METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 18, No. 4, pp. 417–426, December 2011

004

GEVREY REGULARITY OF CERTAIN SOLUTIONS TO THE CAHN-HILLIARD EQUATION WITH ROUGH INITIAL DATA*

DAVID SWANSON†

Abstract. In this article we consider the problem of local Gevrey regularity of periodic solutions to the Cahn-Hilliard equation with initial data in a space of distributions. The method presented in this paper is based on the analysis of the Navier-Stokes system presented in [2] and makes use of a convolution inequality due to Kerman [3].

Key words. Cahn-Hilliard equation, Gevrey regularity.

AMS subject classifications. 35Q35, 35B65, 35K35.

1. Introduction.

The Cahn-Hilliard equation we shall study is

$$u_t = -\Delta^2 u - \alpha\Delta u + \beta\Delta(u^3),\ x \in \Omega = [0, L]^n,\ t > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x),\ x \in \Omega, \tag{1.2}$$

where $\alpha \geq 0$, and $\beta > 0$ and u satisfies a periodic boundary condition on $\partial\Omega$. We assume for simplicity that $L = 2\pi$.

We let ϕ_k, $k \in \mathbb{Z}^n$, denote the k^{th} Fourier mode of a space periodic function ϕ, so that ϕ may be identified with its Fourier series

$$\phi \sim \sum_{k\in\mathbb{Z}^n} \phi_k e^{\imath kx}.$$

The sequence of Fourier coefficients of ϕ is denoted by $\vec{\phi} = \{\phi_k\}$. Equation (1.1) may be cast as an infinite dimensional complex-valued dynamical system taking the form

$$\frac{d}{dt}u_k = -|k|^4 u_k + \alpha|k|^2 u_k - \beta B[\vec{u}]_k, \quad k \in \mathbb{Z}^n, \tag{1.3}$$

where $B[\vec{u}] = B[\vec{u}, \vec{u}, \vec{u}]$ is a trilinear term satisfying

$$B[\vec{u}, \vec{v}, \vec{w}]_k = |k|^2 \sum_j \sum_h u_j v_h w_{k-j-h}. \tag{1.4}$$

We denote by A the operator $\vec{v} \mapsto A\vec{v}$ given by $(A\vec{v})_k = |k|^2 v_k$, so that the system (1.1)–(1.2) takes the form

$$\vec{u}_t = -A^2\vec{u} + \alpha A\vec{u} - \beta B[\vec{u}]. \tag{1.5}$$

$$\vec{u}(0) = \vec{u}_0. \tag{1.6}$$

2. Main result.

DEFINITION 2.1. We denote by V the vector space of all complex valued sequences $\vec{v} = \{v_k\}_{k\in\mathbb{Z}^n}$ defined on $\mathbb{Z}^n$. Given $\vec{v} \in V$, $\lambda \geq 0$, $\theta \in \mathbb{R}$, and $1 \leq p < \infty$ we define

$$\|\vec{v}\|_{\lambda,\theta,p} = \left(\sum_{k\in\mathbb{Z}^n} e^{\lambda p|k|}(1 + |k|)^{\theta p}|v_k|^p\right)^{1/p}$$

*Received January 19, 2012; accepted for publication June 11, 2012.
†Department of Mathematics, 328 Natural Science Building, University of Louisville, Louisville KY 40292, USA (david.swanson@louisville.edu).

and

$$V_{\lambda,\theta,p} = V \cap \{\vec{v} : \|\vec{v}\|_{\lambda,\theta,p} < \infty\}.$$

In case $\lambda = 0$ we write $\|\vec{v}\|_{\theta,p}$ and $V_{\theta,p}$ in place of $\|\vec{v}\|_{0,\theta,p}$ and $V_{0,\theta,p}$, respectively. $V_{\lambda,\theta,p}$ is a Banach space with norm $\|\cdot\|_{\lambda,\theta,p}$. We denote by $\dot{V}$ the vector space

$$\dot{V} = V \cap \{\vec{v} : v_0 = 0\}$$

and define

$$\dot{V}_{\lambda,\theta,p} = \dot{V} \cap V_{\lambda,\theta,p}.$$

For any $\vec{v} \in \dot{V}_{\lambda,\theta,p}$ the norms

$$\|\vec{v}\|_{\lambda,\theta,p} \quad \text{and} \quad \left(\sum_{k\in\mathbb{Z}^n} e^{\lambda p|k|}|k|^{\theta p}|v_k|^p\right)^{1/p}$$

are equivalent.

