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Spanning trees and orientations of graphs *

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A conjecture of Merino and Welsh says that the number of spanning trees $\tau(G)$ of a loopless and bridgeless multigraph G is always less than or equal to either the number $a(G)$ of acyclic orientations, or the number $c(G)$ of totally cyclic orientations, that is, orientations in which every edge is in a directed cycle. We prove that $\tau(G) \leq c(G)$ if G has at least $4n$ edges, and that $\tau(G) \leq a(G)$ if G has at most $16n/15$ edges. We also prove that $\tau(G) \leq a(G)$ for all multigraphs of maximum degree at most 3 and consequently $\tau(G) \leq c(G)$ for any planar triangulation.

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1. Introduction

One of the most fundamental properties of a connected graph is the existence of a spanning tree. Also the number $\tau(G)$ of spanning trees is an important graph invariant. The number of spanning trees plays a crucial role in Kirchhoff's classical theory of electrical networks, for example in computing driving point resistances. More recently, $\tau(G)$ is one of the values of the Tutte polynomial which now plays a central role in statistical mechanics. So are $a(G)$ and $c(G)$ defined in the abstract, and as a first step towards convexity properties of the Tutte polynomial, Merino and Welsh [10] conjectured that $\tau(G) \leq max\{a(G), c(G)\}$ for every loopless and bridgeless multigraph G, see also [6]. We shall here prove that $\tau(G) \leq c(G)$ for all loopless and bridgeless multigraphs with n vertices and at least $4n$ edges and that $\tau(G) \leq a(G)$ for all graphs (with no loops or multiple edges) with n vertices and at most $16n/15$ edges. We also investigate cubic graphs (which are in between these

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two bounds). McKay [9] proved that the maximum number of spanning trees in cubic multigraphs with n vertices is at most $2^{5/3}((16/3)^{n/2})$. For large cubic graphs, McKay [9] proved a stronger result which was proved to be best possible, within a constant multiplicative factor, by Chung and Yau [4].

We prove that

$$a(G) \geq (2/3) \cdot 6^{n/2} - 2 \cdot 3^{n/2-1}$$

for every 3-connected cubic graph with n vertices. This inequality, combined with McKay's upper bound on the number of spanning trees proves the conjecture of Merino and Welsh for all cubic graphs with at least 26 vertices. Brendan McKay (private communication) has kindly refined his result for $n < 26$ and thereby obtained an upper bound for $\tau(G)$ where G is cubic and with fewer than 26 vertices which is smaller than our lower bound on $a(G)$. Hence the conjecture by Merino and Welsh holds for all cubic graphs. This easily extends to all multigraphs of maximum degree at most 3.

The maximum number of oriented trees in oriented graphs was investigated by Lonc [8]. For diregular graphs with indegree and outdegree 2, the maximum is attained by certain circulants. The underlying undirected graphs do not have particularly many spanning trees.

The number of cycles in cubic graphs was investigated in [1, 2]. Intuitively, many spanning trees imply many cycles and vice versa. But, we have no indication that the extremal graphs for the two problems are the same or even similar. In fact, the prism (that is, the cartesian product of a K_2 and a cycle) is perhaps a cubic graph which has almost as many cycles as possible among the cubic graphs of that order, and it does not have particularly many spanning trees.

When we say *graph* we mean a graph with no loops and no multiple edges. When we say *multigraph* we allow multiple edges but no loops. If e is an edge in a multigraph G, then $G - e$ denotes the graph obtained by deleting e, and G/e denotes the graph obtained by contracting e, that is, we identify the ends of e and delete the loops that may arise. We shall apply the first of the well-know formulas

$$\begin{aligned} \tau(G) &= \tau(G-e) + \tau(G/e) \\ a(G) &= a(G-e) + a(G/e) \\ c(G) &= c(G-e) + c(G/e). \end{aligned}$$

The first of these hold for every edge. The second holds when e is not part of a multiple edge. The third holds when e is not part of a multiple edge, and e is not a bridge (cut-edge).

The *Tutte polynomial* $T(G,x,y)$ satisfies a similar recursion formula, and this can be used to prove that $\tau(G) = T(G,1,1)$, $c(G) = T(G,2,0)$, $a(G) = T(G,0,2)$ which was part of the motivation for the Merino-Welsh conjecture.

2. Degrees and spanning trees

Kostochka [7] proved that, if G is a graph with n vertices and vertex-degrees $d_1, d_2, \dots, d_n$, then

$$\tau(G) \le d_1 d_2 \dots d_n/(n-1).$$

We shall here prove a similar result for multigraphs.

