WEIGHTED NORM INEQUALITIES FOR THE CONJUGATE FUNCTION ON $a$-ADIC SOLENOIDS

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ABSTRACT. In this paper we generalize a theorem of Hunt, Muckenhoupt, and Wheeden on weighted norm inequalities for the conjugate function. Our generalization to the cases of $a$-adic solenoids is formulated in terms of the ergodic $A_p$-condition.

1. Introduction

We consider an arbitrary noncyclic subgroup of $\mathbb{Q}$ and its compact dual group $\Sigma_a$. There is an explicit construction for $\Sigma_a$ which is called the $a$-adic solenoid. Since $\Sigma_a$ is simply a subgroup of $\mathbb{Q}$, $\Sigma_a$ inherits the order from $\mathbb{Q}$; that is, if we let $P = \Sigma_a \cap (0, \infty)$ then $P$ defines the order on $\Sigma_a$. For $f \in L^2(\Sigma_a)$, we use the Fourier transform of $f$ to define the conjugate function $\tilde{f}$ (with respect to the order $P$):

$$\tilde{f}(\chi) = -i \text{sgn}_P(\chi) \hat{f}(\chi) \quad (\chi \in \Sigma_a)$$

(1.1)

where $\text{sgn}_P(\chi) = -1, 0, or 1$ according to $\chi \in (-P) \setminus \{0\}$, $\chi = 0$, or $\chi \in P \setminus \{0\}$. The operator $f \mapsto \tilde{f}$ is clearly a norm-decreasing multiplier on $L^2(\Sigma_a)$. If $1 < p < \infty$, the operator $f \mapsto \tilde{f}$ extends from $L^2(\Sigma_a) \cap L^p(\Sigma_a)$ to a bounded linear operator of $L^p(\Sigma_a)$ such that the identity (1.1) holds, and the inequality

$$\|\tilde{f}\|_p \leq M_p \|f\|_p$$

holds for all $f \in L^p(\Sigma_a)$, where $M_p$ is independent of $f$ (see [3], or [1, Theorem 7.2]). We ask for which measures, other than Haar measure, is the operator $f \mapsto \tilde{f}$ a bounded operator. More precisely, if $1 < p < \infty$, we seek to characterize those finite nonnegative Borel measures $\nu$ for which the operator $f \mapsto \tilde{f}$ is bounded from $L^p(\Sigma_a, \nu) \cap L^1(\Sigma_a)$ into $L^p(\Sigma_a, \nu)$.

By way of background, we recall that Forelli [7] studied this problem in the case $G = \mathbb{T}$ (henceforth, $\mathbb{T}$ is parameterized by $[-\pi, \pi]$). He showed that if the operator $f \mapsto \tilde{f}$ is bounded from $L^p(\mathbb{T}, \nu) \cap L^1(\mathbb{T})$ into $L^p(\mathbb{T}, \nu)$, then $\nu$ must be absolutely continuous with respect to Lebesgue measure $\lambda$ ($\nu \ll \lambda$), and hence there is a nonnegative function $w$ in $L^1(\nu)$ where $d\nu = w\frac{dt}{2\pi}$. This result was later extended by Hunt, et al. [13], who showed that the operator $f \mapsto \tilde{f}$ is bounded from $L^p(\mathbb{T}, \nu) \cap L^1(\mathbb{T})$ into $L^p(\mathbb{T}, \nu)$ exactly when $w$ satisfies a property called the $A_p$-condition. We state this result in the following definition and theorem:

Definition 1.1. (The $A_p$-condition on $\mathbb{T}$) Let $1 \leq p < \infty$. Let $w$ be a nonnegative $2\pi$-periodic measurable function. The function $w$ satisfies the $A_p$-condition on $\mathbb{T}$ if
there is a constant \( A_p \) independent of all intervals \( I \subseteq \mathbb{R} \) such that
\[
\sup_I \frac{1}{I} \int_I w(t) dt \left( \frac{1}{I} \int_I w^{-1/(p-1)}(t) dt \right)^{p-1} \leq A_p.
\]
(1.2)
We say that \( w \in A_p(\mathbb{T}) \) if \( w \) satisfies (1.2), and we let \( A_p(w) \) denote the least constant such that (1.2) holds. When \( p = 1 \), (1.2) is of the form \( \sup_I \frac{1}{I} \int_I w(t) dt \) and \( \sup_{t \in I} w(t) \leq A_1 \).

**Theorem 1.1.** Let \( w \) be a nonnegative 2\( \pi \)-periodic measurable function. If \( 1 < p < \infty \), then \( w \in A_p(\mathbb{T}) \) if and only if for all \( f \in L_p(\mathbb{T}, w) \),
\[
\left( \int_{-\pi}^{\pi} |\hat{f}(t)|^p w(t) dt \right)^{1/p} \leq K_p \left( \int_{-\pi}^{\pi} |f(t)|^p w(t) dt \right)^{1/p}
\]
(1.3)
where \( K_p \) is independent of \( f \). If \( 1 \leq p < \infty \), \( w \in A_p(\mathbb{T}) \) if and only if for all \( f \in L_p(\mathbb{T}, w) \),
\[
\sup_{\tau > 0} \int_{-\pi}^{\pi} 1_{\{ t \in [-\pi, \pi]: |\hat{f}(t)| > \tau \}}(t) w(t) dt \leq K_p \int_{-\pi}^{\pi} |f(t)|^p w(t) dt
\]
(1.4)
where \( K_p \) is independent of \( f \).

