SYMPLECTIC DEFORMATIONS OF KÄHLER MANIFOLDS

PAOLO DE BARTOLOMEOIS

Given a compact symplectic manifold \((M, \kappa)\), \(H^2(M, \mathbb{R})\) represents, in a natural sense, the tangent space of the moduli space of germs of deformations of the symplectic structure. In the case \((M, \kappa, J)\) is a compact Kähler manifold, the author provides a complete description of the subset of \(H^2(M, \mathbb{R})\) corresponding to Kähler deformations, including the non-generic case, where (at least locally) some hyperkähler manifold factors out from \(M\). Several examples are also discussed.

1. Introduction

The naïf deformation theory of symplectic manifolds is quite simple: let \((M, \kappa)\) be a compact symplectic manifold and let \(\alpha \in \Lambda^2(M, \mathbb{R}), \ d\alpha = 0\): then

\[
\kappa_t := \kappa + t\alpha
\]

is a (germ of) curve of symplectic structures having tangent \(\alpha\) at 0; moreover, Moser’s lemma (cf. [2]) ensures that \(\kappa_t = \phi_t^\ast(\kappa)\) for a path of diffeomorphisms with \(\phi_0 = id_M \iff \alpha = d\beta\) and so \(H^2(M, \mathbb{R})\) is the tangent space of the moduli space of germs of deformations of symplectic structures and the theory is totally unobstructed (for a non-naïf version, see [1]).

Let \((M, \kappa, J)\) be a compact Kähler manifold: therefore, \(J\) is a \(\kappa\)-calibrated holomorphic structure and so \(g = g_J := \kappa(J\cdot, \cdot)\) is a positive definite Hermitian metric; we want to investigate the subset of \(H^2(M, \mathbb{R})\) corresponding to Kähler deformations of \(\kappa\).

We have the following

**Theorem 1.1.** Let \((M, \kappa, J)\) be a compact Kähler manifold; let \(\mathcal{K}\) be the subset of \(H^2(M, \mathbb{R})\) corresponding to Kähler deformations of \(\kappa\); i.e., \([\alpha] \in \mathcal{K}\) if and only if there exists a curve of Kähler structures \((\kappa_t, J_t)\) with \(\kappa_t = \kappa + t\alpha + o(t), \ J_0 = J;\)
then:
\[ \mathcal{K} = \mathcal{P}^{2,0+0,2} \oplus H^{1,1}(M, \mathbb{R}) \]

where
\[ \mathcal{P}^{2,0+0,2}(M) := \{ a \in H^{2,0+0,2}(M, \mathbb{R}) | \nabla^M h(a) = 0 \} \]

and \( h(a) \) is the \( g \)-harmonic representative of \( a \).

Note that, clearly, \( \mathcal{P}^{2,0+0,2}(M) \) is generically reduced to \( \{0\} \) and, if it is not the case, then (at least locally) some hyperkähler manifold factors out from \( M \).

The author is pleased to thank the referee for valuable remarks and suggestions for a better presentation of the results.

2. Reduction to the \((2, 0 + 0, 2)\)-case

We have first the following

**Lemma 2.1.** Let \((M, \kappa, J)\) be a compact Kähler manifold; then:
\[ \mathcal{K} + H^{1,1}(M, \mathbb{R}) = \mathcal{K} \]

i.e., for every \( a \in \mathcal{K} \), every \( c \in H^{1,1}(M, \mathbb{R}) \), we have \( a + c \in \mathcal{K} \).

**Proof.** Let \( \alpha \in \wedge^2(M, \mathbb{R}), d\alpha = 0 \), such that \([\alpha] \in \mathcal{K}\).
Given \( c \in H^{1,1}(M, \mathbb{R}) \), let \( \gamma \in \wedge^{1,1}_J(M) \) be its harmonic representative; by assumption, there is a curve of Kähler structures \((\kappa_t, J_t)\) with \( \kappa_t = \kappa + t\alpha + o(t) \); by Kodaira–Spencer theory, the projection
\[ P_t : \wedge^{1,1}_J(M) \rightarrow \mathcal{H}^{1,1}_{g_{J_t}}(M) \]
(where, of course, \( \mathcal{H}^{1,1}_{g_{J_t}}(M) \) is the space of \( g_{J_t}\)-harmonic \((1, 1)\)-forms on \( M \)) is smooth in \( t \) (see e.g., [3], p. 184).
Let
\[ \tilde{\kappa}_t := \kappa_t + \frac{1}{2} t(\gamma + J_t\gamma), \]
i.e.,
\[ \tilde{\kappa}(X, Y) = \kappa_t(X, Y) + \frac{1}{2} t(\gamma(X, Y) + \gamma(J_tX, J_tY)) \]
and
\[ \tilde{\kappa}_t := P_t(\tilde{\kappa}_t) = \kappa_t + \frac{1}{2} tP_t(\gamma + J_t\gamma). \]
Clearly \((\tilde{\kappa}_t, J_t)\) is a curve of Kähler structures (note: the same \( J_t\)'s!) and
\[ \frac{d\tilde{\kappa}_t}{dt}_{|t=0} = \alpha + \frac{1}{2} P_0(\gamma + J\gamma) = \alpha + \gamma. \]
3. The main result

