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WEAKLY WELL POSED HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS WITH NON CHARACTERISTIC BOUNDARY*

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Abstract. We study the mixed initial-boundary value problem for a linear hyperbolic system with non characteristic boundary. We assume the problem to be "weakly" well posed, in the sense that a unique L^2-solution exists, for sufficiently smooth data, and obeys an a priori energy estimate with a finite loss of regularity. This is the case of problems that do not satisfy the uniform Kreiss-Lopatinskiĭ condition. Under the assumption of the loss of one tangential derivative, we obtain the Sobolev regularity of solutions, provided the data are sufficiently smooth.

Key words. Symmetrizable systems, symmetric hyperbolic systems, mixed initial-boundary value problem, weak well posedness, loss of derivatives, Sobolev spaces.

AMS subject classifications. 35L40, 35L50.

1. Introduction and main results. For $n \geq 2$, let $\mathbb{R}^n_+$ denote the $n-$dimensional positive half-space

$$\mathbb{R}^n_+ := \{x = (x_1, x'),\ x_1 > 0,\ x' := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}.$$

The boundary of $\mathbb{R}^n_+$ will be sistematically identified with $\mathbb{R}^{n-1}_{x'}$. For $T > 0$ we set $Q_T := \mathbb{R}^n_+ \times]0, T[$ and $\Sigma_T := \mathbb{R}^{n-1} \times]0, T[$; we also set $\Omega_T := \mathbb{R}^n_+ \times] - \infty, T[$ and $\omega_T := \mathbb{R}^{n-1} \times] - \infty, T[$. If time t spans the whole real line $\mathbb{R}$, we set $Q := \mathbb{R}^n_+ \times \mathbb{R}_t$ and $\Sigma := \mathbb{R}^{n-1} \times \mathbb{R}_t$. We are interested in the following initial-boundary value problem (shortly written IBVP)

$$(L + B)u = F \quad \text{in } Q_T, \tag{1}$$
$$Mu = G \quad \text{on } \Sigma_T, \tag{2}$$
$$u_{|t=0} = f \quad \text{in } \mathbb{R}^n_+, \tag{3}$$

where L is a first order linear partial differential operator

$$L = \partial_t + \sum_{j=1}^{n} A_j \partial_j, \tag{4}$$

$\partial_t := \frac{\partial}{\partial t}$ and $\partial_j := \frac{\partial}{\partial x_j}$, $j = 1, \dots, n$.

The coefficients $A_j = A_j(x,t)$ of L, for $j = 1, \dots, n$, and $B = B(x,t)$ are real $N \times N$ matrix-valued functions, defined on $\overline{Q}$. The unknown $u = u(x,t)$ and the data $F = F(x,t)$, $G = G(x,t)$, $f = f(x)$ are vector-valued functions with N components. M is a given real $d \times N$ matrix of maximal rank $d \leq N$ (the value of d will be specified below).

We study the problem (1)-(3) under the following assumptions. The function spaces involved in (D) and in the statement of Theorems 1, 2 below, as well as the norms appearing in (7) (10), (12), will be described in the next Section 2.

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(A) L is *Friedrichs symmetrizable*, namely there exists a matrix-valued function S_0, definite positive on $\overline{Q}$ (i.e. there exists a constant $\rho > 0$ such that $S_0(x,t) \geq \rho I_N$ for every $(x,t) \in \overline{Q}$), symmetric and such that the matrices $S_0 A_j$, for $j = 1, \dots, n$, are also symmetric.
(B) There exists $\mu > 0$ such that $|\det A_1(x,t)| \geq \mu$, for all $(x,t) \in \overline{Q}$.
(C) The matrix M has the form $M = (I_d\ 0)$, where I_d denotes the identity matrix of order d, 0 is the zero matrix of size $d \times (N-d)$, and $d \leq N$ is the (constant) number of positive eigenvalues of $A_{1\,|\,\{x_1=0\}}$ (the so-called *incoming characteristics* of problem (1)-(3)).
(D) *Existence of the L^2-weak solution.* Assume that $S_0, A_j \in C_b^\infty(\overline{Q})$, for $j = 1, \dots, n$. For all $T > 0$ and matrices $B \in C_b^\infty(\overline{\Omega_T})$, there exist constants $\gamma_0 \geq 1$ and $C_0 > 0$ (that depend on $T, \rho, \mu, S_0, A_j, B$) such that for all $\gamma \geq \gamma_0$, $F \in L^2(0, +\infty; H^1_\gamma(\omega_T))$ and $G \in H^1_\gamma(\omega_T)$, vanishing for $t < 0$, the boundary value problem (shortly written BVP)

$$(L+B)u = F \quad \text{in } \Omega_T, \tag{5}$$

$$Mu = G \quad \text{on } \omega_T \tag{6}$$

admits a unique solution $u \in L^2(\Omega_T)$, vanishing for $t < 0$, with $u_{|\,\omega_T} \in L^2(\omega_T)$. Furthermore $u \in C([0,T]; L^2(\mathbb{R}^n_+))$ and satisfies an a priori estimate of the form

$$\begin{aligned} &\gamma \|u_\gamma\|^2_{L^2(\Omega_t)} + \|u_\gamma(t)\|^2_{L^2(\mathbb{R}^n_+)} + ||u_{\gamma|\,\omega_t}||^2_{L^2(\omega_t)} \\ &\leq C_0 \left(\frac{1}{\gamma^3} ||F_\gamma||^2_{L^2(H^1_\gamma(\omega_t))} + \frac{1}{\gamma^2} ||G_\gamma||^2_{H^1_\gamma(\omega_t)} \right) \end{aligned} \tag{7}$$

for all $\gamma \geq \gamma_0$ and $0 < t \leq T$, where we have set $u_\gamma := e^{-\gamma t} u$, $F_\gamma := e^{-\gamma t} F$, $G_\gamma := e^{-\gamma t} G$.
Furthermore, if $T = +\infty$, for all matrices $B_1 \in C_b^\infty(\overline{Q})$ and all tangential pseudo-differential operators B_2 with symbol in Γ^0, there exist constants $\gamma_0' \geq 1$ and $C_0' > 0$ (that depend on ρ, μ, S_0, A_j, B_1 and a finite number of seminorms of the symbol of B_2) such that for all $\gamma \geq \gamma_0'$ and for all $F \in L^2(0, +\infty; e^{\gamma t} H^1_\gamma(\Sigma))$, $G \in e^{\gamma t} H^1_\gamma(\Sigma)$, the BVP (5), (6) on Q, with $B = B_1 + B_2$ in (5), admits a unique solution $u \in e^{\gamma t} L^2(Q)$ such that $u_{|\,\Sigma} \in e^{\gamma t} L^2(\Sigma)$. Furthermore u satisfies the a priori estimate

$$\gamma \|u_\gamma\|^2_{L^2(Q)} + ||u_{\gamma|\,\Sigma}||^2_{L^2(\Sigma)} \leq C_0' \left(\frac{1}{\gamma^3} ||F_\gamma||^2_{L^2(H^1_\gamma(\Sigma))} + \frac{1}{\gamma^2} ||G_\gamma||^2_{H^1_\gamma(\Sigma)} \right). \tag{8}$$

The tangential pseudo-differential operators B_2, involved in the statement of the assumption (D), are integral operators acting *tangentially* on smooth functions $u = u(x_1, x', t)$, defined on $\overline{Q}$, by the following formula

$$B_2 u(x_1, x', t) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix' \cdot \xi' + t(\gamma + i\delta)} b_2(x_1, x', t, \xi', \delta, \gamma) \widehat{u_\gamma(x_1, \cdot, \cdot)}(\xi', \delta)\, d\xi'\, d\delta\,. \tag{9}$$

In (9), $f\widehat{(x_1, \cdot, \cdot)}$ denotes the Fourier transform of a function $f = f(x_1, x', t)$ with respect to x', t; ξ' and δ denote the Fourier dual variables of x' and t respectively, and $b_2 = b_2(x_1, x', t, \xi', \delta, \gamma)$ is a suitable function, called the *symbol* of B_2, that must fulfil some convenient growth conditions. The class of symbols Γ^0 and the related pseudo-differential operators allowed in (D) will be introduced in the next Section 3.2

(even thought slightly different notations will be used there). For a thorough analysis of operators of the form (9), the reader is addressed to [3].
When an IBVP admits the solution u, enjoying an a priori estimate of type (7) or (8) with $F = (L + B)u$, $G = Mu$, the IBVP is *weakly L^2-well posed.* This is the case of problems that do not satisfy the *uniform Kreiss-Lopatinskiĭ condition.* Examples of physically interesting problems where the uniform Kreiss-Lopatinskiĭ condition breaks down are provided by elastodynamics (with the well-known Rayleigh waves [22]), shock waves or contact discontinuities in compressible fluid mechanics, see e.g. [12, 13], [7, 8, 9]. For such *nonstable problems*, where the so-called *Lopatinskiĭ determinant* vanishes at the order one in the *hyperbolic region* of the frequency space, one can prove an a priori estimate with a loss of one tangential derivative from the source term to the solution. For noncharacteristic problems, such energy estimates with loss of one tangential derivative have been derived by Coulombel in [5] and, for uniformly characteristic problems, by Coulombel-Secchi in [7, 8]. Under an a priori estimate of this form, Coulombel [6] has proven the well posedness of the problem (1)-(3) with zero initial datum, namely the existence of the L^2-solution for all H^1-data. When a loss of derivatives greater than one appears in the energy estimate, no well posedness result is known yet.
Under the assumptions (A)-(D) it is not hard to obtain the L^2-solvability of the nonhomogeneous IBVP (1)-(3) on $[0, T]$, with initial data $f \neq 0$, that we state in the following theorem.