Definition 2.2. A **mild solution** to equation (1.1) with initial data in $\dot{V}_{\theta,p}$ is a map $\vec{u}(\cdot) \in C([0,T]; \dot{V}_{\theta,p})$ satisfying

$$\vec{u}(t) = e^{-t(A^2-\alpha A)}\vec{u}_0 - \beta \int_0^t e^{-(t-s)(A^2-\alpha A)} B[\vec{u}(s)]\, ds, \quad 0 \le t \le T. \tag{2.1}$$

for some $T > 0$. A mild solution $\vec{u}$ is said to be **Gevrey regular** if, in addition to satisfying (2.1), there exists $\lambda > 0$ with the property that

$$\sup_{0\le t\le T} \|\vec{u}(t)\|_{\lambda t,\theta,p} < \infty. \tag{2.2}$$

Theorem 2.3. *Let $1 < p < \infty$, let $\theta = \frac{n}{p'} - 1$ where $p' = p/(p-1)$, and let $\lambda > 0$. Then*

1. *For any $T > 0$ there exists a solution $\vec{u} \in C([0,T]; V_{\theta,p})$ to (2.1) satisfying (2.2) provided that $\beta > 0$ is sufficiently small.*
2. *For any $\beta > 0$ there exist $T > 0$ and a solution $\vec{u} \in C([0,T]; V_{\theta,p})$ to (2.1) satisfying (2.2).*

It is well-known that weak solutions to (1.1) exist for all times $T > 0$ provided that the initial data belongs to $L^2(\Omega)$, cf. [5, Ch. III, Theorem 4.2]. Solutions in the sense of (2.1) are known to exist for initial data in $V_{2,2}$, cf. [4, Lemma 55.3]. In case $n = 1, 2, 3$ the initial data spaces allowed in Theorem 2.3 allow for rougher initial data than the classical existence theory. Recently Biswas and Bae [1] obtained general estimates which when $p = 2$ yield Theorem 2.3 as a special case. The result presented in this paper is valid for general p and is based on the analysis of convolution inequalities rather than methods of harmonic analysis.

To prove Theorem 2.3 we let $T > 0$ and fix a parameter $\gamma > 0$ satisfying

$$\min\left\{1 - \tfrac{2n}{3p'}, 1 - \tfrac{n}{3p'} - \tfrac{n}{3p}\right\} < \gamma < 1 \tag{2.3}$$

where p' denotes the Hölder conjugate of p. We employ the path space

$$E = C([0,T];\dot{V}_{\theta,p})\cap\left\{\vec{u}: \sup_{0\le t\le T}\|\vec{u}(t)\|_{\lambda t,\theta,p} < \infty\right\}\cap\left\{\vec{u}: \sup_{0<t\le T} t^{\frac{\gamma}{4}}\|\vec{u}(t)\|_{\lambda t,\theta+\gamma,p} < \infty\right\}$$

with norm

$$\|\vec{u}(\cdot)\|_E = \sup_{0\le t\le T}\|\vec{u}(t)\|_{\lambda t,\theta,p} + \sup_{0<t\le T} t^{\frac{1}{4}}\|\vec{u}(t)\|_{\lambda t,\theta+1,p}.$$

We will obtain a solution to (2.1), (2.2) by showing that for $\vec{u}_0 \in V_{\theta,p}$ the map $t \mapsto e^{-t(A^2-\alpha A)}\vec{u}_0$ belongs to E and that the trilinear functional S on $E\times E\times E$ defined by

$$S(\vec{u}(t),\vec{v}(t),\vec{w}(t)) = \beta\int_0^t e^{-(t-s)(A^2-\alpha A)}B[\vec{u}(s),\vec{v}(s),\vec{w}(s)]\,ds \tag{2.4}$$

is bounded from $E\times E\times E$ to E.