Theorem 1. *Let G be a multigraph with n vertices and vertex-degrees $d_1, d_2, \dots, d_n$. Then*

$$\tau(G) \le d_1 d_2 \dots d_{n-1}$$

with equality if and only if either some d_i is zero or the vertex of degree d_n is incident with all edges.

Moreover, if M is any matching in G, then for each edge in M joining the vertices of degrees d_i, d_j, say, (both distinct from the vertex of degree d_n), the term $d_i d_j$ may be replaced by $d_i d_j - 1$ in the above product.

Proof of Theorem 1. We prove the theorem by induction on n, the number of vertices. The theorem is clearly true for multigraphs with one or two vertices. If some d_i is zero or if the vertex of degree d_n is incident with all edges, then clearly $\tau(G) = d_1 d_2 \dots d_{n-1}$. So assume that G has an edge e such that e is not incident with the vertex of degree d_n. Assume the notation has been chosen such that e joins the vertices v_1, v_2 of degrees d_1, d_2, respectively. Then

$$\begin{aligned}\tau(G) &= \tau(G-e) + \tau(G/e) \\ &\le (d_1-1)(d_2-1)d_3 d_4 \dots d_{n-1} + (d_1 - 1 + d_2 - 1)d_3 d_4 \dots d_{n-1} \\ &= (d_1 d_2 - 1) d_3 d_4 \dots d_{n-1} \\ &< d_1 d_2 \dots d_{n-1}.\end{aligned}$$

This proves the first part of Theorem 1. The last part is proved by the same argument where e is an edge of M. □

The sum of the vertex-degrees of a multigraph G is $2m$ where m is the number of edges of G. If we fix m and let d_n be the maximum degree, then the product of the $n-1$ smallest degrees is maximized when the n degrees are nearly equal. So we get

Corollary 1. *Let G be a multigraph with n vertices and m edges. Then*

$$\tau(G) \leq (2m/n)^{n-1}.$$

For dense graphs, Kostochka's inequality and its modified version in Theorem 1 are quite good. Cayley's formula says that the complete graph K_n has n^{n-2} spanning trees (see e.g. [3], page 103) whereas the first inequality in Theorem 1 gives the upper bound $(n-1)^{n-1}$, and the last inequality (using a matching) gives an even better result. Scoin's formula (see e.g. [3], page 108) says the complete bipartite graph $K_{p,q}$ has $p^{q-1}q^{p-1}$ spanning trees whereas the first inequality in Theorem 1 gives $p^{q-1}q^p$. Perhaps for all dense graphs, that is, graphs with $\Omega(n^2)$ edges, the ratio of the product of vertex-degrees and $\tau(G)$ is bounded above by a polynomial of n. A weaker result was proved by Kostochka [7]. He proved that, for graphs with n vertices and minimum degree k, the above-mentioned ratio is at most $k^{nO(log(k)/k)}$.

3. Spanning trees and totally cyclic orientations in multigraphs with many edges

In this section we verify the conjecture of Merino and Welsh for dense graphs. We begin with an observation on totally cyclic orientations.

Theorem 2. *Let G be a connected, bridgeless multigraph with n vertices and m edges. Then*

$$c(G) \geq 2^{m-n+1}.$$

Proof of Theorem 2. We prove the theorem by induction on m, the number of edges. If G is a cycle, then G has two totally cyclic orientations. So assume that G is not a cycle. We consider a cycle in G and we extend that cycle to a maximal connected, bridgeless proper subgraph H of G. Then G has a path or cycle P which begins and ends in H such that each intermediate vertex of P (if any) is outside of H. (Here we think of a cycle as a walk which starts and ends at the same vertex.) As $H \cup P$ is bridgeless, it follows that $G = H \cup P$. We now apply the induction hypothesis to H. As $c(G) \geq 2c(H)$, Theorem 2 follows. □

By combining Theorem 2 with Corollary 1 we get

Corollary 2. *Let G be a bridgeless multigraph with n vertices and m edges. If $m \geq 4n - 4$, then*

$$\tau(G) < c(G).$$

4. Spanning trees and acyclic orientations in graphs with few edges

Theorem 3. *Let G be a graph with n vertices and m edges. If $m \leq 16n/15$, then*

$$\tau(G) < a(G).$$

Proof of Theorem 3. For technical reasons we prove a slightly stronger statement: If s is a nonnegative real number and $m \leq 16n/15 + s/15$, then

$$\tau(G) < 2^s a(G).$$

We prove this statement by induction on m, the number of edges. We may assume that G is connected.