THEOREM 1. *Assume that problem* (1)-(3) *obeys the assumptions* (A)-(D)*. For all $T > 0$ and matrices $B \in C^\infty_b(\overline{Q_T})$, there exist constants (denoted as above) $\gamma_0 \geq 1$ and $C_0 > 0$ (that depend on T, ρ, μ, S_0, A_j and B) such that for all $F \in L^2(0, +\infty; H^1_\gamma(\Sigma_T))$, $G \in H^1_\gamma(\Sigma_T)$, $f \in H^1_\gamma(\mathbb{R}^n_+)$, such that $Mf = G_{|\,t=0}$ on $\mathbb{R}^{n-1}$, the problem* (1)-(3)*, with data (F, G, f), admits a unique solution $u \in L^2(Q_T)$ with $u_{|\,\Sigma_T} \in L^2(\Sigma_T)$. Furthermore $u \in C([0, T]; L^2(\mathbb{R}^n_+))$, and it satisfies an a priori estimate of the form*

$$\begin{aligned}&\gamma\|u_\gamma\|^2_{L^2(Q_t)} + ||u_\gamma(t)||^2_{L^2(\mathbb{R}^n_+)} + ||u_{\gamma|\,\Sigma_t}||^2_{L^2(\Sigma_t)}\\&\leq C_0\left(\frac{1}{\gamma^3}||F_\gamma||^2_{L^2(H^1_\gamma(\Sigma_t))} + \frac{1}{\gamma^2}||f||^2_{H^1_\gamma(\mathbb{R}^n_+))} + \frac{1}{\gamma^2}||G_\gamma||^2_{H^1_\gamma(\Sigma_t)}\right),\end{aligned} \tag{10}$$

for all $\gamma \geq \gamma_0$ and $0 < t \leq T$.

The proof of Theorem 1 will be given in Appendix A. This paper is concerned with the regularity of solutions of the IBVP (1)-(3). In order to study such regularity, we need to impose some *compatibility conditions* on the data F, G, f. The compatibility conditions are defined in the usual way, see [19]. For the initial data f, one firstly defines the *time-derivatives* $f^{(h)}$, $h \geq 0$, by setting recursively $f^{(0)} := f$ and, for $h \geq 1$,

$$f^{(h)} = \partial_t^{h-1}F_{|t=0} - \sum_{j=1}^{n}\sum_{q=0}^{h-1}\begin{pmatrix} h-1 \\ q \end{pmatrix}\partial_t^{h-1-q}A_{j|t=0}\partial^j f^{(q)} - \sum_{q=0}^{h-1}\begin{pmatrix} h-1 \\ q \end{pmatrix}\partial_t^{h-1-q}B_{|t=0}f^{(q)}.$$

Given the equation (1), the above formula for $f^{(h)}$ is obtained by formally taking $h-1$ time derivatives of $Lu = F$, solving for $\partial_t^h u$ and evaluating it at $t = 0$.
The *compatibility condition* of order $m \geq 0$ for the IBVP (1)-(3) reads as

$$Mf^{(h)} = \partial_t^h G_{|\,t=0}\,, \quad \text{on } \mathbb{R}^{n-1}\,,\ h = 0, \dots, m\,. \tag{11}$$

The aim of this paper is to prove the following regularity theorem.

THEOREM 2. *Let $m \in \mathbb{N}$, $m \geq 1$, $A_j, B \in C_b^\infty(\overline{Q})$, for $1 \leq j \leq n$, and $T > 0$. Assume also that the assumptions* (A)-(D) *are satisfied. Then there exist constants $C_m > 0$, $\gamma_m \geq 1$, depending only on A_j, B, such that for all $\gamma \geq \gamma_m$, for all $F \in L^2(0,+\infty; H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T)$ for which $\partial_t^j F_{|t=0} \in H^{m-j}_\gamma(\mathbb{R}^n_+)$ $(j = 0, \dots, m-1)$, $G \in H^{m+1}_\gamma(\Sigma_T)$, $f \in H^{m+1}_\gamma(\mathbb{R}^n_+)$, satisfying the compatibility condition* (11) *of order m, the unique solution u to* (1)–(3)*, with data (F, G, f), belongs to $\mathcal{C}_T(H^m_\gamma)$, $u_{|\,\Sigma_T} \in H^m_\gamma(\Sigma_T)$ and the following a priori estimate*

$$\gamma||u_\gamma||^2_{H^m_\gamma(Q_t)} + ||u_{\gamma|\,\Sigma_t}||^2_{H^m_\gamma(\Sigma_t)} + \sum_{j=0}^{m} ||\partial_t^j u_\gamma(t)||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}$$
$$\leq C_m \left(\frac{1}{\gamma^3}||F_\gamma||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))\cap H^{m-1}_\gamma(Q_t)} + \frac{1}{\gamma^2}\sum_{j=0}^{m-1} ||\partial_t^j F_{|\,t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)} + \right. \tag{12}$$
$$\left. + \frac{1}{\gamma^2}||G_\gamma||^2_{H^{m+1}_\gamma(\Sigma_t)} + \frac{1}{\gamma^2}||f||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} \right)$$

holds true for all $0 < t \leq T$.

In [16], the regularity of weak solutions to the IBVP (1)-(3) is studied, under the assumption that the problem is *strongly L^2-well posed*, namely that a unique L^2-solution exists for arbitrarily given L^2-data, and the solution obeys an a priori energy estimate *without loss of regularity with respect to the data*; this means that the L^2-norms of the interior and boundary values of the solution are measured by the L^2-norms of the corresponding data F, G, f. The statement of Theorems 1, 2 deals with the case where only a *weak well posedness* property is satisfied by the IBVP (1)–(3). In Theorem 1, the L^2-solvability of (1)–(3) requires an additional regularity of the data F, G, f, cfr. (D). Correspondingly, in Theorem 2 the regularity of the solution of order m is achieved provided the data have a regularity of order $m + 1$. In [17] we derive similar results for the case of a noncharacteristic BVP, with finitely smooth coefficients. In [14], [15], regularity results analogous to that of Theorem 2 were established for weakly well posed problems, when the boundary is *characteristic with constant multiplicity*.

Differently from the characteristic case, where the regularity of the solution is established in the framework of the *anisotropic Sobolev spaces* (see [15]), in the present paper the regularity is sought in the usual Sobolev spaces. Firstly we study the tangential regularity of the solutions and then we derive from it the full regularity. This approach allows us to underline the gap between the tangential and the normal regularity required on the data, in order to derive the full regularity of the solution at any prescribed order. Indeed, due to the non characteristicity of the boundary, one normal derivative of the solution is estimated directly from the equation without any control on the normal derivatives of the data. The result of Theorem 2, compared with the one obtained in the characteristic case (see [15]), requires less regularity on the data.

Here, we follow the analysis developed in [14], [15]. One firstly considers the case of a problem (1)-(3) with a homogeneous initial condition ($f = 0$). The study of such a *homogeneous problem* is reduced to that of a BVP (5)-(6) (with $T = +\infty$) by a suitable time-extension of the solution u and the nonzero data F, G through the whole real line, see Section 4. Then, one studies the tangential regularity of the "extended" solution u to the BVP (5)-(6); roughly speaking, this is made by acting on such a problem by a suitable family of tangential pseudo-differential operators $\{\lambda_\delta^{m-1,\gamma}(D')\}_{0<\delta\leq 1}$, that occurs to characterize the (tangential) Sobolev regularity of functions, by means of its uniform boundedness properties as δ goes to zero (see Propositions 3, 4). By using the rules of the functional calculus for pseudo-differential operators (relying on the symbolic calculus), and taking advantage from the invertibility of the operator $\lambda_\delta^{m-1,\gamma}(D')$, one can show that the function $\lambda_\delta^{m-1,\gamma}(D')u$ solves the same problem (5)-(6) as u, modulo some lower order operator of the type (9), for each $0<\delta\leq 1$. Hence, a $\delta-$uniform bound of the L^2-norm of the function $\lambda_\delta^{m-1,\gamma}(D')u$ follows from applying to the associated problem the well posedness assumption (D) (see the energy estimate (8)), that finally yields the desired tangential regularity of u. The normal regularity of u directly follows by an induction argument which rests on the possibility to express the normal derivative of u as a function of tangential derivatives of u itself and the source term F, following from the invertibility of the coefficient A_1 (see (49)). Eventually, the general case of problem (1)-(3), with nonzero initial datum f, can be treated by firstly smoothing the original data and then reducing to the already studied case of a problem with homogeneous initial condition, by "subtracting" to the solution of the approximated problem, with regularized data, a suitable smooth function, see Section 6.

The paper is organized as follows. In Section 2 we introduce the function spaces and some notations. In Section 3 we give some technical results useful for the proof of the regularity of the homogeneous IBVP ($f=0$), discussed in Sections 4, 5. Section 6 contains the proof of the regularity for the general IBVP ($f\neq 0$). The proof of Theorem 1 is given in Appendix A. The Appendix B finally contains the statement of a few technical tools needed along the exposition.

2. Function spaces.

The purpose of this section is to introduce the main function spaces to be used in the following and collect their basic properties.

2.1. Weighted Sobolev spaces.

For $\gamma\geq 1$ and $s\in\mathbb{R}$, we set

$$\lambda^{s,\gamma}(\xi):=(\gamma^2+|\xi|^2)^{s/2} \tag{13}$$

and, in particular, $\lambda^s:=\lambda^{s,1}$.

Throughout the paper, for real $\gamma\geq 1$, $H^s_\gamma(\mathbb{R}^n)$ will denote the Sobolev space of order s, equipped with the $\gamma-$depending norm $||\cdot||_{s,\gamma}$ defined by

$$||u||^2_{s,\gamma}:=(2\pi)^{-n}\int_{\mathbb{R}^n}\lambda^{2s,\gamma}(\xi)|\widehat{u}(\xi)|^2d\xi\,, \tag{14}$$

$\widehat{u}$ being the Fourier transform of u. The norms defined by (14), with different values of the parameter γ, are equivalent each other. For $\gamma=1$ we set for brevity $||\cdot||_s:=||\cdot||_{s,1}$ (and, accordingly, $H^s(\mathbb{R}^n):=H^s_1(\mathbb{R}^n)$ for the standard Sobolev space).

For $s\in\mathbb{N}$, the norm in (14) turns to be equivalent, *uniformly with respect to* γ, to the norm $||\cdot||_{H^s_\gamma(\mathbb{R}^n)}$ defined by

$$||u||^2_{H^s_\gamma(\mathbb{R}^n)}:=\sum_{|\alpha|\leq s}\gamma^{2(s-|\alpha|)}||\partial^\alpha u||^2_{L^2(\mathbb{R}^n)}\,, \tag{15}$$

where, for every multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we set $\partial^\alpha:=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$ and $|\alpha|:=\alpha_1+\cdots+\alpha_n$ as usual.