**Remark 1.1.** We note that from the proof of Theorem 1.1 ([13]), when (1.4) holds, \( w \in A_p(\mathbb{T}) \) with \( A_p(w) \) less than or equal to \( K_p^2 (4\pi)^{2p} \). Also, by a modification of the proof in [13], it is enough to assume that (1.4) holds for all \( f \in L_p(\mathbb{T}, w) \cap L_1(\mathbb{T}) \).

Hewitt and Ritter in [8] and [9] make an extensive study of conjugate Fourier series on \( a \)-adic solenoids. In this paper, we study weighted norm inequalities on \( a \)-adic solenoids. Our main theorem (Theorem 4.4) gives a generalization of Theorem 1.1 in terms of the conjugate function on \( \Sigma_a \), obtaining a similar characterization as Hunt et al. [13] of those finite nonnegative Borel measures \( \nu \) for which the operator \( f \mapsto \tilde{f} \) is bounded from \( L_p(\Sigma_a, \nu) \cap L_1(\Sigma_a) \) into \( L_p(\Sigma_a, \nu) \).

The plan of the paper is as follows. In Section 2, we give an explicit representation of \( \Sigma_a \) and define some other terms needed in our analysis. In Section 3, we show that if \( \nu \) is a nonnegative Borel measure on \( \Sigma_a \), and the operator \( f \mapsto \tilde{f} \) is bounded from \( L_p(\Sigma_a, \nu) \cap L_1(\Sigma_a) \) into \( L_p(\Sigma_a, \nu) \), then \( \nu \) is absolutely continuous with respect to Haar measure \( \mu \). This shows that we need only characterize those weights \( w \in L_1(\Sigma_a) \) that satisfy the property that the operator \( f \mapsto \tilde{f} \) is bounded from \( L_p(\Sigma_a, w) \cap L_1(\Sigma_a) \) into \( L_p(\Sigma_a, w) \). In Section 4, we state and prove a characterization of those weights that satisfy this last property (Theorem 4.4).

2. Preliminaries

2.1. The \( a \)-adic solenoid and its character group. Up to isomorphism, any non-cyclic subgroup of \( \mathbb{Q} \) can be described as follows. Let \( a = (a_0, a_1, \ldots) \) be a fixed infinite sequence of integers all greater than 1. Let
\[
A_0 = 1, \quad A_1 = a_0, \quad A_2 = a_0a_1, \ldots, \quad A_n = a_0a_1 \cdots a_{n-1}, \ldots
\]
Let \( Q_a \) be the set of all rational numbers \( \frac{1}{A_k} \), where \( l \in \mathbb{Z} \) and \( k \in \mathbb{Z}^+ \). Clearly \( Q_a \) is a non-cyclic additive subgroup of \( \mathbb{Q} \), and as shown in [2], any non-cyclic subgroup of \( \mathbb{Q} \) is of this form.

According to the Pontrjagin duality ([10, 24.8, p. 378]), the character group of \( Q_a \) is a compact abelian group, which we denote by \( \Sigma_a \), and the character group of \( \Sigma_a \) is again \( Q_a \). We let \( \mu \) denote normalized Haar measure on \( \Sigma_a \). The group \( \Sigma_a \) can be realized as the set \([0,1) \times \Delta_a \), which is described in detail in [10, Section
The group \( \Delta_a \) consists of all infinite sequences \( x = (x_0, x_1, \ldots, x_k, \ldots) \) where each \( x_k \in \{0,1,\ldots,a^k - 1\} \). Addition in \( \Delta_a \) is defined coordinate-wise and carrying quotients (see [10, 10.2]). Also, the elements \( u = (1,0,0,\ldots) \) and \( 0 = (0,0,0,\ldots) \) are both in \( \Delta_a \), and addition on \([0,1) \times \Delta_a\) is defined by

\[
(\xi, x) + (\eta, y) = (\xi + \eta - \lfloor \xi + \eta \rfloor, x + y - \lfloor \xi + \eta \rfloor)
\]

where \( \lfloor \cdot \rfloor \) is the greatest integer function. The group \( \Sigma_a \) is a compact connected Abelian group admitting a continuous homomorphism \( \varphi : \mathbb{R} \to \Sigma_a \), where \( \varphi(\mathbb{R}) \) is a dense subgroup of \( \Sigma_a \) and

\[
\varphi(s) = (s - \lfloor s \rfloor, \lfloor s \rfloor)
\]

([10, Theorem 10.13], and [9, 3.2]). For \( k = 1,2,\ldots \), define the sets

\[
\Lambda_k = \{(0, x) \in \Sigma_a : x_0 = x_1 = \cdots = x_{k-1} = 0\}.
\]

The sets \( \Lambda_k \) are compact, closed subgroups of \( \Sigma_a \) ([10, Theorem 10.5, p. 110]), and we let \( \mu_k \) denote normalized Haar measure on \( \Lambda_k \). The measure \( \mu_k \) is a singular Borel measure on \( \Sigma_a \), and the Fourier transform is equal to the indicator function of \((1/A_k)\mathbb{Z}:

\[
\hat{\mu}_k = 1_{(1/A_k)\mathbb{Z}}
\]

