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Asymptotics of Sobolev orthogonal polynomials for a Jacobi weight

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ABSTRACT. In this paper, we obtain the strong asymptotics for the sequence of orthogonal polynomials with respect to the inner product

$$(f,g)_S = \int f(x)g(x)\,d\mu(x) + \lambda \int_{-1}^{1} f'(x)g'(x)(1-x)^\alpha(1+x)^\beta dx$$

with α, β, $\lambda > 0$, where μ is an admissible measure on $[-1,1]$.

1. Introduction and statement of results

The study of the orthogonal polynomials with respect to the inner products involving derivatives has been very active for the last ten years. In particular, the asymptotic behavior of the orthogonal polynomials with respect to the inner product of the form

$$(f,g)_S = \int fg d\mu_1 + \int f'g'd\mu_2 \tag{1}$$

was investigated in [1] and [3] for the case when μ_2 is discrete (a combination of a finite number of masses). If both measures have absolutely continuous components, the asymptotic results are very scarce. Probably the first non-trivial step in this direction was done under the assumption that the measures form a so-called coherent pair or symmetrically coherent pair (see [2, 4]). The goal of this paper is to establish the strong asymptotics of the orthogonal polynomials with respect to the inner product (1) when the second measure μ_2 is given by the classical Jacobi weight and μ_1 satisfies some restrictions.

Hence, we consider the Sobolev inner product

$$(f,g)_S = \langle f,g\rangle_1 + \lambda\langle f',g'\rangle_2, \quad \lambda > 0, \tag{2}$$

where

$$\langle f,g\rangle_1 = \int f(x)g(x)\,d\mu(x),$$

μ is a finite Borel measure supported on $[-1,1]$, and

$$\langle f,g\rangle_2 = \int_{-1}^{1} f(x)g(x)\rho(\alpha,\beta;x)\,dx$$

with

$$\rho(\alpha,\beta;x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha,\beta > 0.$$

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We will use the following notation. For a measure μ, we denote the nth monic orthogonal polynomial with respect to μ by $P_n(\mu; x)$, and the $L^2(\mu)$-norm by

$$\|f\|_\mu^2 = \int |f(x)|^2 \, d\mu(x).$$

Then $P_n(\mu; x)$ is the solution of the extremal problem

$$k_n(\mu) = \|P_n(\mu; x)\|_\mu^2 = \min\{ \, \|P_n\|_\mu^2 : P_n(x) = x^n + \dots \}. \tag{3}$$

For the measure given by the Jacobi weight $\rho(\alpha, \beta; \cdot)$, we abbreviate

$$P_n(\alpha, \beta; x) = P_n(\rho(\alpha, \beta; t)dt; x) \quad \text{and} \quad k_n(\alpha, \beta) = k_n(\rho(\alpha, \beta; t)dt).$$

Moreover, $p_n(\mu; x) = k_n^{-1/2}(\mu) P_n(\mu; x)$ will denote the nth orthonormal polynomial with respect to μ.

Finally, $\{Q_n(x)\}$ is the monic orthogonal polynomial sequence with respect to the inner product $(\cdot, \cdot)_S$ and $\tilde{k}_n = (Q_n, Q_n)_S$. As usual, the key to the asymptotics of $\{Q_n\}$ resides in the behavior of the norms $\tilde{k}_n$.

Theorem 1. *With the notation introduced above, for $n \geq 1$,*

$$k_n(\mu) + \lambda n^2 k_{n-1}(\alpha, \beta) \leq \tilde{k}_n \leq \|P_n(\alpha - 1, \beta - 1; \cdot)\|_\mu^2 + \lambda n^2 k_{n-1}(\alpha, \beta). \tag{4}$$

Furthermore, if μ satisfies the condition

$$\|p_n(\alpha - 1, \beta - 1; \cdot)\|_\mu = o(n), \quad n \to \infty, \tag{5}$$

then

$$\lim_{n \to \infty} \frac{\tilde{k}_n}{n^2 k_{n-1}(\alpha, \beta)} = \lambda. \tag{6}$$

Limit (6) will be essential for the proof of the asymptotics of the Sobolev polynomials Q_n. Hence, for applications, it is convenient to formulate some sufficient conditions for (5).