An useful remark is that

$$||u||_{s,\gamma}\leq\gamma^{s-r}||u||_{r,\gamma}\,, \tag{16}$$

for arbitrary $s \leq r$ and $\gamma \geq 1$.
Accordingly to the previous notations, for every $m \in \mathbb{N}$, $T > 0$, $\gamma \geq 1$, we write $H^m_\gamma(\mathbb{R}^n_+)$, $H^m_\gamma(Q_T)$, $H^m_\gamma(Q)$, $H^m_\gamma(\Sigma_T)$, $H^m_\gamma(\Sigma)$ to mean the usual Sobolev spaces of order m on $\mathbb{R}^n_+$, Q_T, Q, Σ_T and Σ respectively, equipped with the $\gamma-$weighted norms defined similarly to (15). We also denote by $L^2\left(0,+\infty; H^m_\gamma(\mathbb{R}^{n-1})\right)$ the set of functions $u = u(x_1, x')$ such that

$$||u||^2_{L^2(H^m_\gamma)} := \int_0^{+\infty} ||u(x_1,\cdot)||^2_{H^m_\gamma(\mathbb{R}^{n-1})}\, dx_1 < +\infty\,. \tag{17}$$

Similarly, for $T > 0$, by $L^2\left(0,+\infty; H^m_\gamma(\Sigma_T)\right)$ we mean the set of functions $u = u(x_1, x', t)$ such that

$$||u||^2_{L^2(H^m_\gamma(\Sigma_T))} := \int_0^{+\infty} ||u(x_1,\cdot,\cdot)||^2_{H^m_\gamma(\Sigma_T)}\, dx_1 < +\infty\,. \tag{18}$$

For a given Banach space X and $j \in \mathbb{N}$, let $C^j\left([0,T];X\right)$ denote the space of all X-valued j-times continuously differentiable functions on $[0,T]$. For every $m \in \mathbb{N}$ we define the space

$$\mathcal{C}_T(H^m_\gamma) := \bigcap_{j=0}^{m} C^j\left([0,T]; H^{m-j}_\gamma(\mathbb{R}^n_+)\right)\,,$$

with the norm

$$||u||_{\mathcal{C}_T(H^m_\gamma)} := \sum_{j=0}^{m} \sup_{t\in[0,T]} ||\partial_t^j u(t)||_{H^{m-j}_\gamma(\mathbb{R}^n_+)}\,. \tag{19}$$

Let $C^\infty_b(\mathbb{R}^\nu)$ be the set of smooth functions on $\mathbb{R}^\nu$ with bounded derivatives of any order; $C^\infty_b(\overline{\mathbb{R}^n_+})$ will denote the space of functions which are restrictions to $\mathbb{R}^n_+$ of functions belonging to $C^\infty_b(\mathbb{R}^n)$, provided with the natural topology. In the similar way, we define the spaces $C^\infty_b(\overline{Q})$, $C^\infty_b(\overline{\Omega_T})$, $C^\infty_b(\overline{Q_T})$. Finally, we denote by $\mathcal{S}(\mathbb{R}^n_+)$ the spaces of restrictions to $\mathbb{R}^n_+$ of functions belonging to $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions in $\mathbb{R}^n$.

3. Preliminaries and technical tools. In this section, we collect several technical tools that will be used in the subsequent analysis (cf. the next Section 4).

3.1. Parameter depending norms on Sobolev spaces. We start by recalling a classical characterization of ordinary Sobolev spaces in $\mathbb{R}^n$, due to Hörmander [10], based upon the uniform boundedness of a suitable family of parameter-depending norms.
For given $s \in \mathbb{R}$, $\gamma \geq 1$ and for each $\delta \in]0,1]$ a norm in $H^{s-1}(\mathbb{R}^n)$ is defined by setting

$$||u||^2_{s-1,\gamma,\delta} := (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{2s,\gamma}(\xi)\lambda^{-2,\gamma}(\delta\xi)|\widehat{u}(\xi)|^2 d\xi. \tag{20}$$

According to Section 2, for $\gamma = 1$ and any $0 < \delta \leq 1$ we set $||\cdot||_{s-1,\delta} := ||\cdot||_{s-1,1,\delta}$; the family of $\delta-$weighted norms $\{||\cdot||_{s-1,\delta}\}_{0<\delta\leq 1}$ was deeply studied in [10]; easy arguments (relying essentially on a $\gamma-$rescaling of functions) lead to get the same properties for the norms $\{||\cdot||_{s-1,\gamma,\delta}\}_{0<\delta\leq 1}$ defined in (20) with an arbitrary $\gamma \geq 1$. Of course, one has $||\cdot||_{s-1,\gamma,1} = ||\cdot||_{s-1,\gamma}$ (cf. (14), with $s-1$ instead of s). It is also

clear that, for each fixed $\delta \in]0,1[$, the norm $||\cdot||_{s-1,\gamma,\delta}$ is equivalent to $||\cdot||_{s-1,\gamma}$ in $H^{s-1}_\gamma(\mathbb{R}^n)$, *uniformly with respect to* γ; notice, however, that the constants appearing in the equivalence inequalities will generally depend on δ (see (26)).
The next characterization of Sobolev spaces readily follows by taking account of the parameter γ into the arguments used in [10, Thm. 2.4.1].

Proposition 3. *For every $s \in \mathbb{R}$ and $\gamma \geq 1$, $u \in H^s_\gamma(\mathbb{R}^n)$ if and only if $u \in H^{s-1}_\gamma(\mathbb{R}^n)$, and the set $\{||u||_{s-1,\gamma,\delta}\}_{0<\delta\leq 1}$ is bounded. In this case, we have*

$$||u||_{s-1,\gamma,\delta} \uparrow ||u||_{s,\gamma} \quad \text{as } \delta \downarrow 0.$$

In the sequel, we will make use of a tangential counterpart of the preceding characterization adapted to the spaces $L^2(0,+\infty; H^m_\gamma(\mathbb{R}^{n-1}))$. To be definite, we state the result here below.

Proposition 4. *For all $m \in \mathbb{N}$, $m \geq 1$, and $\gamma \geq 1$, $u \in L^2(0,+\infty; H^m_\gamma(\mathbb{R}^{n-1}))$ if and only if $u \in L^2(0,+\infty; H^{m-1}_\gamma(\mathbb{R}^{n-1}))$ and the norm*

$$\int_0^{+\infty} ||u(x_1,\cdot)||^2_{m-1,\gamma,\delta}\, dx_1$$

remains bounded with respect to $\delta \in]0,1]$; moreover, in this case one has

$$\int_0^{+\infty} ||u(x_1,\cdot)||^2_{m-1,\gamma,\delta}\, dx_1 \uparrow \int_0^{+\infty} ||u(x_1,\cdot)||^2_{m,\gamma}\, dx_1 \quad \text{as } \delta \downarrow 0.$$

3.2. Tangential pseudo-differential operators. Let us introduce the pseudo-differential symbols, with a parameter, to be used later; here we closely follow the terminology and notations of [4].

Definition 5. *We say that a real (or complex) valued measurable function $a(x_1, x', \xi', \gamma)$ on $\mathbb{R}^n_+ \times \mathbb{R}^{n-1} \times [1,+\infty[$, such that a is C^∞ with respect to $x' = (x_2,\dots,x_n)$ and $\xi' = (\xi_2,\dots,\xi_n)$, is a parameter-depending symbol of order $m \in \mathbb{R}$ if for all multi-indices $\alpha', \beta' \in \mathbb{N}^{n-1}$, there exists a positive constant $C_{\alpha',\beta'}$ such that*

$$|\partial^{\alpha'}_{\xi'}\partial^{\beta'}_{x'} a(x,\xi',\gamma)| \leq C_{\alpha',\beta'}\lambda^{m-|\alpha'|,\gamma}(\xi')\,, \tag{21}$$

for all $x = (x_1,x') \in \mathbb{R}^n_+$, $\xi' \in \mathbb{R}^{n-1}$ and $\gamma \geq 1$ (recall that $\lambda^{m-|\alpha'|,\gamma}(\xi') := (\gamma^2 + |\xi'|^2)^{(m-|\alpha'|)/2}$).

The same definition as above extends to functions $a(x,\xi',\gamma)$ taking values in the space $\mathbb{R}^{N\times N}$ (resp. $\mathbb{C}^{N\times N}$) of $N\times N$ real (resp. complex)-valued matrices, for all integers $N > 1$ (where the module $|\cdot|$ is replaced in (21) by any equivalent norm in $\mathbb{R}^{N\times N}$ (resp. $\mathbb{C}^{N\times N}$)). We denote by Γ^m the set of $\gamma-$depending symbols of order $m \in \mathbb{R}$ (the same notation being used for both scalar or matrix-valued symbols). Γ^m is equipped with the norms

$$|a|_{m,k} := \max_{|\alpha'|+|\beta'|\leq k} \sup_{\substack{(x,\xi')\in\mathbb{R}^n_+\times\mathbb{R}^{n-1},\\ \gamma\geq 1}} \lambda^{-m+|\alpha'|,\gamma}(\xi')|\partial^{\alpha'}_{\xi'}\partial^{\beta'}_{x'} a(x,\xi',\gamma)|\,, \quad \forall\, k \in \mathbb{N}\,, \tag{22}$$

which turn it into a Fréchet space. For all $m, m' \in \mathbb{R}$, with $m \leq m'$, the continuous imbedding $\Gamma^m \subset \Gamma^{m'}$ can be easily proven.

For all $m\in\mathbb{R}$, the function $\lambda^{m,\gamma}(\xi')$ is of course a (scalar-valued) symbol in Γ^m.
To perform the analysis of Section 4, it is important to consider the behavior of the weight function $\lambda^{m,\gamma}(\xi')\lambda^{-1,\gamma}(\delta\xi')$, involved in the definition of the parameter-depending norms in (20), as a $\gamma-$depending symbol according to Definition 5.
Following [14], henceforth the following short notations will be used

$$\begin{aligned}&\lambda_\delta^{m-1,\gamma}(\xi'):=\lambda^{m,\gamma}(\xi')\lambda^{-1,\gamma}(\delta\xi')\\&\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi'):=\left(\lambda_\delta^{m-1,\gamma}(\xi')\right)^{-1}\left(=\lambda^{-m,\gamma}(\xi')\lambda^{1,\gamma}(\delta\xi')\right),\end{aligned}\tag{23}$$

for all real numbers $m\in\mathbb{R}$, $\gamma\geq 1$ and $\delta\in]0,1]$. One has the obvious identities

$$\lambda_1^{m-1,\gamma}(\xi')\equiv\lambda^{m-1,\gamma}(\xi')\,,\quad \widetilde{\lambda}_1^{-m+1,\gamma}(\xi')\equiv\lambda_1^{-m+1,\gamma}(\xi')\equiv\lambda^{-m+1,\gamma}(\xi')\,.$$

However, to avoid confusion in the following, we remark that functions $\lambda_\delta^{-m+1,\gamma}(\xi')$ and $\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi')$ are no longer the same as soon as δ becomes strictly smaller than 1; indeed (23) gives $\lambda_\delta^{-m+1,\gamma}(\xi')=\lambda^{-m+2,\gamma}(\xi')\lambda^{-1,\gamma}(\delta\xi')$.
A straightforward application of Leibniz's rule leads to the following result.

