Raoul Bott at Harvard[*,†]

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The hiring of Raoul Bott also paid off immensely for Harvard, although it is fair to say that as a young man, Bott did not exhibit great mathematical flair, nor did he show much academic promise in general. Born in Budapest in 1923—and raised mainly in Slovakia (until his family immigrated to Canada in 1938)—Bott was, at best, a mediocre student throughout childhood. In five years of schooling in Bratislava, Slovakia, he did not earn a single A, except in singing and German. In mathematics, he typically got Cs and the occasional B, which should make him a hero among late bloomers.

As a youth of about twelve to fourteen, Bott and a friend had fun playing around with electricity—creating sparks, wiring together fuse boxes, transformers, and vacuum tubes, and, in the process, figuring out how various gadgets work. This experimentation eventually served him well. A mathematician, Bott later explained, is "someone who likes to get to the root of things."[1]

Although Bott frequently told his Harvard students that he never would have made it into the school as an undergraduate, he somehow managed to get into McGill University, where he majored in electrical engineering.[2] Upon graduating in 1945, he joined the Canadian army but left after four months when World War II came to a sudden and unexpected end.

Bott then enrolled in a one-year master's program in engineering at McGill, but after completing that he was still unsure of his future course. So he met with the dean of McGill's medical school, as he contemplated a career shift. In response to questions posed by the dean, Bott confessed to disliking the dissection of animals and hating chemistry even more, showing little enthusiasm for the standard subjects taken up in medical school. Finally, the dean asked him: "Is it that maybe you want to do good for humanity…? Because they make the worst doctors."[3]

That ended Bott's thoughts of a medical career. "I thanked him," Bott said. "And as I walked out of his door I knew that I would start afresh and with God's grace try and become a mathematician."[4]

Initially, he wanted to pursue mathematics at McGill, until he was told that his background was so thin he would have to get a bachelor's degree first—a process that could take three years. He turned instead to the Carnegie Institute of Technology (since renamed Carnegie Mellon University), interested in the master's program in applied mathematics. But the course requirements were so extensive that it would have taken him three years to get a master's degree. Carnegie's math chair, John Lighton Synge, suggested their new doctoral program, which had hardly any requirements at all. Bott liked the idea and was assigned Richard J. Duffin as his adviser.

Bott and Duffin took on, and eventually solved, what was then one of the most challenging problems in electrical network theory. The resultant Bott-Duffin theorem not only was of great theoretical significance but also had important practical applications in the electronics industry. The paper that Bott and Duffin coauthored had important consequences for Bott's career, as well, because the work impressed Hermann

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Weyl, who arranged for Bott to spend the 1949–50 academic year at the Institute for Advanced Study in Princeton.[5] (Weyl, as you may recall, came up earlier in this chapter, having secured a position at the Institute for Advanced Study for Richard Brauer; Weyl later did the same for Richard's brother, Alfred.)

"The general plan of my appointment [at the institute], as I understood it," Bott wrote, "was that I was to write a book on network theory at the Institute." On his first day at work in Princeton, Bott met with Marston Morse, who was in charge of the temporary members that year. "[Morse] immediately dismissed my fears of having to write a book. It was a matter of course to him that at the Institute a young man should only do what he wanted to do; that was the place where a young man should find himself and the last place in the world for performing a chore… I remember leaving this interview with a light heart, newly liberated, and buoyed by the energy and optimism I had just encountered."[6]

Bott immediately went to his office and started working on the four-color problem, thinking that the trick he had used to solve the network problem—a function that he considered his "secret weapon," might crack this problem as well. Morse dropped by for a chat a few weeks later. "When he heard what I was doing, he didn't really object," Bott wrote. "Instead, he first spoke of the great interest in the question, but then started to talk about the many good men he had seen start to work on it, never to reappear again. After he left, I threw *all* my computations in the waste basket and never thought about the question again!"[7]

What he did think about—perhaps owing to the presence of Morse, Norman Steenrod, and others—was topology. Surrounded by giants in the field, Bott studied the subject deeply, though he appeared to be in no great rush to publish anything. In fact, he said, "I didn't write a single paper in my first year there. So I was very delighted when Marston Morse called me up at the end of that year and said, 'Do you want to stay another year?' And I said, 'Of course, yes!' He said, 'Is your salary enough?' (It was $300 a month.) I said, 'Certainly!' because I was so delighted to be able to stay another year. My wife took a dimmer view, but we managed."[8]

In 1951, Bott joined the faculty of the University of Michigan, where he continued to focus on topology, paying particular attention to Morse's theory of critical points. A standard picture from Morse theory, as discussed in Chapter 4, involves a doughnut (or "torus") standing upright. This surface has four "critical points"—a maximum on top of the doughnut, a minimum on the bottom, and two saddle points on the top and bottom of the doughnut's inner ring. "Generally, the critical points of a function are isolated," Bott wrote, but he realized these points could come in "bigger aggregates" and could even be the special kinds of spaces we call manifolds.[9]

One way to picture this is to take the upright doughnut from the previous example and topple it over, so that it is now lying on its side, flat on a tabletop. The maximum of this newly configured object is no longer a point—it is a circle. The minimum, similarly, is a circle too. One could determine the topology of the space—and correctly identify it as a torus—by knowing that the critical manifolds are two circles, aligned one above the other.