([9, 5.1ff, p. 825]). For all \( k \in \mathbb{N} \), the quotient group \( \Sigma_a/\Lambda_k \) is topologically isomorphic to the circle group \( \mathbb{T} \) (see [8, 3.1]). Indeed, the mapping

\[
\pi_k(t, x) = \chi_{\frac{1}{A_k}}(t, x)
\]

is a continuous homomorphism of \( \Sigma_a \) onto \( \mathbb{T} \) with kernel \( \Lambda_k \) where

\[
\chi_{\frac{1}{A_k}}((t, x)) = \exp \left( 2\pi i \frac{1}{A_k} \left( t + \sum_{h=0}^{k-1} x_h A_h \right) \right)
\]

is the character corresponding to the element \( \frac{1}{A_k} \) of \( \mathbb{Q}_a \). Also, if \( f \in \mathcal{L}_1(\Sigma_a) \) and \( f \) is constant on cosets of \( \Lambda_k \), then \( f = f \ast \mu_k \), and there is a function \( f_k \in \mathcal{L}_1(\mathbb{T}) \) that satisfies \( f = f \ast \mu_k = f_k \circ \pi_k \) and

\[
\int_{\Sigma_a} f d\mu = \int_{\Sigma_a} f_k \circ \pi_k d\mu = \int_{\mathbb{T}} f_k dx
\]

([11, 28.55] and [9, 5.1.3]).

**Martingales on \( \Sigma_a \).** If \( f \in \mathcal{L}_1(\Sigma_a) \), then the sequence \( (f \ast \mu_k)_{k \geq 0} \) is a martingale relative to a sequence of \( \sigma \)-algebras \( (\mathcal{F}_k)_{k \geq 0} \) where \( \mathcal{F}_k \) consists of those Borel sets \( F \subset \Sigma_a \) such that \( F + \Lambda_k = F \) (see [6, Theorem 5.4.1]). The functions \( f \ast \mu_k \) also are known as the conditional expectations of \( f \) relative to \( \mathcal{F}_k \). It is a well-known theorem of Doob's that if \( f \in \mathcal{L}_p(\Sigma_a) \), then \( f \ast \mu_k \to f \) in \( \mathcal{L}_p(\Sigma_a) \) as \( k \to \infty \) (see [5], or [6, Theorem 5.2.6]).

**The conjugate function on \( \Sigma_a \).** It is easy to see that \( \mathbb{Q}_a \) admits exactly one order \( \mathcal{P} \) under which \( 1 \) is in \( \mathcal{P} \); the order is the one inherited from the usual order on \( \mathbb{R} \). We take this ordering on \( \mathbb{Q}_a \) where \( P = \{x_\alpha \in \mathbb{Q}_a : \alpha \geq 0\} \). For \( f \in \mathcal{L}_2(\Sigma_a) \), we use the Fourier transform and the order generated by \( P \) to define the conjugate function \( \tilde{f} \):

\[
\tilde{f}^\mathcal{P}(x_\alpha) = -i \text{sgn}_{\mathcal{P}}(x_\alpha) \tilde{f}(x_\alpha) \quad (x_\alpha \in \mathbb{Q}_a)
\]

where \( \text{sgn}_{\mathcal{P}}(x_\alpha) = -1, 0, \) or \( 1, \) according to \( \alpha < 0, \alpha = 0, \) or \( \alpha > 0, \) respectively. As noted before, if \( 1 < p < \infty \), the operator \( f \mapsto \tilde{f} \) extends from \( \mathcal{L}_2(\Sigma_a) \cap \mathcal{L}_p(\Sigma_a) \)
to a bounded linear operator of $L_p(S_\alpha)$ ([1, Theorem 7.2]). In addition, the conjugate function $\tilde{f}$ has an integral representation that exists $\mu$-almost everywhere for all functions $f \in L_1(S_\alpha)$ ([1, 6.11(c) and Theorem 6.5]).

The ergodic $A_p$-condition on $S_\alpha$. Let $\varphi : \mathbb{R} \to S_\alpha$ be the continuous homomorphism defined in (2.1). If $1 \leq p < \infty$ and $w$ is a nonnegative function in $L_1(S_\alpha)$, we say that $w$ is in $A_p(S_\alpha)$ if the following condition is satisfied: for almost every $x \in S_\alpha$,

$$\sup_I \frac{1}{|I|} \int_I w(x - \varphi(t))dt \left( \frac{1}{|I|} \int_I w^{-1/(p-1)}(x - \varphi(t))dt \right)^{p-1} \leq K_p$$

(2.4)

where $K_p$ is a constant independent of $x$. We let $A_p(w)$ denote the least constant such that (2.4) holds.

3. The continuity of the conjugate function
with respect to Borel measures

In this section, we show that if $\nu$ is a finite nonnegative Borel measure on $S_\alpha$, the continuity of the operator $f \mapsto \tilde{f}$ from $L_p(S_\alpha, \nu) \cap L_1(S_\alpha)$ into $L_p(S_\alpha, \nu)$ implies that $\nu \ll \mu$ where $\mu$ is Haar measure on $S_\alpha$.

**Theorem 3.1.** Let $1 < p < \infty$. Let $\nu$ be a finite nonnegative Borel measure on $S_\alpha$. Suppose that the inequality

$$\|\tilde{f}\|_{L_p(S_\alpha, \nu)} \leq K_p \|f\|_{L_p(S_\alpha, \nu)}$$

(3.1)

is valid for all $f \in L_p(S_\alpha, \nu) \cap L_1(S_\alpha)$ where $K_p$ is independent of $f$. Then $\nu \ll \mu$, and hence there is a nonnegative function $w$ in $L_1(S_\alpha)$ such that $d\nu = wd\mu$.