Proposition 1. *If μ is a finite Borel measure supported on $[-1, 1]$, each of the following conditions is sufficient for* (5):

(C.1) *there exist $\alpha' < 3/2 - \alpha$ and $\beta' < 3/2 - \beta$ such that*

$$(1 - x)^{\alpha'} (1 + x)^{\beta'} \in L^1(\mu); \tag{7}$$

(C.2) $\alpha, \, \beta \in (0, 3/2)$;

(C.3) *μ is an absolutely continuous measure with respect to Lebesgue measure on $[-1, 1]$ and*

$$\mu'(x) = h(x)(1 - x)^{\alpha - 1}(1 + x)^{\beta - 1} \tag{8}$$

with

$$\int_{-1}^{1} (1 - x^2)^{\varkappa} h(x) \, dx < \infty$$

for some $\varkappa < 1/2$;

(C.4) $\operatorname{supp} \mu \subset (-1, 1)$.

In what follows, we write $\mu \in S$ to denote that the finite measure μ with $\operatorname{supp} \mu = [-1,1]$ satisfies the Szegő condition, i.e.,

$$\int_{-1}^{1} \frac{\ln \mu'(x)}{\sqrt{1-x^2}}\, dx > -\infty. \tag{9}$$

A finite measure μ is *admissible* if $\mu \in S$ and (5) holds.

Now we state the asymptotic result for the monic polynomials Q_n.

Theorem 2. *If μ is admissible, then locally uniformly in $\Omega = \overline{\mathbb{C}} \setminus [-1,1]$,*

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(\alpha,\beta;x)} = \frac{2}{\Phi'(x)} \tag{10}$$

where $\Phi(x) = x + \sqrt{x^2-1}$ with $|\Phi(x)| > 1$ for $x \in \Omega$.

Since

$$P_n(\alpha,\beta;x) = 2^{-n-\alpha-\beta-1/2} \frac{(\sqrt{x-1}+\sqrt{x+1})^{\alpha+\beta}}{(x-1)^{\alpha/2}(x+1)^{\beta/2}} \frac{\Phi^{n+1/2}(x)}{(x^2-1)^{1/4}} (1+o(1)) \tag{11}$$

locally uniformly in Ω (see below), Theorem 2 gives the desired strong outer asymptotics of $\{Q_n\}$. Now, the following corollary follows immediately.

Corollary 1. *The zeros of Sobolev orthogonal polynomials Q_n distribute on $[-1,1]$ according to the* arcsin *law. All of them accumulate at $[-1,1]$, i.e.,*

$$\bigcap_{n\geq 1} \overline{\bigcup_{k=n}^{\infty} \{x : Q_k(x) = 0\}} = [-1,1]. \tag{12}$$

Finally, we make the following remarks.

1. Besides Proposition 1, some other sufficient conditions for (5) can be produced. For example, an addition to μ of a finite number of mass points on $(-1,1)$ does not affect (8), although masses at -1 and 1 might spoil the admissibility.

2. We can take $h(x) = |x-\xi|$ in (8) with $\xi \in \mathbb{R} \setminus (-1,1)$. Then measures μ and $\rho(\alpha,\beta;x)dx$ constitute a coherent pair. Analogously, a symmetrically coherent pair can be obtained if $\alpha = \beta$ and in (8) we choose either $h(x) = x^2 + \xi^2$ with $\xi \in \mathbb{R} \setminus \{0\}$ or $h(x) = \xi^2 - x^2$ with $\xi \in \mathbb{R} \setminus (-1,1)$. In these cases, (10) has been proved in [4] and [2], respectively.

3. The admissibility of μ is not necessary for the asymptotics (10). Consider, for example, $\mu = \delta_0$. Then, easy computation shows that

$$Q_n(x) = P_n(\alpha-1,\beta-1;x) - P_n(\alpha-1,\beta-1;0),$$

and (10) certainly takes place.

2. Proofs

Proof of Theorem 1. First, from the extremal property (3) for $k_n(\mu)$ and $k_n(\alpha,\beta)$, it follows that $k_n(\mu) \leq \|Q_n\|_\mu^2$ and $k_{n-1}(\alpha,\beta) \leq \langle Q_n'/n, Q_n'/n\rangle_2$, so

$$\tilde{k}_n = (Q_n,Q_n)_S = \|Q_n\|_\mu^2 + \lambda\langle Q_n', Q_n'\rangle_2 \geq k_n(\mu) + \lambda n^2 k_{n-1}(\alpha,\beta).$$

On the other hand, using the extremal property of $\tilde{k}_n$ and the relation

$$P_n'(\alpha-1,\beta-1;x) = nP_{n-1}(\alpha,\beta;x), \quad n \geq 1, \quad \alpha,\beta > 0$$

(see [6]), we have

$$\tilde{k}_n \le \left(P_n(\alpha-1,\beta-1;\,\cdot\,), P_n(\alpha-1,\beta-1;\,\cdot\,)\right)_S$$
$$= \|P_n(\alpha-1,\beta-1;\,\cdot\,)\|_\mu^2 + \lambda n^2 k_{n-1}(\alpha,\beta).$$

Hence, (4) is proved.

