spaces. $R$ will be an $E_\infty$ ring spectrum.

The Thom spectrum of a map $G \to BGL_1 R$ is a twisted $R$-module generated by $G$. If this is a loop map, the construction is that of a twisted group ring. Recall that the Hochschild Homology of group rings over a field is given by

$$HH_*(k[G]) \cong k \otimes H_*(G, G),$$

where $G$ acts on itself by conjugation. This is the homology of the Borel construction $G_{hG} \cong EG \times_G G \simeq LBG$, the free loop space of $BG$, and so, $HH_*(k[G]) \cong k \otimes H_*(LBG)$. The analogous statement for topological Hochschild homology is the classical result of Bökstedt and Waldhausen:

$$THH(\Sigma^\infty \Omega X_+) \cong \Sigma^\infty LX_+.$$

In the category of $R$-modules, the theorem is $THH^R(R \wedge \Omega X_+) \cong R \wedge LX_+$, computing the topological Hochschild homology of the Thom spectrum of the constant map. More generally, let $f : X \to BGL_1 R$ and $\zeta \simeq \Omega f : G \to BGL_1 R$, the Thom spectrum has an $A_\infty$-ring structure, and the topological Hochschild homology is the Thom spectrum of a map from $LX$ to $BGL_1 R$.

In the second part of the section, we apply the theorem for $R = K_p^\wedge$ and $G = S^1$, in the computation of the previous section. This implies that the Thom spectrum is homotopy equivalent to the cofibre of a certain map $K_p^\wedge \wedge CP^\infty_+ \to K_p^\wedge \wedge CP^\infty_+$. 
3.1. Identifying topological Hochschild homology as a Thom spectrum

Recall that the free loop space $LY$ fits into a fibration

$$\Omega Y \to LY \to Y.$$ 

If $Y$ is an $H$-space, then the fibration splits as $LY \simeq Y \times \Omega Y$. This is an equivalence of $H$-spaces if $Y$ is homotopy commutative.

Let $f$ be a map from $X$ to $B^2GL_1R$ and $\eta: B^2GL_1R \to \Omega B^2GL_1R$ be induced from the Hopf map by

$$B^2GL_1R \simeq Maps(S^2, B^4GL_1R) \xrightarrow{\eta^*} Maps(S^3, B^4GL_1R) \simeq Maps(S^1, \Omega B^4GL_1R) \simeq Maps(S^1, B^2GL_1R) \simeq \Omega B^2GL_1R.$$ 

Let $L^nf$ be the map from $LX$ to $BGL_1R$ defined by the diagram

$$\begin{array}{c}
LX \\ \downarrow L^n f \\
LB^2GL_1(R) \\ \downarrow \eta \times id \\
B^2GL_1(R) \times \Omega B^2GL_1(R) \\ \downarrow \simeq \\
\Omega B^2GL_1(R) \\ \downarrow \simeq \\
BGL_1(R).
\end{array}$$ 

The map $\eta \times id: B^2GL_1R \times \Omega B^2GL_1R \to \Omega B^2GL_1R$ is the product of the maps $\eta$ and $id$ using the $H$-space structure of $\Omega B^2GL_1R$. Without proof, we state:

**Theorem 3.1.** There is a homotopy equivalence

$$THH^R(G_\wedge) \simeq (LX)^{L^n f}.$$ 

This was proved in the case of the sphere spectrum in [5, 11]. The argument for general $E_\infty$-ring spectra $R$ is given in [3, Theorem 1.7].

3.2. The example of $G = S^1$ and $R = K_\wedge_p$

By Proposition 2.8, we have the commutative diagram

$$\begin{array}{c}
S^2 \\ \downarrow \sigma \\
\Sigma BGL_1(K_\wedge_p) \\ \downarrow \sigma \\
CP^\infty \xrightarrow{-f} B^2GL_1(K_\wedge_p)
\end{array}$$ 

and write $THH^{K_\wedge_p}(K/p, f)$ for the topological Hochschild homology corresponding to this $A_\infty$-ring structure.
Proposition 3.2.

\[ \text{THH}^K\wedge (K/p, f) \simeq (\text{LCP} \sim) \hat{f}, \]

where \( \hat{f} \) is the composite

\[ \text{LCP} \to \text{LB}^2 \text{GL}_1 K^\wedge \simeq B^2 \text{GL}_1 K^\wedge \times \text{BGL}_1 K^\wedge \to B\text{GL}_1 K^\wedge. \]

Proof. By Theorem 3.1, \( \text{THH}^K\wedge (K/p, f) \simeq (\text{LCP} \sim) \eta \). Since \( \pi_1(K^\wedge) = 0 \), \( \eta = 0 \) in this case. Hence, the proposition.

The focus of the rest of the paper will be the calculation of \( \pi_* ((\text{LCP} \sim) \hat{f}) \simeq \text{THH}^K\wedge (K/p, f) \). First of all we note that:

Proposition 3.3. There is a long exact sequence

\[ K^\wedge_\ast CP \to K^\wedge_\ast CP \to \pi_* \text{THH}^K\wedge (K/p, f) \to K^\wedge_\ast CP \to \ldots. \]

Proof. Note that \( CP \) is an infinite loop space, and hence homotopy commutative, which implies that \( \text{LCP} \sim \simeq \Omega CP \times CP \simeq S^1 \times CP \). The space \( S^1 \) is a union of two contractible open sets whose intersection is \( S^0 \), so, there is a homotopy pushout

\[ \xymatrix{ CP \coprod CP \ar[r] \ar[d] & CP \ar[d] \\
CP \ar[r] & \text{LCP} \sim } \]

and hence, a homotopy pushout square of Thom spectra

\[ \xymatrix{ (CP \coprod CP) \hat{f} \ar[r] \ar[d] & (CP) \hat{f} \ar[d] \\
(CP) \hat{f} \ar[r] & (LCP) \hat{f} } \]

The two maps \( CP \to \text{LCP} \sim \) in (3.1) are the inclusion of constant loops, so, the two compositions \( CP \to \text{LCP} \sim \to \text{LB}^2 \text{GL}_1 K^\wedge \to \text{BGL}_1 K^\wedge \) are nullhomotopic and the Thom spectra are \( \simeq K^\wedge \wedge CP^\sim. \) The map from \( CP \coprod CP \) to \( \text{BGL}_1 K^\wedge \) factors through \( CP \to \text{BGL}_1 K^\wedge \), so, the Thom spectrum \( (CP \coprod CP) \hat{f} \simeq K^\wedge \wedge CP^\sim \vee K^\wedge \wedge CP^\sim. \) Therefore, the pushout can be written as

\[ \xymatrix{ K^\wedge \wedge CP^\sim \vee K^\wedge \wedge CP^\sim \ar[r] \ar[d] & K^\wedge \wedge CP^\sim \ar[d] \\
K^\wedge \wedge CP^\sim \ar[r] & (LCP) \sim \hat{f}. } \]

This gives a Mayer Vietoris sequence on homotopy groups

\[ \ldots \to K^\wedge_\ast (CP^\sim) \oplus K^\wedge_\ast (CP^\sim) \to K^\wedge_\ast (CP^\sim) \oplus K^\wedge_\ast (CP^\sim) \to \pi_* ((LCP) \sim ) \ldots \]

To simplify, one needs to understand the left hand map i.e., how \( K^\wedge \wedge CP^\sim \vee K^\wedge \wedge CP^\sim \) maps to the two different copies of \( K^\wedge \wedge CP^\sim \) in the pushout square. For
that one needs to examine the structure of \( P^\wedge \), the \( GL_1 K_p^\wedge \)-bundle over \( S^1 \times CP^\infty \) classified by \( \tilde{f} \).

Following the pushout square (3.1), we see that \( P^\wedge \) is obtained by identifying two trivial bundles over \( CP^\infty \) after restricting over \( CP^\infty \sqcup CP^\infty \), via a map \( u: CP^\infty \sqcup CP^\infty \to GL_1 K_p^\wedge \). The adjoint of \( u \) is the map \( \tilde{u} \) in the diagram:

\[
\begin{array}{cccc}
CP^\infty \sqcup CP^\infty & \longrightarrow & CP^\infty \vee CP^\infty & \longrightarrow & S^1 \times CP^\infty & \longrightarrow & \Sigma CP^\wedge_+ \vee \Sigma CP^\wedge_+ \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \longrightarrow & BGL_1 K_p^\wedge.
\end{array}
\]

The top row is the cofibre sequence associated to the pushout (3.1). Since the map \( S^1 \times CP^\infty \to BGL_1 K_p^\wedge \) is nullhomotopic on \( CP^\infty \vee CP^\infty \), it factors through \( \Sigma CP^\wedge_+ \vee \Sigma CP^\wedge_+ \) as \( \tilde{u} \).