Let us first recall the basic linear algebraic frame: let \((T, J, g)\) be a Hermitian vector space, i.e., a real vector space \(T\) equipped with \(J \in \text{End}(T)\) satisfying \(J^2 = -I\) and a positive definite scalar product \(g\) on \(T\) satisfying
\[g(JX, JY) = g(X, Y);\]
then
\[T^\mathbb{C} = T^{1,0} \oplus T^{0,1}\]
and
\[\nu: T \longrightarrow T^{1,0}, \quad \nu(X) := \frac{1}{2}(X - iJX)\]
is a linear isomorphism such that \(\nu(JX) = i\nu(X)\).

Let \(V \in \text{End}(T)\) with \(VJ + JV = 0\);
then, we obtain
\[V: T^{0,1} \longrightarrow T^{1,0} \quad \mathbb{C} - \text{linear}\]
simply setting
\[V(X + iJX) = V(X) - iJV(X)\]
(i.e., \(V\) acts now as \(\nu \circ V \circ \nu^{-1}\)); this identifies canonically \((T^*)^{0,1} \otimes T^{1,0}\)
with \(\{V \in \text{End}(T) \mid VJ + JV = 0\}\).

If, moreover, \(V = -\overline{V}\), then, setting
\[\alpha(X, Y) := g(V(X), Y),\]
we obtain \(\alpha \in \wedge^{2,0+0.2}T^*\) and
\[\alpha^{2,0}(X, Y) = \frac{1}{2}(\alpha(X, Y) - i\alpha(JX, Y)),\]
i.e., in terms of the complexified space,
\[\alpha = \gamma + \overline{\gamma},\]
with
\[\gamma \in \wedge^{2,0}T^* \quad \gamma(Z, W) = \overline{g(V(Z), W)}.\]

Let \((M, \kappa, J)\) be a compact Kähler manifold and let \((A, [,], \bar{\partial}_J)\) be the DGLA governing the holomorphic deformation theory of \((M, J)\):
\[A = \bigoplus_{p \in \mathbb{Z}} A_p\]
where
\[A_p = \begin{cases} \wedge^p_{\bar{\partial}_J}(M) \otimes T^{1,0}M, & \text{if } 0 \leq p \leq n \\ 0, & \text{otherwise} \end{cases}\]
and \([,]\) is the (complex) Schouten–Nijenhuis bracket (see e.g., [3], p. 152);
in particular, if \( U, V \in \mathcal{A}_1 \) and, in terms of local holomorphic coordinates \( z_1, \ldots, z_n \),
\[
U = \sum_{j,k=1}^{n} a_{j,k} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} a^{(j)} \frac{\partial}{\partial z_j}
\]
\[
V = \sum_{j,k=1}^{n} b_{j,k} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} b^{(j)} \frac{\partial}{\partial z_j},
\]
with:
\[
a^{(j)} = \sum_{k=1}^{n} a_{j,k} d\bar{z}_k, \quad 1 \leq j \leq n
\]
\[
b^{(j)} = \sum_{k=1}^{n} b_{j,k} d\bar{z}_k, \quad 1 \leq j \leq n,
\]
then:
\[
[U, V] = \sum_{j,k=1}^{n} \left( a^{(j)} \wedge \frac{\partial b^{(k)}}{\partial z_j} + b^{(j)} \wedge \frac{\partial a^{(k)}}{\partial z_j} \right) \frac{\partial}{\partial z_k}
\]
\[
= \sum_{j,k=1}^{n} \sum_{r<s} \left( a_{jr} \frac{\partial b_{ks}}{\partial z_j} - b_{js} \frac{\partial a_{kr}}{\partial z_j} \right) d\bar{z}_r \wedge d\bar{z}_s \otimes \frac{\partial}{\partial z_k}
\]
and so
\[
(1) \quad [U, V] \left( \frac{\partial}{\partial z_r}, \frac{\partial}{\partial \bar{z}_s} \right) = \left[ U \left( \frac{\partial}{\partial z_r} \right), V \left( \frac{\partial}{\partial \bar{z}_s} \right) \right].
\]
(Of course, for general vector fields \( X, Y \), \([U, V](X, Y) \neq [U(X), V(Y)]\)).