We now recall briefly Szegő's extremal problem in $\Omega = \overline{\mathbb{C}} \setminus [-1,1]$, following the work of Widom [7]. Given $\mu \in S$, $\rho(x) = \mu'(x)$, the unique function $R(z)$ holomorphic in Ω can be constructed such that

1. $R(z) \neq 0$ for $z \in \Omega$,
2. $R(\infty) > 0$,
3. for almost every $x \in (-1,1)$,

$$\lim_{y\to 0} |R(x+iy)| = \rho(x).$$

Then, as usual, $H^2(\rho)$ is the Hardy space of functions f analytic in Ω such that $|f^2(z)R(z)|$ admits a harmonic majorant in Ω. It is a Hilbert space with the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\rho(x)dx$. It is well known that any $f \in H^2(\rho)$ has a non-tangential limit

$$\lim_{z\to x} f(z) = f(x)$$

for almost all $x \in (-1,1)$, and the following extremal problem has sense:

$$\nu(\rho) = \inf\left\{ \int_{-1}^{1} |F(x)|^2\rho(x)dx : F \in H^2(\rho), F(\infty) = 1 \right\}. \tag{13}$$

For $\mu \in S$, $\nu(\rho) > 0$ and there exists the unique extremal function

$$F(z) = F(\rho; z)$$

solving (13).

Furthermore,

$$F^2(z) = \frac{\Phi'(z)}{2}\frac{R(\infty)}{R(z)} \tag{14}$$

where Φ and R were defined above (see e.g., [7, Theorem 6.2]).

Function F and the extremal constant $\nu(\rho)$ are closely related with the asymptotics of $P_n(\mu;\,\cdot\,)$. In fact,

$$\lim_{n\to\infty} 4^n k_n(\mu) = \nu(\rho) \tag{15}$$

and

$$\lim_{n\to\infty} 2^n \frac{P_n(\mu; z)}{\Phi^n(z)} = F(\rho; z) \tag{16}$$

locally uniformly in Ω.

In the particular case when $\rho(x) = \rho(\alpha,\beta;x) = (1-x)^\alpha(1+x)^\beta$, it is easy to verify using (14) that

$$F(z) := F(\rho; z) = 2^{-\alpha-\beta-1/2}\frac{\Phi^{1/2}(z)}{(z^2-1)^{1/4}}\frac{\left(\sqrt{z-1}+\sqrt{z+1}\right)^{\alpha+\beta}}{(z-1)^{\alpha/2}(z+1)^{\beta/2}} \tag{17}$$

where the selection of the branch in Ω is obvious, hence the well-known formula (11).

As above, in order to simplify notation in what follows, we write $\nu(\alpha,\beta)$ instead of $\nu(\rho(\alpha,\beta;\,\cdot\,))$.

Assume that (5) holds. Then

$$\lim_n \frac{\|P_n(\alpha-1,\beta-1;\,\cdot\,)\|_\mu^2}{n^2 k_n(\alpha-1,\beta-1)} = 0.$$

Since, by (15),

$$\lim_n \frac{k_n(\alpha-1,\beta-1)}{k_{n-1}(\alpha,\beta)} = \frac{\nu(\alpha-1,\beta-1)}{4\nu(\alpha,\beta)} \in (0,\infty), \tag{18}$$

(6) immediately follows from (4). In particular, by (15),

$$\lim_n \frac{4^{n-1}\tilde{k}_n}{n^2} = \lambda\nu(\alpha,\beta). \tag{19}$$

Theorem 1 is proved. □

Proof of Proposition 1. To establish condition (C.1), we use the uniform bound

$$|p_n(\alpha,\beta;x)|^2 \le C(\sqrt{1-x}+1/n)^{-1-2\alpha}(\sqrt{1+x}+1/n)^{-1-2\beta}, \quad x\in[-1,1],$$