The map \( u \) gives two units \( u_1, u_2 \) in the \( K_p^\wedge \) of \( CP^\infty \). In the Mayer Vietoris sequence for the Thom spectrum, these describe the map \( K_p^\wedge \wedge CP^\wedge_+ \vee K_p^\wedge \wedge CP^\wedge_+ \to K_p^\wedge \wedge CP^\wedge_+ \vee K_p^\wedge \wedge CP^\wedge_+ \) as the matrix

\[
\begin{pmatrix}
1 & u_2 \\
0 & 1
\end{pmatrix}.
\]

In fact, \( u_1 \) and \( u_2 \) are equal because each summand in \( \Sigma CP^\wedge_+ \) of \( \Sigma CP^\wedge_+ \vee \Sigma CP^\wedge_+ \) is the cofibre of the map \( CP^\infty \to LCP^\infty = S^1 \times CP^\infty \) given by the inclusion of the constant loops and both can be defined by the same diagram

\[
\begin{array}{cccc}
CP^\infty & \longrightarrow & S^1 \times CP^\infty & \longrightarrow & BGL_1(K_p^\wedge) \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \longrightarrow & \Sigma CP^\wedge_+.
\end{array}
\]

In terms of \( u \), we can rewrite the Mayer Vietoris sequence as the long exact sequence

\[
\cdots \longrightarrow K_{p_+}((CP^\infty)^{u-1}) K_{p_+}(CP^\infty) \longrightarrow \pi_*(((LCP^\infty)^{L\tilde{f}}) \longrightarrow \cdots \quad (\alpha)
\]

which proves the required statement.

To calculate \( \pi_*(THH_{K^\wedge}(K/p, f)) \), it remains to understand the map \( u \). This is done as follows:

**Proposition 3.4.** The adjoint of the map \( u: \Sigma CP^\wedge_+ \to BGL_1 R \), is homotopy equivalent to the composite \( \Sigma^2 CP^\wedge_+ \mu \to CP^\wedge \tilde{f} \to B^2 GL_1 K_p^\wedge \), where \( \mu \) is the composition \( \Sigma^2 CP^\wedge_+ \simeq S^2 \wedge CP^\wedge_+ \sigma \to CP^\wedge \wedge CP^\wedge_+ \to CP^\wedge \).
Proof. The following diagram commutes:

\[
\begin{array}{ccc}
S^1 \wedge (S^1 \times C P^\infty) & \xrightarrow{\sim} & S^1 \wedge L C P^\infty \\
& & \xrightarrow{S^1 \wedge L f} S^1 \wedge L B^2 G L_1(K_p^\wedge) \\
& ev & ev \\
C P^\infty & \xrightarrow{f} & B^2 G L_1(K_p^\wedge).
\end{array}
\]

Consider the inclusion of the based loops \( B G L_1 K_p^\wedge \hookrightarrow L B^2 G L_1 K_p^\wedge \). Under the composite

\[ S^1 \times B G L_1 K_p^\wedge \rightarrow S^1 \times L B^2 G L_1 K_p^\wedge \rightarrow B^2 G L_1 K_p^\wedge, \]

the copies \( S^1 \times * \) and \(* \times B G L_1 K_p^\wedge\) map trivially. Thus, it factors through \( S^1 \wedge B G L_1 K_p^\wedge \) as \( \Sigma B G L_1 K_p^\wedge \rightarrow B^2 G L_1 K_p^\wedge \). We are trying to figure out the map

\[ S^1 \times L C P^\infty \rightarrow S^1 \times L B^2 G L_1 K_p^\wedge \rightarrow S^1 \times B G L_1 K_p^\wedge \rightarrow B^2 G L_1 K_p^\wedge. \]

Then, this factors through

\[ S^1 \wedge L C P^\infty \rightarrow S^1 \wedge L B^2 G L_1 K_p^\wedge \rightarrow S^1 \wedge B G L_1 K_p^\wedge \rightarrow B^2 G L_1 K_p^\wedge. \]

Also \( L C P^\infty \rightarrow B G L_1 K_p^\wedge \) factors through \( S^1 \wedge C P^\infty \) as \( u \). Putting all the remarks together, we have a commutative diagram

\[
\begin{array}{ccc}
S^2 \wedge C P^\infty & \xrightarrow{\Sigma u} & S^1 \wedge B G L_1 K_p^\wedge \\
& & \downarrow \Sigma f \\
S^1 \wedge (S^1 \times C P^\infty) & \xrightarrow{\sim} & S^1 \wedge L C P^\infty \\
& ev & \xrightarrow{S^1 \wedge L f} S^1 \wedge L B^2 G L_1(K_p^\wedge) \\
& & ev \\
C P^\infty & \xrightarrow{f} & B^2 G L_1(K_p^\wedge).
\end{array}
\]

The left hand vertical map from \( S^2 \wedge C P^\infty \) to \( S^1 \wedge (S^1 \times C P^\infty) \) is the inclusion of a factor in the splitting of the suspension of \( S^1 \wedge (S^1 \times C P^\infty) \simeq (S^2 \wedge C P^\infty) \vee (S^1 \wedge C P^\infty) \). It follows that \( \tilde{u} \simeq \sigma \circ \Sigma u \simeq f \circ g \), where

\[ g: S^2 \times \Sigma C P^\infty \rightarrow S^1 \wedge (S^1 \times C P^\infty) \simeq S^1 \wedge L C P^\infty \rightarrow C P^\infty \]

and the composition \( g \simeq \mu \).

4. The structure of \( G L_1(K_p^\wedge) \)

In this section, we prove a splitting of \( G L_1(K_p^\wedge) \) using the logarithm \( l_p: gl_1 K_p^\wedge \rightarrow K_p^\wedge \) defined by Rezk (see [10]). Throughout this section, we assume that \( p \) is an odd prime.
Proposition 4.1 (Rezk, [10]). Let $R$ be an $E_{\infty}$ ring spectrum. Then there is a logarithmic cohomology operation, $l_{p,n}$, from $gl_1(R)$ to $L_{K(n)}(R)$ for every $n$, and prime $p$. If $R$ is $K(n)$-local, this is a map from $gl_1(R)$ to $R$. When $n = 1$, $l_p : gl_1R \to R$ is given by the formula:

$$l_p(x) = -\frac{1}{p} \log \left( \frac{\psi(x)}{x^p} \right).$$

[Recall that a $\theta$-algebra structure is described by operations $\psi$ and $\theta$ ($\psi$ is a ring homomorphism) such that $\psi(x) = x^p + p\theta(x)$.]

Proposition 4.2. Suppose that $R = K_p^{\wedge}$. The operation $l_p : gl_1K_p^{\wedge} \to K_p^{\wedge}$ factors through $ku_p^{\wedge}$, the connective cover of $K_p^{\wedge}$. On homotopy groups, the map is an isomorphism on $\pi_n$ for $n > 2$. At $n = 2$, it is 0. And for $n = 0$, this is the map

$$Z_p^\times \cong Z/(p-1) \times Z_p \overset{p^2}{\to} Z_p.$$

Proof. The spectrum $K_p^{\wedge}$ is $K(1)$-local, and the operation $\psi$ is the Adams operation $\psi_p$. Since $gl_1K_p^{\wedge}$ is connective, the map $l_p$ factors through $ku_p^{\wedge}$. Recall, that the homotopy groups of $gl_1K_p^{\wedge}$ are given by

$$\pi_n(gl_1K_p^{\wedge}) = \begin{cases} (K_p^{\wedge,0}(S^n))^\times = \pi_n(K_p^{\wedge}) & \text{if } n > 0, \\ (K_p^{\wedge,0}(S^0))^\times = \pi_0(K_p^{\wedge})^\times & \text{if } n = 0. \end{cases}$$

