# A Calabi-Yau Cartography

## by Yang-Hui He*

### 1. Prologus Terræ Sanctæ

"La Géométrie, qui ne doit qu'obéir à la Physique quand elle se réunit avec elle, lui commande quelquefois" (Geometry, which should only obey Physics, when united with it sometimes commands it), so wrote the great philosopher and mathematician Jean d'Alembert in his *Essai d'une nouvelle théorie de la résistance des fluides* (1752). This coextensivity between natural philosophy and geometry, conceived in antiquity, fashioned in Early Modernity and shaped in the Industrial Age, fully blossomed in the XXth century.

The two corner-stones of modern physics – *general relativity* which describes the large structure of space-time and *quantum field theory*, the elementary particles which constitute all matter – are well understood as geometrical in nature. The former, is the study of how the Ricci curvature of spacetime is induced by the energy-momentum tensor in an equality dictated by the Einstein-Hilbert action and the latter, how particles are realizations of connections and representations of appropriate principal bundles governed by the Standard Model action. The brain-child of this tradition of geometrization and unification, dominating mathematical and theoretical physics as we enter the XXIst century, is string theory.

It is well-known by now that string theory is a quantum theory unifying gravity and field theory in ten spacetime dimensions (or, equivalently, a conjectural M-theory in 11 dimensions). We must therefore account for $10 - 4 = 6$ "missing" dimensions. Over the years, there has emerged a plethora of scenarios in the interplay between the physics of our four dimensions and these 6 dimensions, our story here will focus on the most traditional and the most richly developed, viz., the geometry of Calabi-Yau threefolds. Indeed, so great is the number of possible scenarios that it poses as one of the greatest theoretical challenges to modern physics, in what has become known as the "vacuum degeneracy problem", where one is confronted with how to select *our* universe amidst a *landscape* – a word which has become a technical term – of solutions.

I shall not discuss the landscape here, nor its philosophical, anthropological or statistical implications. For this purpose, I have carefully chosen the word *cartography* in the title and will discuss some of the progressive uncovering of the still mysterious space of Calabi-Yau geometries, as if charting the geography of a vast and fertile land, and, with a historical outlook, review the explicit construction of Calabi-Yau threefolds for the sake of physics. For this purpose too I have named this introductory section after the prologue of the famous medieval treatise on the Holy Land [1] by Burchard, who ventured to produce the best early maps of that alluring place.

### 2. Triadophilia

Our story begins with 1985, when Gross, Harvey, Martinec, and Rhom – jocundly called the "Princeton string quartet" – formulated[1] the *heterotic string*. By fusing the bosonic string (whose critical dimension is 26) with the superstring (whose critical dimension is 10) in the process of "heterosis" by assigning them to be respectively left and right moving modes of the string, the quartet obtained something quite revelatory: an interesting gauge group. Using the fact that

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[1] For an entertaining account of the history of string theory, aimed at the public and the specialist alike, the reader is referred to [3]. For the history of Calabi-Yau manifolds, especially in relation to string theory, the book [4] is highly recommended.

$26-10=16$ and that in 16 dimensions there are only two even self-dual integral lattices in which quantized momenta could take value, viz., the root lattices of $E_8 \times E_8$ and of $D_{16} = \mathfrak{so}(32)$, two heterotic string theories were constructed.

## 2.1 String Phenomenology

All at once, the possibility of obtaining chiral fermions in spacetime, crucial to the Standard Model, is realized. Now, $E_8$ is of particular significance, because of the following sequence of embeddings of Lie groups

(2.1)
$$SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8$$

The first group $G_{SM} = SU(3) \times SU(2) \times U(1)$ is, of course, that of the Standard Model of particles where the $SU(3)$ factor is the gauge group of QCD governing the dynamics of baryons and the $SU(2) \times U(1)$, that of QED, governing the leptons. Oftentimes, we add one more $U(1)$ factor, denoted as $U(1)_{B-L}$, to record the difference between baryon and lepton number, in which case $G'_{SM} = SU(3) \times SU(2) \times U(1) \times U(1)_{B-L}$ and the above sequence of embeddings skips $SU(5)$.

In terms of the representation of $G'_{SM}$, denoted as $(\mathbf{a}, \mathbf{b})_{(c,d)}$ where $\mathbf{a}$ is a representation of $SU(3)$, $\mathbf{b}$, that of $SU(2)$, and $(a,b)$ are the charges of the two Abelian $U(1)$ groups, the Standard Model elementary particles (all are fermions except the scalar Higgs) are as follows

| $SU(3) \times SU(2) \times U(1) \times U(1)_{B-L}$ | Multiplicity | Particle |
|---|---|---|
| $(\mathbf{3}, \mathbf{2})_{1,1}$ | 3 | left-handed quark |
| $(\mathbf{1}, \mathbf{1})_{6,3}$ | 3 | left-handed anti-lepton |
| $(\bar{\mathbf{3}}, \mathbf{1})_{-4,-1}$ | 3 | left-handed anti-up |
| $(\bar{\mathbf{3}}, \mathbf{1})_{2,-1}$ | 3 | left-handed anti-down |
| $(\mathbf{1}, \mathbf{2})_{-3,-3}$ | 3 | left-handed lepton |
| $(\mathbf{1}, \mathbf{1})_{0,3}$ | 3 | left-handed anti-neutrino |
| $(\mathbf{1}, \mathbf{2})_{3,0}$ | 1 | up Higgs |
| $(\mathbf{1}, \mathbf{2})_{-3,0}$ | 1 | down Higgs |

In addition to these are vector bosons: (I) the connection associated to the group $SU(3)$, called the gluons, of which there are 8, corresponding to the dimension of $SU(3)$, and (II) the connection associated to $SU(2) \times U(1)$, called $W^{\pm}$, $Z$, and the photon; a total of 4 corresponding to the its dimension. We point out that henceforth by the Standard Model – and indeed likewise for all ensuing gauge theories – we shall actually mean the (minimal) supersymmetric extension thereof, dubbed the **MSSM**, and to each of the fermions above there is a bosonic partner and vice versa.