3. Estimate on the linear term. Define

$$\kappa = \kappa_{\alpha,\lambda} = -\min\left\{\tfrac{1}{2}x^4 - \alpha x^2 - \lambda x:\ x\in\mathbb{R}\right\}. \tag{3.1}$$

We will show that for $\vec{u}_0 \in \dot{V}_{\theta,p}$ the map $t\mapsto e^{-t(A^2-\alpha A)}\vec{u}_0$ belongs to E. We make repeated use of the elementary inequality

$$e^{-atx^4}x^b \le C_{a,b}t^{-\frac{b}{4}}, \quad a,b,t,x>0. \tag{3.2}$$

PROPOSITION 3.1. *If $\vec{v}\in\dot{V}_{\theta,p}$, then $t\mapsto e^{-t(A^2-\alpha A)}\vec{v}$ belongs to E.*

Proof. Let $\vec{v}\in\dot{V}_{\theta,p}$. Equation (3.1) implies

$$\|e^{-t(A^2-\alpha A)}\vec{v}\|^p_{\lambda t,\theta,p} = \sum_k e^{\lambda pt|k|}e^{-pt(|k|^4-\alpha|k|^2)}|k|^{\theta p}|v_k|^p \le e^{p\kappa t}\sum_k e^{-\frac{p}{2}t|k|^4}|k|^{\theta p}|v_k|^p$$

so that

$$\|e^{-t(A^2-\alpha A)}\vec{v}\|_{\theta,p} \le e^{\kappa t}\|\vec{v}\|_{\theta,p} \tag{3.3}$$

for all $0\le t\le T$. Similarly, equations (3.1) and (3.2) imply

$$\|e^{-t(A^2-\alpha A)}\vec{v}\|^p_{\lambda t,\theta+\gamma,p} \le e^{p\kappa t}\sum_k e^{-\frac{p}{2}t|k|^4}|k|^{\theta+\gamma}|v_k|^p \le C_{\gamma,p}e^{p\kappa t}t^{-\frac{p\gamma}{4}}\sum_k |k|^{\theta p}|v_k|^p$$

so that

$$t^{\frac{\gamma}{4}}\|e^{-t(A^2-\alpha A)}\vec{v}\|_{\theta+\gamma,p} \le C_{\gamma,p}e^{\kappa t}\|\vec{v}\|_{\theta,p} \tag{3.4}$$

for all $0<t\le T$. Finally we may combine the estimates (3.3) and (3.4) to obtain $\|e^{-t(A^2-\alpha A)}\vec{v}\|_E < \infty$. ∎

4. Weighted convolution inequalities. The analysis of the nonlinear term will be based on the following convolution theorem due to Kerman [3].

THEOREM 4.1. *Assume that $1 < p, q, r < \infty$ and that the following eight conditions hold:*

1. $\gamma = \alpha + \beta + n\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r} - 1\right)$,
2. $\frac{1}{r} \le \frac{1}{p} + \frac{1}{q}$,
3. $\alpha < \frac{n}{p'}$
4. $\beta < \frac{n}{q'}$
5. $\gamma > -\frac{n}{r}$,
6. $\alpha + \beta \ge 0$,
7. $\alpha \ge \gamma$,
8. $\beta \ge \gamma$.

Then

$$\left(\int_{\mathbb{R}^n} |f * g(x)|^r |x|^{\gamma r}\,dx\right)^{1/r} \le C\left(\int |f(x)|^p |x|^{\alpha p}\right)^{1/p}\left(\int_{\mathbb{R}^n} |g(x)|^q |x|^{\beta q}\,dx\right)^{1/q}$$

for all measurable f and g, where C does not depend on either f or g.

Our primary interest is in the case $p = q = r$ and the corresponding inequality

$$\left(\int_{\mathbb{R}^n} |f * g(x)|^p |x|^{\gamma p}\,dx\right)^{1/p} \le C\left(\int |f(x)|^p |x|^{\alpha p}\right)^{1/p}\left(\int_{\mathbb{R}^n} |g(x)|^p |x|^{\beta p}\,dx\right)^{1/p}.$$

In this case, condition (2) is superfluous and conditions (7) and (8) are implied by conditions (1), (3), and (4). From now on we denote by $C_{a,b,\ldots}$ a constant whose precise value depends only on $a, b, \ldots$. The preceding remarks are summarized in the following theorem:

THEOREM 4.2. *Let $1 < p < \infty$. If $\alpha, \beta < \frac{n}{p'}$, $\alpha + \beta \ge 0$, and $\alpha + \beta > \frac{n}{p'} - \frac{n}{p}$, then*

$$\left(\int_{\mathbb{R}^n} |f * g(x)|^p |x|^{(\alpha+\beta-\frac{n}{p'})p}\,dx\right)^{1/p} \le C_{\alpha,\beta,n,p}\left(\int |f(x)|^p |x|^{\alpha p}\right)^{1/p}\left(\int_{\mathbb{R}^n} |g(x)|^p |x|^{\beta p}\,dx\right)^{1/p}$$

for all measurable functions f and g.