Since $\pi_nK_p^{\wedge}$ is nonzero only for even $n$, it suffices to restrict our attention to even dimensional spheres. The $K$-theory of $S^{2n}$ is generated by $\epsilon$ where for the map $p : (S^2)^n \to S^{2n}$ which quotients out the lower cells, $\epsilon$ splits as the product

$$p^*(\epsilon) = \prod (1 - L_i),$$

where $L_i$ is the canonical line bundle over the $i^{th}$ copy of $S^2 = CP^1$. We have

$$\pi_{2n}(gl_1(K_p^{\wedge})) = gl_1(K_p^{\wedge})^0(S^{2n}) = (\widehat{K_p^{\wedge}}(S^{2n}))^\times = 1 + \epsilon \pi_{2n}(K_p^{\wedge}).$$

To calculate $l_p$ on $\pi_{2n}gl_1K_p^{\wedge}$, one needs to compute $l_p(1 + k\epsilon)$ for $1 + k\epsilon$ in $gl_1K_p^{\wedge}(S^{2n}) = \pi_0(gl_1(K_p^{\wedge}S^{2n}))$. To accomplish this, we need to calculate $\psi_p(\epsilon)$. We do this by calculating $\psi_p(p^*(\epsilon))$ and using that $p^*$ induces an injection in $K$-theory. Since the Adams operation $\psi_p$ raises line bundles to the $p^{th}$ power,

$$\psi_p(L_i) = L_i^p \implies \psi_p(1 - L_i) = 1 - L_i^p = 1 - (1 - (1 - L_i))^p.$$  

The element $1 - L_i$ lies in the $K$-theory of $S^2$, so it squares to 0. Therefore,

$$\psi_p(1 - L_i) = 1 - (1 - p(1 - L_i)) = p(1 - L_i) \implies \psi_p(\epsilon) = p^n \epsilon \implies \psi_p(1 + \epsilon) = 1 + p^n \epsilon.$$
Hence,
\[ l_p(1 + k\epsilon) = -\frac{1}{p} \log \left( \frac{\psi(1 + k\epsilon)}{(1 + k\epsilon)^p} \right) \]
\[ = -\frac{1}{p} \log \left( \frac{1 + p^n k\epsilon}{(1 + k\epsilon)^p} \right) \]
\[ \equiv -\frac{1}{p} \log(1 + (p^n - p)k\epsilon) \pmod{p}, \]
which becomes multiplication by \(1 - p^{n-1} \pmod{p}\) if \(n > 0\). Since the homotopy group \(\pi_{2n}(gl_1 K_p^\wedge) = Z_p\) for \(n > 0\), this is an isomorphism for \(n > 1\). For \(n = 1\), this map is 0. For \(n = 0\), the map \(l_p: Z_p^\times \cong \mu_{p-1} \times Z_p \to Z_p\) is given by
\[ -\frac{1}{p} \log(x^{1-p}). \]
This map has kernel \(\nu_{p-1}\), the group of \((p-1)^{st}\) roots of unity, as it takes \(p\)-adic integers of the form \(1 + pk\) to
\[ l_p(1 + pk) = -\frac{1}{p} \log((1 + pk)^{1-p}) \]
\[ = -\frac{1}{p} \log(1 + p(1 - p)k) \]
\[ = -(1 - p)k + O(p) \]
\[ \equiv -k \pmod{p}. \]
Therefore, the map \(l_p\) on \(Z_p^\times = \nu_{p-1} \times Z_p\), has kernel \(\nu_{p-1}\) and is an isomorphism ontto \(Z_p\). \(\square\)

Recall that the spectrum \(ku_p^\wedge\) splits into Adams summands,
\[ ku_p^\wedge \simeq B \vee \Sigma^2 B \ldots \Sigma^{2^{g-4}} B, \]
where \(B\) is the \(p\)-adic Adams summand(\(\pi_*(B) = Z_p[v_1]\)). Using this, we identify the image of the logarithmic cohomology operation. We construct \(K_p(\hat{2})\) from the spectrum \(ku_p^\wedge\) by killing the \(2^{nd}\) homotopy group:

**Definition 4.3.** Let \(B_2\) be the 2-connective cover of \(B\). Define
\[ K_p(\hat{2}) = B \vee \Sigma^2 B_2 \ldots \vee \Sigma^{2^{g-4}} B. \]

**Proposition 4.4.** There is a split cofibre sequence
\[ H\nu_{p-1} \vee \Sigma^2 HZ_p \to gl_1 (K_p^\wedge) \to K_p(\hat{2}). \]

**Proof.** From the definition above, note that \(gl_1 K_p^\wedge \overset{\nu}{\to} ku_p^\wedge \to K_p(\hat{2})\) is surjective on homotopy groups. The fibre \(F\) has homotopy only in dimensions 0 and 2. The Postnikov tower of \(F\) then is a cofibre sequence
\[ \Sigma^2 HZ_p \to F \to H\nu_{p-1} \to \Sigma^3 HZ_p. \]
Since the group \(H^3(H\nu_{p-1}; Z_p) = 0\), the sequence splits and one obtains
Hence, the cofibre sequence splits and

\[ H_{\nu p-1} \vee \Sigma^2 HZ_p \rightarrow gl_1(K_p^\wedge) \rightarrow K_p(\hat{2}). \]

The next term in this sequence is

\[ \Sigma(H_{\nu p-1} \vee \Sigma^2 HZ_p) \simeq \Sigma H_{\nu p-1} \vee \Sigma^3 HZ_p \]

and the next map is \( K_p(\hat{2}) \rightarrow \Sigma H_{\nu p-1} \vee \Sigma^3 HZ_p \). Since the spaces in the Adams summands are retracts of \( bu_p \), their homology concentrated in even dimensions. Therefore,

\[ [\Sigma^2 B, \Sigma H_{\nu p-1} \vee \Sigma^3 HZ_p] \cong H^1(B; \nu_{p-1}) \oplus H^3(B; Z_p) \cong 0. \]

Since the spectrum \( B_2 \) is 3-connected,

\[ [\Sigma^2 B_2, \Sigma H_{\nu p-1} \vee \Sigma^3 HZ_p] \cong H^{-1}(B_2; \nu_{p-1}) \oplus H^1(B_2; Z_p) \cong 0 \]

\[ \implies [K_p(\hat{2}), H^1(B; \nu_{p-1}) \oplus H^3(B; Z_p)] = 0. \]

Hence, the cofibre sequence splits and

\[ gl_1(K_p^\wedge) \simeq K_p(\hat{2}) \vee H_{\nu p-1} \vee \Sigma^2 HZ_p \]

completing the proof. \( \square \)

We will use this decomposition later to calculate homotopy classes of extensions. For that, we also have to understand how the splitting looks like when we map a space \( X \) to \( GL_1(K_p^\wedge) \). Recall, \([X, GL_1(K_p^\wedge)] = K_p^{\wedge 0}(X)^\times \). The map \( l_p \) gives the way to map this to \([X, K_p(\hat{2})] \). The map \( K_p^{\wedge 0}(X)^\times \rightarrow H^0(X; \nu_{p-1}) \) is the composite

\[ X \rightarrow GL_1(K_p^\wedge) \rightarrow \pi_0GL_1(K_p^\wedge) \cong Z_p^\times \cong \nu_{p-1} \times Z_p \rightarrow \nu_{p-1} \simeq K(\nu_{p-1}, 0). \]

The third factor is \( \Sigma^2 HZ_p \), and we have to understand the map from \( H^2(X; Z_p) \) to \( K_p^{\wedge 0}(X)^\times \). Now, \( H^2(X; Z_p) = [X, K(Z_p, 2)] = [X, CP_{p \infty \wedge}] \). The space \( CP_{p \infty} \) classifies line bundles which are invertible elements in \( K \)-theory.

**Proposition 4.5.** The map \( H^2(X; Z_p) \rightarrow K_p^{\wedge 0}(X)^\times \) is given by \( f \in [X, CP_{p \infty \wedge}] \mapsto L^f \) where \( L^f \) is the line bundle classified by \( f \).