Of note in the table is the number 3, signifying that the particles replicate themselves in three families, or **generations**, except for the recently discovered Higgs boson, of which is only a single doublet under $SU(2)$. That there should be 3 and only 3 generations, with vastly disparate masses, is an experimental fact with confidence [5] level $\sigma = 5.3$ and has no satisfactory theoretical explanation to date. The possible symmetry amongst them, called flavour symmetry, is independent of the gauge symmetry of $G_{SM}$.

The fact that $G_{SM}$ is not simple has troubled many physicists since the early days: it would be more pleasant to place the baryons and leptons in the same footing by allowing them to be in the same representation of a larger simple gauge group. This is the motivation for the sequence in (2.1): starting from $SU(5)$, theories whose gauge groups are simple are called **grand unified theories** (GUTs), the most popular historically had been $SU(5)$, $SO(10)$ and $E_6$, long before string theory came onto the scene in theoretical physics.

That the heterotic string could produce potentially realistic (supersymmetric) grand unified theories, in addition to the natural incorporation of the graviton, gave the first glimpse of string theory as a candidate for ToE, the *Theory of Everything*. Together with anomaly cancellation by Green-Schwarz [6] which showed the quantum consistency of string theory in the previous year, and the subsequent paper by Candelas-Horowitz-Strominger-Witten (CHSW) [7] in the following year, to which we now turn, this constituted the "First String Revolution".

The paper of CHSW, gave the conditions for which the heterotic string, when compactified – i.e., its 10-dimensional background is taken to be of the form $\mathbb{R}^{1,3} \times X_6$ with $\mathbb{R}^{1,3}$ our familiar space-time and $X_6$ some small curled up 6-manifold, endowed with a vector bundle $V$, at the Planck scale too small to be currently observed directly – would give a supersymmetric gauge theory in $\mathbb{R}^{1,3}$ with potentially realistic particle spectrum. In short, the paradigm is simply

$$\boxed{\text{Geometry of } X_6 \longleftrightarrow \text{physics of } \mathbb{R}^{1,3}.}$$

What better realization of that noble goal of the geometrization of nature which has been exalted for so long!

Specifically, with more generality, the set of conditions, known as the *Strominger System* [8], for the low energy low-dimensional theory on $\mathbb{R}^{1,3}$ to be a supersymmetric gauge theory are

1. $X_6$ is complex;
2. The Hermitian metric $\omega$ on $X_6$ and $h$ on $V$ satisfy

(a) $\partial\bar{\partial}\omega = i\,\mathrm{Tr}\,F\wedge F - i\,\mathrm{Tr}\,R\wedge R$ where $F$ is the curvature (field strength) 2-form for $h$ and $R$ the (Hull) curvature 2-form for $\omega$;

(b) $d^\dagger\omega = i(\partial - \bar{\partial})\ln||\Omega||$, where $\Omega$ is a holomorphic 3-form on $X_6$ which must exist. Recently, Li-Yau [9] showed that this is equivalent to $\omega$ being balanced, i.e., $d\left(||\Omega||\omega^2\right) = 0$;

3. $F$ satisfies the Hermitian Yang-Mills equations

$$\omega^{a\bar{b}}F_{a\bar{b}} = 0\ , \quad F_{ab} = F_{\bar{a}\bar{b}} = 0\ .$$

We will not discuss the technicalities of the above concepts in detail here but have included them for completeness. Suffice it to say that the general solutions to this system continue to inspire research today, engendering more geometric structures that contribute to the landscape. The simplest and most famous solution, of course, is when $X_6$ is a Calabi-Yau threefold (CY3), which we now address.

### 2.2 Calabi-Yau Manifolds

The history of Calabi-Yau manifolds is a distinguished one and dates long before string theory or even the conception of the Standard Model (Glashow-Salam-Weinberg's electroweak theory was finalized in 1967). This is a golden example of a magical aspect of string theory: it consistently infringes, almost always unexpectedly rather than forcibly, upon the most profound mathematics of paramount concern, and then quickly proceeds to contribute to and even revolutionize it.