Note that, via \( \nu \), we can put the theory in a completely real setting, where:
\[
\mathcal{A}_p = \wedge_J^{0,p}(M) \otimes TM = \{ R \in \wedge^p(M) \otimes TM \mid R(X_1, \ldots, JX_h, \ldots, X_p) = -JR(X_1, \ldots, X_h, \ldots, X_p), \ 1 \leq h \leq p \} \text{ and, with a slight abuse of notation,}
\]
\[
[R \ast S] = \nu^{-1}[\nu(R), \nu(S)];
\]
e.g., for \( p = 0 \):
\[
[X \ast Y] = \frac{1}{2}([X, Y] - [JX, JY]).
\]

We shall confine to the complex form of the theory.

Let
\[
\Box := \partial_j \partial^*_j + \partial^*_j \partial_j : \mathcal{A} \to \mathcal{A}
\]
and let
\[
\nabla^{TM} = \nabla' + \nabla'' : \text{End}(TM) \to \wedge^1(M) \otimes \text{End}(TM)
\]
\[
\nabla^M : \wedge^* (M) \to \wedge^1(M) \otimes \wedge^* (M)
\]
be the exterior covariant differential operators with respect to the Levi–Civita connection (which coincides, in the Kähler case, with the Hermitian canonical connection).
Let $V \in \text{End}(TM)$ such that $JV + VJ = 0$ and so, in particular

$$V \in \wedge^{0,1}(M) \otimes T^{1,0}M;$$

let $\alpha = \gamma + \bar{\gamma} \in \wedge^{2,0+0,2}(M, \mathbb{R})$ be defined by:

$$\alpha(X, Y) = \frac{1}{2}g((V - \bar{V})X, Y);$$

therefore, in terms of normal local holomorphic coordinates $z_1, \ldots, z_n$, we have

$$V = \sum_{j,k=1}^{n} b_{jk}d\bar{z}_k \otimes \frac{\partial}{\partial z_j}$$

$$\bar{V} = \sum_{j,k=1}^{n} p_{jk}d\bar{z}_k \otimes \frac{\partial}{\partial z_j}$$

with

$$p_{jk} = \frac{1}{n} \sum_{r,s} g^{r\bar{j}}b_{sr}g_{sk}$$

and

$$\gamma = \sum_{j<k} c_{jk}dz_j \wedge dz_k$$

with

$$\frac{1}{2}(b_{jk} - \bar{p}_{jk}) = \sum_{r=1}^{n} g^{r\bar{j}}c_{kr}. $$

Therefore, if $B = (b_{jk})$, $P = (p_{jk})$, $G = (g_{jk})$, $C = (c_{jk})$, then:

(2) $$P = \bar{G}^{-1}BG$$

(3) $$p_{jk} = b_{kj} + o(|z|)$$

(4) $$c_{kj} = \frac{1}{2}(\bar{b}_{jk} - \bar{b}_{kj}) + o(|z|)$$
Note first that, performing the computation at the origin 0 of the system of normal holomorphic coordinates,

\[
\bar{\partial} J V = 0 \iff \frac{\partial b_{\bar{j} \bar{k}}}{\partial z_r} = \frac{\partial b_{\bar{j} \bar{p}}}{\partial z_k}, \quad 1 \leq j, k, r \leq n
\]

\[
\iff \frac{\partial b_{\bar{j} \bar{k}}}{\partial z_r} = \frac{\partial b_{\bar{j} \bar{p}}}{\partial z_k}, \quad 1 \leq j, k, r \leq n
\]

\[
\implies 2 \frac{\partial c_{kj}}{\partial z_r} + 2 \frac{\partial c_{jr}}{\partial z_k} + 2 \frac{\partial c_{rk}}{\partial z_j} = (\text{by (4)})
\]

\[
\frac{\partial b_{\bar{j} \bar{k}}}{\partial z_r} - \frac{\partial b_{\bar{k} \bar{j}}}{\partial z_r} = \frac{\partial b_{\bar{j} \bar{p}}}{\partial z_k} - \frac{\partial b_{\bar{p} \bar{j}}}{\partial z_k} + \frac{\partial b_{\bar{k} \bar{p}}}{\partial z_j} - \frac{\partial b_{\bar{p} \bar{k}}}{\partial z_j} = 0
\]

\[
1 \leq j, k, r \leq n
\]

\[
\iff \partial J \gamma = 0.
\]

We have now the following.