(see e.g., [5, Lemma 16, p. 83]). Since μ is finite, a sufficient condition for (5) is the existence of an $\varepsilon>0$ such that

$$\lim_n \frac{1}{n^2}\int_{-1}^{-1+\varepsilon}\left(\sqrt{1+x}+\frac{1}{n}\right)^{1-2\beta} d\mu(x) = 0 \tag{20}$$

and

$$\lim_n \frac{1}{n^2}\int_{1-\varepsilon}^{1}\left(\sqrt{1-x}+\frac{1}{n}\right)^{1-2\alpha} d\mu(x) = 0.$$

Consider the first limit (the second one can be handled in a similar way). Put $\beta' = 3/2-\beta-\delta$ for a $\delta>0$; without loss of generality, we may assume $\delta<1$. Then,

$$\begin{aligned}
&\frac{1}{n^2}\int_{-1}^{-1+\varepsilon}\left(\sqrt{1+x}+\frac{1}{n}\right)^{1-2\beta} d\mu(x)\\
&\quad= \frac{1}{n^{2\delta}}\int_{-1}^{-1+\varepsilon}\left(n\sqrt{1+x}+1\right)^{2\delta-2}\left(\sqrt{1+x}+\frac{1}{n}\right)^{2\beta'} d\mu(x)\\
&\quad\le \frac{1}{n^{2\delta}}\int_{-1}^{-1+\varepsilon}\left(\sqrt{1+x}+\frac{1}{n}\right)^{2\beta'} d\mu(x)\,.
\end{aligned}$$

If $\beta'\ge 0$, then both $(1+x)^{\beta'}\in L^1(\mu)$ and the left-hand side of the last inequality tends to zero. Otherwise,

$$\frac{1}{n^{2\delta}}\int_{-1}^{-1+\varepsilon}\left(\sqrt{1+x}+\frac{1}{n}\right)^{2\beta'} d\mu(x) \le \frac{1}{n^{2\delta}}\int_{-1}^{-1+\varepsilon}(1+x)^{\beta'}\,d\mu(x),$$

and (7) implies that the last integral is finite, which is sufficient for (20). This proves (C.1). Clearly, if α, $\beta\in(0,3/2)$, we can take α', $\beta'>0$ and (C.2) follows from (7).

Under assumptions of (C.3), choose

$$\alpha' = 5/4-\alpha+\varkappa/2 \quad\text{and}\quad \beta' = 5/4-\beta+\varkappa/2.$$

Then (7) holds. Condition (C.4) is also a straightforward consequence of (C.1). Proposition 1 is established. □

Proof of Theorem 2. By (16), the asymptotics (10) is equivalent to

$$\lim_{n\to\infty} \frac{2^{n-1}Q_n(z)}{\Phi^n(z)} = \frac{F(z)}{\Phi'(z)} \tag{21}$$

locally uniformly in Ω with F given in (17). Since $\lambda\langle Q'_n, Q'_n\rangle_2 \le \tilde{k}_n$, by (4) and (15),

$$\left\langle \frac{Q'_n}{n}, \frac{Q'_n}{n} \right\rangle_2 \le k_{n-1}(\alpha,\beta)\,(1+o(1)). \tag{22}$$

Then, Q'_n/n is an extremal sequence for the problem (3), and its asymptotic behavior is determined. In fact, the well-known argument using the parallelogram law can be used (see [7, p.194]). In order to simplify notation, we denote

$$\|p_n\|^2 = \langle p_n, p_n\rangle_2.$$

Then,

$$2\left\|\frac{2^{n-1}Q'_n}{n}\right\|^2 + 2\left\|\Phi^{n-1}F\right\|^2 = \left\|\frac{2^{n-1}Q'_n}{n} - \Phi^{n-1}F\right\|^2 + \left\|\frac{2^{n-1}Q'_n}{n} + \Phi^{n-1}F\right\|^2. \tag{23}$$

On the other hand, by (15) and (22),

$$\begin{aligned}\limsup_{n\to\infty}\left\|\frac{2^{n-1}Q'_n}{n}\right\|^2 &= \limsup_{n\to\infty} 4^{n-1}\left\|\frac{Q'_n}{n}\right\|^2 \\ &\le \limsup_{n\to\infty} 4^{n-1}k_{n-1}(\alpha,\beta)\,(1+o(1)) = \nu(\alpha,\beta).\end{aligned}$$

Furthermore, since $|\Phi(x)| = 1$ for $x \in [-1,1]$,

$$\left\|\Phi^{n-1}F\right\|^2 = \|F\|^2 = \nu(\alpha,\beta)$$

because F is the solution of the extremal problem (13). Hence, the left-hand side of (23) is asymptotically not greater than $4\nu(\alpha,\beta)$ as $n\to\infty$.