Lemma 6. *For every $m\in\mathbb{R}$ and all $\alpha'\in\mathbb{N}^{n-1}$ there exists a positive constant $C_{m,\alpha'}$ such that*

$$|\partial_{\xi'}^{\alpha'}\lambda_\delta^{m-1,\gamma}(\xi')|\leq C_{m,\alpha'}\lambda_\delta^{m-1-|\alpha'|,\gamma}(\xi')\,,\quad\forall\,\xi'\in\mathbb{R}^{n-1}\,,\gamma\geq 1\,,\delta\in]0,1]\,.\tag{24}$$

Because of estimates (24), $\lambda_\delta^{m-1,\gamma}(\xi')$ can be regarded as a $\gamma-$depending symbol, in two different ways. On the one hand, combining estimates (24) with the trivial inequality

$$\lambda^{-1,\gamma}(\delta\xi')\leq 1\tag{25}$$

immediately gives that $\{\lambda_\delta^{m-1,\gamma}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^m.
On the other hand, the left inequality in

$$\delta\lambda^{1,\gamma}(\xi')\leq\lambda^{1,\gamma}(\delta\xi')\leq\lambda^{1,\gamma}(\xi')\,,\quad\forall\,\xi'\in\mathbb{R}^{n-1}\,,\ \forall\,\delta\in]0,1]\,,\tag{26}$$

together with (24), also gives

$$|\partial_{\xi'}^{\alpha'}\lambda_\delta^{m-1,\gamma}(\xi')|\leq C_{m,\alpha'}\delta^{-1}\lambda^{m-1-|\alpha'|,\gamma}(\xi')\,,\quad\forall\,\xi'\in\mathbb{R}^{n-1}\,,\gamma\geq 1\,.\tag{27}$$

According to Definition 5, (27) means that $\lambda_\delta^{m-1,\gamma}$ actually belongs to Γ^{m-1} for each fixed δ; nevertheless, the family $\{\lambda_\delta^{m-1,\gamma}\}_{0<\delta\leq 1}$ is generally unbounded as a subset of Γ^{m-1}.
For later use, we also need to study the behavior of functions $\widetilde{\lambda}_\delta^{-m+1,\gamma}$ as $\gamma-$depending symbols.
Analogously to Lemma 6, one can prove the following result.

Lemma 7. *For all $m\in\mathbb{R}$ and $\alpha'\in\mathbb{N}^{n-1}$ there exists $\widetilde{C}_{m,\alpha'}>0$ such that*

$$|\partial_{\xi'}^{\alpha'}\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi')|\leq\widetilde{C}_{m,\alpha'}\widetilde{\lambda}_\delta^{-m+1-|\alpha'|,\gamma}(\xi')\,,\quad\forall\,\xi'\in\mathbb{R}^{n-1}\,,\ \gamma\geq 1\,,\ \delta\in]0,1]\,.\tag{28}$$

In particular, Lemma 7 implies that the family $\{\widetilde{\lambda}_\delta^{-m+1,\gamma}\}_{0<\delta\leq 1}$ is a bounded subset of Γ^{-m+1} (it suffices to combine (28) with the right inequality in (26)).

Any symbol $a = a(x,\xi',\gamma) \in \Gamma^m$ defines a pseudo-differential operator $\mathrm{Op}^\gamma_{\tan}(a) = a(x,D',\gamma)$ acting *tangentially* on $\mathbb{R}^{n-1}_{x'}$ by the standard formula

$$\mathrm{Op}^\gamma_{\tan}(a)u(x) = a(x,D',\gamma)u(x) := (2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}a(x,\xi',\gamma)\widehat{u(x_1,\cdot)}(\xi')d\xi', \tag{29}$$

for all $u \in \mathcal{S}(\mathbb{R}^n_+)$ and $x = (x_1,x') \in \mathbb{R}^n_+$, where $x'\cdot\xi' := \sum_{j=2}^n x_j\xi_j$ and $\widehat{u(x_1,\cdot)}$ denotes the Fourier transform of $u = u(x_1,x')$ with respect to x'. $\mathrm{Op}^\gamma_{\tan}(a)$ is called the *tangential pseudo-differential operator with symbol* a; m is the order of $\mathrm{Op}^\gamma_{\tan}(a)$. It comes from the classical theory that $\mathrm{Op}^\gamma_{\tan}(a)$ defines a linear bounded operator

$$\mathrm{Op}^\gamma_{\tan}(a) : \mathcal{S}(\mathbb{R}^n_+) \to \mathcal{S}(\mathbb{R}^n_+).$$

An exhaustive account of the symbolic calculus for pseudo-differential operators with symbols in Γ^m can be found in [3]. In the next Section 4, we will be mainly interested to the family of tangential operators whose symbols are the functions $\lambda^{m-1,\gamma}_\delta(\xi')$, $\widetilde{\lambda}^{-m+1,\gamma}_\delta(\xi')$. According to (23), (29), henceforth we write:

$$\lambda^{m-1,\gamma}_\delta(D') := \mathrm{Op}^\gamma_{\tan}(\lambda^{m-1,\gamma}_\delta), \quad \widetilde{\lambda}^{-m+1,\gamma}_\delta(D') := \mathrm{Op}^\gamma_{\tan}(\widetilde{\lambda}^{-m+1,\gamma}_\delta). \tag{30}$$

It is worth noticing that the tangential operators $\lambda^{m-1,\gamma}_\delta(D')$ are involved in the characterization of tangential Sobolev regularity provided by Proposition 4 (remember that, after Lemma 6, $\lambda^{m-1,\gamma}_\delta \in \Gamma^{m-1}$). Indeed, from Plancherel's formula the following identity

$$\int_0^{+\infty} ||u(x_1,\cdot)||^2_{m-1,\gamma,\delta}\,dx_1 \equiv ||\lambda^{m-1,\gamma}_\delta(D')u||^2_{L^2(\mathbb{R}^n_+)} \tag{31}$$

can be straightforwardly established; hence, Proposition 4 can be restated in terms of the boundedness, with respect to δ, of the L^2-norms of functions $\lambda^{m-1,\gamma}_\delta(D')u$. This observation is the key point that leads to the analysis performed in Section 4.
Notice also that the operator $\lambda^{m-1,\gamma}_\delta(D')$ is invertible, and its two-sided inverse is given by $\widetilde{\lambda}^{-m+1,\gamma}_\delta(D')$.
We end this section by stating a technical result concerning the symbolic calculus for tangential pseudo-differential operators, that will be used in the sequel. For the proof we refer to [14, Lemma 4.5]

LEMMA 8. *Let $\{a_\delta\}_{0<\delta\le 1}$ be a family of symbols $a_\delta = a_\delta(x,\xi',\gamma) \in \Gamma^{r-1}$, $r \in \mathbb{R}$, such that for all multi-indices $\alpha',\beta' \in \mathbb{N}^{n-1}$ there exists a positive constant $C_{r,\alpha',\beta'}$, independent of γ and δ, for which:*

$$|\partial^{\alpha'}_{\xi'}\partial^{\beta'}_{x'}a_\delta(x,\xi',\gamma)| \le C_{r,\alpha',\beta'}\lambda^{r-1-|\alpha'|,\gamma}_\delta(\xi'), \quad \forall\, x = (x_1,x') \in \mathbb{R}^n_+,\ \forall\, \xi' \in \mathbb{R}^{n-1}. \tag{32}$$

Let $b = b(x,\xi',\gamma)$ be another symbol in Γ^l, for $l \in \mathbb{R}$.
Then, for every $\delta \in]0,1]$ the product $\mathrm{Op_{tan}}^\gamma(a_\delta)\mathrm{Op_{tan}}^\gamma(b)$ is a pseudo-differential operator whose symbol $a_\delta\# b$ belongs to Γ^{l+r-1}. Moreover, for all multi-indices $\alpha',\beta' \in \mathbb{N}^{n-1}$ there exists a constant $C_{r,l,\alpha',\beta'}$, independent of γ and δ, such that

$$|\partial^{\alpha'}_{\xi'}\partial^{\beta'}_{x'}(a_\delta\# b)(x,\xi',\gamma)| \le C_{r,l,\alpha',\beta'}\lambda^{l+r-1-|\alpha'|,\gamma}_\delta(\xi'), \quad \forall\, x = (x_1,x') \in \mathbb{R}^n_+,\ \forall\, \xi \in \mathbb{R}^{n-1}. \tag{33}$$

Under the same hypotheses, ${\mathrm{Op}_{\tan}}^{\gamma}(a_\delta){\mathrm{Op}_{\tan}}^{\gamma}(b)\widetilde{\lambda}_\delta^{-m+1,\gamma}(D')$ *is a pseudo-differential operator whose symbol* $(a_\delta\#b)\widetilde{\lambda}_\delta^{-m+1,\gamma}$ *belongs to* Γ^{l+r-m}*; moreover,* $\{(a_\delta\#b)\widetilde{\lambda}_\delta^{-m+1,\gamma}\}_{0<\delta\le 1}$ *is a bounded subset of* Γ^{l+r-m}*. Eventually, if the symbol* a_δ *is scalar-valued,* $[{\mathrm{Op}_{\tan}}^{\gamma}(a_\delta),{\mathrm{Op}_{\tan}}^{\gamma}(b)]\widetilde{\lambda}_\delta^{-m+1,\gamma}(D')$ *is a pseudo-differential operator with symbol* $c_\delta\in\Gamma^{l+r-m-1}$*, and* $\{c_\delta\}_{0<\delta\le 1}$ *is a bounded subset of* $\Gamma^{l+r-m-1}$*.*

3.3. Sobolev continuity of tangential operators.

Proposition 9. *If* $s\in\mathbb{N}$ *and* $m\in\mathbb{Z}$ *are such that* $s+m\in\mathbb{N}$*, then for all* $a\in\Gamma^m$ *the pseudo-differential operator* $\mathrm{Op}^\gamma_{\tan}(a)$ *extends as a linear bounded operator*

$$\mathrm{Op}^\gamma_{\tan}(a): L^2\left(0,+\infty; H^{s+m}_\gamma(\mathbb{R}^{n-1})\right)\to L^2\left(0,+\infty; H^{s}_\gamma(\mathbb{R}^{n-1})\right)$$

and the operator norm of such an extension is uniformly bounded with respect to γ*.*

We refer the reader to [3] for a detailed proof of Proposition 9; a sharp calculation shows that the norm of $\mathrm{Op}^\gamma_{\tan}(a)$, as a linear bounded operator from $L^2\left(0,+\infty; H^{s+m}_\gamma(\mathbb{R}^{n-1})\right)$ to $L^2\left(0,+\infty; H^{s}_\gamma(\mathbb{R}^{n-1})\right)$, actually depends only on a semi-norm of type (22) of the symbol a, besides the Sobolev order s and the symbolic order m (cf. [3] for detailed calculations). This observation entails, in particular, that the operator norm is uniformly bounded with respect to γ and other additional parameters from which the symbol of the operator should possibly depend as a bounded map.