In 1954, Calabi conjectured [10] of the existence of certain nicely behaved Riemannian metrics on complex manifolds, that for $X$ a compact Kähler manifold[2] with Kähler metric $g$ and Kähler form $\omega$, and $R$ a $(1,1)$-form representing the first Chern class of $X$, then

**PROPOSITION 1.** *There exists a unique Kähler metric $\tilde{g}$ with Kähler form $\tilde{\omega}$ such that $\omega$ and $\tilde{\omega}$ are cohomologous in $H^2(X,\mathbb{R})$ and the Ricci form of $\tilde{\omega}$ is $R$.*

This conjecture was proven by S.-T. Yau in 1977-8 in his Fields-winning treatise [11]. In particular, the case of vanishing first Chern class is also that of zero Ricci curvature $R$, and such $X$ is appropriately dubbed **Calabi-Yau** and its unique Kähler metric inducing this flat curvature is the Calabi-Yau metric. In summary, we will take the following as *equivalent* definitions, some more differential and some more algebraic, of a Calabi-Yau $n$-fold $X$: that it is a compact Kähler manifold of complex dimension $n$ such that

- The canonical bundle, i.e., the higher exterior power $\bigwedge^n T_X^*$ of the cotangent bundle is the trivial bundle $\mathcal{O}_X$;
- $X$ admits a nowhere vanishing holomorphic $n$-form $\Omega$;
- The first Chern class of the tangent bundle $T_X$ vanishes: $c_1(T_X) = 0$;
- $X$ has a Kähler metric with vanishing Ricci curvature;
- The holonomy of the Kähler metric is contained in $SU(n)$.

We will briefly mention the non-compact case towards the end of this review but for now, and indeed as far as the heterotic compactification is concerned, we restrict our attention to compact, smooth Calabi-Yau manifolds.

While Calabi-Yau manifolds are certainly of interest in pure mathematics, what [7] showed is that, remarkably, geometries of Calabi-Yau threefolds are a concrete realization of $X_6$ as a string compactification scenario which could attain potentially realistic physics.[3] Thus is born the subject of **string phenomenology**.

### 2.3 Triadophilia: Three Generations

Recall that the Strominger system also has a vector bundle $V$ on the Calabi-Yau threefold $X$, luckily, this can be chosen to be simply the tangent bundle $T_X$ whose first Chern class we have already seen to vanish. What is the gauge theory in $\mathbb{R}^{1,3}$? The rule turns out to be simple: the low-energy (grand unified) gauge group is simply the commutant of the structure group of $V$ in $E_8$. The astute reader may wonder about the "other" $E_8$ factor. Indeed, in heterotic compactifications, only one of the $E_8$ is declared "visible" and the other, called "hidden", is placed at the end of the universe and actually has rich physics in its own right [12–14, 16].

The structure group for $V = T_X$ is here simply the holonomy group $SU(3)$ and its commutant in $E_8$ is $E_6$. In other words, we naturally have an $E_6$ GUT theory in $\mathbb{R}^{1,3}$. By taking $V$ not being $T_X$, but, for example, a stable $SU(4)$ or $SU(5)$ bundle, one could obtain the more interesting commutant $SU(10)$ or $SU(5)$ GUTs. This has come to be known as "non-standard" embedding and has with the developments in the theory

[2] We briefly remind the reader that for a Hermitian (complex) manifold $X$ with metric $h$, one can extract a Riemannian metric $g_R = \frac{1}{2}(h+\bar{h})$ as the real part. If, in addition the Hermitian form $\omega = \frac{i}{2}(h-\bar{h})$ written as the imaginary part, closes (i.e., $d\omega = 0$), then $X$ is Kähler and $g_K = \omega$ is the Kähler metric. In such a case, there is a potential $K$ whence the metric can be locally derived: $g_K = \frac{i}{2}\partial\bar{\partial}K$. In a sense, a Kähler manifold is one so well endowed that it has three compatible structures: (complex) Hermitian, symplectic and Riemannian.

[3] Heuristically, one might conceive of this as follows: one needs $X_6$ to be complex in order to have chiral fermions, Kähler, for supersymmetry and Ricci-flat, to solve vacuum Einstein's equations.

of stable bundles on Calabi-Yau manifolds become an industry of realistic model building [15].

The particle content is readily determined from group theory. More importantly, this in turn determines the vector bundle cohomology group which is associated with the particles. In short, we have that[4]

$$\begin{aligned} &\text{generations of particles} \sim H^1(T_X)\ ,\\ &\text{anti-generations of particles} \sim H^1(T_X^*)\ . \end{aligned}$$

In general, the lesson is that

| |
|---|
| Particle content in $\mathbb{R}^{1,3}$ $\longleftrightarrow$ cohomology groups of $V, V^*$ and their exterior/tensor powers |

The cubic Yukawa couplings in the Lagrangian constituted by these particles (fermion-fermion-Higgs) are tri-linear maps[5] taking the cohomology groups to $\mathbb{C}$.

An immediate *constraint* is, of course, that there be 3 net generations, meaning that

$$\left|h^1(X,T_X) - h^1(X,T_X^*)\right| = 3\ . \tag{2.2}$$

Thus, the endeavour of finding Calabi-Yau threefolds with the property (2.2) began in 1986. This geometrical "love for threeness", much in the same spirit as triskaidekaphobia, has been dubbed by Candelas et al. as **Triadophilia** [17]. Recently, independent of string theory or any unified theories, why there might be geometrical reasons for three generations to be built into the very geometry of the Standard Model has been explored [18].

# 3. De Practica Geometriæ

Having extracted a mathematical problem from physical constraints, it therefore becomes a practical quest in geometry which has prompted some 30 years of research. We are indeed reminded of Fibonacci's tome of 1220 after which I have named this section in his honour.