**Proof.** The formula in the statement of the proposition defines a map of infinite loop spaces \( CP_{p \infty \wedge} \rightarrow GL_1K_p^\wedge \), and hence, a map of spectra \( \Sigma^2 HZ_p \rightarrow gl_1K_p^\wedge \). Composing it with \( l_p \), we get

\[
l_p(L^f) = -\frac{1}{p} \log \left( \frac{\psi_p(L^f)}{(L^f)^p} \right) = -\frac{1}{p} \log \left( \frac{(L^f)^p}{(L)^p} \right) = -\frac{1}{p} \log(1) = 0.
\]

The computation above shows that the composition \( \Sigma^2 HZ_p \rightarrow gl_1(K_p^\wedge) \rightarrow K_p(\hat{2}) \) equals 0. Therefore, it factors through \( \nu_{p-1} \times \Sigma^2 HZ_p \) in the diagram:
To complete this proof, we need to show that the map \( \Sigma^2 HZ_p \to H\nu_{p-1} \vee \Sigma^2 HZ_p \to \Sigma^2 HZ_p \) is an equivalence. The only non-zero homotopy group of \( \Sigma^2 HZ_p \) is \( \pi_2 \), so it suffices to check that the map \( [S^2, CP^\infty] \to H^2(S^2; Z_p) \) as described by the statement is an isomorphism. The left group is isomorphic to \( Z_p \), via \( k \mapsto L^k \), \( L = \) the tangent bundle of \( S^2 \). The right group is \( H^2(S^2; Z_p) \approx Z_p \) inside \( K_p^\wedge(S^2) \times \) as elements \( 1 + k\epsilon, \epsilon = 1 - L \). The map between the two is \( L^k \mapsto (1 - \epsilon)^k = 1 - k\epsilon \) because \( \epsilon^2 = 0 \), and is evidently an isomorphism.

\[
\begin{array}{c}
\Sigma^2 HZ_p \\
H\nu_{p-1} \times \Sigma^2 HZ_p \\
gl_1(K_p^\wedge) \\
K_p^\wedge(\hat{2}).
\end{array}
\]

5. Calculation of \( \text{THH} \)

In this section, we complete the computation of \( \text{THH} \) for odd primes \( p \). We first parameterise the homotopy classes of extensions \( f \)

\[
\begin{array}{ccc}
S^2 & \xrightarrow{\Sigma 1-p} & \Sigma BGL_1(K_p^\wedge) \\
\sigma & \downarrow & \downarrow \sigma \\
CP^\infty & \xrightarrow{f} & B^2 GL_1(K_p^\wedge)
\end{array}
\]

using the results of the previous section.

Recall that

\[
GL_1(K_p^\wedge) = \nu_{p-1} \times K(Z_p, 2) \times \Omega^\infty K_p^\wedge(\hat{2})
\]

\[
\implies B^2 GL_1(K_p^\wedge) = B^2 \nu_{p-1} \times K(Z_p, 4) \times \Omega^\infty \Sigma^2 K_p^\wedge(\hat{2}).
\]

The condition on the map \( f \) is that its restriction to \( S^2 \) is \( 1 - p \). The homotopy classes of maps from \( S^2 \) to \( B^2 GL_1(K_p^\wedge) \) is split into three factors:

1. \( [S^2, B^2 \nu_{p-1}] = H^2(S^2; \nu_{p-1}) \approx \nu_{p-1} \),
2. \( [S^2, K(Z_p, 4)] = H^4(S^2; Z_p) = 0 \),
3. \( [S^2, \Omega^\infty \Sigma^2 K_p^\wedge(\hat{2})] = [S^2, \Omega^\infty (\Sigma^2 B \vee \Sigma^4 B_2 \vee \Sigma^6 B \ldots \vee \Sigma^{2p-4} B)] = [S^2, \Omega^\infty \Sigma^2 B] = B^2(S^2) \approx Z_p \).

In the splitting

\[
[S^2, B^2 GL_1(K_p^\wedge)] = \nu_{p-1} \oplus B^2(S^2) \oplus H^4(S^2; Z_p) = \nu_{p-1} \oplus Z_p \oplus 0,
\]

\( 1 - p \) is in the factor \( Z_p \), where it equals \( \ell_p(1 - p) = \alpha_p \) and

\[
\alpha_p = -\frac{1}{p} \log((1 - p)^{1-p})
\]

\[
\equiv -\frac{1}{p} \log(1 - (1 - p)p)
\]

\[
\equiv -1 \pmod{p}.
\]
5.1. Calculation at the prime 3

Let us begin the calculation at the prime 3. The cofibre sequence for \( gl_1 K_3^\wedge \) is

\[
HZ/2 \vee \Sigma^2 HZ_3 \to gl_1(K_3^\wedge) \to K_3(\hat{2})
\]

and

\[
K_3(\hat{2}) = B \vee \Sigma^2 B_2.
\]

Therefore,

\[
GL_1 K^\wedge / p = Z/2 \times K(Z_p, 2) \times \Omega^\infty B \times \Omega^\infty B_2.
\]

We will study the extension to \( \mathbb{CP}^\infty \) of the map \( 1 - p \), to the four factors \( Z/2, K(Z_3, 2), \Omega^\infty B, \Omega^\infty B_2 \) one by one. Let us start with the factor \( B \). The Adams summands are the eigenspaces of the action of the \( (p - 1)^{st} \) roots of unity by Adams operations. The spectrum \( B \) is fixed by all the Adams operations. The projection from \( K_p^{\wedge,*}(X) \) to \( B^*(X) \) is given by

\[
\pi = \frac{1}{p - 1}(1 + \psi + \psi^2 + \cdots + \psi^{p-2}),
\]

where \( \zeta \in \nu_{p-1} \subset Z_p^\times \).

For the prime 3, we can take \( \zeta = -1 \) and then the projection operator is

\[
\pi = \frac{1 + \psi^{-1}}{2}.
\]

Let us start by working out an example.

**Example 5.1.** Consider the element \( \beta L \in K_3^{\wedge,2}(\mathbb{CP}^\infty) \) where \( \beta \) is the Bott element. Applying the projection, we get

\[
\pi(\beta L) = \frac{\beta(L - L^{-1})}{2}.
\]

Restricting to \( S^2 \), using \( L = 1 - \epsilon \) and \( \epsilon^2 = 0 \), we obtain

\[
\frac{\beta((1 - \epsilon) - (1 - \epsilon)^{-1})}{2} = \frac{\beta((1 - \epsilon) - (1 + \epsilon))}{2} = -\beta \epsilon = -1.
\]

In order for it to be an extension of the kind required, this restriction must be \( \alpha_3 \), so we multiply by \( -\alpha_3 \). This defines

\[
f = -\alpha_3 \frac{\beta(L - L^{-1})}{2}.
\]

Recall that, \( THH^{K_3^\wedge}(K/3, f) \) is the cofibre of

\[
K_3^\wedge \wedge CP^\infty \xrightarrow{\alpha_3 - 1} K_3^\wedge \wedge CP^\infty \tag{\beta}
\]
where \( u \in K_3^0(CP^\infty)^\times = [CP^\infty_+, GL_1(K_3^\wedge)] \) is the adjoint of

\[ S^2 \wedge CP_+ \xrightarrow{\tilde{u}} CP^\infty \xrightarrow{f} B^2GL_1(K_3^\wedge). \]

The group structure of \( CP^\infty \) classifies tensor product of line bundles so, \( \mu^* L = L \otimes L \). This implies

\[ \mu^*(f) = -\alpha_3 \frac{\beta((1 - \epsilon) \otimes L - (1 + \epsilon) \otimes L^{-1})}{2} = -\alpha_3 \frac{\beta \epsilon \otimes (L + L^{-1})}{2}. \]

Using the suspension isomorphism (given by \( \beta \epsilon = 1 \)) we get

\[ \mu^*(f) = -\alpha_3 \frac{L + L^{-1}}{2}. \]

To get \( u \) we need to invert the logarithmic cohomology operation. Suppose that \( u = h(x) \in K_3^\wedge^0(CP^\infty)^\times \). Then, we have to solve

\[ \frac{1}{3} \log \left( \frac{\psi_3(h(x))}{h(x)^3} \right) = -\alpha_3 \frac{L + L^{-1}}{2} \]

\[ \Rightarrow \frac{\psi_3(h(x))}{h(x)^3} = \exp \left( 3 \alpha_3 \frac{L + L^{-1}}{2} \right). \]  

Note that \( \psi_3(x) = 1 - (1 - x)^3 \) and hence,

\[ \frac{h(1 - (1 - x)^3)}{h(x)^3} = \exp \left( 3 \alpha_3 \frac{L + L^{-1}}{2} \right). \]

Let us look at the equation \( \pmod{(3^2, x^3)} \). The right side of the equation can be written in terms of \( x \) using \( L = 1 - x \), and then, \( L^{-1} = 1 + x + x^2 \pmod{(3^2, x^3)} \). Therefore, the right side simplifies to

\[ \exp \left( 3 \alpha_3 \frac{L + L^{-1}}{2} \right) = \exp \left( 3 \alpha_3 \frac{2 + x^2}{2} \right) = 1 + 3 \alpha_3 + 3 \alpha_3^2 x^2. \]

Now we will simplify the left side of (5.1). Suppose that \( h(x) = a + bx + cx^2 \). In order to solve the equation, we have to invert \( l_3 \). We know that \( l_3 \) has a kernel \( Z/2 \vee K(Z_3, 2) \), so the equation can be solved once we know the restriction to these.