4. The homogeneous IBVP. We introduce the new unknown $u_\gamma(x,t):=e^{-\gamma t}u(x,t)$ and the new data $F_\gamma:=e^{-\gamma t}F(x,t)$, $G_\gamma:=e^{-\gamma t}G(x,t)$. Then problem (1)-(3) becomes equivalent to

$$\begin{array}{ll}(L_\gamma+B)u_\gamma=F_\gamma\,, & \text{in } Q_T,\\ Mu_\gamma=G_\gamma\,, & \text{on } \Sigma_T,\\ u_{\gamma\,|t=0}=f\,, & \text{in } \mathbb{R}^n_+,\end{array}\tag{34}$$

with

$$L_\gamma:=\gamma I_N+L.$$

In this section we concentrate on the study of the tangential regularity of the solution to the IBVP (34), where the initial datum f is identically zero and the data F_γ and G_γ satisfy the compatibility conditions in a more restrictive form than (11). More precisely, we concentrate on the *homogeneous* IBVP

$$\begin{array}{ll}(L_\gamma+B)u_\gamma=F_\gamma\,, & \text{in } Q_T\,,\\ Mu_\gamma=G_\gamma\,, & \text{on } \Sigma_T\,,\\ u_{\gamma|\,t=0}=0\,, & \text{in } \mathbb{R}^n_+\,.\end{array}\tag{35}$$

We remark that here and in the following the word *homogeneous* is referred by convention to the initial datum f.

For a given integer $m\ge 1$, we assume that F_γ and G_γ satisfy the following conditions

$$\partial_t^h F_{\gamma|\,t=0}=0\,,\quad \partial_t^h G_{\gamma|\,t=0}=0\,,\quad h=0,\dots,m\,.\tag{36}$$

It is worth to notice that conditions (36) imply the compatibility conditions (11), in the case $f=0$. We prove the following theorem.

THEOREM 10. *Assume that* S_0, A_j, B, *for* $j = 1, \dots, n$, *are in* $C_b^\infty(\overline{Q})$, *and that problem* (35) *satisfies assumptions* (A)–(D)*; then for all* $T > 0$ *and* $m \in \mathbb{N}$ *there exist constants* $C_m > 0$ *and* γ_m, *with* $\gamma_m \geq \gamma_{m-1}$, *such that for all* $\gamma \geq \gamma_m$, $F_\gamma \in L^2(0, +\infty; H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T)$ *and* $G_\gamma \in H^{m+1}_\gamma(\Sigma_T)$ *satisfying* (36) *the unique* L^2-*solution* u_γ *to* (35) *belongs to* $H^m_\gamma(Q_T)$, *the trace of* u_γ *on* Σ_T *belongs to* $H^m_\gamma(\Sigma_T)$, *and the a priori estimate*

$$\begin{aligned}
&\gamma ||u_\gamma||^2_{H^m_\gamma(Q_T)} + ||u_{\gamma|\,\Sigma_T}||^2_{H^m_\gamma(\Sigma_T)} \\
&\leq C_m \Big(\tfrac{1}{\gamma^3} ||F_\gamma||^2_{L^2(H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T)} + \tfrac{1}{\gamma^2} ||G_\gamma||^2_{H^{m+1}_\gamma(\Sigma_T)} \Big)
\end{aligned} \tag{37}$$

is fulfilled, where we set

$$||F_\gamma||^2_{L^2(H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T)} := \gamma^4 ||F_\gamma||^2_{H^{m-1}_\gamma(Q_T)} + ||F_\gamma||^2_{L^2(H^{m+1}(\Sigma_T))}. \tag{38}$$

The first step to prove Theorem 10 is reducing the original problem (35) to a boundary value problem where the time is allowed to span the whole real line and is treated, consequently, as an additional tangential variable. To make this reduction, we extend the data F_γ, G_γ and the unknown u_γ of (35) to all positive and negative times. In the sequel, for the sake of simplicity, we remove the subscript γ from the unknown u_γ and the data F_γ, G_γ.
Because of conditions (36), we may extend F, G by setting them equal to zero for all negative times and for $t > T$ by "reflection" (see [2, Chap. 9], [11, Theorem 2.2]), so that the extended data, say $\breve{F}$ and $\breve{G}$, vanish also for all $t > T$ sufficiently large. Moreover, we get $\breve{F} \in L^2(H^{m+1}_\gamma(\Sigma)) \cap H^{m-1}_\gamma(Q)$, $\breve{G} \in H^{m+1}_\gamma(\Sigma)$ and there exists some positive constant C_m, independent of γ, such that

$$\begin{aligned}
&||\breve{F}||_{L^2(H^{m+1}_\gamma(\Sigma)) \cap H^{m-1}_\gamma(Q)} \leq C_m ||F||_{L^2(H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T)} \\
&||\breve{G}||_{H^{m+1}_\gamma(\Sigma))} \leq C_m ||G||_{H^{m+1}_\gamma(\Sigma_T)}\,.
\end{aligned} \tag{39}$$

As we did for the data, the solution u to (35) is extended to all negative times, by setting it equal to zero. We extend u also for times $t > T$, by following the argument of [16], where we make use of assumption (D). Let us denote by $\breve{u}$ the extended solution; by construction, $\breve{u}$ solves the BVP

$$\begin{aligned}
&(L_\gamma + B)\breve{u} = \breve{F}\,, \quad \text{in } Q\,, \\
&M\breve{u} = \breve{G}\,, \quad \text{on } \Sigma\,.
\end{aligned} \tag{40}$$

In (40), the time t is involved with the same role of the tangential space variables, as it spans the whole real line $\mathbb{R}$. Therefore, (40) is a stationary problem posed in Q, with data $\breve{F} \in L^2(0, +\infty; H^{m+1}_\gamma(\Sigma)) \cap H^{m-1}_\gamma(Q)$, $\breve{G} \in H^{m+1}_\gamma(\Sigma)$. Furthermore, $\breve{u}$ enjoys the estimate (8), that is

$$\gamma ||\breve{u}||^2_{L^2(Q)} + ||\breve{u}_{|\,\Sigma}||^2_{L^2(\Sigma)} \leq \breve{C}_0 \left(\frac{1}{\gamma^3} ||\breve{F}||^2_{L^2(H^1_\gamma(\Sigma))} + \frac{1}{\gamma^2} ||\breve{G}||^2_{H^1_\gamma(\Sigma)} \right), \tag{41}$$

for all $\gamma \geq \breve{\gamma}_0$ and a suitable $\breve{\gamma}_0 \geq 1$.
The proof of Theorem 10 will be derived from the study of the regularity of solutions

to the BVP (40), that will be performed in the next Section 5. In view of the analysis performed there (see Theorem 11), we are able to prove that there exist constants $\breve{\gamma}_m \geq 1$, $\breve{C}_m > 0$ such that for $\gamma \geq \breve{\gamma}_m$ and under the regularity of the data $\breve{F}$, $\breve{G}$, the solution $\breve{u}$ of the BVP (40) actually belongs to $H^m_\gamma(Q)$, $\breve{u}_{|\,\Sigma} \in H^m_\gamma(\Sigma)$ and the following a priopri estimate

$$\gamma||\breve{u}||^2_{H^m_\gamma(Q)} + ||\breve{u}_{|\,\Sigma}||^2_{H^m_\gamma(\Sigma)} \leq \breve{C}_m \left(\frac{1}{\gamma^3}||\breve{F}||^2_{L^2(H^{m+1}_\gamma(\Sigma))\cap H^{m-1}_\gamma(Q)} + \frac{1}{\gamma^2}||\breve{G}||^2_{H^{m+1}_\gamma(\Sigma)} \right) \tag{42}$$

holds.
Since the solution u of the homogeneous IBVP (35) is $u = \breve{u}_{|\,[0,T]}$ we immediately obtain that $u \in H^m_\gamma(Q_T)$, $u_{|\,\Sigma_T} \in H^m_\gamma(\Sigma_T)$ with

$$||u||_{H^m_\gamma(Q_T)} \leq ||\breve{u}||_{H^m_\gamma(Q)}\,, \quad ||u_{|\,\Sigma_T}||_{H^m_\gamma(\Sigma_T)} \leq ||\breve{u}_{|\,\Sigma}||_{H^m_\gamma(\Sigma)}\,. \tag{43}$$

The energy estimate (37) also comes at once from (42) combined with (39) and (43).