**Proof.** Assuming that the linear operator $f \mapsto \tilde{f}$ is bounded from $L_p(S_\alpha, \nu) \cap L_1(S_\alpha)$ into $L_p(S_\alpha, \nu)$, we can continuously extend the operator to all of $L_p(S_\alpha, \nu)$. Let $T$ denote the extended linear operator. Fix a real-valued function $g$ in $L_q(S_\alpha, \nu) \cap L_1(S_\alpha)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder's inequality, we have for all $f \in L_p(S_\alpha, \nu)$,

$$\left| \int_{S_\alpha} (Tf)g d\nu \right| \leq \|Tf\|_{L_p(S_\alpha, \nu)} \|g\|_{L_q(S_\alpha, \nu)} \leq K_p \|g\|_{L_q(S_\alpha, \nu)} \|f\|_{L_p(S_\alpha, \nu)}.$$

Hence, if we define the linear functional $L_g : L_p(S_\alpha, \nu) \to \mathbb{C}$ by $L_g f = \int_{S_\alpha} (Tf)g d\nu$, then $L_g$ is bounded. By the Riesz Representation Theorem ([15, p. 284]), there is a function $h \in L_q(S_\alpha, \nu)$ such that

$$L_g f = \int_{S_\alpha} (Tf)g d\nu = \int_{S_\alpha} hf d\nu$$

(3.2)

for all $f \in L_p(S_\alpha, \nu)$.

We claim that $h$ is real-valued $\nu$-a.e. To see this, consider a continuous character $\chi_\alpha \in P \setminus \{0\}$ (so then $\alpha > 0$ and $\overline{\chi}_\alpha \in (-P) \setminus \{0\}$). By (3.2), we have

$$\int_{S_\alpha} \Re(\chi_\alpha) h d\nu = \frac{1}{2} \int_{S_\alpha} (\chi_\alpha + \overline{\chi}_\alpha) h d\nu = \frac{1}{2} \int_{S_\alpha} (T\chi_\alpha + T\overline{\chi}_\alpha) g d\nu$$

$$= \frac{1}{2} \int_{S_\alpha} (-i\chi_\alpha + i\overline{\chi}_\alpha) g d\nu$$

$$= \int_{S_\alpha} (\Im(\chi_\alpha)) g d\nu.$$

Since the last integral is real-valued, $\int_{S_\alpha} \Re(\chi_\alpha) h d\nu = 0$. Similarly, $\int_{S_\alpha} \Im(\chi_\alpha) h d\nu = 0$, so that $\int_{S_\alpha} \chi_\alpha \Im h d\nu = 0$. This is also true if $\chi_\alpha$ is replaced by any
trigonometric polynomial $\sum_{j=1}^{m} X_{\alpha_j} \chi_{\alpha_j} \in \mathbb{Q}_a$. By the Stone-Weierstrass Theorem, the set of trigonometric polynomials on $\Sigma_a$ is dense in the set of all continuous functions on $\Sigma_a$; hence for all continuous functions $f$ on $\Sigma_a$, we have $\int_{\Sigma_a} f \text{Im} h d\nu = 0$. But then the signed measure $\text{Im} h d\nu \equiv 0$, which means that $\text{Im} h = 0 \nu$-a.e. and $h$ is real-valued $\nu$-a.e.

We also claim that $(h + ig) d\nu$ is of analytic type in the sense that $(h + ig) d\nu$ has a Fourier transform vanishing for the negative characters in $\mathbb{Q}_a$ (see [16, p. 197]). By (3.2), we have

$$-i \int_{\Sigma_a} \chi_{\alpha} g d\nu = \int_{\Sigma_a} h \chi_{\alpha} d\nu, \text{ for all } \chi_{\alpha} \in P \setminus \{0\}.$$ 

Thus,

$$\int_{\Sigma_a} \chi_{\alpha} (h + ig) d\nu = 0, \text{ for all } \chi_{\alpha} \in P \setminus \{0\},$$

and equivalently

$$\int_{\Sigma_a} \overline{\chi}_{\alpha} (h + ig) d\nu = 0, \text{ for all } \chi_{\alpha} \in (-P) \setminus \{0\}.$$ 

Thus, $(h + ig) d\nu$ is of analytic type.

By [12, Theorem 19.42, p. 326], we can write the Lebesgue decomposition of $d\nu$ as $d\nu = d\nu_s + d\nu_a$ where $d\nu_s$ is singular with respect to $d\mu$ ($d\nu_s \perp d\mu$), and $d\nu_a$ is absolutely continuous with respect to $d\mu$ ($d\nu_a \ll d\mu$). Then it is clear that the Lebesgue decomposition of $(h + ig) d\nu$ is

$$(h + ig) d\nu = (h + ig) d\nu_s + (h + ig) d\nu_a. \quad (3.3)$$

Since $(h + ig) d\nu$ is of analytic type,

$$\int_{\Sigma_a} (h + ig) d\nu_s = 0$$

([16, Theorem 8.2.3, p. 200]). Since $g$ and $h$ are real-valued $\nu$-a.e., $\int_{\Sigma_a} g d\nu_s = 0$. This is true for every continuous real-valued $g \in L_p(\Sigma_a, w) \cap L_1(\Sigma_a)$, so it is also true for the real and imaginary parts of every continuous complex-valued function $g$, and hence $\nu_s \equiv 0$. But then $\nu \ll \mu$, and by the Radon-Nikodym Theorem, ([12, Theorem 19.23, p. 315]), there is a nonnegative measurable function $w \in L_1^+(\Sigma_a)$ such that $\nu(A) = \int_A w d\mu$ for all Borel measurable subsets $A$ of $\Sigma_a$.