In the part of \( HZ/2, \sigma : S^2 \to CP^\infty \) induces an isomorphism in \( H^2(-; Z/2) \). Therefore, the extension is 0 here. The map \( K_3^\wedge^0(CP^\infty)^\times \to H^0(CP^\infty; Z/2) \) sends \( a \mapsto a \)
(mod 3) (identifying \( \mathbb{Z}/2 \) with the group of units in \( \mathbb{F}_3 \)). Therefore, since \( \mu^*(0) = 0 \), we get the equation

\[ a \equiv 1 \pmod{3}. \]

In the factor \( K(\mathbb{Z}_3, 2) \), there is no restriction on \( f \). Assume that it is trivial, so \( \mu^*(0) = 0 \). This maps into \( GL_1(K_3^\wedge) \) by taking a line bundle over \( CP^\infty \) to the corresponding unit in \( K \)-theory. If we look at \( k \in \mathbb{Z}_p = H^2(CP^\infty, \mathbb{Z}_3) = [CP^\infty, K(\mathbb{Z}_3, 2)] \), this is the line bundle \( L = (1 - x)^k = 1 - kx + \frac{k(k-1)}{2}x^2 \pmod{x^3} \). This is the only factor that gives a non zero coefficient of \( x \) so, we get that \( b = 0 \).

Therefore, \( h(x) = a + cx^2 \pmod{3^2, x^3} \) and \( a \equiv 1 \pmod{3} \). The left side of (5.1) is

\[ \frac{\psi(h(1 - (1 - x)^3))}{h(x)^3} \equiv \frac{h(3x - 3x^2 + x^3)}{h(x)^3} \]

\[ \equiv \frac{a}{a^3 + 3ca^2x^2} \]

\[ \equiv a^{-2} \left( 1 - \frac{3c}{a}x^2 \right) \pmod{3^2, x^3}. \]

Working \( \pmod{3^2, x^3} \), we have

\[ a^{-2} \left( 1 - \frac{3c}{a}x^2 \right) = 1 + 3\alpha_3 + \frac{3\alpha_3x^2}{2} \]

\[ \implies a \equiv 1 + 3\alpha_3 \pmod{3^2} \quad \text{and} \quad c \equiv \alpha_3 \pmod{3}. \]

Thus \( a = 1 + 3(\text{unit}) \) and \( c \) is a unit (since \( \alpha_3 \) is a unit). Therefore, \( u - 1 \) looks like \( 3(\text{unit}) + x^2(\text{unit}) \). We can choose a different parameterisation for \( K \)-theory of \( CP^\infty \) to assume that \( u - 1 = 3 + x^2 \).

Now \( K_3^\wedge(\mathbb{CP}^\infty) = K_3^\wedge \{\beta_0, \beta_1, \ldots\} \) where \( \beta_i \) is dual to \( x^i \). Therefore,

\[ \langle (u - 1)(\beta_i), x^j \rangle = \langle \beta_i, x^j(3 + x^2) \rangle = \begin{cases} 3 & \text{if } j = i, \\ 1 & \text{if } j = i - 2, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore, the map \( u - 1 \) on \( K_3^\wedge(\mathbb{CP}^\infty) \) is given by

\[ (u - 1)(\beta_i) = \begin{cases} 3\beta_i & \text{if } i = 0, 1, \\ 3\beta_i + \beta_{i-2} & \text{if } i > 1. \end{cases} \]

Following the cofibre \( (\beta) \), we understand that \( u - 1 \) is injective, and its cokernel has two copies of \( \mathbb{Z}/(3^\infty) \) in even dimensions. Thus,

\[ \pi_k(THH K_3^\wedge(K/3), f) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ \mathbb{Z}/(3^\infty) \oplus \mathbb{Z}/(3\infty) & \text{if } k \text{ is even}, \end{cases} \]

completing the calculation in this example.

Now we perform the calculation at the prime 3 for all extensions that are non trivial only on the factor \( \Omega^\infty B \) of \( GL_1(K_3^\wedge) \). The extension in the example was of this kind. So, we are looking at elements in \( B^2(CP^\infty) \) which restrict to \( \alpha_3 \) in \( S^2 \).
An element in $K^\wedge_3^2(CP^\infty)$ is given by $\beta g(x)$. Therefore, an element in $B^2(CP^\infty)$ is

$$\pi(\beta g(x)) = \frac{\beta (g(x) - g\left(1 - \frac{1}{1-x}\right))}{2}.$$

Suppose that $g(x) = a' + b' x + c' x^2 \pmod{(3^2, x^3)}$. Restricting to $S^2$ (using $x = \epsilon$ and $\epsilon^2 = 0$) we get $b'$. We need to get $\alpha_3$. Thus, to get an extension we must have $b' = \alpha_3$. This gives us all possible extensions $f$ on the factor $B$. Let us work as before (mod $(3^2, x^3)$). Then,

$$f = \frac{\beta \left(g(x) - g\left(1 - \frac{1}{1-x}\right)\right)}{2}$$

$$= \frac{\beta (a' + b' x + c' x^2 - g(-x - x^2))}{2}$$

$$= \frac{\beta (a' + b' x + c' x^2 - (a' - b' x - b' x^2 + c' x^2))}{2}$$

$$= \frac{\beta (2b' x + b' x^2)}{2}$$

$$= \beta b' x + \frac{\beta b'}{2} x^2.$$

We have to calculate $u$ using

$$S^2 \times CP^\infty \xrightarrow{\mu} CP^\infty \xrightarrow{f} B^2 GL_1(K^\wedge_3^1).$$

By definition, the multiplication map takes $x$ to the formal group, which for $K$-theory is the multiplicative group. Therefore,

$$x \mapsto \epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x$$

$$\implies \ x^2 \mapsto (\epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x)^2$$

$$= 1 \otimes x^2 + 2\epsilon \otimes x - 2\epsilon \otimes x^2.$$  

To get $\mu^*$ we must project onto the factor $S^2 \wedge CP^\infty_\pm$. Thus, we obtain

$$\mu^*(x) = \epsilon \otimes 1 - \epsilon \otimes x, \ \mu^*(x^2) = 2\epsilon \otimes x - 2\epsilon \otimes x^2.$$  

Using these formulae and the suspension isomorphism $\beta \epsilon = 1$ we calculate $\mu^*(f)$.

$$\mu^*(f) = \beta b'(\epsilon \otimes 1 - \epsilon \otimes x) + \frac{\beta b'}{2} (2\epsilon \otimes x - 2\epsilon \otimes x^2)$$

$$= b'(1 - x) + \frac{b'}{2} (2x - 2x^2)$$

$$= b' - b' x^2$$

To get $u$, we have to invert the logarithmic cohomology operation $l_3$, as in the example.
Suppose that \( u = h(x) \). Then, we need to solve
\[
l_3(u) = \frac{\psi_3(h(x))}{h(x)^3} = \exp(-3b'(1 - x^2)).
\]
We have the formula \( \psi_3(x) = 1 - (1 - x)^3 \). Similar to the example, we assume that in our extension the contribution from \( HZ/2 \) is 1 and \( HZ_3 \) is 0. In the same way, this implies that if \( h(x) = a + bx + cx^2 \),
\[
a \equiv 1 \pmod{3}, \quad b = 0.
\]
Then, the equation becomes
\[
a^{-2}(1 - \frac{3c}{a}x^2) = \exp(-3b'(1 - x^2))
\]
\[
= 1 - 3b' + 3b'x^2.
\]
In the same way, we understand that the unit \( u = 1 + 3.\text{unit} + x^2.\text{unit} \), and so, we obtain the same computation
\[
\pi_k(\mathrm{THH}^{K^3}(K/3), f) = \begin{cases} 
0 & \text{if } k \text{ is odd}, \\
Z/(3^\infty) \oplus Z/(3^\infty) & \text{if } k \text{ is even}.
\end{cases}
\]

Now we want to see what happens if we allow extensions with non trivial contributions from the other 3 factors of \( GL_1(K^3) = Z/2 \times K(Z_3, 2) \times \Omega^\infty B \times \Omega^\infty B_2 \). In the part \( Z/2 \), the restriction \( H^2(CP^\infty; Z/2) \rightarrow H^2(S^2; Z/2) \) is an isomorphism. So, this factor always contributes trivially.