5. Regularity of the BVP (40)**.** In this section we deal with the study of the regularity of the BVP (40), where for simplicity we will write u, F, G instead of $\breve{u}$, $\breve{F}$, $\breve{G}$. Within the problem (40), the "extended" time-variable t plays the same role of the tangential space variables $(x_2, \dots, x_n)$. Hence, to avoid overloading formulas, it is convenient to change our notations, by setting $x_{n+1} := t$ and $x := (x_1, x')$ with $x' := (x_2, \dots, x_n, x_{n+1})$; accordingly, we will denote $A_{n+1} := I_N, \partial_{n+1} := \partial_t$, so that the differential operator L_γ takes the form

$$L_\gamma = \gamma I_N + \sum_{j=1}^{n+1} A_j \partial_j, \tag{44}$$

where the coefficients $A_j = A_j(x)$ are given matrices in $C^\infty_b(\overline{Q})$.
The Fourier dual variables of the tangential *space-time* variables x' will be again denoted by the vector $\xi' = (\xi_2, \dots, \xi_{n+1}) \in \mathbb{R}^n$ in a compact form.
From now on, troughout the whole section, we will use the tools introduced in Section 3 where, due to the use of the time-variable as an additional space variable, the dimension n will be substituted by $n + 1$.
We observe that from hypothesis (D) the BVP (40) enjoys the following:

(WWP) *Weak well posedness.* For every tangential pseudo-differential operator $\mathcal{B} = \mathrm{Op}^\gamma_{\tan}(b)$ with symbol $b = b(x_1, x', \xi', \gamma) \in \Gamma_0$ there exist constants $C_0 > 0$, $\gamma_0 \geq 1$, depending boundedly on a finite number of seminorms of b (and the coefficients A_j of L_γ in (44)), such that for all $\gamma \geq \gamma_0$, for all $F \in L^2(0, +\infty; H^1_\gamma(\Sigma))$, $G \in H^1_\gamma(\Sigma)$, the BVP (40) (with data F, G and with the zero order term B replaced by $\mathcal{B}$) admits a unique solution $u \in L^2(Q)$, with $u_{|\,\Sigma} \in L^2(\Sigma)$ and the a priori estimate (41) is fulfilled (with u instead of $\breve{u}$).

After the change of unknown made at the beginning of Section 4, the assumption (WWP) on (40) follows at once by restating the assumption (D) about (5)-(6) (with $T = +\infty$) in terms of the functions $e^{-\gamma t}u$, $e^{-\gamma t}F$ and $e^{-\gamma t}G$. Let us notice, in particular, that the pseudo-differential operators of type (9), involved in (D), actually reduce to the form in (29), under the above change of unknown function.
The aim of this section will be the proof of the following Theorems (for the sake of clarity, we prefer to treat separately the tangential and the full regularity of the BVP).

THEOREM 11 (Tangential regularity). *Let the BVP (40) satisfy the assumption (WWP). Then for every $m \in \mathbb{N}$ and $B \in C_b^\infty(\overline{Q})$ there exist constants $C_m > 0$, γ_m (with $\gamma_m \geq \gamma_{m-1} \geq 1$) such that for all $\gamma \geq \gamma_m$, for all $F \in L^2(0,+\infty; H_\gamma^{m+1}(\Sigma))$ and $G \in H_\gamma^{m+1}(\Sigma)$ the unique L^2-solution of the BVP (40)(with data F, G and lower order term B) belongs to $L^2(0,+\infty; H_\gamma^m(\Sigma))$ with $u_{|\Sigma} \in H_\gamma^m(\Sigma)$ and the a priori estimate of order m*

$$\gamma||u||^2_{L^2(H^m_\gamma(\Sigma))} + ||u_{|\Sigma}||^2_{H^m_\gamma(\Sigma)} \leq C_m \left(\frac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))} + \frac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)} \right) \quad (45)$$

is satisfied.

Proof. We proceed by induction on m. The case $m = 0$ comes at once from hypotesis (WWP). Assume now the result is true for $m-1$, with given integer $m \geq 1$; thus we already know that $u \in L^2(0,+\infty; H_\gamma^{m-1}(\Sigma))$. In order to increase the tangential regularity of order one for u, we apply the tangential operator $\lambda_\delta^{m-1,\gamma}(D')$ to the system (40) and try to write a similar BVP for the new unknown $\lambda_\delta^{m-1,\gamma}(D')u$. Notice that, in view of Lemma 6 and Proposition 9, we know that $\lambda_\delta^{m-1,\gamma}(D')u \in L^2(Q)$.
Now we follow the strategy explained in [14] to derive the new BVP satisfied by $\lambda_\delta^{m-1,\gamma}(D')u$.
1^{st} *Step: Internal equation.* We decompose the operator $L_\gamma + B$ as

$$L_\gamma + B = A_1\partial_1 + L_{\tan,\gamma}\,,$$

where $L_{\tan,\gamma} := \gamma I_N + \sum_{j=2}^{n+1} A_j\partial_j + B$ and apply $\lambda_\delta^{m-1,\gamma}(D')$ to $(40)_1$ to get

$$\begin{aligned}
&\lambda_\delta^{m-1,\gamma}(D')(L_\gamma + B)u = \lambda_\delta^{m-1,\gamma}(D')(A_1\partial_1 u + L_{\tan,\gamma}u)\\
&= A_1\lambda_\delta^{m-1,\gamma}(D')\partial_1 u + [\lambda_\delta^{m-1,\gamma}(D'), A_1]\partial_1 u + L_{\tan,\gamma}\lambda_\delta^{m-1,\gamma}(D')u\\
&\qquad + [\lambda_\delta^{m-1,\gamma}(D'), L_{\tan,\gamma}]u\\
&= L_\gamma(\lambda_\delta^{m-1,\gamma}(D')u) + [\lambda_\delta^{m-1,\gamma}(D'), A_1]\partial_1 u + [\lambda_\delta^{m-1,\gamma}(D'), L_{\tan,\gamma}]u\,,
\end{aligned} \quad (46)$$

where, to find the last row, we have exploited that the tangential operator $\lambda_\delta^{m-1,\gamma}(D')$ commutes with the normal derivative ∂_1.
Since $\lambda_\delta^{m-1,\gamma}(D')$ is invertible, with inverse operator given by $\widetilde{\lambda}_\delta^{-m+1,\gamma}(D')$, using the symbolic calculus for tangential pseudo-differential operators we get

$$[\lambda_\delta^{m-1,\gamma}(D'), L_{\tan,\gamma}]u = \mathcal{B}^1_\delta(x, D', \gamma)\left(\lambda_\delta^{m-1,\gamma}(D')u\right)\,, \quad (47)$$

where

$$\mathcal{B}^1_\delta(x, D', \gamma) := [\lambda_\delta^{m-1,\gamma}(D'), L_{\tan,\gamma}]\widetilde{\lambda}_\delta^{-m+1,\gamma}(D') \quad (48)$$

is a tangential pseudo-differential operator with a $\delta-$uniformly bounded symbol in Γ^0 (see Lemma 8).
We treat now the other commutator term $[\lambda_\delta^{m-1,\gamma}(D'), A_1]\partial_1 u$. Firstly, directly from

the system $(40)_1$, we can express the normal derivative of u as a function of tangential derivatives of u itself and the datum F, as

$$\partial_1 u = A_1^{-1} F - A_1^{-1} L_{\tan,\gamma} u. \tag{49}$$

Then we insert (49) into the normal commutator to get

$$[\lambda_\delta^{m-1,\gamma}(D'), A_1]\partial_1 u = [\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}F - [\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}L_{\tan,\gamma}u$$

and using again the invertibility of $\lambda_\delta^{m-1,\gamma}(D')$ finally gives

$$[\lambda_\delta^{m-1,\gamma}(D'), A_1]\partial_1 u = [\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}F + \mathcal{B}_\delta^2(x, D', \gamma)\left(\lambda_\delta^{m-1,\gamma}(D')u\right), \tag{50}$$

where

$$\mathcal{B}_\delta^2(x, D', \gamma) := [\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}L_{\tan,\gamma}\widetilde{\lambda}_\delta^{-m+1,\gamma}(D') \tag{51}$$

is again a tangential pseudo-differential operator with a $\delta-$uniformly bounded symbol in Γ^0 (see again Lemma 8). Combining the formulas (47) and (50) with (46) we find that $\lambda_\delta^{m-1,\gamma}(D')u$ solves the system

$$\left(L_\gamma + \mathcal{B}_\delta(x, D', \gamma)\right)\lambda_\delta^{m-1,\gamma}(D')u = \lambda_\delta^{m-1,\gamma}(D')F - [\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}F \quad \text{in } Q, \tag{52}$$

where

$$\mathcal{B}_\delta(x, D', \gamma) := B(x) + \mathcal{B}_\delta^1(x, D', \gamma) + \mathcal{B}_\delta^2(x, D', \gamma) \tag{53}$$

is a tangential pseudo-differential operator with δ-uniformly bounded symbol in Γ^0.
2^{nd} *Step: Boundary condition.* Now we are going to seek for an appropriate boundary condition to be coupled with the interior equation (52). Note that for continuous functions with respect to x_1, say $u \in C([0, +\infty); \mathcal{S}'(\Sigma))$, we easily see that the tangential operator $\lambda_\delta^{m-1,\gamma}(D')$, acting only on the tangential variables x', trivially commutes with the restriction of u to the boundary $\Sigma = \{x_1 = 0\}$; indeed

$$\begin{aligned} \left(\lambda_\delta^{m-1,\gamma}(D')u\right)_{|\Sigma} &= \left(\lambda_\delta^{m-1,\gamma}(D')u(x_1, \cdot)\right)_{|x_1=0} \\ &= \lambda_\delta^{m-1,\gamma}(D')u(0, \cdot) = \lambda_\delta^{m-1,\gamma}(D')\left(u_{|\,\Sigma}\right). \end{aligned} \tag{54}$$

We note that, with a slight abuse, in (54) and in the following the same notation $\lambda_\delta^{m-1,\gamma}(D')$ is used to mean either the tangential operator with symbol $\lambda_\delta^{m-1,\gamma}(\xi')$ as defined in (30), or the ordinary pseudo-differential operator on $\mathbb{R}^n_{x'}$ acting on functions which depend only on x'. More precisely, for $u = u(x')$, we set

$$\lambda_\delta^{m-1,\gamma}(D')u = (2\pi)^{-n+1}\int_{\mathbb{R}^n} e^{ix'\cdot\xi'}\lambda_\delta^{m-1,\gamma}(\xi')\widehat{u}(\xi')\,d\xi' = \mathcal{F}^{-1}\left(\lambda_\delta^{m-1,\gamma}\widehat{u}\right). \tag{55}$$

From now on when $\lambda_\delta^{m-1,\gamma}(D')$ acts on functions defined on Σ we intend it as in (55). We need to apply the previous formula when u is the solution of the original BVP (40). In principle u is not sufficiently smooth with respect to x_1 since, by the inductive hypothesis, it belongs to $L^2(0, +\infty; H_\gamma^{m-1}(\Sigma))$. However, because of the invertibility of A_1, writing $\partial_1 u$ as in (49), we find that $\partial_1 u \in L^2(0, +\infty; H_\gamma^{m-2}(\Sigma)) \hookrightarrow L^2(0, +\infty; H_\gamma^{-1}(\Sigma))$; hence, by Sobolev Imbedding Theorem, we get $u \in H^1(0,+\infty; H^{-1}_\gamma(\Sigma)) \hookrightarrow C([0,+\infty); H^{-1}_\gamma(\Sigma))$ and formula (54) follows. We consider now $M = (I_d, 0)$. From (54) we immediately get

$$\left(M\lambda_\delta^{m-1,\gamma}(D')u\right)_{|\,\Sigma} = \lambda_\delta^{m-1,\gamma}(D')(Mu_{|\,\Sigma}) = \lambda_\delta^{m-1,\gamma}(D')G\,. \tag{56}$$