If G has a vertex of degree 1, we delete that vertex and use induction. So assume all vertices of G have degree at least 2.

If G is a cycle, it has n spanning trees and $2^n - 2$ acyclic orientations, so assume that G is not a cycle.

Now there exists a unique multigraph H such that G is a subdivision of H and H has no vertex of degree 2 except possibly vertices incident with double edges. As G is not a cycle, H is not a cycle of length 2. Let p, q denote the number of vertices and edges, respectively, of H.

We claim that $q \geq 4p/3 - 1/3$ with an equality holding if and only if H is obtained from a tree with vertices of degree $1, 3$ by adding, for each vertex x of degree 1 a new vertex x' joined to x by a double edge. We prove this claim by induction on p. It is easy to verify the statement when $p \leq 4$. So assume that $p > 4$. If all vertices have degree at least 3, then $q \geq 3p/2 > 4p/3 - 1/3$. So assume that x has degree 2. Then x has precisely one neighbor y. We apply induction to $G - x$ unless y has degree 1 or 2 and has two distinct neighbors in the latter case. In the latter case we replace the two edges leaving y in $G - x$ by one edge and use induction. In the former case we apply induction to $G - x - y$ possibly after replacing two edges by one edge if $G - x - y$ has a vertex of degree 2. (Only in this case we can have equality in the inequality we are proving.)

If H is a graph with precisely $4p/3 - 1/3$ edges described in the previous paragraph, then it is easy to verify Theorem 3. So assume that $q \geq 4p/3$. Let $e_1, e_2, \ldots, e_q$ denote the edges of H. Then G is obtained from H by inserting p_i vertices, say, on edge e_i for $i = 1, 2, \ldots, q$, where some p_i may be zero. Put $r = p_1 + p_2 + \cdots + p_q = n - p$. Then a spanning tree T of H can be chosen in less than 2^q ways. Consider now any spanning tree T of H. We use

T to construct a spanning tree in G by omitting, for each edge e_i outside T, one of the corresponding $p_i + 1$ edges in G. Hence

$$\tau(G) < 2^q(p_1 + 1)(p_2 + 1)\dots(p_q + 1) \le 2^q(r/q + 1)^q.$$

On the other hand, we may orient all of the $p_i + 1$ edges, except one, at random and still extend the resulting orientation to an acyclic orientation. Hence

$$a(G) > 2^{p_1} 2^{p_2} \dots 2^{p_q} = 2^r.$$

By the assumption of the theorem,

$$\begin{aligned} q + r = m &\le 16n/15 + s/15 = 16(p + r)/15 + s/15 \\ &\le (16/15)((3/4)q + r) + s/15 = (4/5)q + 16r/15 + s/15. \end{aligned}$$

and hence $s + r \ge 3q$. This implies that

$$\tau(G) \le 2^q(r/q + 1)^q \le 2^q((r + s)/q + 1)^q \le 2^{r+s} < 2^s a(G).$$

The second last inequality holds because $2^q(x/q + 1)^q \le 2^x$ for all real $x \ge 3q$. □

5. Acyclic orientations of graphs of maximum degree 3

We shall now describe lower bounds for the number of acyclic orientations of a cubic graph. If G is a graph, then a *suspended path* in G is a path such that the ends have degree at least 3 in G and all intermediate vertices have degree 2 in G.

Theorem 4. *If G is a 3-connected cubic graph with n vertices, then for all n,*

$$a(G) \ge (2/3) \cdot 6^{n/2} - 2 \cdot 3^{n/2-1}.$$

Proof of Theorem 4. We first delete an edge e from G. We claim that we can successively delete edges of suspended paths of length at least 2 in such a way that, at each stage, the current graph has only one component with edges, and this component is bridgeless.

To prove this claim, suppose we have deleted a number of suspended paths of length at least 2 such that the resulting graph has only one component G' containing edges, and this G' is bridgeless. Let H be the unique cubic multigraph such that G' is a subdivision of H. If H is 3-connected, we can use any suspended path of length at least 2 in G'. So assume that H

contains two edges e_1, e_2 such that $H - e_1 - e_2$ is disconnected. Choose e_1, e_2 such that the smallest component H' of $H - e_1 - e_2$ is smallest possible. As H is cubic, H' contains at least one edge. As G is 3-connected, some edge e of H' must correspond to a suspended path P of length at least 2 in G'. The minimality of H' implies that $H - e$ is bridgeless. Hence $G' - E(P)$ has only one component containing edges, and this component is bridgeless. This proves the above claim.