The convolution of two sequences $\vec{u}, \vec{v} \in V$ is given by

$$(\vec{u} * \vec{v})_k = \sum_h u_{k-h} v_h$$

whenever the sum is convergent. Theorem 4.2 will be used to prove the following inequality in weighted sequence spaces.

THEOREM 4.3. *Under the same hypotheses as Theorem 4.2 we have that*

$$\|\vec{u} * \vec{v}\|_{\alpha+\beta-\frac{n}{p'},p} \le C_{\alpha,\beta,n,p}\|\vec{u}\|_{\alpha,p}\|\vec{v}\|_{\beta,p}$$

for all $\vec{u} \in V_{\alpha,p}$ *and* $\vec{v} \in V_{\beta,p}$.

Proof. For each $k \in \mathbb{Z}^n$ we denote by Q_k the open n-cube in $\mathbb{R}^n$ with edges parallel to the coordinate axes, sidelength one, and center k. Two cubes Q_k and Q_l are said to be adjacent if $|k-l|_\infty = 1$. First we observe that if $t > -n$ then

$$\int_{Q_k} |x|^t \, dx \approx (1+|k|)^t. \tag{4.1}$$

If $k \neq 0$, then $|x| \geq C_n$ for all $x \in Q_k$. In this instance we have

$$|x| \leq |x-k| + |k| \leq C_n(1+|k|)$$

and

$$|k| \leq |x-k| + |x| \leq C_n|x|,$$

hence

$$1 + |k| \leq C_n|x|.$$

Since

$$\int_{Q_k} (1+|k|)^t \, dx = (1+|k|)^t$$

the equivalence (4.1) is valid for any real t. On the other hand, if $k=0$, then

$$\int_{Q_0} |x|^t \, dx = C_{n,t}$$

provided that $t > -n$, establishing (4.1). Now let us examine the structure of the convolution. Let $\gamma = \alpha + \beta - \frac{n}{p'}$. Since $\gamma p > -n$ we have

$$\begin{aligned} \|\vec{u} * \vec{v}\|_{\gamma,p}^p &= \sum_k (1+|k|)^{\gamma p} |(\vec{u} * \vec{v})_k|^p \\ &\leq \sum_k (1+|k|)^{\gamma p} \left(\sum_h |u_h||v_{k-h}| \right)^p \\ &\lesssim \sum_k \int_{Q_k} |x|^{\gamma p} \left(\sum_h |u_h||v_{k-h}| \right)^p dx. \qquad (4.2) \end{aligned}$$

Define functions $f, g : \mathbb{R}^n \to [0, \infty)$ by

$$f(x) = |u_k|, \quad g(x) = |v_k|, \qquad x \in Q_k.$$

Fix $k \in \mathbb{Z}^n$. Then

$$\sum_h |u_h||v_{k-h}| = \sum_h \int_{Q_h} f(y)|v_{k-h}| \, dy.$$

Note that $y \in Q_h$ if and only if $k - y \in Q_{k-h}$. Therefore $g(k-y) = |v_{k-h}|$ for all $y \in Q_h$ and

$$\sum_h \int_{Q_h} f(y)|v_{k-h}| \, dy = \sum_h \int_{Q_h} f(y)g(k-y) \, dy.$$

Now let $x \in Q_k$. If $y \in Q_h$, then $k-y$ and $x-y$ belong to adjacent n-cubes. Thus there exists $j \in \mathbb{Z}^n$ with $|j|_\infty \le 1$ such that $g(k-y) = g(x+j-y)$. It follows that

$$g(k-y) \le \sum_{|j|_\infty \le 1} g(x+j-y)$$

and consequently

$$\sum_h |u_h||v_{k-h}| \le \sum_{|j|\le 1} \int_{\mathbb{R}^n} f(y) g(x+j-y)\, dy, \quad x \in Q_k. \tag{4.3}$$

Denote by τ_z the translation operator given by $(\tau_z \psi)(y) = \psi(y-z)$, and define

$$G = \sum_{|j|_\infty \le 1} \tau_{-j} g.$$

Then (4.3) may be written as

$$\sum_h |u_h||v_{k-h}| \le \sum_{|j|\le 1} \int_{\mathbb{R}^n} f(y) \tau_{-j} g(x-y)\, dy = f * G(x).$$