Bott thus provided a generalization of classical Morse theory, often called Morse-Bott theory, in which critical manifolds replaced the critical points of the original theory. The critical manifolds of this theory could be individual points, which is the zero-dimensional special case. Or they could be one-dimensional manifolds, like the circles in the tipped doughnut. They could be higher-dimensional objects, too—manifolds of any finite dimension, in fact.

Bott used this generalized version of Morse theory to compute the homotopy groups of a manifold, and from there he proved the periodicity theorem. That, admittedly, is a bit of a mouthful, so we will try to break down that statement, explaining in simplified terms what he did.

Whereas Morse was primarily interested in using topology to solve problems in analysis—to solve differential equations, in other words—Bott turned that around, using Morse theory to solve problems in topology. And one of the main problems in topology—as in other areas of mathematics and throughout science, in general—is the classification problem. "Just as scientists want to classify plants and animals to understand how biology works and how life is or organized, mathematicians also strive to find some order among mathematical objects," says Tufts University mathematician Loring Tu, a former Harvard graduate student who coauthored a book on algebraic topology with Bott.[10] Group theorists, therefore, are interested in classifying groups, such as the finite simple groups discussed earlier in this chapter. Topologists, similarly, want to be able to look at various spaces—seeing which ones are equivalent, which ones are different—and then sort them into their proper bins.

Mathematicians define invariants—fixed, intrinsic features of a space—in order to distinguish among different topological spaces. If two spaces are "homeomorphic"—meaning that one can be deformed into the other by stretching, bending, or squishing but not cutting—they must have the same topological invariants. One of the simplest topological invariants to define is the homotopy group, there being one for each dimension. If two spaces (or manifolds) have different homotopy groups, they really are different and cannot be homeomorphic.

Computing the homotopy groups of a manifold constitutes an important step toward understanding the topology of that manifold. The first homotopy group, also called the fundamental group, relates to the kinds of loops you can draw in a space that cannot be shrunk down to a point. A two-dimensional sphere, for example, has a trivial fundamental group, because any loop you can draw on the surface of a sphere can be shrunk down to a point without impediment. A sphere's first homotopy group is therefore zero. On a doughnut, there are two different kinds of circles that cannot be constricted to a point. One would be a circle that starts on the outside of the doughnut, goes through the hole in the center, and loops around. This cannot be shrunk to a point without cutting the doughnut. There is also another kind of circle that wraps around the circumference of the doughnut, sticking to what might be called the "equator." It, too, cannot be shrunk to a point without crushing the doughnut so that it no longer has a hole and, therefore, is no longer a doughnut—just some amorphous, mashed-up pastry.

The fundamental group of the doughnut thus has "two generators," two distinct circles, explains Tu, "but you can go around a circle any number of times, in a positive or negative direction, so we say the fundamental group of the doughnut is two copies of the integers." Tu adds that "the homotopy groups are very easy to define, but they are very difficult to compute, even for a two-dimensional sphere, which seems like a simple enough object."[11]

One puzzling feature was that the homotopy groups appeared to follow no pattern whatsoever. The first homotopy group for the sphere (or the fundamental group) is zero, as mentioned before. The second homotopy group contains all the integers, and the third homotopy group contains all the integers, too, whereas the fourth and fifth groups have just two elements, and the sixth group has twelve elements. There was no apparent rhyme or reason to it.

This perplexing situation made Bott curious enough to try computing some of the homotopy groups on his own. Among other things, he was interested in determining the homotopy groups associated with a rotation group of arbitrary dimension, or SO(n), as it is called. Keep in mind, however, that these rotation groups are Lie groups (as discussed in Chapter 6), which means that they are also manifolds. A manifold, in turn, has homotopy groups. So if you want to understand the structure of rotations in n-dimensional space, one of the first things you might try is computing the homotopy groups.