**Lemma 3.1.** Let \((M, \kappa, J)\) be a compact Kähler manifold; let \(\alpha \in \wedge^{2,0+0,2}(M, \mathbb{R})\), \(\alpha = \gamma + \bar{\gamma} = g(V \cdot, \cdot)\) (and so, in particular, \(V \in \text{End}(TM)\) with \(JV + VJ = 0\) and \(V = -tV\)); then:

1. \(\Box V = 0 \iff \nabla^{TM} V = 0\)
2. \(\nabla^M \gamma = 0 \iff \nabla^{TM} V = 0\)
3. \(\nabla^{TM} V = 0 \implies [V, V] = 0\)

**Proof.** 1: we have

\[
\Box V = 0 \iff \begin{cases}
\partial_j V = 0 \\
\bar{\partial}_j V = 0;
\end{cases}
\]

now, in terms of previous notations, and so, once more at 0 of our system of normal local holomorphic coordinates:

\[
b_{jk} = \sum_{r=1}^{n} g^{\bar{r} \bar{j}} c_{kr}
\]

with:

\[
\frac{\partial b_{\bar{j} \bar{k}}}{\partial z_r} = \frac{\partial b_{\bar{j} \bar{p}}}{\partial z_k}, \quad 1 \leq j, k, r \leq n
\]

and the extra condition

\[
b_{kj} = -b_{\bar{j} \bar{k}} + o(|z|)
\]

and so, setting

\[
A_{jk}^r := \frac{\partial b_{\bar{j} \bar{k}}}{\partial z_r},
\]
we obtain, by (5) and (6):
\[ A^r_{jk} = -A^r_{kj} = A^k_{jr} = -A^k_{jr} = A^j_{rk} = -A^j_{rk} = 0, \]
i.e.,
\[ \bar{\partial}_j V = 0 \& V = \bar{\partial} V \implies \nabla'' V = 0; \]
also,
\[ \bar{\partial}_j V = 0 \iff \sum_{k=1}^{n} \frac{\partial b_{jk}}{\partial z_k} = 0 \iff \partial_j \gamma = 0, \quad 1 \leq j \leq n; \]
consequently,
\[ \Box V = 0 \implies \Box M \gamma = 0 \iff \Delta \gamma = 0; \]
finally,
\[ \partial \gamma = 0 \iff \partial_{cjk} = 0 \iff \frac{\partial b_{jk}}{\partial z_r} = 0 \iff \nabla' V = 0 \]
clearly \( \nabla' TM V = 0 \implies \Box V = 0 \) and so all the arrows can be reversed;
2: it’s a general Riemannian fact that
\[ (\nabla_X \alpha)(Y, Z) = g((\nabla_X V)Y, Z) \]
3: we have (cf. (1)):
\[ [V, V] \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right) = \left[ V \left( \frac{\partial}{\partial z_j} \right), V \left( \frac{\partial}{\partial z_k} \right) \right] \]
\[ = \sum_{r,s=1}^{n} \left[ b_{rj} \frac{\partial}{\partial z_r}, b_{sk} \frac{\partial}{\partial z_s} \right] \]
\[ = \sum_{r,s=1}^{n} \left( b_{rj} \frac{\partial b_{sk}}{\partial z_s} - b_{sk} \frac{\partial b_{rj}}{\partial z_s} \right) = 0 \]

We have now the following

**Lemma 3.2.** Let \((M, \kappa, J)\) be a Kähler manifold; given \(L \in \wedge^{0,1}(M) \otimes T^{1,0} M\), we have:
\[ \Box L = L \Box L \]

**Proof.** Let \(z_1, \ldots, z_n\) be local normal holomorphic coordinates; then at 0 the curvature tensor of \(g\) is given by:
\[ R_{abjk} = \frac{\partial^2 g_{ab}}{\partial z_j \partial \bar{z}_k} \]
and
\[ R_{abjk} = R_{jba} = R_{jka} = R_{akjb} \]
moreover, the Ricci tensor is given by
\[ R_{j\bar{k}} = -\sum_{r=1}^{n} R_{jkr\bar{r}}, \]
i.e., setting \( R = (R_{j\bar{k}}) \), we have:
\[ R = \frac{1}{2} \Delta_M G. \]