Inequality (4.2), Theorem 4.2, and Minkowski's inequality imply that

$$\begin{aligned} \|\vec{u} * \vec{v}\|_{\gamma,p} &\lesssim \left(\int_{\mathbb{R}^n} |x|^{\gamma p} \left(f * G(x)\right)^p \, dx \right)^{1/p} \\ &\lesssim \left(\int_{\mathbb{R}^n} |x|^{\alpha p} f(x)^p \, dx \right)^{1/p} \left(\int_{\mathbb{R}^n} |x|^{\beta p} G(x)^p \, dx \right)^{1/p} \\ &\lesssim \sum_{|j|\le 1} \left(\int_{\mathbb{R}^n} |x|^{\alpha p} f(x)^p \, dx \right)^{1/p} \left(\int_{\mathbb{R}^n} |x|^{\beta p} \tau_{-j} g(x)^p \, dx \right)^{1/p}. \end{aligned}$$

If $|j|_\infty \le 1$, (4.1) and the definition of g imply that

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{\beta p} (\tau_{-j} g)(x)^p \, dx &\lesssim \sum_k (1+|k|)^{\beta p} \int_{Q_k} (\tau_{-j} g)(x)^p \, dx \\ &= \sum_k (1+|k|)^{\beta p} \int_{Q_{k-j}} g(x)^p \, dx \\ &= \sum_k (1+|k|)^{\beta p} |v_{k-j}|^p \\ &\lesssim \sum_k (1+|k|)^{\beta p} |v_k|^p \end{aligned}$$

so that

$$\left(\int_{\mathbb{R}^n} |x|^{\beta p} \tau_{-j} g(x)^p \, dx \right)^{1/p} \lesssim \|\vec{v}\|_{\beta,p}.$$

Likewise we have that

$$\left(\int_{\mathbb{R}^n} |x|^{\alpha p} f(x)^p \, dx \right)^{1/p} \lesssim \|\vec{u}\|_{\alpha,p},$$

which yields the desired result. □

The following corollary extends Theorem 4.3 to three-term convolutions.

Corollary 4.4. *If* $\alpha, \beta, \gamma < \frac{n}{p'}$, $\alpha+\beta+\gamma \geq \frac{n}{p'}$, *and* $\alpha+\beta+\gamma > \frac{2n}{p'} - \frac{n}{p}$, *then*

$$\|\vec{u} * \vec{v} * \vec{w}\|_{\alpha+\beta+\gamma-\frac{2n}{p'},p} \leq C_{\alpha,\beta,n,p}\|\vec{u}\|_{\alpha,p}\|\vec{v}\|_{\beta,p}\|\vec{w}\|_{\gamma,p}.$$

Proof. The stated assumptions imply that

$$(\alpha+\beta-\tfrac{n}{p'}), \gamma < \tfrac{n}{p'},\ (\alpha+\beta-\tfrac{n}{p'})+\gamma \geq 0, \text{ and } (\alpha+\beta-\tfrac{n}{p'})+\gamma > \tfrac{n}{p'} - \tfrac{n}{p}$$

so that Theorem 4.3 implies

$$\begin{aligned}\|\vec{u} * \vec{v} * \vec{w}\|_{\alpha+\beta+\gamma-\frac{2n}{p'},p} &= \|(\vec{u} * \vec{v}) * \vec{w}\|_{(\alpha+\beta-\frac{n}{p'})+\gamma-\frac{n}{p'},p} \\ &\leq C_{\alpha,\beta,n,p}\|\vec{u} * \vec{v}\|_{\alpha+\beta-\frac{n}{p'},p}\|\vec{w}\|_{\gamma,p}.\end{aligned}$$

Finally note that the hypotheses imply $\alpha+\beta \geq 0$ and $\alpha+\beta > \frac{n}{p'} - \frac{n}{p}$ and apply Theorem 4.3 again. □

The following simplification to Corollary 4.4 will be used in the sequel.

Corollary 4.5. *If* $\frac{n}{3p'} < \alpha < \frac{n}{p'}$ *and* $\alpha > \frac{2n}{3p'} - \frac{n}{3p}$, *then*

$$\|\vec{u} * \vec{v} * \vec{w}\|_{3\alpha-\frac{2n}{p'},p} \leq C_{\alpha,\beta,n,p}\|\vec{u}\|_{\alpha,p}\|\vec{v}\|_{\alpha,p}\|\vec{w}\|_{\alpha,p}.$$

5. Estimate on the nonlinear term.

Proposition 5.1. *The mapping* S *defined by (2.4) is a bounded mapping from* $E \times E \times E$ *to* E.