**Remark 3.1.** We note that the proof of Theorem 3.1 does not depend on the structure of the $a$-adic solenoid $\Sigma_a$. In fact, using the same argument, we can show that Theorem 3.1 holds for any compact connected abelian group $G$ where the dual is ordered and the conjugate function $\bar{f}$ is defined as in (1.1).

4. The $A_p$-condition on $a$-adic solenoids

We seek to characterize those finite nonnegative Borel measures $\nu$ for which the operator $f \mapsto \bar{f}$ is bounded from $L_p(\Sigma_a, \nu) \cap L_1(\Sigma_a)$ into $L_p(\Sigma_a, \nu)$. By Theorem 3.1, it suffices to characterize those weights $w \in L_1(\Sigma_a)$ for which the operator $f \mapsto \bar{f}$ is bounded from $L_p(\Sigma_a, w) \cap L_1(\Sigma_a)$ into $L_p(\Sigma_a, w)$. In Theorem 4.4, we show that this property holds if and only if $w$ satisfies the ergodic $A_p$-condition in (2.4).
We prove some propositions before proving Theorem 4.4. It is essential for our analysis to define the following classes of functions for \(1 < p < \infty\) and \(w \in \mathcal{L}_1(\Sigma_a)\):

\[\mathcal{L}_p(\Sigma_a, \mu) \ast \mu_k = \{f \ast \mu_k : f \in \mathcal{L}_p(\Sigma_a)\},\]

\[\mathcal{L}_p(\Sigma_a, w \ast \mu_k) \ast \mu_k = \{f \ast \mu_k : f \in \mathcal{L}_p(\Sigma_a, w \ast \mu_k)\}.\]

From Hewitt and Ross [11, p. 95, Theorem 28.55], \(\mathcal{L}_p(\Sigma_a) \ast \mu_k\) is isometrically isomorphic to \(\mathcal{L}_p(\Sigma_a / \Lambda_k) \approx \mathcal{L}_p(\mathbb{T})\). By a modification of the proof in [11], we also have \((\mathcal{L}_p(\Sigma_a, w \ast \mu_k) \cap \mathcal{L}_1(\Sigma_a)) \ast \mu_k\) isometrically isomorphic to \(\mathcal{L}_p(\mathbb{T}, w_k) \cap \mathcal{L}_1(\mathbb{T})\) where \(w_k\) is the function in \(\mathcal{L}_1(\mathbb{T})\) such that \(w \ast \mu_k = w_k \circ \pi_k\) (see (2.3)).

We use the following notation. If \(\nu\) is a nonnegative Borel measure on \(\Sigma_a\) and \(1 < p < \infty\), we define the Lorentz \(\mathcal{L}_{p, \infty}\) quasi-norm for a measurable function \(f\) as

\[\|f\|_{\mathcal{L}_{p, \infty}(\nu)} = \sup_{\tau > 0} \tau^{1/p} \nu(\{x \in \Sigma_a : |f(x)| > \tau\}).\]

(See [17, Ch.5, Sect.3]; note that \(\|\cdot\|_{\mathcal{L}_{p, \infty}(\nu)}\) actually defines a norm when \(1 < p < \infty\).)

**Proposition 4.1.** Let \(1 \leq p < \infty\). Let \(T\) denote the operator \(f \mapsto \tilde{f}\), and let \(w\) be a nonnegative function in \(\mathcal{L}_1(\Sigma_a)\). Then the following are equivalent:

(i) The inequality

\[\|Tf\|_{\mathcal{L}_{p, \infty}(\Sigma_a, w)} \leq K_p \|f\|_{\mathcal{L}_p(\Sigma_a, w)}\]

is valid for every \(f \in \mathcal{L}_p(\Sigma_a, w) \cap \mathcal{L}_1(\Sigma_a)\) where \(K_p\) is independent of \(f\).

(ii) For each \(k = 1, 2, \ldots\), the inequality

\[\|T(f \ast \mu_k)\|_{\mathcal{L}_{p, \infty}(\Sigma_a, w \ast \mu_k)} \leq K_p \|f \ast \mu_k\|_{\mathcal{L}_p(\Sigma_a, w \ast \mu_k)}\]

is valid for every \(f \in \mathcal{L}_p(\Sigma_a, w \ast \mu_k) \cap \mathcal{L}_1(\Sigma_a)\) where \(K_p\) is independent of \(f\) and \(k\).

**Proof.** (i)\(\Rightarrow\) (ii) Let \(f\) be a trigonometric polynomial on \(\Sigma_a\) and fix an integer \(1 \leq k < \infty\). Since \(f\) is bounded, \(f \ast \mu_k\) is bounded and \(f \ast \mu_k \in \mathcal{L}_p(\Sigma_a, w) \cap \mathcal{L}_1(\Sigma_a)\). Then we have by Fubini's Theorem and the translation invariance of \(\mu\),

\[
\sup_{\tau > 0} \tau^p \int_{\Sigma_a} 1_{\{x \in \Sigma_a : |T(f \ast \mu_k)(x)| > \tau\}}(x)w \ast \mu_k(x) d\mu(x)
\]

\[
= \sup_{\tau > 0} \tau^p \int_{\Sigma_a} \int_{\Sigma_a} 1_{\{x \in \Sigma_a : |T(f \ast \mu_k)(x)| > \tau\}}(x)w(x-y) d\mu_k(y) d\mu(x)
\]

\[
= \sup_{\tau > 0} \tau^p \int_{\Sigma_a} \int_{\Sigma_a} 1_{\{x \in \Sigma_a : |T(f \ast \mu_k)(x)| > \tau\}}(x+y)w(x) d\mu(x)d\mu_k(y)
\]

\[= \sup_{\tau > 0} \tau^p \int_{\Sigma_a} \int_{\Sigma_a} 1_{\{x \in \Sigma_a : |T(f \ast \mu_k)(x+y)| > \tau\}}(x+y)w(x) d\mu(x)d\mu_k(y).\] (4.1)