Let now
\[ L = \sum_{j,k=1}^{n} l_{j\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} \lambda^{(j)} \otimes \frac{\partial}{\partial z_j} \]
with, clearly, \( \lambda^{(j)} = \sum_{k=1}^{n} l_{j\bar{k}} d\bar{z}_k, \ 1 \leq j \leq n; \)
then we have (cf. e.g., [3], pp. 101–102):
\[ \bar{\partial}^* L = \sum_{j=1}^{n} \left( -* \partial * \lambda^{(j)} - \sum_{r,s=1}^{n} g^{\bar{s}j} *(\partial g_{r\bar{s}} \wedge *\lambda^{(r)}) \right) \frac{\partial}{\partial z_j} \]
and so, at 0:
\[ \bar{\partial} \bar{\partial}^* L = \sum_{j=1}^{n} \left( -\bar{\partial} (*\partial *) \lambda^{(j)} - \sum_{r=1}^{n} \bar{\partial} * (\partial g_{r\bar{j}} \wedge *\lambda^{(r)}) \right) \otimes \frac{\partial}{\partial z_j}, \]
while
\[ \bar{\partial}^* \bar{\partial} L = \sum_{j=1}^{n} \left( -* \partial * \bar{\partial} \lambda^{(j)} \right) \otimes \frac{\partial}{\partial z_j}; \]
therefore,
\[ \Box L = \sum_{j=1}^{n} \left( \frac{1}{2} \Delta \lambda^{(j)} - \sum_{r=1}^{n} \bar{\partial} * (\partial g_{r\bar{j}} \wedge *\lambda^{(r)}) \right) \otimes \frac{\partial}{\partial z_j}; \]
now:
\[ \frac{1}{2} \Delta \lambda^{(j)} = \left( \frac{1}{2} \Delta l_{j\bar{k}} - \sum_{r=1}^{n} R_{r\bar{k}l_{j\bar{r}}} \right) d\bar{z}_k \]
and
\[ \bar{\partial} * (\partial g_{r\bar{j}} \wedge *\lambda^{(r)}) = \sum_{k,p=1}^{n} R_{rj\bar{k}}l_{r\bar{p}}d\bar{z}_k; \]
consequently:
\[ \Box L = \sum_{j,k=1}^{n} \frac{1}{2} \Delta_M l_{jk} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \]
\[ - \sum_{j,k=1}^{n} \left( \sum_{r=1}^{n} R_{rk,lj} \right) d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \]
\[ - \sum_{j,k=1}^{n} \left( \sum_{r,s=1}^{n} R_{rjsk} l_{rs} \right) d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \]
\[ \frac{1}{2} \Delta_M A - AR - C(A) \]
where \( A := (l_{jk}) \);

now:
\[ ^tL = \sum_{j,k=1}^{n} p_{jk} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \]
with
\[ p_{jk} = \sum_{r,s=1}^{n} g^{rj} l_{sp} g_{sk} . \]

Set \( B := (p_{jk}) \); then:
\[ B = G^{-1} A G \quad \text{and} \quad B(0) = ^tA(0) ; \]

now (always at 0):
\[ \Delta_M B = (\Delta_M G^{-1}) A + \Delta_M ^tA + A \Delta_M G = \Delta_M ^tA + 2^tAR - 2\bar{R}A ; \]

consequently:
\[ \Box ^tL = \frac{1}{2} \Delta_M ^tA + ^tAR - \bar{R}A - ^tAR - C(\text{t}A) \]
\[ = \frac{1}{2} \Delta_M ^tA - \bar{R}A - C(\text{t}A) . \]

Finally, from
\[ \bullet \quad ^tR = \bar{R}, \]
\[ \bullet \quad C(\text{t}A) = C(A) = ^tC(A) , \]
we obtain the result.  

**Corollary 3.3.** If \( L \in \Lambda^{0,1}(M) \otimes T^{1,0} M \) is \( \Box \)-harmonic, so are \( 1/2(L - ^tL) \) and \( 1/2(L + ^tL) \).

We are now in position to prove our main result, i.e., Theorem (1.1);

**Proof. (a):** let \( \alpha \in \mathcal{P}^{2,0+0.2} ; \) write \( \alpha = g(V \cdot , \cdot ) ; \)

therefore, by Lemma 3.1, \( V \) satisfies

\[ \nabla^{TM} V = 0 \]
• $[V, V] = 0$

and clearly this is also the case for $L := \frac{1}{2} V J$;

in fact:

$t(VJ) = JV = -VJ$

and

$\nabla^{TM} V J = (\nabla^{TM} V) J + V (\nabla^{TM} J) = 0$;

then apply Lemma 3.1 (3).