For the factor \( K(Z_3, 2) \), the group \( [S^2, B^2 K(Z_3, 2)] = [S^2, K(Z_3, 4)] = H^4(S^2; Z_3) = 0 \). Therefore, there is no condition on \( f \) here. The group \( H^4(CP^\infty; Z_3) \) is generated by \( x^2 \) and \( f \) is given by \( ax^2 \) for some \( a \in Z_p \). To compute \( u \), consider:
\[
\begin{array}{ccc}
S^2 \times CP^\infty & \xrightarrow{\mu} & CP^\infty \\
| & \downarrow{u} & \downarrow{f} \\
& CP^\infty & \rightarrow K(Z_3, 4).
\end{array}
\]

Note that in this case, \( \mu^*(x) = \epsilon \otimes 1 + 1 \otimes x \), which implies
\[
\mu^*(x^2) = (\epsilon \otimes 1 + 1 \otimes x)^2
\]
\[
= 2\epsilon \otimes x + 1 \otimes x^2.
\]
To get \( u \) we have to project to \( S^2 \wedge CP^\infty \) and apply the suspension isomorphism. Then, we get \( 2ax \in H^2(CP^\infty; Z_3) \). Recall from the previous section that, from this we get the unit by taking \( L^{2a} \), where \( L = (1 - x) \) is the canonical line bundle. Therefore, the contribution to \( u \) from this factor is \( (1 - x)^{2a} \).

Now if \( a \) is divisible by 3 then, we still get that our \( u = 1 + 3.\text{unit} + x^2.\text{unit} \) which results in the same calculation for \( \pi_k(\mathrm{THH}^{K^3}(K/3, f)) \). If \( a \) is not divisible by 3 then, it is a unit, so that \( u = 1 + 3.\text{unit} + x.\text{unit} \). Therefore, by reparameterising we can write \( u - 1 = 3 + x \).
\[
\implies \langle (u - 1)(\beta_i), x^j \rangle = \langle \beta_i, x^j(3 + x) \rangle = \begin{cases} 
3 & \text{if } j = i, \\
1 & \text{if } j = i - 1, \\
0 & \text{otherwise}.
\end{cases}
\]
\[ (u-1)(\beta_i) = \begin{cases} 
3\beta_i & \text{if } i = 0, \\
3\beta_i + \beta_{i-1} & \text{if } i > 0 
\end{cases} \]

Therefore, in this case,
\[
\pi_k(THH^{K_{\hat{3}}}(K/3), f) = \begin{cases} 
0 & \text{if } k \text{ is odd,} \\
\mathbb{Z}/(3^\infty) & \text{if } k \text{ is even.} 
\end{cases}
\]

Now consider the factor \( \Sigma^2B_2 \). We know that this is 5-connected. So, if we look at extensions we know that they always restrict to \( 0 \in K_{\hat{3}}^\wedge(CP^2) \). Since we are working \( (mod \ x^3) \), this means that these extensions always give 0.

Therefore, we get that, depending on \( f \) either \( \pi_*(THH^{K_{\hat{3}}}(K/3, f)) = (\mathbb{Z}/(3^\infty))^i \) in even degrees where \( i = 1 \) or 2 depending on \( f \). This finishes our calculation at the prime 3.

### 5.2. Calculation at primes \( \geq 5 \)

Let us now look at the other odd primes and work \( (mod \ (x^p, p^2)) \). Recall that there is a splitting

\[ GL_1(K_{p^\wedge}^\wedge) = nu_{p-1} \times K(Z_p, 2) \times \Omega^\infty K_p(\hat{2}), \]

\[ K_p(\hat{2}) = B \vee \Sigma^2B_2 \vee \ldots \Sigma^{2p-4}B. \]

We start by working in the factor \( B \) of \( K_p(\hat{2}) \). The projection operator from \( K_{p^\wedge}^\wedge(X) \) to \( B^*(X) \) is given by

\[ \pi = \frac{1 + \psi + \psi^2 + \ldots \psi^{p-2}}{p-1}. \]

Define \( \kappa \) to be the composite

\[ K_{p^\wedge}^\wedge(CP^\infty) \xrightarrow{\mu^*} K_{p^\wedge}^\wedge(S^2 \wedge CP^\infty) \xrightarrow{\Sigma^p} K_{p^\wedge}^\wedge-2(CP^\infty). \]

First observe that the following diagram commutes:

\[
\begin{array}{ccc}
K_{p^\wedge}^\wedge-2(CP^\infty) & \xrightarrow{\psi_a} & K_{p^\wedge}^\wedge(CP^\infty) \\
\downarrow{\kappa} & & \downarrow{\kappa} \\
K_{p^\wedge}^\wedge0(CP^\infty) & \xrightarrow{\psi_a} & K_{p^\wedge}^\wedge0(CP^\infty). 
\end{array}
\]

This implies all Adams operations hence \( \pi \), commutes with \( \kappa \).

Write \( x = 1 - L \) for the generator in \( K_{p^\wedge}^\wedge2(CP^\infty) \) and \( \epsilon \) its restriction to \( S^2 \). We have to look for \( f \) as in the diagram:

\[
\begin{array}{ccc}
S^2 & \xrightarrow{\Sigma^{1-p}} & \Sigma BGL_1(K_{p^\wedge}) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
CP^\infty & \xrightarrow{f} & B^2GL_1(K_{p^\wedge}). 
\end{array}
\]

Suppose that \( f \) is given by \( \pi(\beta g(x)) \), where

\[ g(x) = a_0 + a_1 x + \cdots + a_{p-1} x^{p-1} \quad (mod \ (x^p, p^2)). \]
Claim 5.2.

\[ \kappa(\beta g(x)) = g'(x)(1 - x) \]

**Proof.** It is enough to check this on the generators \( x^n \). The multiplication takes \( x \) to the formal group of \( K \)-theory, which is the multiplicative formal group.

\[ \mu^*(x) = \epsilon \otimes 1 - \epsilon \otimes x \]

Therefore,

\[ \mu^*(\beta x^n) = \beta(\epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x)^n \]

\[ = \beta(1 \otimes x^n + n\epsilon \otimes x^{n-1} - n\epsilon \otimes x^n). \]

\( \kappa \) is obtained by projecting this onto the factor \( S^2 \wedge CP_\infty^\infty \) of the product, and then applying the suspension isomorphism \( (\beta \epsilon = 1) \). Therefore, we obtain

\[ \kappa(\beta x^n) = nx^{n-1} - nx^n \]

\[ = nx^{n-1}(1 - x) \]

\[ = (x^n)'(1 - x), \]

which proves the claim. \( \square \)

If we restrict \( f \) to \( S^2 \), we get

\[ \pi(\beta g(\epsilon)) = \pi(\beta(a_0 + a_1\epsilon)) \]

\[ = \left( \frac{1 + \psi_1 + \psi_2 + \ldots + \psi_{p-2}}{p-1} \right) (\beta(a_0 + a_1\epsilon)). \]

The action of the Adams operations on the Bott element and \( \epsilon \) are given by

\[ \psi_a(\beta) = \frac{\beta}{a}, \quad \psi_a(\epsilon) = 1 - (1 - \epsilon)^a = a\epsilon. \]

Therefore,

\[ \pi(\beta g(\epsilon)) = \left( \frac{1 + \psi_1 + \psi_2 + \ldots + \psi_{p-2}}{p-1} \right) (\beta(a_0 + a_1\epsilon)) \]

\[ = \beta(a_0(1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{2-p}) + (p - 1)a_1\epsilon) \]

\[ = \beta(a_0(1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{2-p})) + a_1\epsilon. \]

Since \( \zeta \) is a \((p - 1)^{st}\) root of unity, we get

\[ 1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{2-p} = \zeta^{p-1} + \zeta^{p-2} + \ldots + \zeta \]

\[ = 0. \]