First, it is well-known that the two terms in (2.2) are topological quantities associated with $X$, in particular, by Hodge decomposition we have that $h^1(X,T_X) \simeq h^{2,1}$ and $h^1(X,T_X) \simeq h^{1,1}$ where the latter are the so-called Hodge numbers, which is a refined (complex) version of Betti numbers counting (holomorphic) cycles in the Kähler manifold. The following diagram will illustrate the relevant points:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{2,1} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

$$= \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \rightsquigarrow \begin{array}{c} b^0 \\ b^1 \\ b^2 \\ b^3 \\ b^1 \\ b^2 \\ b^0 \end{array} = \begin{array}{c} 1 \\ 0 \\ h^{1,1} \\ 2h^{2,1}+2 \\ h^{1,1} \\ 0 \\ 1 \end{array}$$

Here it is customary to represent the matrix of Hodge numbers in **diamond** form. The refinement of the Betti numbers is in the sense that $b^i = \sum\limits_{j+k=i} h^{j,k}$. The symmetry about the middle horizontal line is simply Poincaré duality and that about the middle vertical line is essentially complex conjugation (of the Laplacian). Moreover, we assume that $X$ is connected so that $b^0 = h^{0,0} = 1$. Furthermore, for the case of $X$ being simply-connected, the fundamental group $\pi_1(X)$ and hence its Abelianization $H^1(X)$ also vanishes. Finally, the Calabi-Yau property that there be a unique holomorphic 3-form implies that $h^{3,0} = h^{0,3} = 1$. Looking at the Hodge diamond, we see that topologically, a CY3 is characterized by only two integers $h^{1,1}$ and $h^{2,1}$, essentially counting the number of Kähler and complex structure deformations respectively. We emphasize that these two are not refined enough: two CY3 with the same pair of $(h^{1,1}, h^{2,1})$ need not be isomorphic. Importantly, we have that

$$\chi = \sum_{i=0}^{6} (-1)^i b^i = 2(h^{1,1} - h^{2,1}) \tag{3.3}$$

is the standard topological Euler number. Our question (2.2) thus becomes: *does there exist a Calabi-Yau threefold with $\chi = \pm 6$?*

## 3.1 The Quintic

How then, does one explicitly construct a Calabi-Yau manifold? A cursory look at low dimension will give us not only some experience but also another reason why Calabi-Yau manifolds are of central importance to mathematics. What is a Calabi-Yau one-

[4] Specifically, we have that the decomposition of the adjoint 248 of $E_8$ breaks into $SU(3)\times E_6$ as $248 \to (1,78)\oplus(3,27)\oplus(\bar{3},\overline{27})\oplus(8,1)$. Thus the Standard Model particles, which in an $E_6$ GUT all reside in its 27 representation, is associated with the fundamental 3 of $SU(3)$. The 10-dimensional fermions are eigenfunctions of the Dirac operator, which then splits into the 4-dimensional one, giving the fermions we see and that on the Calabi-Yau threefold, the low-energy particles are then dictated by the zero-eigenvalues of the Dirac operator on $X$, and thence, via the Atiyah-Singer index theorem, by the cohomology of appropriate bundles on $V$. Here, for example, the 27 representation is thus associated to $H^1(T_X)$ and the conjugate $\overline{27}$, to $H^1(T_X^*)$. Similarly, the 1 representation of $E_6$ is associated with the 8 of $SU(3)$, and thus to $H^1(T_X \otimes T_X^*)$.

[5] This works out perfectly for a Calabi-Yau threefold: for example, $H^1(X,V)\times H^1(X,V)\times H^1(X,V) \to H^3(X,\mathcal{O}_X) \simeq \mathbb{C}$.

fold? This is nothing but a Riemann surface of zero curvature, which is classically well-known to be the torus $T^2 = S^1 \times S^1$. Algebro-geometrically, this can be realized as a cubic[6] in $\mathbb{CP}^2$. Hence, the study of Calabi-Yau one-folds is that of the elliptic curve! No wonder we are in the very heart of modern mathematics.

Moving onto complex dimension 2, one could so generalize and have a (smooth) quartic algebraic surface in $\mathbb{P}^3$. This is called a **K3 surface** and is again one of the classical objects studied at the end of the XIXth century. It turns out that the only other Calabi-Yau two-fold is the rather trivial case of the 4-torus $T^4 = (S^1)^4$, which is simply the direct product of two elliptic curves.

This construction, of having a degree $n+1$ hypersurface in $\mathbb{CP}^n$ as an *algebraic variety* is indeed valid in general. One can show that the number of projective coordinates, here $n+1$, equaling to the degree of the hypersurface implies the vanishing of the first Chern class. Thus we arrive at our first, and perhaps most famous, example of a Calabi-Yau threefold: the quintic hypersurface in $\mathbb{CP}^4$. There are many degree 5 monomials one could compose of 5 coordinates, the most well-studied is the so-called Fermat quintic:

$$Q := \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \psi x_0 x_1 x_2 x_3 x_4 = 0\} \subset \mathbb{CP}^4_{[x_0:x_1:x_2:x_3:x_4]}$$

where $\psi$ is some complex coefficient. What are the topological numbers of $Q$? It turns out that $h^{2,1}(Q) = 101$ and $h^{1,1}(Q) = 1$ so that $\chi(Q) = -200$ and this is quite far from $\pm 6$.