Notice that

$$\frac{1}{2}\left(\frac{2^{n-1}Q'_n}{n\Phi^{n-1}} + F\right) \in H^2\left(\rho(\alpha,\beta;\cdot)\right)$$

and is equal to 1 at ∞, so by extremality of $\nu(\alpha,\beta)$,

$$\left\|\frac{2^{n-1}Q'_n}{n} + \Phi^{n-1}F\right\|^2 = 4\left\|\frac{1}{2}\left(\frac{2^{n-1}Q'_n}{n\Phi^{n-1}} + F\right)\right\|^2 \ge 4\nu(\alpha,\beta).$$

This shows that

$$\lim_{n\to\infty}\left\|\frac{2^{n-1}Q'_n}{n} - \Phi^{n-1}F\right\| = 0. \tag{24}$$

We can use the following result (see [7, Corollary 7.4]).

Lemma 1. *Given a weight $\rho \in S$ and a compact subset $K \subset \Omega$, there exists a constant $C = C(K)$ such that*

$$\max_{z\in K} |f(z)|^2 \le C\int_{-1}^{1} |f(x)|^2\rho(x)dx \quad \textit{for all} \quad f \in H^2(\rho).$$

Then, from (24), we have that

$$\lim_{n\to\infty} \frac{2^{n-1}Q_n'}{n\Phi^{n-1}}(z) = F(z), \tag{25}$$

locally uniformly in Ω. We should prove now that (25) implies (21).

We have that

$$\tilde{k}_n = \|Q_n\|_\mu^2 + \lambda n^2 \left\| \frac{Q_n'}{n} \right\|^2,$$

so that

$$\frac{4^{n-1}\tilde{k}_n}{n^2} = \left\| \frac{2^{n-1}Q_n}{n\Phi^n} \right\|_\mu^2 + \lambda \left\| \frac{2^{n-1}Q_n'}{n\Phi^{n-1}} \right\|^2.$$

From (24),

$$\lim_{n\to\infty} \left\| \frac{2^{n-1}Q_n'}{n\Phi^{n-1}} \right\|^2 = \|F\|^2 = \nu(\alpha,\beta),$$

and by (19),

$$\lim_{n\to\infty} \frac{4^{n-1}\tilde{k}_n}{n^2} = \lambda\nu(\alpha,\beta).$$

Hence,

$$\lim_{n\to\infty} \left\| \frac{2^{n-1}Q_n}{n\Phi^n} \right\|_\mu = 0.$$

Applying Lemma 1 again, we have that

$$\lim_{n\to\infty} \frac{2^{n-1}Q_n}{n\Phi^n}(z) = 0,$$

locally uniformly in Ω. Then,

$$\lim_{n\to\infty} \left(\frac{2^{n-1}Q_n}{n\Phi^n} \right)'(z) = 0,$$

also locally uniformly in Ω. It remains to use the identity

$$\frac{2^{n-1}Q_n(z)}{\Phi^{n+1}(z)}\Phi'(z) = \frac{2^{n-1}Q_n'(z)}{n\Phi^n(z)} - \left(\frac{2^{n-1}Q_n(z)}{n\Phi^n(z)} \right)'$$

and (25) to obtain (21). The theorem is proved. □

Proof of Corollary 1. Corollary 1 is a direct consequence of the classical *nth* root asymptotics of $|Q_n|$ or $\tilde{k}_n$ (see (21) and (19)). That is, if we associate with each $Q_n(x)$ the discrete unit measure with equal positive masses at its zeros (accounting for multiplicity)

$$\omega_n = \frac{1}{n} \sum_{Q_n(\xi)=0} \delta_\xi,$$

then

$$d\omega_n(x) \longrightarrow \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$$

in the weak-* topology.

Now, it is sufficient to observe that $Q_n(x)/P_n(\alpha, \beta; x)$ is a sequence of analytic functions in Ω, $\Phi(x)$ is analytic and has no zeros in Ω, and the zero asymptotic of $P_n(\alpha, \beta; x)$ is known. Hence, the zeros of $Q_n(x)$ cannot accumulate outside $[-1, 1]$. On the other hand, the weak asymptotics shows that they must be dense in $[-1, 1]$. □

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