That is what Bott set out to do. Applying Morse theory to study the homotopy groups of Lie groups, he uncovered an astonishing pattern: the "stable" homotopy groups of SO(n)—which is to say, the homotopy groups of SO(n) when n is sufficiently large—literally repeat in cycles of eight. For large values of n—or for stable homotopy groups, in other words—the first homotopy group for SO(n) is the same as the ninth homotopy group, and the second homotopy group is the same as the tenth homotopy group, and so on. The same thing happens when one looks at rotations in "complex space," a space with complex number coordinates. These rotations form a group called SU(n), which stands for "special unitary group." When n, once again, is sufficiently large, the homotopy groups of SU(n) repeat in cycles of two: the first homotopy group for SU(n) is the same as the third homotopy group, the second homotopy group is the same as the fourth, and so on.

Bott's 1957 paper—which established this finding and was expanded upon in later work—came as a "bombshell," according Michael Atiyah, a longtime collaborator of Bott's presently based at the University of Edinburgh. "The results were beautiful, far-reaching and totally unexpected."[12]

Some mathematicians have compared the periodicity theorem to the periodic table of elements in chemistry. Hans Samelson, a University of Michigan colleague whom Bott considered a "kindred spirit,"[13] called the "periodicity result… the loveliest fact in all topology, with its endlessly repeatable 'mantras'… The discovery had a tremendous effect and started a flood of developments."[14]

Some of the developments Samelson alluded to included K-theory, the study of vector bundles that was pioneered by Grothendieck, Serre, Atiyah, and Friedrich Hirzebruch. (A cylinder is a simple example of a vector bundle consisting of vectors—in this case, vertical line segments or "arrows" endowed with both a direction and magnitude—attached to a circle lying in a horizontal plane.) A 1959 paper by Bott provided a "K-theoretic formulation of the periodicity theorem," and several years later he and Atiyah provided a new proof of periodicity, which fit into the K-theory framework.[15] The periodicity theorem was extremely useful in this context, because it provided an expeditious way for mathematicians to classify vector bundles. As Harvard mathematician Michael Hopkins explains, "The periodicity theorem accounts for the computability of K-theory (and in some sense K-theory itself)."[16] Consequently, Bott added, "K-theory then took off, and it was great fun to be involved in its development."[17]

Thanks to his proof of the periodicity theorem, Bott received offers from four universities and did not know what to do. Harvard mathematician John

Tate urged his school—which had no topologist at the time—to hire Bott. Zariski, who was the mathematics chair at the time, liked the idea, figuring that "Bott was just the man to enliven what often seemed to him a rather stodgy department."[18] Bott accepted the offer in 1959 and stayed for the rest of his career.

Five years later, at a conference in Woods Hole, Massachusetts, in 1964, Atiyah and Bott came up with a formula that Bott considered "among my favorites in all of mathematics."[19] Their work was a broad generalization of the Lefschetz fixed-point theorem, which Princeton mathematician Solomon Lefschetz proved in 1937. Lefschetz's formula (which is expressed in terms of cohomology) involves the number of fixed points of a map from a space to itself. A map, as stated previously, is like a function that takes a point in one space and assigns it to a point in another space (although the "other" space could, in fact, be the same space).

As a simple example, suppose we have the function $g(x) = 3x^4 + 2x + 1$ and want to solve the equation $3x^4 + 2x + 1 = 0$. One thing we can do is add an x to both sides of the equation: $g(x) + x = x$. Next, we invent a map $h(x)$ that is equal to $g(x) + x$. Then the original equation $g(x) = 0$ is equivalent to $h(x) = x$, which means that h maps x to itself. Thus, a solution x of the original equation is a fixed point of the map h. It is called a fixed point because x does not change during the mapping from one space to another; its position remains fixed. (Technically speaking, this example concerns an algebraic equation on the real line; the same idea of transforming a solution of an equation to a fixed point of a map applies to a differential equation on a manifold.)

To see the Lefschetz fixed-point theorem in action, rotate a sphere around its vertical axis. That is an example of a transformation, or map, that takes a sphere to a sphere. In this case, there are just two fixed points—the north and south poles—as every other point moves during the rotation. One can also use Lefschetz's formula to determine algebraically that there are two fixed points. A virtue of the latter approach is that you can use it to figure out the number of fixed points in more complicated situations where you cannot draw a simple picture of a spinning globe.

"Algebra is almost always easier to do than geometry and topology, and that's the basic idea behind cohomology, which involves converting geometric and topological problems into algebraic ones," explains Tu. Atiyah and Bott went further still, Tu adds, "deriving a far-reaching generalization that, in one special case, gives you back the Lefschetz theorem, but it also gives you many other fixed-point theorems—some new theorems and some classical ones, as well."[20]

In 1982, Atiyah and Bott came up with another formula involving fixed points, this one on the subject of "equivariant cohomology." Cohomology, a term that comes up many times throughout this book, is an algebraic invariant that mathematicians assign to spaces, which means it is one of the tools they use to study spaces. When the space you are studying has symmetries of a particular sort, there is a kind of cohomology you can study called "equivariant cohomology."