3^{rd} *Step: regularity of order* m. Collecting (52), (56), $\lambda_\delta^{m-1,\gamma}(D')u$ solves the BVP

$$\begin{aligned} &(L_\gamma + \mathcal{B}_\delta(x,D',\gamma))\,\lambda_\delta^{m-1,\gamma}(D')u = \mathcal{F}_\delta \quad \text{in}\, Q,\\ &M\lambda_\delta^{m-1,\gamma}(D')u = \mathcal{G}_\delta\,, \quad \text{on}\ \Sigma\,, \end{aligned} \tag{57}$$

with

$$\begin{aligned} &\mathcal{F}_\delta := \lambda_\delta^{m-1,\gamma}(D')F - [\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}F\,,\\ &\mathcal{G}_\delta := \lambda_\delta^{m-1,\gamma}(D')G\,. \end{aligned}$$

In order to get $\delta-$uniform estimates of the data $\mathcal{F}_\delta$ and $\mathcal{G}_\delta$ in terms of the orginal data F, G, we need to regard $\lambda_\delta^{m-1,\gamma}(D')$ as an operator of order m. Hence, from the symbolic calculus, $[\lambda_\delta^{m-1,\gamma}(D'), A_1]A_1^{-1}$ is a tangential pseudo-differential operator with $\delta-$uniformly bounded symbol in Γ^{m-1}; moreover, after Proposition 9 (and Sobolev continuity of ordinary pseudo-differential operators on Σ), we have that $\mathcal{F}_\delta \in L^2(0,+\infty; H^1_\gamma(\Sigma))$, $\mathcal{G}_\delta \in H^1_\gamma(\Sigma)$ and there exists a positive constant $C_m > 0$ such that

$$\begin{aligned} &||\mathcal{F}_\delta||_{L^2(H^1_\gamma(\Sigma))} \le C_m ||F||_{L^2(H^{m+1}_\gamma(\Sigma))}\,,\\ &||\mathcal{G}_\delta||_{H^1_\gamma(\Sigma)} \le C_m ||G||_{H^{m+1}_\gamma(\Sigma)}\,, \quad \forall\,\delta \in]0,1]\,,\ \gamma \ge 1\,. \end{aligned} \tag{58}$$

By assumption (WWP) applied to (57) and exploiting estimates (58), we may find constants $\tilde{C}_m > 0$, $\tilde{\gamma}_m \ge 1$ independent of $\delta \in]0,1]$ [1], such that for all $\gamma \ge \tilde{\gamma}_m$, $\lambda_\delta^{m-1,\gamma}(D')u$ is the unique L^2-solution of (57) and it obeys the energy estimate

$$\begin{aligned} &\gamma||\lambda_\delta^{m-1,\gamma}(D')u||^2_{L^2(Q)} + ||\lambda_\delta^{m-1,\gamma}(D')u_{|\,\Sigma}||^2_{L^2(\Sigma)}\\ &\le \tilde{C}_m \left(\tfrac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))} + \tfrac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)}\right)\,, \end{aligned} \tag{59}$$

which gives a $\delta-$uniform estimate of the norms $||\lambda_\delta^{m-1,\gamma}(D')u||_{L^2(Q)}$ and $||\lambda_\delta^{m-1,\gamma}(D')u_{|\,\Sigma}||_{L^2(\Sigma)}$. Together with the inductive hypothesis and in view of Proposition 3 and Proposition 4 (see also (31)), these estimates give that $u \in L^2(0,+\infty; H^m_\gamma(\Sigma))$ and $u_{|\,\Sigma} \in H^m_\gamma(\Sigma)$.
To derive the a priori estimate (45) of order m, we pass to the limit as $\delta \downarrow 0$ in (59) and use again Propositions 3 and 4 to conclude. □

Theorem 12 (Full regularity). *Under the same assumptions of Theorem 11, let us suppose now $m \ge 1$. For every $B \in C^\infty_b(\overline{Q})$ there exist constants $C_m > 0$ and $\gamma_m \ge 1$ such that for all $\gamma \ge \gamma_m$, $F \in L^2(0,+\infty; H^{m+1}_\gamma(\Sigma)) \cap H^{m-1}_\gamma(Q)$ and*

[1]Recall that in assumption (WWP) if the seminorms of b are uniformly bounded with respect to a parameter, then the corresponding constants C_0 and γ_0 can be choosen to be independent of this parameter.

$G \in H^{m+1}_\gamma(\Sigma)$, the L^2-solution of the BVP (40) *(with data F, G and lower order term B) belongs to $H^m_\gamma(Q)$ with $u_{|\,\Sigma} \in H^m_\gamma(\Sigma)$ and the a priori estimate of order m*

$$\begin{aligned}&\gamma||u||^2_{H^m_\gamma(Q)} + ||u_{|\,\Sigma}||^2_{H^m_\gamma(\Sigma)}\\ &\le C_m\left(\tfrac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))\cap H^{m-1}_\gamma(Q)} + \tfrac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)}\right)\end{aligned} \tag{60}$$

is satisfied.

Proof. We proceed again by induction on the integer order $m \ge 1$.

1^{st} *Step*: For $m = 1$, the assumption on the regularity of the data is just the same as for Theorem 11, namely $F \in L^2(0,+\infty; H^2_\gamma(\Sigma))$ and $G \in H^2_\gamma(\Sigma)$. From that theorem, we already know that $u \in L^2(0,+\infty; H^1_\gamma(\Sigma))$ and $u_{|\,\Sigma} \in H^1_\gamma(\Sigma)$ for all γ sufficiently large, and the estimate (45) with $m = 1$ holds true. We use again (49) to write the normal derivative $\partial_1 u$ in terms of tangential derivatives of u and the source term F in the interior equation $(40)_1$. The tangential regularity of u gives that $\partial_1 u \in L^2(Q)$ which implies that $u \in H^1_\gamma(Q)$. From the same equation (49) we also derive

$$\begin{aligned}&||\partial_1 u||_{L^2(Q)} \le C\left(||F||_{L^2(Q)} + \gamma||u||_{L^2(Q)} + \sum_{j\ge 2}||\partial_j u||_{L^2(Q)}\right)\\ &\le C\left(||F||_{L^2(Q)} + ||u||_{L^2(H^1_\gamma(\Sigma))}\right).\end{aligned} \tag{61}$$

Recalling that $||u||^2_{H^1(Q)} = \gamma^2||u||^2_{L^2(Q)} + \sum_{j=1}^{n+1}||\partial_j u||^2_{L^2(Q)}$, from (61) we get

$$||u||_{H^1_\gamma(Q)} \le C\left(||F||_{L^2(Q)} + ||u||_{L^2(H^1_\gamma(\Sigma))}\right). \tag{62}$$

Combining the above estimate with the tangential estimate of order 1 and using (16) finally gives

$$\begin{aligned}&\gamma||u||^2_{H^1_\gamma(Q)} + ||u_{|\,\Sigma}||^2_{H^1_\gamma(\Sigma)} \le C\left(\gamma||F||^2_{L^2(Q)} + \gamma||u||^2_{L^2(H^1_\gamma(\Sigma))}\right) + ||u_{|\,\Sigma}||^2_{H^1_\gamma(\Sigma)}\\ &\le C_1\left(\frac{1}{\gamma^3}||F||^2_{L^2(H^2_\gamma(\Sigma))} + \frac{1}{\gamma^2}||G||^2_{H^2_\gamma(\Sigma)}\right)\end{aligned} \tag{63}$$

for $\gamma \ge 1$ large enough and a suitable constant $C_1 > 0$ independent of γ.

2^{nd} *Step: Inductive step.* We assume now the result is true for $m-1$ and let the data F and G belong to $L^2(0,+\infty; H^{m+1}_\gamma(\Sigma)) \cap H^{m-1}_\gamma(Q)$ and $H^{m+1}_\gamma(\Sigma)$, respectively. Again, from the tangential regularity (see Theorem 11) we know that $u \in L^2(0,+\infty; H^m_\gamma(\Sigma))$ and $u_{|\Sigma} \in H^m_\gamma(\Sigma)$, provided that γ is sufficiently large, and the a priori tangential estimate (45) of order m is satisfied; this already proves the desired regularity of the trace $u_{|\,\Sigma}$. On the other hand, from the inductive hypothesis we also know that $u \in H^{m-1}_\gamma(Q)$ and the norm $||u||_{H^{m-1}_\gamma(Q)}$ is bounded by the right-hand side of (60), with $m-1$ instead of m. In order to prove that $u \in H^m_\gamma(Q)$ it is sufficient to show that all the derivatives of u of order m that contain at least one normal derivative are in $L^2(Q)$, that is all derivatives of the form:

$$\partial^{\alpha'}_{x'}\partial^h_1 u\,, \quad \forall\,(\alpha', h) \in \mathbb{N}^{n+1}\,,\ |\alpha'| + h = m\,,\ h \ge 1\,. \tag{64}$$

Let us firstly assume that the normal order h is 1; in this case, we need to estimate in $L^2(Q)$ the derivatives of type $\partial_{x'}^{\alpha'}\partial_1 u$ with $|\alpha'| = m-1$. Applying $\partial_{x'}^{\alpha'}$, with $|\alpha'| = m-1$, to (49) allows to express each derivative $\partial_{x'}^{\alpha'}\partial_1 u$ only in terms of tangential derivatives of F of order at most $m-1$ and tangential derivatives of u of order at most m, which have been already estimated by Theorem 11. Then from the tangential estimate (45) we get

$$\gamma \sum_{|\alpha'|=m-1} ||\partial_{x'}^{\alpha'}\partial_1 u||^2_{L^2(Q)} \leq C_m \left(\frac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))} + \frac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)} \right). \quad (65)$$