Using this claim, we delete successively edges of suspended paths of length at least 2 until the current graph is a cycle with r_1 edges, say. This cycle has $2^{r_1} - 2$ acyclic orientations. Then we put the suspended paths back in reverse order. Let $r_2, r_3, \ldots, r_k$ be their numbers of edges, respectively. If we put back a suspended path with r_i edges, then, for every orientation of the current graph, the r_i edges can be oriented in 2^{r_i} ways and at most one of these orientations create a directed cycle. Each time we add a suspended path, the number of edges minus the number of vertices increases by 1. Hence $k = n/2$. (Note that the single edge we deleted to begin with should not be counted.) Also,

$$r_1 + r_2 + \cdots + r_k = 3n/2 - 1.$$

The product of orientations counted above is minimized when $r_2 = r_3 = \cdots = r_k = 2$ and consequently $r_1 = n/2 + 1$.

This proves Theorem 4. □

Theorem 5. *If G is a multigraph of maximum degree 3, then $\tau(G) \leq a(G)$.*

Proof of Theorem 5. The proof is by induction on the number of edges. We may assume that G is connected.

If G has a bridge, we delete it and apply induction to the components of the remaining graph. If G has a vertex v of degree 2 incident with a double edge, we delete v and use induction. If G has a vertex v of degree 2 incident with two edges vv_1, vv_2, where v_1, v_2 are distinct, then we delete v and add the edge v_1v_2 instead, and we call the resulting graph H. Clearly $\tau(G) < 2\tau(H)$, and $a(G) \geq 3a(H)$, so we complete the proof by applying induction to H. If G has a path (or cycle) $xyzu$, where y, z are joined by a double edge, then we delete y, z. If x, u are distinct we add the edge xu instead, and we call the resulting graph H. If x, u are distinct, then clearly $\tau(G) < 5\tau(H)$, and $a(G) \geq 7a(H)$, so we complete the proof by applying induction to H. If $x = u$, then $\tau(G) = 5\tau(H)$, and $a(G) = 6a(H)$, so we complete the proof by applying induction to H.

So we may assume that G is a connected bridgeless cubic graph. Consider now the case where G is not 3-connected. That is, G contains two edges

$e_1 = x_1x_2$ and $e_2 = y_1y_2$ such that $G - e_1 - e_2$ has two components H_1, H_2 such that H_i contains x_i, y_i for $i = 1, 2$. Let G_i be obtained from H_i by adding the edge x_iy_i, also if that edge is already present. If a spanning tree in G contains e_1, e_2 then that tree has a path between e_1 and e_2. Suppose this path is in H_1. Then the part of the spanning tree which is in H_2 is a forest which becomes a spanning tree when we add the edge x_2y_2. Thus the number of such spanning trees is $\tau(H_1)\tau'(G_2)$ where $\tau'(G_2)$ is the number of spanning trees in G_2 containing the edge x_2y_2. Similarly the number of spanning trees containing e_1, e_2 and also containing a path in H_2 connecting these two edges is $\tau(H_2)\tau'(G_1)$. The number of spanning trees containing precisely one of e_1, e_2 is $2\tau(H_1)\tau(H_2)$. Thus

$$\begin{aligned}\tau(G) &= \tau(H_1)\tau'(G_2) + \tau(H_2)\tau'(G_1) + 2\tau(H_1)\tau(H_2)\\ &= \tau(H_1)\tau(G_2) + \tau(H_2)\tau(G_1).\end{aligned}$$

For $i = 1, 2$, we let $a_1(H_i)$ (respectively $a_0(H_i)$) denote the number of acyclic orientations of H_i which contain (respectively do not contain) a directed path between x_i, y_i. Then $a(H_i) = a_0(H_i) + a_1(H_i)$, and $a(G_i) = 2a_0(H_i) + a_1(H_i)$, for $i = 1, 2$.

Also $a(G) = 4a(H_1)a(H_2) - 2(a_1(H_1)/2)(a_1(H_2)/2)$ because e_1, e_2 can be oriented in 4 ways and we must subtract those orientations which create directed cycles. So, $a(G) = 4a_0(H_1)a_0(H_2) + 4a_0(H_1)a_1(H_2) + 4a_1(H_1)a_0(H_2) + (7/2)a_1(H_1)a_1(H_2)$.