Consequently, we have

$\bar{\partial} J t L + \frac{1}{2} t^2 [L, L] = 0$

and so

$J_t := (id_M + t L) J (id_M + t L)^{-1}$

is a holomorphic structure satisfying $\nabla^{TM} J_t = 0$;

consequently,

$\kappa_t := \frac{1}{2} (\kappa + J_t \kappa)$

is parallel, thus it is closed, and $\kappa_t = \kappa + t \alpha + o(t)$;

therefore, $\alpha$ is tangent to the curve of Kähler structures $(\kappa_t, J_t)$;

(b): let $(\kappa_t, J_t)$ be a curve of Kähler structures with

$\kappa_t = \kappa + t \alpha + o(t), \quad \alpha \in \Lambda^{2,0+0,2}(M)$;

by the basic features of holomorphic deformation theory, we can choose

the $\Box$-harmonic representative of the class corresponding to the tangent endomorphism to the curve $J_t$;

i.e., up to diffeomorphisms, we can assume

$J_t := (id_M + t L + o(t)) J (id_M + t L + o(t))^{-1}$,

with $\Box L = 0$ and $L J + J L = 0$;

it follows from Lemma 3.2 that, if $V = 2 J (L - t L)$, then $\Box V = 0$ and thus, by Lemma 3.1, $\nabla^{TM} V = 0$ and finally, $\nabla^{M} \alpha = 0$.

4. Further Remarks

As we have already remarked, deformation theory of holomorphic structures ensures that, up to diffeomorphisms, the general (germ of) curve of holomorphic structures on a Hermitian manifold $(M, J, g)$ is of the form

$J_t := (id_M + t L + o(t)) J (id_M + t L + o(t))^{-1}$

for $L \in \text{End}(TM)$ satisfying $J L + L J = 0$, $\Box L = 0$;

but, in general, not every such an $L$ gives rise to an actual deformation; in other words, in general, the deformation theory is obstructed.

Let $\mathcal{M}$ be the subset of $\text{Ker} \Box$ of elements providing actual deformations.
In the Kähler case, the situation looks somehow neater; in fact:

1) by Corollary 3.3, $\text{Ker} \Box$ splits as

$$\text{Ker} \Box = \mathfrak{A} \oplus \mathfrak{S}$$

where, clearly:

$$\mathfrak{A} = \{ L \in \text{Ker} \Box | L = -tL \}$$

and

$$\mathfrak{S} = \{ L \in \text{Ker} \Box | L = tL \}$$

2) $\mathfrak{A} \subset M$ and to every $L \in \mathfrak{A}$, we can associate a canonical curve of holomorphic structures:

$$J_t := (id_M + tL)J(id_M + tL)^{-1};$$

3) note that, in general, for a curve of almost symplectic structures,

$$\kappa_t = \kappa + t\alpha + o(t)$$

and a curve of $\kappa_t$-calibrated complex structures,

$$J_t := (id_M + tL + o(t))J(id_M + tL + o(t))^{-1}, \quad (LJ + JL = 0),$$

from

$$\kappa_t - J_t \kappa_t = 0,$$

by taking the $t$-derivative at 0, we obtain

$$\alpha_{2,0+0,2}^2 = g((JL + tLJ) \cdot, \cdot);$$

therefore, if $L \in \mathfrak{S} \cap M$, and

$$J_t := (id_M + tL + o(t))J(id_M + tL + o(t))^{-1}$$

is a curve of holomorphic structures, then any curve

$$\kappa_t = \kappa + t\alpha + o(t)$$

of $J_t$-Kähler structures satisfies $\alpha \in \Lambda_{1,1}^1(M)$; consequently, by Lemma 2.1, it is possible to choose $\kappa_t$ of the form

$$\kappa_t = \kappa + o(t)$$

therefore, the map

$$\lambda: M \rightarrow H^2(M, \mathbb{R}) \quad L \mapsto [g((2J(L - tL) \cdot, \cdot)]$$

is a linear surjection over $\mathcal{K}/H^{1,1}(M, \mathbb{R})$, which is one-to-one when restricted to $\mathfrak{A}$.

Note that, generically,

$$\text{Ker} \Box = \{0\},$$

but, within the exceptional range $\dim \mathbb{C} \text{Ker} \Box > 0$, then, generically,

$$\mathfrak{A} = \{0\}.$$
i.e., non-trivial \( L's \in \wedge^{0,1}(M) \otimes T^{1,0}M \) satisfying \( \Box L = 0 \) are generically symmetric: this is a sort of Ayers Rock snow flake principle.