## 3.2 The CICY Database

To continue to address the question raised by triadophilia, an algorithmic generalization of the construction for the quintic was undertaken: instead of a single $\mathbb{CP}^n$, what about embedding a collection of (homogeneous) polynomials into a product $A$ of projective spaces? For further simplification, let us consider only *complete intersections* which means the optimal case where the number of equations is 3 less than the dimension of the ambient space $A$ so that each polynomial slices out exactly one new degree of freedom. In other words, let $A = \mathbb{CP}^{n_1} \times \ldots \times \mathbb{CP}^{n_m}$, of dimension $n = n_1 + n_2 + \ldots + n_m$ and each having homogeneous coordinates $[x_1^{(r)} : x_2^{(r)} : \ldots : x_{n_r}^{(r)}]$ with the superscript $(r)$ indexing the projective space factors. Our CY3 is then defined as the intersection of $K = n-3$ homogeneous polynomials in the coordinates $x_j^{(r)}$. Clearly this is a generalization of the quintic, for which $r = m$, $n_r = 4$ and $K = 1$. The Calabi-Yau condition of the vanishing of $c_1(T_X)$ generalizes analogously to the condition that for each $r = 1, \ldots, m$, we have $\sum_{j=1}^{K} q_j^r = n_r + 1$. Succinctly, one can write this information into an $m \times K$ configuration matrix (to which we frequently adjoin the first column, designating the ambient product of projective spaces, for clarity; this is redundant because one can extract $n_r$ from one less than the row sum):

$$X = \left[\begin{array}{c|cccc} \mathbb{CP}^{n_1} & q_1^1 & q_2^1 & \ldots & q_K^1 \\ \mathbb{CP}^{n_2} & q_1^2 & q_2^2 & \ldots & q_K^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{CP}^{n_m} & q_1^m & q_2^m & \ldots & q_K^m \end{array}\right]_{m \times K} , \quad (3.4)$$
$$K = \sum_{r=1}^{m} n_r - 3 ,$$
$$\sum_{j=1}^{K} q_j^r = n_r + 1 , \ \forall\, r = 1, \ldots, m .$$

For example, [5], or [4|5], denotes the quintic. Two more immediate examples are

$$S = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} , \qquad \tilde{S} = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} .$$

The first is called the Schoen Manifold and the second, constructed by Yau et al. Specifically, the configuration $S$ means that the ambient space is $\mathbb{CP}^1 \times \mathbb{CP}^2 \times \mathbb{CP}^2$, of dimension 5, so that indeed the two polynomials, being complete intersection, give a threefold. The first column $(1,3,0)$ means that the first polynomial is linear in $\mathbb{CP}^1$ and cubic in the first $\mathbb{CP}^2$ while having no dependence on the second $\mathbb{CP}^2$. The second column is likewise defined. Another lesson, as can be induced from $S$ and $\tilde{S}$, is the rather cute fact that the transpose gives a new valid configuration which could be completely different in topology. Importantly, the Chern classes and thence the Euler number can be read off the matrix configuration directly.[7] We often attach the topological numbers as $X_\chi^{h^{1,1}, h^{2,1}}$ for completeness, so we can write, for instance, $[5]^{1,101}_{-200}$, $S_0^{19,19}$ and $\tilde{S}^{14,23}_{-18}$.

Such manifolds were considered and explicitly constructed by Candelas et al. [19] in the early 1990s and were affectionately called CICYs (complete intersection Calabi-Yau manifolds). Classifying these

[6] The beginning student is perhaps more familiar with the Weierstraß model: $\{x, y \in \mathbb{C} | y^2 = x^3 - 4g_2 x - g_6\}$; we simply projectivize by having homogeneous coordinates $[x : y : z]$ of $\mathbb{CP}^2$.

[7] We have that $c_1(T_X) = 0$ and moreover,

$$c_2^{rs}(T_X) = \frac{1}{2}\left[-\delta^{rs}(n_r + 1) + \sum_{j=1}^{K} q_j^r q_j^s\right] ,$$
$$c_3^{rst}(T_X) = \frac{1}{3}\left[\delta^{rst}(n_r + 1) - \sum_{j=1}^{K} q_j^r q_j^s q_j^t\right] ,$$

where we have written the coefficients of the total Chern class $c = c_1^r J_r + c_2^{rs} J_r J_s + c_3^{rst} J_r J_s J_t$ explicitly, with $J_r$ being the Kähler form in $\mathbb{P}^{n_r}$. The triple-intersection form $d_{rst} = \int_X J_r \wedge J_s \wedge J_t$ is a totally symmetric tensor on $X$ and the Euler number is simply $\chi(X) = d_{rst} c_3^{rst}$.

above matrices, up to topological equivalence, would then classify the CICYs. One could then read off the Euler number to see whether any of them had magnitude of 6. The combinatorial problem for these integer matrices turned out to be rather non-trivial and one of the most powerful super-computers then available was recruited. Philip Candelas often recounts to me his fond memories of running the code on the computer at CERN and the print-out still sits in a compile in his office. This was perhaps the first time when heavy machine computation was done for the sake of algebraic geometry. In all, CICYs were shown to be finite in number, a total of 7890 inequivalent configurations. Recently, CICY4, the four-fold version of this was completed in the nice work [20] and 921,497 were found.

Unfortunately, none of the 7890 had $\chi = \pm 6$. While this was initially disappointing, it was soon realized that circumventing this problem gave rise to the resolution of another important physical question. A freely acting order 3 symmetry was found on $\tilde{S}$; the freely acting is important, because it means that the quotient $S' = \tilde{S}/\mathbb{Z}_3$ is also a smooth CY3, albeit not a CICY. For such smooth quotients, the Euler divides[8] and $S'$ became the first three-generation manifold!