Proof. We will make a general estimate of the term $\|S(\vec{u},\vec{v},\vec{w})\|_{\delta,p}$ for $\delta \in \mathbb{R}$ satisfying

$$0 < \delta + 2 - 3\theta - 3\gamma + \tfrac{2n}{p'} < 4. \tag{5.1}$$

The restrictions placed on γ in (2.3) imply that both θ, $\theta+\gamma$ are admissible values of δ satisfying (5.1). Let $\vec{u}(\cdot), \vec{v}(\cdot), \vec{w}(\cdot) \in E$. The triangle inequality implies

$$\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\lambda t,\delta,p} \leq \beta \int_0^t \|e^{-(t-s)(A^2-\alpha A)}B[\vec{u}(s),\vec{v}(s),\vec{w}(s)]\|_{\lambda t,\delta,p}\,ds.$$

For simplicity of notation we will suppress the variable s from the trilinear term. Let $0 < s < t$. Employing (3.1) above we have

$$\begin{aligned}\|e^{-(t-s)(A^2-\alpha A)}B[\vec{u},\vec{v},\vec{w}]\|^p_{\lambda t,\delta,p} &\leq \sum_k e^{\lambda pt|k|}e^{-p(t-s)(|k|^4-\alpha|k|^2)}|k|^{\delta p}|B[\vec{u},\vec{v},\vec{w}]_k|^p \\ &= \sum_k e^{\lambda ps|k|}e^{-p(t-s)(|k|^4-\alpha|k|^2-\lambda|k|)}|k|^{\delta p}|B[\vec{u},\vec{v},\vec{w}]_k|^p \\ &\leq e^{p\kappa(t-s)}\sum_k e^{\lambda ps|k|}e^{-\frac{1}{2}p(t-s)|k|^4}|k|^{\delta p}|B[\vec{u},\vec{v},\vec{w}]_k|^p \\ &\leq e^{p\kappa(t-s)}\sum_k e^{\lambda ps|k|}e^{-\frac{1}{2}p(t-s)|k|^4}|k|^{(\delta+2)p}(|\vec{u}| * |\vec{v}| * |\vec{w}|)^p_k\end{aligned}$$

where we used (1.4) on the last line. An application of (3.2) and (5.1) yields

$$e^{-\frac{1}{2}p(t-s)|k|^4}|k|^{(\delta+2)p} \leq C_{\delta,\gamma,\theta,n,p}(t-s)^{-\frac{p}{4}(\delta+2-3\theta-3\gamma+\frac{2n}{p'})}|k|^{(3\theta+3\gamma-\frac{2n}{p'})p}$$

so that

$$\begin{aligned}&\|e^{-(t-s)(A^2-\alpha A)}B[\vec{u},\vec{v},\vec{w}]\|_{\lambda t,\delta,p}\\ &\qquad \leq C_{\delta,\gamma,\theta,n,p}e^{\kappa(t-s)}(t-s)^{-\frac{1}{4}(\delta+2-3\theta-3\gamma+\frac{2n}{p'})}\||\vec{u}|*|\vec{v}|*|\vec{w}|\|_{\lambda s,3\theta+3\gamma-\frac{2n}{p'},p}.\end{aligned}$$

The definition $\theta = \frac{n}{p'} - 1$ and assumption (2.3) imply

$$\frac{n}{3p'} < \theta+\gamma < \frac{n}{p'} \quad \text{and} \quad \theta+\gamma > \frac{2n}{3p'} - \frac{n}{3p}. \tag{5.2}$$

Since

$$e^{\lambda s|k|}\left(|\vec{u}|*|\vec{v}|*|\vec{w}|\right)_k \leq \sum_h \sum_j e^{\lambda s|k-h-j|}|u_{k-h-j}|\, e^{\lambda sh}|v_h|\, e^{\lambda sj}|w_j|,$$

we may apply Corollary 4.5 to obtain

$$\||\vec{u}|*|\vec{v}|*|\vec{w}|\|_{\lambda s,3\theta+3\gamma-\frac{2n}{p'},p} \leq C_{\gamma,\theta,n,p}\|\vec{u}\|_{\lambda s,\theta+\gamma,p}\|\vec{v}\|_{\theta+\gamma,p}\|\vec{w}\|_{\lambda s,\theta+\gamma,p}.$$