This shows that \( \pi(\beta g(x)) \) restricts to \( a_1 \in B^2(S^2) \). Thus, we have that \( a_1 = l_p(1 - p) = \alpha_p. \)
We need to calculate $u$ from the extension $\pi(\beta g(x))$ by solving

$$l_p(u) = \kappa \pi(\beta g(x)) = \pi \kappa(\beta g(x)) = \pi(g'(x)(1-x)).$$

Suppose that $h(x) = g'(x)(1-x) = c_0 + c_1 x + \cdots + c_{p-1} x^{p-1} \pmod{(x^p, p^2)}$. Then

$$\pi(h(x)) = \frac{1 + \psi \zeta + \psi \zeta^2 + \cdots + \psi \zeta^{p-2}}{p-1}(h(x)) = \sum_{i=0}^{p-2} \frac{h(1 - (1-x)^i)}{p-1}.$$

Let us look at the coefficient of $x^a$ in the above equation.

$$[\pi(x^n)]_a = \left[ \sum_{i=0}^{p-2} \frac{(1 - (1-x)^i)^n}{p-1} \right]_a$$

Since $\zeta$ is a $(p-1)st$ root of unity,

$$(\zeta)^i + (\zeta^2)^i + \cdots + (\zeta^{p-1})^i = \begin{cases} 0 & \text{if } i = 1, 2, \ldots, p-2, \\ p-1 & \text{if } i = 0, p-1. \end{cases}$$

The binomial coefficient $\binom{y}{a}$ is a polynomial in $y$ of degree $a$ with the constant term 0 and the top coefficient $1/a!$. Therefore,

$$\binom{l(\zeta)}{a} + \binom{l(\zeta^2)}{a} + \cdots + \binom{l(\zeta^{p-1})}{a} = \begin{cases} 0 & \text{if } a = 1, 2, \ldots, p-2, \\ \frac{1}{(p-1)!} l^{p-1} & \text{if } a = p-1, \\ \frac{1}{a!} & \text{if } a = 0. \end{cases}$$

Therefore, we get

$$[\pi(x^n)]_a = \begin{cases} 0 & \text{if } a = 1, 2, \ldots, p-2, \\ \frac{1}{(p-1)!} \sum\binom{n}{l} (-1)^l l^{p-1} & \text{if } a = p-1, \\ \sum\binom{n}{l} (-1)^l & \text{if } a = 0 \end{cases}$$

$$\implies [\pi(x^n)]_0 = \sum\binom{n}{l} (-1)^l = \begin{cases} (1-1)^n = 0 & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

The other possible non zero coefficient is $[\pi(x^n)]_{p-1}$. If $n = 0$, this must be 0. If $n > 0$
this gives
\[
[\pi(x^n)]_{p-1} \equiv \frac{1}{(p-1)!} \sum \binom{n}{l} (-1)^{l(p-1)} \\
\equiv \frac{1}{(p-1)!} \sum \binom{n}{l} (-1)^{l(p-1)} \\
\equiv -\sum \binom{n}{l} (-1)^l \\
\equiv -(1-1)^n + 1 \\
\equiv 1 \pmod{p}.
\]

Summarising the calculation \((\mod p)\), we get
\[
[\pi(x^n)]_a = \begin{cases} 
1 & \text{if } a = 0, n = 0, \\
1 & \text{if } a = p - 1, n > 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Now we are in a position to calculate \(\pi(h(x)) \pmod{p}\)
\[
\pi(h(x)) = \pi(c_0 + c_1 x + \cdots + c_{p-1} x^{p-1}) \\
= c_0 \pi(1) + c_1 \pi(x) + \cdots + c_{p-1} \pi(x^{p-1}) \\
= c_0 + c_1 x^{p-1} + \cdots + c_{p-1} x^{p-1} \\
= c_0 + bx^{p-1},
\]
where \(c_0 = a_1\) and
\[
b = c_1 + \cdots + c_{p-1} = a_1 - 2a_2 + 2a_2 - 3a_3 \ldots - (p-1)a_{p-1} + (p-1)a_{p-1} - pa_p \\
\equiv a_1 \pmod{p}.
\]

Thus the equation for \(u \pmod{p}\) reduces to
\[
l_p(u) = a_1 + bx^{p-1} \pmod{p} \implies -\frac{1}{p} \log \left( \frac{\psi_p(u(x))}{u^p} \right) = a_1 + bx^{p-1} \pmod{p} \\
\implies \frac{\psi_p(u(x))}{u^p} = \exp(-p(a_1 + bx^{p-1})) = 1 - pa_1 + pbx^{p-1} \pmod{p^2}.
\]
We are looking at extensions which are non trivial only on the factor \(B\). This implies \(u(x) \in B^0(CP^\infty)\) which implies \(u\) is in the image of \(\pi\). By the calculations above, this implies that \(u(x) = d_0 + d_1 x^{p-1} \pmod{x^p}\). Then
\[
\frac{\psi_p(u(x))}{u^p} = \frac{d_0}{d_0^p + pd_0^{p-1}d_1 x^{p-1}} \\
= (d_0)^{1-p} \left( 1 - p \frac{d_1}{d_0} x^{p-1} \right).
\]
Therefore, we obtain
\[
d_0^{1-p} = 1 - pa_1 \\
\Rightarrow d_0 = (1 - pa_1)^{\frac{1}{1-p}} \\
= 1 - \frac{p}{1-p}a_1 \\
= 1 - pa_1 \pmod{p^2} \\
\Rightarrow d_1 = -d_0^p b \\
\equiv -1 \pmod{p}.
\]
Therefore, \(d_0 = 1 + p \cdot \text{unit}\) and \(d_1 = \text{unit}\). Thus, \(u = 1 + p \cdot \text{unit} + \text{unit} \cdot x^{p-1}\). We can reparameterise so that \(u = 1 + p + x^{p-1}\).

\[
\langle (u - 1)(\beta_i), x^j \rangle = \langle \beta_i, x^j(p + x^{p-1}) \rangle = \begin{cases} 
p & \text{if } j = i, \\
1 & \text{if } j = i - (p - 1), \\
0 & \text{otherwise}
\end{cases}
\]
\[
\Rightarrow (u - 1)(\beta_i) = \begin{cases} 
p \beta_i & \text{if } i = 0, \\
p \beta_i + \beta_i - (p - 1) & \text{if } i > 0
\end{cases}
\]

Inputting this in the long exact sequence (\(\alpha\)), we get
\[
\pi_k(THH_{K^+}(K/p, f)) = \begin{cases} 
0 & \text{if } k \text{ is odd,} \\
(Z/(p^\infty))^{p-1} & \text{if } k \text{ is even.}
\end{cases}
\]

Now let us look at what happens if we allow non trivial extensions in the other factors. Under restriction to \(S^2\), \(H_{p-1}^2(S^2) \cong H_{p-1}^2(CP^\infty) = \nu_{p-1}\). The element \(1 - p\) gives \(1 \in \nu_{p-1}\). So, this part always contributes trivially.

The factor \(\Sigma^2 B_2\) is \((2p - 1)\)-connected. So, \([CP^p-1, \Sigma^2 B_2] = 0\). Thus \(\pmod{x^p}\) this factor is always trivial.

Next let us look at the factor \(\Sigma^2 HZ_p\). Since \([S^2, \Sigma^4 HZ_p] = HZ^4_p(S^2) = 0\), we have no condition on \(f\) from this factor. The group \(HZ^4_p(CP^\infty)\) is generated by \(x^2\). Suppose that \(f\) is given by \(ax^2 \in HZ^4_p(CP^\infty)\). To compute the contribution to \(u\), we have the diagram
\[
\begin{array}{ccc}
S^2 \times CP^\infty & \xrightarrow{\mu} & CP^\infty \\
\downarrow u & & \downarrow f \\
CP^\infty & \rightarrow & K(Z_p, 4)
\end{array}
\]

Under \(\mu\), \(x\) pulls back to the formal group and thus
\[
\mu^*(x^2) = (\epsilon \otimes 1 + 1 \otimes x)^2 \\
= 2\epsilon \otimes x + 1 \otimes x^2.
\]

Projecting this to the factor \(S^2 \wedge CP^\infty\), and applying the suspension isomorphism we get \(2ax \in HZ^2_p(CP^\infty)\). The map from \(HZ^2_p(CP^\infty) \rightarrow [CP^\infty, GL_1(K_p^\infty)] = K_p^{\wedge 0}(CP^\infty)^x\) is given by \(ax \rightarrow (1 + x)^a\).