Now, the quotienting is crucial for another reason. In the sequence (2.1), we have focused on GUTs. What about the Standard Model itself? It so happens that one standard way of obtaining $G_{SM} = SU(3) \times SU(2) \times U(1)$ from any of the GUT groups is precisely by quotienting. Group theoretically, this amounts to finding a discrete group whose generators can be embedded into the GUT group, so that the commutant is the the desired $G_{SM}$. Geometrically, this is the action of the **Wilson Line**, where a CY3 with non-trivial fundamental group admits a non-trivial loop which, coupled with the discrete group action, decomposes the $E_8$ further from the structure group $V$. In our example above, $\tilde{S}$ has trivial $\pi_1$ but the quotient $S'$ has, by construction $\pi_1(S') \simeq \mathbb{Z}_3$, whereby admitting a $\mathbb{Z}_3$ Wilson line. This, for the early models, can be used to break the $E_6$ GUT down to the Standard Model.

Not surprisingly, the manifold $S'$ became central to string phenomenology in the period after String Revolution [21]. Some even believed that they have found the geometry of the universe. More recently, using non-standard embedding, and by studying stable $SU(5)$ and $SO(10)$ bundles on non-simply-connected CY3, the first heterotic compactification with *exact* MSSM particle content were constructed, whereby realizing a 20-year old dream [24, 25]. Thus, once again mathematics and physics conspire to a parallel and co-extensive development.

[8] The individual Hodge numbers do not and it turns out that we have $(h^{1,1}, h^{2,1}) = (6,9)$ for $S'$.

### 3.3 A Plethora of CY3

Of the 7890, one could proceed to find various freely acting discrete symmetries and this was only lately accomplished [22], using today's desktop computer which is already far more powerful than the best super-computer back in the day. Many more candidates have been found.

Indeed, this brings us to the heart of a question, in nature both mathematical and physical: *how many CY3s are there*? In a way, we have an interesting sequence: in complex dimension 1, there is only the elliptic curve which is CY1, in dimension 2, as mentioned, there are 2. Starting in dimension 3, we till this day still have no idea how many distinct smooth manifolds are there, even though we have found literally billions. It was conjectured by Yau in the early days, that the number might be finite for CY3 (or indeed for Calabi-Yau manifolds of any dimension) [23] and it was moreover a fantasy of Miles Reid that they are all connected via topology-changing processes exemplified by conifold transitions. Sometimes, one is tempted to speculate how much more convenient if our spacetime were 8 or even 6 dimensional, in which case the compactification of superstrings would be quite facile.

#### 3.3.1 Weighted Hypersurfaces

After the success story of CICYs, the search continued. Another natural ambient space to have, in view of the quintic, is to take *weighted* projective space $\mathbb{CP}^4_{[d_0:\ldots:d_4]}$ where we recall this to be the quotient $\mathbb{C}^5 \backslash \{\vec{0}\} / \left( (z_0, z_1, \ldots, z_5) \sim (\lambda^{d_0} z_0, \ldots, \lambda^{d_5} z_5) \right)$ for some non-zero complex $\lambda$. Of course, taking all weights $d_i = 1$ is the ordinary $\mathbb{CP}^4$. We then embed a hypersurface of degree $d_0 + d_1 + \ldots + d_4$ therein, which defines a CY3. One caviat is that unlike ordinary projective space, weighted projective spaces are generically singular, and care must be taken to make sure the hypersurface avoids these singularities. The classification of such manifolds was performed in [26] and a total of 7555 is found, of which 28 have Euler number $\pm 6$. Of course, with the importance of Wilson Lines, we should no longer be limited $|\chi| = 6$, but rather those with non-trivial fundamental group and those with freely acting discrete groups of order $k$ which divides $\chi$.

#### 3.3.2 Elliptic Fibrations

The mid 1990s saw the "Second String Revolution" and with the advent of dualities and branes which linked the various string theories, the traditional heterotic compactification scenario subsequently experienced a period of relative cool compared to its incipience a decade earlier. Nevertheless, Calabi-Yau manifolds continued to occupy the center stage. String dualities rely on the equivalence of effective field theories after compactification on different

Calabi-Yau spaces. A true gem which emerged from this paradigm is clearly **mirror symmetry** which, due to such equivalences, predicted that to each CY3 with Hodge number $(h^{1,1}, h^{2,1})$, there should be a mirror pair with these exchanged. The geometrical implications are immense and the limitations of space cannot allow me to expound upon what clearly deserves a separate account.

Another highlight of this revolution on dualities and perhaps should be categorized as a Third String Revolution by itself, is the celebrated **AdS/CFT** Correspondence of Maldacena. This bring us to the closely neighbouring geography of *non-compact* Calabi-Yau manifolds, which are affine cones over Sasaki-Einstein 5-folds and which complement the AdS factor in the 10-dimensional metric. This is again a vast land which I do not have space here to describe and I shall, as mentioned in the introduction confine myself to smooth, compact CY3 and their constructions.

Due to the web of dualities, in particular that between the heterotic string and F-theory, there emerged another family of CY3 studied in the 1990s which has recently been investigated with renewed zest, this is the class of *elliptically fibred CY3* [27, 28]. This is a generalization of CY1, by allowing the elliptic curve to fibre over a complex base surface, by allowing the coefficients in the Weierstraß to take values in dual of the canonical bundle of the base, we would arrive at an overall trivial first class of the CY3 as total space.