Letting \((f \ast \mu_k)_y\) denote the function \(x \mapsto (f \ast \mu_k)(x+y)\) and applying the hypothesis to \((f \ast \mu_k)_y\), we have from (4.1),

\[
\sup_{\tau > 0} \tau^p \int_{\Sigma_a} 1_{\{x \in \Sigma_a : |T(f \ast \mu_k)(x)| > \tau\}}(x)w \ast \mu_k(x) d\mu(x)
\]

\[\leq K_p^p \int_{\Sigma_a} \int_{\Sigma_a} |(f \ast \mu_k)_y(x)|^p w(x) d\mu(x)d\mu_k(y)
\]

\[= K_p^p \int_{\Sigma_a} |f \ast \mu_k(x)|^p w \ast \mu_k(x) d\mu(x).\]

It is easy to see that this is enough to show that (ii) holds.
(ii)$\rightarrow$(i) Consider a trigonometric polynomial $f$ on $\Sigma_a$. Then it is clear that there is an integer $N \geq 1$ such that $f = f * \mu_k$ for all $k \geq N$. Fix $\tau > 0$. Since $\|w * \mu_k - w\|_{L^p(\Sigma_a)} \rightarrow 0$, and $f$ is a bounded function, we have by the hypothesis

$$
\tau^p \int_{\Sigma_a} 1_{\{x \in \Sigma_a \mid Tf(x) > \tau\}}(x)w(x)d\mu(x) \\
= \lim_{k \rightarrow \infty} \tau^p \int_{\Sigma_a} 1_{\{x \in \Sigma_a \mid T(f * \mu_k)(x) > \tau\}}(x)w * \mu_k(x)d\mu(x) \\
\leq K_p^p \lim_{k \rightarrow \infty} \int_{\Sigma_a} |f * \mu_k(x)|^p w * \mu_k(x)d\mu(x) \\
= K_p^p \int_{\Sigma_a} |f(x)|^p w(x)d\mu(x).
$$

It is easy to see that this is enough to show that (i) holds.

**Remark 4.1.** We note that by slightly modifying the proof of Proposition (4.1), we can show that similar strong-type estimates hold.

**Proposition 4.2.** Let $1 \leq p < \infty$. Suppose $w$ is a nonnegative function in $L^1(\Sigma_a)$ and $w$ is constant on the cosets of $\Lambda_k$ for some positive integer $k$. Let $w_k$ denote the function in $L^1(\mathbb{T})$ such that $w = w * \mu_k = w_k \circ \pi_k$ (see (2.3)). Then $w$ is in $A_p(\Sigma_a)$ if and only if $w_k$ is in $A_p(\mathbb{T})$. Moreover, in this case $A_p(w_k) \leq A_p(w)$.

**Proof.** We show that the necessity part of the proposition holds. The sufficiency part follows by a similar argument. Assume that $w$ is in $A_p(\Sigma_a)$ with bound $A_p(w)$. Let $I = (a, b)$ be an interval in $\mathbb{R}$. Let $(t, x)$ be an element in $\Sigma_a$ such that (2.4) holds. As noted in (2.2) and the following, we have $\pi_k((t, x)) = \chi_{\Lambda_k}((t, x)) = \exp(2\pi i \frac{1}{A_k} t_0)$ where $t_0 = t + \sum_{h=0}^{k-1} x_h A_h$. We consider the expression

$$
\frac{1}{|I|} \int_I w_k(\exp(is))ds \left( \frac{1}{|I|} \int_I w_k^{-1/(p-1)}(\exp(is))ds \right)^{p-1},
$$

(4.2)

Let $s = \frac{2\pi}{A_k}(t_0 - u)$, $ds = -\frac{2\pi}{A_k} du$, $a' = t_0 - \frac{A_k}{2\pi} a$, $b' = t_0 - \frac{A_k}{2\pi} b$, and $I' = (b', a')$. It is easily observed that $\pi_k(\varphi(u)) = \chi_{\Lambda_k}(\varphi(u)) = \exp(2\pi i \frac{1}{A_k} u)$ for all $u \in \mathbb{R}$ (see [9, 3.2.4 ff]). Since $w$ is in $A_p(\Sigma_a)$ with bound $A_p(w)$ and $w = w * \mu_k = w_k \circ \pi_k$, we can use a change of variables to see that (4.2) is bounded by $A_p(w)$:

$$
\frac{1}{|I|} \int_I w_k(\exp(is))ds \left( \frac{1}{|I|} \int_I w_k^{-1/(p-1)}(\exp(is))ds \right)^{p-1} \\
= \frac{1}{|I|} \left(-\frac{2\pi}{A_k}\right) \int_{a'}^{b'} w_k(\exp(2\pi i \frac{1}{A_k}(t_0 - u)))du \\
\times \left( \frac{1}{|I|} \frac{-2\pi}{A_k} \int_{a'}^{b'} w_k^{-1/(p-1)}(\exp(2\pi i \frac{1}{A_k}(t_0 - u)))du \right)^{p-1} \\
= \frac{1}{|I|} \int_{I'} w_k(\pi_k((t, x) - \varphi(u)))du \\
\times \left( \frac{1}{|I|} \int_{I'} w_k^{-1/(p-1)}(\pi_k((t, x) - \varphi(u)))du \right)^{p-1} \\
\leq A_p(w).
$$
This is true for any interval $I$, hence $w_k$ is in $A_p(T)$ with bound less than or equal to $A_p(w)$. \hfill \Box