Therefore, if \(a\) is divisible by \(p\) then we still get that \(u = 1 + p \cdot \text{unit} + x^{p-1} \cdot \text{unit}\). This does not change the calculation of \(THH_{K^+}(K/p, f)\). If \(a\) is not divisible by
$p$, then it is a unit. Then, $u = 1 + p \cdot \text{unit} + x \cdot \text{unit}$. This can be reparameterised to $u = 1 + p + x$. Then

$$\langle (u - 1)(\beta_i), x^j \rangle = \langle \beta_i, x^j(p + x) \rangle$$

$$= \begin{cases} p & \text{if } j = i, \\ 1 & \text{if } j = i - 1, \\ 0 & \text{otherwise} \end{cases}$$

$$\implies (u - 1)(\beta_i) = \begin{cases} p\beta_i & \text{if } i = 0, \\ p\beta_i + \beta_{i-1} & \text{if } i > 0. \end{cases}$$

Therefore, we obtain

$$\pi_k(\text{THH}^{K_p^\infty}(K/p, f)) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ \mathbb{Z}/(p^\infty) & \text{if } k \text{ is even}. \end{cases}$$

The other factors are $\Sigma^{2k}B$ for $k = 2, 3, \ldots, p - 2$. These correspond to the eigenspaces of the action of the Adams operations where $\psi_\xi$ acts as $\zeta^{ki}$. The projection operator is given by

$$\pi_k = \frac{1 + \zeta^{-k}\psi_\xi + \zeta^{-2k}\psi_\xi^2 + \cdots + \zeta^{-(p-2)k}\psi_\xi^{p-2}}{p-1}.$$

The group $[S^2, \Omega^\infty \Sigma^2 \Sigma^{2k}B] = B^{2k+2}(S^2) = 0$, so, there is no condition on restriction to $S^2$. Then, we may choose any $\pi_k(\beta h(x))$ for $f$, and $u$ must satisfy

$$l_p(u) = \kappa(\pi_k(\beta h(x)))$$

$$= \pi_k(\kappa(\beta h(x)))$$

$$= \pi_k(h'(x)(1 - x)).$$

Now assume $g(x) = h'(x)(1 - x) = c_0 + c_1 x + \cdots + c_{p-1} x^{p-1}$. Then

$$\pi_k(g(x)) = \frac{1 + \zeta^{-k}g_\xi + \zeta^{-2k}g_\xi^2 + \cdots + \zeta^{-(p-2)k}g_\xi^{p-2}}{p-1}(g(x))$$

$$= \frac{1 + \zeta^{-k}g(1 - (1 - x)\xi) + \zeta^{-2k}g(1 - (1 - x)\xi^2) + \cdots + \zeta^{-(p-2)k}g(1 - (1 - x)\xi^{p-2})}{p-1}.$$

The following proposition is useful to complete the calculation

**Proposition 5.3.** There is a polynomial $f_k(x) = x^k + a_{k+1}x^{k+1} + \ldots$ such that, $\text{Im}(\pi_k)$ has polynomials that are multiples of $f_k \pmod{x^p}$.

**Proof.** These polynomials are in the $p$-adic $K$-theory of $CP^\infty$. By looking $\pmod{x^p}$, we are restricting to the $K$-theory of $CP^{p-1}$. It splits into eigenspaces

$$K_p^{\wedge 0}(CP^{p-1}) = \bigoplus_{k=0}^{p-2} \Lambda_k,$$

where $\Lambda_k = [CP^{p-1}, \Omega^\infty \Sigma^{2k}B] = B^{2k}(CP^{p-1})$ is the eigenspace on which the Adams operations $\psi_\xi$ act as multiplication by $\zeta^k$. $\pi_k$ is the projection on to the eigenspace $\Lambda_k$. In this decomposition, $\dim(\Lambda_0) = 2$ and $\dim(\Lambda_k) = 1$ for all $k \geq 1$. Therefore, $\Lambda_k = \text{span}(f_k)$ for some polynomial $f_k$. To see how the polynomial $f_k$ looks we compute
\[ \pi_k(x) = \frac{1 + \zeta^{-k} \psi_1 + \zeta^{-2k} \psi_2 + \cdots + \zeta^{-k(p-2)} \psi_{p-2}}{p-1} (x) \]

\[ = \frac{1}{p-1} \sum_{i=0}^{p-2} \zeta^{-ik} (1 - (1 - x) \zeta^i) \]

\[ = \frac{1}{p-1} \sum_{i=0}^{p-2} \zeta^{-ik} \sum_{n=1}^{\infty} (-1)^{i-1} \binom{\zeta^i}{n} x^n \]

\[ = \frac{1}{p-1} \sum_{i=0}^{p-2} \sum_{n=1}^{\infty} (-1)^{i-1} \zeta^{-ik} \binom{\zeta^i}{n} x^n \]

Let us look at the coefficient of \( x^n \) in the above formula. \( \binom{y}{n} \) is a polynomial of degree \( n \) in \( y \), and therefore, \( (-1)^{i-1} y^{-k} \binom{y}{n} \) has terms of degree \(-k\) to \(-k+n\). So, if we sum the series, it is 0 if \( n < k \). Thus, the first possible non zero coefficient of \( x \) is in degree \( k \). The coefficient of \( x^k \) in \( \pi_k(x) \) is given by

\[ [\pi_k(x)]_k \equiv \frac{1}{p-1} \sum_{i=0}^{p-2} (-1)^{i-1} \zeta^{-ik} \binom{\zeta^i}{k} \]

\[ = \frac{1}{p-1} \sum_{i=0}^{p-2} (-1)^{i-1} \frac{1}{k!} \]

\[ = \frac{1}{(p-1)k!} \neq 0 \pmod{p}. \]

So, this is a unit in \( \mathbb{Z}_p \). Therefore, \( \text{Im}(\pi_k) = \text{Span}(f_k) \) where \( f_k \) looks like \( x^k + O(x^{k+1}) \).

Therefore, \( \pi_k(g(x)) = cf_k(x) \) for some constant \( c \). The equation for \( u \) is

\[ l_p(u) = \pi_k(g(x)) = cf_k(x) \quad \implies \quad \frac{\psi_p(u)}{u^p} = \exp(-pcf_k(x)). \]

If \( c \) is divisible by \( p \), then \( \pmod{p^2} \) the above equation is 0. If \( c \) is not divisible by \( p \), then the coefficient of \( x^k \) in the right side is \( p \) times an unit. We can solve for \( u \) as in the cases before. From here, we get a contribution = \text{unit} \cdot x^k. Therefore, the unit becomes \( u = 1 + p\cdot \text{unit} + x^k \cdot \text{unit} \). As before, we have the long exact sequence (\( \alpha \))

\[ K_{p,*}^\wedge (C^\infty P) \xrightarrow{u-1} K_{p,*}^\wedge (C^\infty P) \xrightarrow{\pi_* (THH_{K_p}^\wedge (K/p,f))} \]

and

\[ \langle (u-1)(\beta_i), x^j \rangle = \langle \beta_i, x^j (p+x^k) \rangle \]

\[ = \begin{cases} 
  p & \text{if } j = i, \\
  1 & \text{if } j = i-k, \\
  0 & \text{otherwise}
\end{cases} \]

\[ \implies (u-1)(\beta_i) = \begin{cases} 
  p\beta_i & \text{if } i = 0, \\
  p\beta_i + \beta_{i-k} & \text{if } i > 0.
\end{cases} \]
Therefore, we obtain that

\[ \pi_n(THH^K_{/p}(K/p, f)) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (\mathbb{Z}/(p^\infty))^k & \text{if } n \text{ is even}. \end{cases} \]

This ends the calculation for all odd primes. The homotopy groups of \( THH^K_{/p}(K/p) \) are 0 in odd degrees and \( (\mathbb{Z}/(p^\infty))^k \) in even degrees, where \( k \) is a number between 1 and \( p - 1 \) depending on the \( A_\infty \) structure on \( K/p \). This result was proved before by Angeltveit [1]. He used the Bökstedt spectral sequence to calculate topological Hochschild homology.

Remark 5.4. This is the calculation identifying \( K/p \) as the Thom spectrum of \( S^1 \). A similar calculation can be carried out for the Thom spectrum of \( S^3 \) to get the same results.

References


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