What are the possible bases? Once again, this turned out to be a finite set: (I) Hirzebruch surfaces $\mathbb{F}_r$ for $r = 0, 1, \ldots, 12$; (II) $\mathbb{P}^1$-blowups of Hirzebruch surfaces $\widehat{\mathbb{F}}_r$ for $r = 0, 1, 2, 3$; (III) Del Pezzo surfaces $d\mathbb{P}_r$ for $r = 0, 1, \ldots, 9$; and (IV) Enriques surface $\mathbb{E}$. These are classical surfaces with whose precise definition we need not presently concern ourselves. Even though the list of possible bases seems limited, by tuning the possible elliptic curve, a diverse range of CY3 can be reached. This was much explored in the 1990s [31], uncovering a wealth of beautiful structure. With the help of modern computing, of the known CY3, many tens of thousands have been identified as elliptic fibrations [29, 30]; the full classification of this rich dataset is still in progress.

#### 3.3.3 Toric Hypersurfaces: The KS Database

The most impressive indentation into the uncharted land of CY3 so far is undoubtedly the so called **Toric Hypersurfaces**. Founded on the theoretic development of [32], Kreuzer and Skarke spent almost a decade compiling this database [33, 34]. Due to the untimely death of Max Kreuzer – who very touchingly was dedicated to the Calabi-Yau cause even in his last hours and continued, on his deathbed, to email us who were collaborating with him at the time – it became a pressing issue to attempt to salvage the data for posterity, a recent version of this legacy project is presented in [35].

In a nutshell, these manifolds extend the weighted projective case. Indeed, a toric variety is a very powerful generalization of weighted projective space in that, instead of having a single list of weights, we have a matrix of weights acting on a higher dimensional $\mathbb{C}^m$. We shall not delve into the elegant combinatorics of the theory of toric varieties but only summarize the key points of the construction here, much as we did for the CICY case above.

The ambient space is a toric 4-fold $A$, which is specified by an *integer polytope* $\Delta \in \mathbb{R}^4$ containing in particular the origin $(0,0,0,0)$ which can be represented by the list of its vertices, which are integer, or equivalently as a $k \times 4$ matrix of $k$ linear inequalities with integer coefficients. From this, one can define[9] the *dual polytope* $\Delta^\circ := \left\{\vec{v} \in \mathbb{R}^4 | \vec{m} \cdot \vec{v} \geq -1 \ \forall \vec{m} \in \Delta\right\}$. For our familiar example of $\mathbb{CP}^4$, an archetypal example of a toric 4-fold, we have that $\Delta = \begin{bmatrix} -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$ and $\Delta^\circ = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$. The idea is that if $\Delta$ is *reflexive*, i.e., if $\Delta^\circ$ has also integer vertices, such as the $\mathbb{CP}^4$ example, then the hypersurface

$$X = \left\{ \sum_{\vec{m} \in \Delta} c_{\vec{m}} \prod_{j=1}^{k} x_j^{\vec{m} \cdot \vec{v}_j + 1} = 0 \right\} \subset A \ ,$$

with $x_j$ coordinates of the ambient toric 4-fold, $c_{\vec{m}}$ complex coefficients, and $\vec{v}_j$ the (integer) vertices of $\Delta^\circ$, defines a CY3. For $\mathbb{CP}^4$, this is precisely the quintic hypersurface.

Thus the question of toric hypersurface CY3 is the question of reflexive integer 4-polytopes. In $\mathbb{R}^{n=1,2,3}$, there are 1, 16 and 4319 such polytopes, with the 16 in the plane famously giving us the toric del Pezzo surfaces and beyond. The computational challenge of Kreuzer and Skarke was to find all reflexive integer polytopes in $\mathbb{R}^4$. The actual calculation was performed on an SGI origin 2000 machine with about 30 processors (quite the state of the art in the 1990s) which took approximately 6 months and 473,800,776 was found. Each of these gives a hypersurface CY3 and thus from the database of tens of thousands established by the early 1990s, the list of CY3 suddenly grew, with this tour de force, to half a billion.

As with the weighted projective case, the ambient space $A$ is not necessarily smooth, so long as the hypersurface CY3 is. Interestingly, in this large family, only 125 have smooth ambient space $A$ and

[9] In the perhaps more familiar definition of a toric variety in terms of fans of cones, the fan $\Sigma$ is simply the faces of $\Delta^\circ$.