The next proposition shows that if $w \in A_p(\Sigma_a)$, then the operator $f \mapsto \tilde{f}$ is bounded from $L_p(\Sigma_a, w) \cap L^1(\Sigma_a)$ into $L_p(\Sigma_a, w)$. The proof is similar to that of [14, Theorem 2.1 and Corollary 2.4], using the transference methods of Coifman and Weiss [4]. We include the proof for completeness.

**Proposition 4.3.** Let $T$ denote the operator $f \mapsto \tilde{f}$ and let $w$ be a nonnegative function in $L^1(\Sigma_a)$. If $1 < p < \infty$ and $w \in A_p(\Sigma_a)$, then the inequality

$$
\|Tf\|_{L^p(\Sigma_a, w)} \leq A_p(w) \|f\|_{L^p(\Sigma_a, w)}
$$

is valid for all $f \in L_p(\Sigma_a, w) \cap L^1(\Sigma_a)$, where $A_p(w)$ is independent of $f$. If $p = 1$ and $w \in A_1(\Sigma_a)$, then the inequality

$$
\|Tf\|_{L^1,\infty(\Sigma_a, w)} \leq A_1(w) \|f\|_{L^1(\Sigma_a, w)}
$$

is valid for all $f \in L_1(\Sigma_a, w) \cap L^1(\Sigma_a)$, where $A_1(w)$ is independent of $f$.

**Proof.** We show the proposition holds for the case $1 < p < \infty$. The case $p = 1$ follows by a similar argument. We assume that $w \in A_p(\Sigma_a)$ with bound $A_p(w)$ and show that the inequality

$$
\|Tf\|_{L^p(\Sigma_a, w)} \leq A_p(w) \|f\|_{L^p(\Sigma_a, w)}
$$

(4.3)

is valid for all $f \in L_p(\Sigma_a, w) \cap L^1(\Sigma_a)$. Let $K_n = \{ \frac{1}{n} \leq |t| \leq n \}$ and $k_n(t) = \frac{1}{\pi} \inf_{K_n}(t)$ and $H_n f(x) = \int_R f(x - \varphi(t))k_n(t)dt$ where $\varphi : R \rightarrow \Sigma_a$ is the homomorphism defined in (2.1). To see that (4.3) holds, it is enough to show that for all $n \geq 1$, the inequality

$$
\int_{\Sigma_a} |H_n f(x)|^p w(x) d\mu(x) \leq A_p(w)(1 + \frac{1}{n}) \int_{\Sigma_a} |f(x)|^p w(x) d\mu(x)
$$

(4.4)

is valid for all $f \in L_p(\Sigma_a, w) \cap L^1(\Sigma_a)$. (By [1, Theorem 6.5 and 6.11(c)], if $f \in L^1(\Sigma_a)$, then $|H_n f(x)| \rightarrow |T f(x)|$ for $\mu$-a.e. $x \in \Sigma_a$. So assuming (4.4) holds, we can use Fatou's lemma to show that (4.3) is valid for all $f \in L_p(\Sigma_a, w) \cap L^1(\Sigma_a)$.)

To see that (4.4) holds, fix $f \in L_p(\Sigma_a, w) \cap L^1(\Sigma_a)$ and let $n > 1$. Since $R$ is amenable, we can choose a compact set $K$ such that $\frac{|K - K_n|}{|K|} < 1 + \frac{1}{n}$ (see [4, 2.1, p. 8]). By the translation invariance of Haar measure $\mu$ and Fubini’s theorem, we have

$$
\int_{\Sigma_a} |H_n f(x)|^p w(x) d\mu(x)
$$

$$
= \frac{1}{|K|} \int_K \int_{\Sigma_a} |H_n f(x - \varphi(t))|^p w(x - \varphi(t)) d\mu(x) dt
$$

$$
= \frac{1}{|K|} \int_K \int_{\Sigma_a} \left[ \int_R |f(x - \varphi(t - s))k_n(s)|^p w(x - \varphi(t)) ds \right] d\mu(x)
$$

$$
= \frac{1}{|K|} \int_K \int_{\Sigma_a} \left[ \int_R |f(x - \varphi(t - s))1_{K - K_n}(t - s)k_n(s)|^p w(x - \varphi(t)) ds \right] d\mu(x).
$$