Let us now give a closer look to the case \( \mathcal{P}^{2,0+0,2}(M) \neq \{0\} \): first of all recall that, if \((M, \kappa, J)\) is a compact Kähler manifold with \( \text{Ric} \geq 0 \), then any holomorphic form on \( M \) is parallel and thus so are harmonic forms in \( H^{2,0+0,2}(M, \mathbb{R}) \); consequently, for such manifolds

\[
\mathcal{K} = H^2(M, \mathbb{R})
\]

(recall also that \( \text{Ric} > 0 \) at some point \( \implies H^{2,0+0,2}(M, \mathbb{R}) = 0 \)).

Moreover, if on a Kähler manifold \((H, J, \kappa)\) there exists \( \alpha \in \wedge^{2,0+0,2}(H, \mathbb{R}) \), non-degenerate, satisfying \( \nabla_H \alpha = 0 \), then \( H \) is hyperkähler, i.e., there exists \( K \in \mathfrak{c}_\kappa(H) \), satisfying \( KJ + JK = 0 \), \( \nabla^H K = 0 \) (\( K \) is nothing but the orthogonal factor of the polar decomposition of the endomorphism representing \( \alpha \) with respect to the given Kähler metric).

Given \( \alpha \in \wedge^{2,0+0,2}(H, \mathbb{R}) \), with \( \nabla^M \alpha = 0 \), write once more \( \alpha = g(V \cdot, \cdot) \), with \( V = -I V, J V + V J = 0 \), and \( \nabla^T M V = 0 \); then set:

\[
E(\alpha) := \text{Ker} V, \quad F(\alpha) := (E(\alpha))^\perp = \text{Im} V;
\]

then:

\[
X \in TM, \quad Y \in E(\alpha) \implies \nabla^M_X Y \in E(\alpha)
\]

\[
X \in TM, \quad Y \in F(\alpha) \implies \nabla^M_X Y \in F(\alpha).
\]

Therefore, the distributions \( E(\alpha) \) and \( F(\alpha) \) are integrable, \( J \)-invariant, parallel, and totally geodesic; moreover, if \( W \in \text{End}(TM) \) satisfies \( \nabla^T M W = 0 \), then

\[
W_{E(\alpha)} := \begin{cases} W & \text{on } E(\alpha) \\ 0 & \text{on } F(\alpha) \end{cases}
\]

satisfies \( \nabla^T M W_{E(\alpha)} = 0 \);

consequently, if \( \alpha = h(a) \) has maximal rank for \( a \in \mathcal{P}^{2,0+0,2}(M) \), then all elements of \( h(\mathcal{P}^{2,0+0,2}(M)) \) vanish on \( E(\alpha) \) and so \( E = E(\alpha) \) is unique;

thus, passing to the universal covering \( \tilde{M} \), we easily obtain that, from the Kählerian viewpoint

\[
\tilde{M} = N \times H
\]

where \( H \) is hyperkähler (and corresponds to \( F \)) and so:

\[
M = N \times \frac{H}{\Gamma}
\]

where \( \Gamma \) is a discrete group of holomorphic isometries of \( \tilde{M} \);
summarizing:

\[ \mathcal{P}^{2,0+0,2}(M) \begin{cases} = \{0\} \implies K = H^{1,1}(M, \mathbb{R}) \\ \neq \{0\} \implies M = N \times \mathbb{H} \end{cases} \]

Note finally that both \( \mathfrak{A} \) and \( \mathcal{P}^{2,0+0,2} \) are complex vector spaces:

- \( \mathcal{J}_\mathfrak{A}L := J\mathcal{L} \)
- more in general, if \( \alpha \in \wedge^{2,0+0,2}(M) \), define
  \((\mathcal{J}\alpha)(X, Y) := \alpha(JX, Y) = \alpha(X, JY)\);
  then set, for \( a \in \mathcal{P}^{2,0+0,2} \):
    \[ \mathcal{J}a = [Jh(a)]; \]
  from \( \nabla J = 0 \), it follows that \( \mathcal{P}^{2,0+0,2} \) is a \( \mathcal{J} \)-complex space;
  it is clear that \( \lambda \circ \mathcal{J}_\mathfrak{A} = \mathcal{J} \circ \lambda \).