We arrive at the estimate

$$\begin{aligned}&\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\delta,p}\\ &\leq C_{\delta,\gamma,\theta,n,p}e^{\kappa t}\beta\int_0^t (t-s)^{-\frac{1}{4}(\delta+2-3\theta-3\gamma+\frac{2n}{p'})}\|\vec{u}\|_{\lambda s,\theta+\gamma,p}\|\vec{v}\|_{\lambda s,\theta+\gamma,p}\|\vec{w}\|_{\lambda s,\theta+\gamma,p}\,ds.\end{aligned}$$

At this point it is convenient to denote, for $\vec{u}(\cdot) \in E$,

$$\|\vec{u}(\cdot)\|_{E'} = \sup_{0<t\leq T}\|\vec{u}(t)\|_{\lambda t,\theta+\gamma,p}$$

so that $s^{\frac{\gamma}{4}}\|\vec{u}(s)\|_{\lambda s,\theta+\gamma,p} \leq \|\vec{u}(\cdot)\|_{E'}$ for all $0 < s \leq T$. This leads to the estimate

$$\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\delta,p} \leq C_{\delta,\gamma,\theta,n,p}e^{\kappa t}\beta\|\vec{u}\|_{E'}\|\vec{v}\|_{E'}\|\vec{w}\|_{E'}\int_0^t (t-s)^{-\frac{1}{4}(\delta+2-3\theta-3\gamma+\frac{2n}{p'})}s^{-\frac{3\gamma}{4}}\,ds.$$

Since

$$\frac{1}{4}\left(\delta+2-3\theta-3\gamma+\frac{2n}{p'}\right) < 1 \quad \text{and} \quad \frac{3\gamma}{4} < 1$$

this integral equals a constant $C_{\delta,\theta,\gamma,n,p}$ times

$$t^{1-\frac{1}{4}(\delta+2-3\theta-3\gamma+\frac{2n}{p'})-\frac{3\gamma}{4}} = t^{\frac{1}{4}(2-\delta+3\theta-\frac{2n}{p'})}.$$

We conclude there exists a constant $C = C_{\delta,\gamma,\theta,n,p}$ with the property that

$$\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\delta,p} \leq Ce^{\kappa T}\beta t^{\frac{1}{4}(2-\delta+3\theta-\frac{2n}{p'})}\|\vec{u}\|_{E'}\|\vec{v}\|_{E'}\|\vec{w}\|_{E'}$$

for all $0 < t \leq T$. When $\delta = \theta$ this gives

$$\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\theta,p} \leq C\beta t^{\frac{1}{4}(2+2\theta-\frac{2n}{p'})}\|\vec{u}\|_{E'}\|\vec{v}\|_{E'}\|\vec{w}\|_{E'}$$

and when $\delta = \theta + \gamma$ this gives

$$\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\theta+\gamma,p} \leq C\beta t^{\frac{1}{4}(2+2\theta-\gamma-\frac{2n}{p'})}\|\vec{u}\|_{E'}\|\vec{v}\|_{E'}\|\vec{w}\|_{E'}$$

so that

$$t^{\frac{\gamma}{4}}\|S(\vec{u},\vec{v},\vec{w})(t)\|_{\theta+\gamma,p} \leq C\beta t^{\frac{1}{4}(2+2\theta-\frac{2n}{p'})}\|\vec{u}\|_{E'}\|\vec{v}\|_{E'}\|\vec{w}\|_{E'}.$$

Since $\theta = \frac{n}{p'} - 1$ we conclude

$$\|S(\vec{u},\vec{v},\vec{w})(\cdot)\|_E \leq C_{\alpha,n,p}\beta\|\vec{u}\|_{E'}\|\vec{v}\|_{E'}\|\vec{w}\|_{E'}. \tag{5.3}$$

It follows that S is a bounded mapping from $E \times E \times E$ to E. ☐

6. Construction of the local solution. Write $E = E_T$ to emphasize the dependence of the space E on T. Let $u_0 \in \dot{V}_{\theta,p}$ and write $\vec{v}(t) = e^{-t(A^2-\alpha A)}\vec{u}_0$. Define

$$B = \{\vec{u}(\cdot) \in E : \|\vec{u}(\cdot) - \vec{v}(\cdot)\|_{E_T'} \leq \|\vec{v}(\cdot)\|_{E_T'}\}$$