Let $g_x(t) = f(x - \varphi(t))1_{K - K_n}(t)$ and $w_x(t) = w(x - \varphi(t))$. We have assumed that $w \in A_p(\Sigma_a)$, which means that for $\mu$-a.e. $x \in \Sigma_a$, $w_x(t)$ satisfies (2.4) with bound
Then by the above equalities and [13, Theorem 9, p. 247], we have

\[
\int_{\Sigma_a} |f(x)|^p w(x) d\mu(x)
\]
\[
= \frac{1}{|K|} \int_{\Sigma_a} \int_{\Sigma} \left| \int_{\mathbb{R}} g_x(t-s)k_n(s) ds \right|^p \frac{w_x(t)}{|t|} dt d\mu(x)
\]
\[
\leq \frac{A_p(w)}{|K|} \int_{\Sigma_a} \int_{\mathbb{R}} |g_x(t)|^p w_x(t) dt d\mu(x)
\]
\[
= \frac{A_p(w)}{|K|} \int_{K-K_n} \int_{\Sigma_a} \left| f(x - \varphi(t)) \right|^p w(x - \varphi(t)) d\mu(x) dt
\]
\[
= A_p(w) \frac{|K-K_n|}{|K|} \int_{\Sigma_a} |f(x)|^p w(x) d\mu(x).
\]

Since \(|K-K_n| < 1 + \frac{1}{n}\), we have shown that (4.4) holds, completing the proof of the proposition.

Now we state and prove our main theorem.

**Theorem 4.4.** Let \(T\) denote the operator \(f \mapsto \tilde{f}\) and let \(w\) be a nonnegative function in \(L_1(\Sigma_a)\). If \(1 < p < \infty\), then \(w \in A_p(\Sigma_a)\) if and only if the inequality

\[
\|Tf\|_{L_p(\Sigma_a,w)} \leq K_p \|f\|_{L_p(\Sigma_a,w)} \tag{4.5}
\]

is valid for all \(f \in L_p(\Sigma_a,w) \cap L_1(\Sigma_a)\), where \(K_p\) is a constant independent of \(f\). If \(1 \leq p < \infty\), then \(w \in A_p(\Sigma_a)\) if and only if the inequality

\[
\|Tf\|_{L_p,\infty(\Sigma_a,w)}^* \leq K_p \|f\|_{L_p(\Sigma_a,w)} \tag{4.6}
\]

is valid for all \(f \in L_p(\Sigma_a,w) \cap L_1(\Sigma_a)\), where \(K_p\) is a constant independent of \(f\).

**Proof.** By Proposition (4.3), the necessity parts of the theorem hold. To prove the sufficiency parts of the theorem, let \(1 \leq p < \infty\) and assume that (4.6) holds. As noted before, \((L_p(\Sigma_a,w*\mu_k) \cap L_1(\Sigma_a)) \ast \mu_k\) is isometrically isomorphic to \(L_p(T,w_k) \cap L_1(T)\) where \(w_k\) is a function in \(L_1(T)\) such that \(w*\mu_k = w_k \circ \pi_k\). So by Proposition (4.1), for \(k = 1,2,\ldots\), the inequality

\[
\|Tf\|_{L_p,\infty(T,w_k)}^* \leq K_p \|f\|_{L_p(T,w_k)}
\]

is valid for all \(f \in L_p(T,w_k) \cap L_1(T)\) where \(Tf\) is the conjugate function of \(f\) defined on the circle. By Theorem 1.1, for \(k = 1,2,\ldots\), we have \(w_k \in A_p(T)\) with bound less than or equal to \(K_p(4\pi)^{2p}\). By Proposition (4.2), for \(k = 1,2,\ldots\), we have \(w*\mu_k \in A_p(\Sigma_a)\) with bound less than or equal to \(K_p(4\pi)^{2p}\). Fix an interval \(I\). Since \(\|w*\mu_k - w\|_{L_1(\Sigma_a)} \to 0\) as \(k \to \infty\), by Fatou’s lemma and Fubini’s Theorem, there is a subsequence \((w*\mu_{k_i})_{i \geq 0}\) such that for \(\mu\)-a.e. \(x\) in \(\Sigma_a\),
\[
\frac{1}{|I|} \int_I w(x - \varphi(s))ds \left( \frac{1}{|I|} \int_I w^{-1/(p-1)}(x - \varphi(s))ds \right)^{p-1} \\
\leq \frac{1}{|I|} \liminf_I \int_I w * \mu_k(x - \varphi(s))ds \\
\times \left( \frac{1}{|I|} \liminf_I \int_I (w * \mu_k)^{-1/(p-1)}(x - \varphi(s))ds \right)^{p-1} \\
\leq \frac{1}{|I|} \limsup_I \int_I w * \mu_k(x - \varphi(s))ds \\
\times \limsup_I \left( \frac{1}{|I|} \int_I (w * \mu_k)^{-1/(p-1)}(x - \varphi(s))ds \right)^{p-1} \\
= \frac{1}{|I|} \limsup_I \int_I w * \mu_k(x - \varphi(s))ds \\
\times \left( \frac{1}{|I|} \int_I (w * \mu_k)^{-1/(p-1)}(x - \varphi(s))ds \right)^{p-1} \\
\leq K_p^2(4\pi)^{2p}.
\]

So, for each interval \( I \), the above inequality holds for \( x \in \Sigma_a \), except possibly on a set of measure 0 (depending on \( I \)). Thus, the inequality holds for \( \mu \)-a.e. \( x \) in \( \Sigma_a \) and for all intervals with rational endpoints (countably many). Approximating an arbitrary interval \( I \) by an interval with rational endpoints, a straightforward argument shows that the above inequality still holds for \( \mu \)-a.e. \( x \) in \( \Sigma_a \) and all intervals \( I \), hence showing that (2.4) holds and \( w \in A_p(\Sigma_a) \). □

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**References**


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