By induction,

$$\begin{aligned}\tau(G) &= \tau(H_1)\tau(G_2) + \tau(H_2)\tau(G_1)\\ &\le (a_0(H_1) + a_1(H_1))(2a_0(H_2) + a_1(H_2))\\ &\quad + (a_0(H_2) + a_1(H_2))(2a_0(H_1) + a_1(H_1))\\ &= 4a_0(H_1)a_0(H_2) + 3a_0(H_1)a_1(H_2) + 3a_1(H_1)a_0(H_2)\\ &\quad + 2a_1(H_1)a_1(H_2)\\ &\le a(G).\end{aligned}$$

So we may assume that G is a 3-connected cubic graph with n vertices, say. If $n > 26$, then the lower bound on $a(G)$ in Theorem 4 combined with the upper bound on $\tau(G)$ by McKay [9] proves that $\tau(G) \le a(G)$.

Gordon Royle (private communication) has found the maximum number of spanning trees in a cubic graph with n vertices for $n = 4, 6, \ldots, 22$ in the first row below.

Brendan McKay (private communication) has refined the methods of [9] to obtain upper bounds for the number of spanning trees in a cubic multigraph with n vertices for $n = 24, 26$. These are the last two numbers in the first row below.

The corresponding lower bounds on $a(G)$ in Theorem 4 are given in the last row below.

n	4	6	8	10	12	14	16	18	20	22	24	26
max $\tau(G)$	16	81	392	2000	9800	50421	248832	1265625	6422000	32710656	$\leq 8.6\cdot 10^8$	$\leq 4.5\cdot 10^9$
min $a(G)$	18	126	810	5000	$3\cdot 10^4$	$1.8\cdot 10^5$	$1.1\cdot 10^6$	$6.7\cdot 10^6$	$4\cdot 10^7$	$2.4\cdot 10^8$	$1.4\cdot 10^9$	$8.7\cdot 10^9$

As the numbers in the last row are bigger than the corresponding numbers in the first row, the proof of Theorem 5 is complete. □

Theorem 6. *If G is a planar triangulation, then*

$$\tau(G) \leq c(G).$$

Proof of Theorem 6. Let H denote the geometric dual graph of G. Then $\tau(G) = \tau(H) \leq a(H) = c(G)$, by Theorem 5. □

6. Open problems

Problem 1. *Let G be a bridgeless multigraph with n vertices and m edges. Is it true that*

$$\tau(G) \leq a(G) \text{ when } m \leq 2n - 2 \text{ and}$$

$$\tau(G) \leq c(G) \text{ when } m \geq 2n - 2?$$

The multigraph obtained from a path with n vertices by replacing every edge by a double edge is a multigraph G for which $\tau(G) = a(G) = c(G) = 2^{n-1}$. By adding a third edge between two consecutive vertices we obtain multigraphs showing that the bound $2n - 2$ cannot be improved in the first inequality. Its planar dual multigraph shows that the bound $2n - 2$ cannot be improved in the second inequality either.

We now focus on acyclic orientations of cubic graphs. We construct a nearly cubic graph by taking a sequence of k pairwise disjoint complete graphs on 4 vertices each. In each copy we delete an edge and obtain a *diamond*. We join a vertex of degree 2 in each diamond with a vertex of degree 2 in the next diamond. The resulting graph has $n = 4k$ vertices, and precisely two vertices have degree 2. The number of acyclic orientations is $18^k 2^{k-1} = 6^{n/2}/2$. If we add an edge between the two vertices of degree 2, we obtain a 2-connected cubic graph with less than $6^{n/2}$ acyclic orientations which is close to the lower bound in Theorem 4. This graph also shows that a 2-connected cubic graph cannot be constructed from a cycle as we did in Theorem 4. Instead one can show that every 2-connected cubic graph (minus an edge) can be constructed from a cycle by adding paths of length at least 2 and lollipops, where addition of a lollipop means addition of a cycle disjoint from the current graph together with a path connecting the new cycle with the current graph. A modification of the proof of Theorem 4 then gives the following which is best possible within a factor 2:

Theorem 7. *Every 2-connected cubic graph has at least $6^{n/2}/2$ acyclic orientations.*

It is also natural to ask how much Theorem 4 can be strengthened for 3-connected graphs.

The Cartesian product of a path with $n/2$ vertices and a K_2 is called a *ladder*. It has n vertices, and it is easy to prove, by induction on n, that it has precisely $2 \cdot 7^{n/2-1}$ acyclic orientations. The Cartesian product of a cycle with $n/2$ vertices and a K_2 is called a *prism*. As it is obtained from a ladder by adding two edges it has less than $8 \cdot 7^{n/2-1}$ acyclic orientations. (The precise number can be found by evaluating the chromatic polynomial of the prism at -1.)

Problem 2. *Does there exist a 3-connected cubic graph which has less than $7^{n/2}n^{-1000}$ acyclic orientations?*

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