and define $L : E \to E$ by

$$L\vec{u}(t) = \vec{v}(t) + S(\vec{u},\vec{u},\vec{u})(t).$$

Let C be the constant from (5.3). It follows that

$$\|L\vec{u}(\cdot) - \vec{v}(\cdot)\|_{E_T'} \leq C\beta\|\vec{u}(\cdot)\|_{E_T'}^3 \leq 8C\beta\|\vec{v}(\cdot)\|_{E_T'}^3 \tag{6.1}$$

for all $\vec{u}(\cdot) \in B$. The trilinearity of S implies

$$S(\vec{u}_1,\vec{u}_1,\vec{u}_1) - S(\vec{u}_2,\vec{u}_2,\vec{u}_2) = S(\vec{u}_1-\vec{u}_2,\vec{u}_2,\vec{u}_2) + S(\vec{u}_1,\vec{u}_1-\vec{u}_2,\vec{u}_2) + S(\vec{u}_1,\vec{u}_1,\vec{u}_1-\vec{u}_2)$$

so that

$$\begin{aligned}\|L\vec{u}_1(\cdot) - L\vec{u}_2(\cdot)\|_{E_T'} &\leq \|S(\vec{u}_1(\cdot),\vec{u}_1(\cdot),\vec{u}_1(\cdot)) - S(\vec{u}_2(\cdot),\vec{u}_2(\cdot),\vec{u}_2(\cdot))\|_{E_T'} \\ &\leq 12C\beta\|\vec{u}_1(\cdot) - \vec{u}_2(\cdot)\|_{E_T'}\|\vec{v}(\cdot)\|_{E_T'}^2.\end{aligned} \tag{6.2}$$

Equations (6.1) and (6.2) show that $L : B \to B$ is a contraction provided that β is sufficiently small. The Banach fixed point theorem provides a solution $\vec{u}(\cdot) \in B$ to the equation

$$\vec{u}(\cdot) = L\vec{u}(\cdot).$$

This fixed point satisfies

$$\vec{u}(t) = \vec{v}(t) + S(\vec{u},\vec{u},\vec{u})(t), \quad 0 \leq t \leq T. \tag{6.3}$$

which is precisely a solution to (2.1).

On the other hand, for any $\vec{w} \in \dot{V}$ having most finitely many nonzero terms we have $\vec{w} \in \dot{V}_{\theta+\gamma,p}$ and thus

$$\begin{aligned}t^{\frac{\gamma}{4}}\|e^{-t(A^2-\alpha A)}\vec{u}_0\|_{\lambda t,\theta+\gamma,p} &\leq t^{\frac{\gamma}{4}}\|e^{-t(A^2-\alpha A)}(\vec{u}_0-\vec{w})\|_{\lambda t,\theta+\gamma,p} + t^{\frac{\gamma}{4}}\|e^{-t(A^2-\alpha A)}\vec{w}\|_{\lambda t,\theta+\gamma,p} \\ &\leq Ce^{\kappa T}\|\vec{u}_0 - \vec{w}\|_{\theta,p} + t^{\frac{\gamma}{4}}e^{\kappa T}\|\vec{w}\|_{\theta+\gamma,p}\end{aligned}$$

for all $0 < t \leq T$. Since $\|\vec{v} - \vec{w}\|_{\theta,p}$ can be made arbitrarily small we conclude that

$$\lim_{t \to 0^+} t^{\frac{\gamma}{4}} \|e^{-t(A^2 - \alpha A)} \vec{u}_0\|_{\lambda t, \theta+\gamma, p} = 0.$$

This implies

$$\lim_{T \to 0^+} \|\vec{v}(\cdot)\|_{E'_T} = 0,$$

so for arbitrary $\beta > 0$ equations (6.1) and (6.2) show that $L : B \to B$ is a contraction provided that T is sufficiently small. As above we obtain a fixed point of L satisfying (6.3), completing the argument.

REFERENCES

[1] A. Biswas and H. Bae, *Gevrey regularity for a class of dissipatice equations with analytic nonlinearity*, preprint.

[2] A. Biswas and D. Swanson, *Gevrey regularity of solutions to the 3D Navier-Stokes equations*, Fluids and Waves, pp. 83–90, Contemp. Math., 440, Amer. Math. Soc., Providence, RI, 2007.

[3] R. A. Kerman, *Convolution theorems with weights*, Trans. Amer. Math. Soc., 280:1 (1983), pp. 207–219.

[4] G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences 143, Springer, 2002.

[5] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd Ed., Applied Mathematical Sciences 68, Springer, 1997.