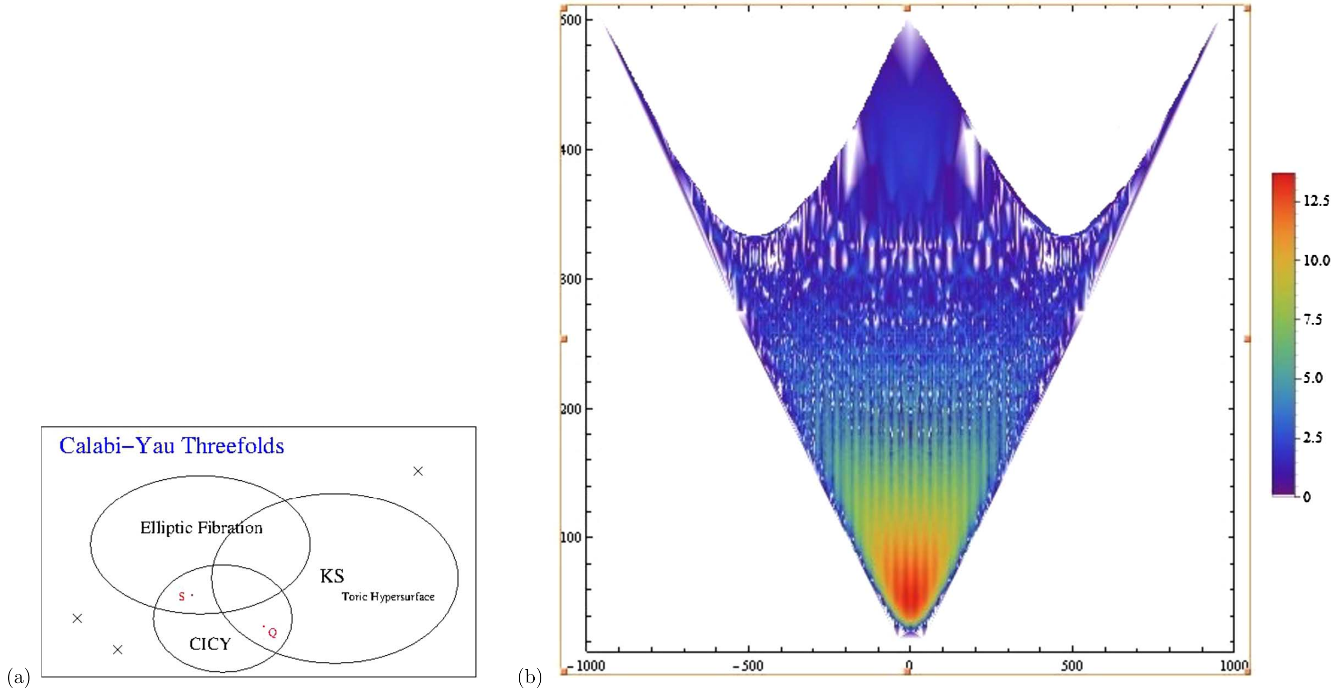


*Figure 1. (a) The space of CY3, with the 3 most studied datasets. There are also some individualized constructions outside the three major databases, symbolically marked as crosses. $Q$ is the quintic, $S$ is the Schoen CY3 and the most "typical" CY3 has Hodge numbers $(27,27)$, totaling almost 1 million. (b) Accumulating $\chi = 2(h^{1,1} - h^{1,2})$ (horizontal) versus $h^{1,1} + h^{1,2}$ (vertical) of all the known Calabi-Yau threefolds in a colour Log-density plot.*

more remarkably, only 16 have non-trivial fundamental group. Though of course a classification of discrete freely-acting symmetries has yet to be systematized from which one could potentially extract many more non-simply-connected CY3 by quotienting, these special 16 are quite interesting [36].

### 3.4 A Statistical Plot

While the cartography of CY3 continues with ever-increasing collaborative effort amongst physicists, mathematicians and computer scientists – for example, the classification of complete intersections in toric varieties, such as double hypersurfaces in 5-folds are well under way and billions have already been found – it is expedient that we draw our short excursion into the *terra sancta* of Calabi-Yau manifolds to a close.

There is an iconic plot: suppose we had $h^{1,1}(X) + h^{2,1}(X)$ in the ordinate versus $\chi = 2(h^{1,1}(X) - h^{2,1}(X))$ in the abscissa, drawn in part (b) of Figure 1. In part (a), we indicate our datasets discussed so far in a Venn diagram.

Let us pause to admire the beauty of this plot, the standard version of which is without the colour-density which I have added here. This standard black and white version is framed and features prominently in Philip Candelas' office. Several properties are of note. There are a total of 30,108 distinct points, meaning the some half-billion CY3 are severely degenerate in $(h^{1,1}, h^{1,2})$. The funnel shape delineating the lower extremes is just due to our plotting difference versus sum of the (non-negative) Hodge numbers. The fact the the figure is left-right symmetric is perhaps the best "experimental" evidence for mirror symmetry: that to each point with $\chi$ there should be one with $-\chi$, coming from the inter-change of the two Hodge numbers. There is a paucity of CY3 near the corners: near the bottom tip and of funnel and the top (note this is a log-density plot) while a huge concentration resides near the bottom center. In fact, the most "typical" CY3 thus far known is one with Hodge numbers $(27,27)$, numbering about 1 million.

There are several observations whose explanation remain mysterious. The largest Euler number in magnitude is 960 and so far no CY3 is known to have anything exceeding this. Is this an upper bound to the topology in the space of smooth CY3? One might note that 960 is twice the difference between the dimension and rank of $E_8 \times E_8$; this observation may not be as frivolous as may appear since the very structure of the exceptional groups seems encoded into these toric CY3 [37]. The parabolic shapes on the top of the shield-like funnel, are they also bounds to possible Hodge values? Recently, these have been identified as elliptically fibered CY3 [30].

## 4. Epilogue

With our tantalizing plot of all known Calabi-Yau threefolds let us pause here. We have taken a small promenade in the land of CY3, mindful of the intricate interplay between the mathematics and physics, emboldened by the plenitude of data and results, and inspired by the glimpses towards the yet inexplicable. The cartography of Calabi-Yau manifolds will certainly continue to provoke further exploration, especially with the advance of ever new mathematics, physics and computing.

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