Two French mathematicians, Nicole Berline and Michèle Vergne, discovered this same formula independently, and almost simultaneously. The Atiyah-Bott-Berline-Vergne formula—or the equivariant localization formula, as it is often called—allows you to compute certain integrals on manifolds with symmetries. This formula is extremely convenient since many important physical quantities can be expressed as integrals, yet computing those integrals can be quite difficult.

Consider again the example of a sphere, this time of radius 1. It has rotational symmetry about a vertical axis and, as before, has exactly two fixed points. The surface area of the sphere is a surface integral. The equivariant localization formula assigns a number, or multiplicity, to each fixed point and says that the surface integral is a constant, 2π, times the sum of the multiplicities at all the fixed points. In this case, the multiplicity assigned to each fixed point is 1, and since there are two fixed points, the area of the unit sphere, according to this reckoning, is 2π times 2, or 4π, as it should be. The approach is "very powerful," says Tu, "because instead of having to compute an integral, you just have to add up a few numbers."[21]

Commenting on their eventual breakthrough, Bott said that "Michael and I had been wrestling with the question of equivariant cohomology since the 1960s," approaching it through the notion of fixed-point theorems. "It is amazing how long it takes for a new idea to penetrate our collective consciousness and how natural and obvious that same idea seems the moment it is properly enunciated."[22]

Speaking of his long-term collaboration with Bott, whom he knew for more than fifty years, Atiyah said: "It was impossible to work with Bott without becoming entranced by his personality. Work became a joy to be shared rather than a burden to bear… His personality overflowed into his work, into his relations with collaborators and students, into his lecturing style, and into his writing. Man and mathematician were happily fused."[23]

Another colleague, the Harvard geometer Clifford Taubes, maintains that Bott had a profound influence on him when he was a graduate student at Harvard, earning his Ph.D. in physics in 1980. "It was just wonderful… to see how this beautiful mathematics flowed," Taubes said of the class he took that was taught by Bott. "I, for one, would have been a physicist if I had not been in his class, but I was seduced by mathematics." Bott's impact, of course, spread far beyond a single graduate student in the late 1970s. All told, says Taubes, "he had a tremendous influence in the development of modern geometry and topology. I would say that his contributions to this were as great as any one person."[24]

Taubes is not alone in that assessment. Bott won the 2000 Wolf Prize in Mathematics, which he shared with Jean-Pierre Serre, for his work in topology that culminated in the periodicity theorem and "provided the foundation for K-theory, to which Bott also contributed greatly."[25]

Bott died in 2005, after having proved many important theorems and leaving an indelible mark on generations of students. Two of his students won the Fields Medal: Stephen Smale, who got his Ph.D. at the University of Michigan in 1957, and Daniel Quillen, who got his Ph.D. from Harvard in 1964. Another of his Harvard students, Robert D. MacPherson—a co-inventor of "intersection homology"—has had a distinguished career at Brown University, MIT, and the Institute for Advanced Study.

Bott was, by all accounts, an imperturbable teacher. Once a five-square-foot chunk of ceiling fell down in the middle of his Math 11 classroom. He calmly waited for the dust to settle and then resumed his discussion, urging his students to ignore the large cracks in the ceiling.[26] A class with Bott, says Benedict Gross, "was an amazing experience, like drinking from the original stream, as a lot of it was his own work."[27]

"I recall him arriving at each class with no notes, puffing on a cigarette right under the No Smoking sign, and simply living the mathematics in our presence," said Washington University mathematician Lawrence Conlon, who was a graduate student when Bott burst onto the Harvard scene. After a mind-blowing class with Bott on algebraic topology, Conlon mustered the courage to ask him to direct his thesis. "Well, Larry," said Bott, "you're a good student, but what we have to find out now is whether you can dream."[28]

Bott's presence, according to Mazur, "radiated friendship of the sort that simply made everyone not only happier but somehow perform better."[29] Bott was that rare person, possessing "such an extraordinary amount of humor and optimism" that he could truthfully claim, as he did, "I can't say that there is any mathematics that I don't like."[30]

Much as his students and colleagues appreciated Bott, he also appreciated them in return. Upon receiving the Steele Prize for Lifetime Achievement in 1990, he offered thanks for what had then been more than thirty years at Harvard, where "there is not a single colleague or student who has not added to my education or uncovered some hidden mystery of our subject."[31]

In hiring Ahlfors, Zariski, Brauer, Bott, and others who followed, Harvard was opening its doors to mathematicians from foreign shores who enriched the department, the field, and the culture. Bott extended thanks in return to "this country, which has accepted so many of us from so many shores with such greatness of spirit and generosity. Accepted us—accent and all—to do the best we can in our craft as we saw fit."[32]

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