### 5. Examples

1. Let \( M = T^{2n} \) be the complex \( n \)-dimensional torus equipped with the standard Kähler structure \((\kappa, J)\):
   in particular, we have the standard global frame
   \[ \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\} \]
   with
   \[ \begin{cases} J \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j} \\ J \left( \frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j} \end{cases} \quad 1 \leq j \leq n \]
   and standard coframe
   \[ \{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n\}; \]
   thus
   \[ \kappa = \sum_{h=1}^{n} dy_h \wedge dx_h; \]
   therefore, if \( \alpha = dx_j \wedge dx_k \), then
   \[ \alpha = \gamma + \tilde{\gamma} + \beta \]
   with
   \[ \gamma = \frac{1}{4} dz_j \wedge dz_k \in \wedge^{2,0}(T^{2n}, \mathbb{C}) \]
   and
   \[ \beta = \frac{1}{4} (dz_j \wedge d\bar{z}_k + d\bar{z}_j \wedge dz_k) \in \wedge^{1,1}(T^{2n}, \mathbb{R}) \]
   Note also that
   \[ \alpha^{2,0+0,2} = \frac{1}{2} (dx_j \wedge dx_k - dy_j \wedge dy_k) \]
(similar formulas for $dy_j \wedge dy_k, dx_j \wedge dy_k$);

thus

$$\gamma = g(V \cdot, \cdot)$$

for

$$V = \frac{1}{4} \left( dz_j \otimes \frac{\partial}{\partial z_k} - dz_k \otimes \frac{\partial}{\partial z_j} \right);$$

or, in real terms,

$$V = \frac{1}{2} \left( dx_j \otimes \frac{\partial}{\partial x_k} - dy_j \otimes \frac{\partial}{\partial y_k} - dx_k \otimes \frac{\partial}{\partial x_j} + dy_k \otimes \frac{\partial}{\partial y_j} \right);$$

consider e.g., $n = 2, \alpha = 4 dx_1 \wedge dx_2$;

then:

$$V = 2 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and so:

$$L = \frac{1}{2} VJ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};$$

therefore,

$$J_t = (I + tL)J(I + tL)^{-1} = \frac{1}{1 + t^2} \begin{pmatrix} 0 & 2t & t^2 - 1 & 0 \\ -2t & 0 & 0 & -2t \\ 1 - t^2 & 0 & 0 & -2t \\ 0 & 1 - t^2 & 2t & 0 \end{pmatrix}$$

and

$$\kappa_t = \frac{1}{2} (\kappa + J_t \kappa) = \kappa + t \frac{1 - t^2}{1 + t^2} \alpha^{2,0+0,2} + \frac{1}{2} t^2 (t^2 - 6) \kappa.$$  

2. First recall that, if $\mathbb{B} := \{z \in \mathbb{C} | |z| < 1\}$, then there are no non-trivial parallel $(1, 0)$-forms on

$$\left( \mathbb{B}, \frac{2}{(1 - |z|^2)^2} dz \wedge d\bar{z} \right);$$

in fact, given $\gamma = adz$, we have:

$$\nabla_{\partial \bar{z}} \gamma = 0 \iff \frac{\partial a}{\partial \bar{z}} = 0$$

$$\nabla_{\partial z} \gamma = 0 \iff \frac{\partial a}{\partial z} - \frac{2\bar{z}}{1 - |z|^2} a = 0 \iff a = 0;$$
consequently, if $\Sigma$ is a Riemann surface covered by $B$ and equipped with the constant $-1$ curvature Kähler metric, then there are no non-trivial parallel $(1, 0)$-forms on $\Sigma$.

Let now $\Sigma_{g_k}$, $k = 1, 2$, be compact Riemann surfaces equipped with the constant $-1$ curvature Kähler metric (and so $g_k \geq 2$); let $M = \Sigma_{g_1} \times \Sigma_{g_2}$; then:

- $H^1(M, \Theta) = H^1(\Sigma_{g_1}, \Theta) \oplus H^1(\Sigma_{g_2}, \Theta)$
- and, although $H^2(M, \Theta) \neq 0$, the holomorphic deformation theory of $M$ is unobstructed and reduces to the deformations of $\Sigma_{g_1}$ and $\Sigma_{g_2}$.
- from the previous remarks, it follows quite easily that there are no non-trivial parallel forms in $\bigwedge^{2,0+0,2}(M, \mathbb{R})$;

therefore, in $M$, we have

\[ \text{Ker} \Box = \mathcal{G} \quad \text{and} \quad \mathcal{K} = H^{1,1}(M, \mathbb{R}). \]

**References**