For every integer $2 \leq h \leq m$, we assume now that all the derivatives of type

$$\partial_{x'}^{\beta'}\partial_1^q u\,, \quad \forall\,(\beta', q) \in \mathbb{N}^{n+1}\,,\ |\beta'| + q = m\,,\ q = 1, \dots, h-1 \quad (66)$$

have already been estimated in $L^2(Q)$ and there holds

$$\gamma \sum_{\substack{|\beta'|+q=m,\\ 1\leq q\leq h-1}} ||\partial_{x'}^{\beta'}\partial_1^q u||^2_{L^2(Q)} \leq C_m \left(\frac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))\cap H^{h-2}_\gamma(Q)} + \frac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)} \right). \quad (67)$$

In order to get the desired estimate of $||u||_{H^m_\gamma(Q)}$ it only remains to prove that all the derivatives $\partial_{x'}^{\alpha'}\partial_1^h u$, with $|\alpha'| + h = m$, also belong to $L^2(Q)$ and they are estimated as in (67). To this end, let us apply the derivative $\partial_{x'}^{\alpha'}\partial_1^{h-1}$ to (49) to get

$$\begin{aligned}
&\partial_{x'}^{\alpha'}\partial_1^h u = \partial_{x'}^{\alpha'}\partial_1^{h-1}\left(A_1^{-1}\left(F - \gamma u - Bu - \sum_{j\geq 2} A_j\partial_j u \right)\right) \\
&= \partial_{x'}^{\alpha'}\partial_1^{h-1}A_1^{-1}F - \gamma\partial_{x'}^{\alpha'}\partial_1^{h-1}A_1^{-1}u - \partial_{x'}^{\alpha'}\partial_1^{h-1}A_1^{-1}Bu - \sum_{j\geq 2}\partial_{x'}^{\alpha'}\partial_1^{h-1}A_1^{-1}A_j\partial_j u \\
&= \sum_{\substack{\mu'+\nu'=\alpha',\\ l+k=h-1}} c_{\mu',\nu',l,k}\partial_{x'}^{\mu'}\partial_1^l A_1^{-1}\partial_{x'}^{\nu'}\partial_1^k F - \gamma \sum_{\substack{\mu'+\nu'=\alpha',\\ l+k=h-1}} c_{\mu',\nu',l,k}\partial_{x'}^{\mu'}\partial_1^l A_1^{-1}\partial_{x'}^{\nu'}\partial_1^k u \\
&- \sum_{\substack{\mu'+\nu'=\alpha',\\ l+k=h-1}} c_{\mu',\nu',l,k}\partial_{x'}^{\mu'}\partial_1^l(A_1^{-1}B)\partial_{x'}^{\nu'}\partial_1^k u - \sum_{j\geq 2}\sum_{\substack{\mu'+\nu'=\alpha',\\ l+k=h-1}} c_{\mu',\nu',l,k}\partial_{x'}^{\mu'}\partial_1^l(A_1^{-1}A_j)\partial_{x'}^{\nu'}\partial_j\partial_1^k u.
\end{aligned} \quad (68)$$

Observe that in the right-hand side of the above equality only mixed derivatives of the source term F of order less than or equal to $m-1$ are involved, where at most $h-1$ normal derivatives appear. As for u, two types of derivatives are involved: the first type, appearing in the third and fourth terms of the last formula, are derivatives $\partial_{x'}^{\nu'}\partial_1^k u$ with $|\nu'| + k \leq m-1$ (already estimated from the inductive hypothesis that $u \in H^{m-1}_\gamma(Q)$), while the second type, appearing in the last term of the above formula, are derivatives of the form $\partial_{x'}^{\beta'}\partial_1^q u$ where $|\beta'| + q \leq m$ and $q \leq h-1$. Observe that the latter are either tangential derivatives of order $\leq m$ (which are estimated since we know that $u \in L^2(0,+\infty; H^m_\gamma(\Sigma))$ or mixed derivatives of order $\leq m$, where at most $h-1$ normal derivatives are involved (which are estimated from the previous assumption, see (66)). This concludes the proof that all derivatives of the type (64), with fixed $2 \leq h \leq m$, belong to $L^2(Q)$, and the similar estimate as (67) for these derivatives

$$\gamma \sum_{|\alpha'|=m-h} ||\partial_{x'}^{\alpha'}\partial_1^h u||^2_{L^2(Q)} \le C_m \left(\frac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))\cap H^{h-1}_\gamma(Q)} + \frac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)} \right) \tag{69}$$

follows at once from (68) combined with the tangential estimate of order m (see (45)), (60) of order $m-1$ and (67). Adding estimates (69) over all integers $2 \le h \le m$ and (65), we obtain

$$\gamma \sum_{\substack{(\alpha',h):\\ |\alpha'|+h=m,\\ h\ge 1}} ||\partial_{x'}^{\alpha'}\partial_1^h u||^2_{L^2(Q)} \le C_m \left(\frac{1}{\gamma^3}||F||^2_{L^2(H^{m+1}_\gamma(\Sigma))\cap H^{m-1}_\gamma(Q)} + \frac{1}{\gamma^2}||G||^2_{H^{m+1}_\gamma(\Sigma)} \right). \tag{70}$$

This proves that all derivatives of order m of the form (64) are estimated in L^2 and gives that $u \in H^m_\gamma(Q)$; the estimate (60) of order m comes from adding the estimates (45), (70) and (60) of order $m-1$, which concludes the proof. ◻

Remark 13. We observe that we have not used the symmetrizability of the operator L_γ in order to prove Theorems 11, 12 for the BVP (40). On the contrary, this hypothesis will be necessary to prove Theorems 1, 2 for the evolution problem (1)-(3).

6. The non-homogeneous IBVP. Proof of Theorem 2. Let us approximate the data F, G, f with the functions F_k, G_k, f_k as in Lemma 14. Now we look for an approximated solution u_k of (1)-(3) with data F_k, G_k, f_k, of the form $u_k = v_k + w_k$, where v_k is solution to the homogeneous IBVP

$$\begin{array}{ll} Lv_k = F_k - Lw_k, & \text{in } Q_T \\ Mv_k = G_k - Mw_k, & \text{on } \Sigma_T \\ v_{k|t=0} = 0, & \text{in } \mathbb{R}^n_+ \end{array} \tag{71}$$

and $w_k \in H^{m+2}_\gamma(Q_T)$ is chosen such that

$$\partial_t^j w_{k\,|t=0} = f_k^{(j)}, \quad j = 0, \dots, m+1. \tag{72}$$

Note that such a function exists by chosing $w_k = \mathcal{R}(f_k^{(0)}, f_k^{(1)}, \dots, f_k^{(m+1)})$, where $\mathcal{R}$ is a lifting operator

$$\mathcal{R} : H^{m+1}_\gamma(\mathbb{R}^n_+) \times H^m_\gamma(\mathbb{R}^n_+) \times \cdots \times L^2(\mathbb{R}^n_+) \longrightarrow H^{m+2}_\gamma(Q_T)$$

defined as in [18], [1].
From the regularity of w_k, F_k, G_k we derive that

$$\begin{array}{l} F_k - Lw_k \in L^2(0, +\infty, H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T), \\ G_k - Mw_k \in H^{m+1}_\gamma(\Sigma_T). \end{array}$$

Moreover, from (72) and the compatibility conditions for the data F, G, f we derive

$$\partial_t^j (F_k - Lw_k)_{|t=0} = 0, \quad \partial_t^j (G_k - Mw_k)_{|t=0} = 0, \quad j = 0, \dots, m.$$

Then, for every k, Theorem 10 applies to the homogeneous problem (71) for γ large enough, and we find that $v_k \in H^m_\gamma(Q_T)$, $v_{k\,|\Sigma_T} \in H^m_\gamma(\Sigma_T)$. Consequently $u_k \in H^m_\gamma(Q_T)$ and $u_{k\,|\Sigma_T} \in H^m_\gamma(\Sigma_T)$ and by construction it is a solution of

$$\begin{array}{ll} Lu_k = F_k, & \text{in } Q_T \\ Mu_k = G_k, & \text{on } \Sigma_T \\ u_{k|t=0} = f_k, & \text{in } \mathbb{R}^n_+. \end{array} \tag{73}$$

Our next goal is to show that $\{u_k\}$ is a Cauchy sequence in $\mathcal{C}_T(H^m_\gamma)$.
1^{st} *Step: tangential regularity.*
To do that we firstly look for the problem solved by the tangential derivatives of u_k of order l

$$\mathcal{U}^l_k := \partial^l_{\tan} u_k := (\partial^\alpha_{\tan} u_k)_{|\alpha|=l}$$

for each $0 \le l \le m$, with

$$\partial^\alpha_{\tan} := \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} \partial_t^{\alpha_{n+1}} = \partial_{x'}^{\alpha'} \partial_t^{\alpha_{n+1}}, \quad \alpha' = (\alpha_2, \dots, \alpha_n), \quad \alpha = (\alpha', \alpha_{n+1}). \tag{74}$$

For $l = 0$, $\mathcal{U}^l_k = u_k$. Since $u_k \in H^m_\gamma(Q_T) \subseteq L^2(Q_T)$ in view of Theorem 1 it is the unique solution of (73), it belongs to $C([0,T]; L^2(\mathbb{R}^n_+))$ and satisfies the estimate (10) with f_k, F_k, G_k instead of f, F, G.
We assume now that we have just proven that all the tangential derivatives up to the order $l-1$, with given $1 \le l \le m$, belong to $C([0,T]; L^2(\mathbb{R}^n_+))$ and for all γ large enough and $0 < t \le T$ obey the a priori estimate

$$\begin{aligned} &\gamma \sum_{j=0}^{l-1} \gamma^{2(l-1-j)} ||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)} + \sum_{j=0}^{l-1} \gamma^{2(l-1-j)} ||\mathcal{U}^j_{k,\gamma\,|\Sigma_t}||^2_{L^2(\Sigma_t)} \\ &\qquad + \sum_{j=0}^{l-1} \gamma^{2(l-1-j)} ||\mathcal{U}^j_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)} \\ &\le C \left(\frac{1}{\gamma^3} ||F_{k,\gamma}||^2_{L^2(H^l_\gamma(\Sigma_t))} + \frac{1}{\gamma^2} \sum_{j=0}^{l-2} ||\partial_t^j F_{k|\,t=0}||^2_{H^{l-1-j}_\gamma(\mathbb{R}^n_+)} \right. \\ &\qquad \left. + \frac{1}{\gamma^2} ||G_{k,\gamma}||^2_{H^l_\gamma(\Sigma_t)} + \frac{1}{\gamma^2} ||f_k||^2_{H^l_\gamma(\mathbb{R}^n_+)} \right), \end{aligned} \tag{75}$$

where for $||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)}$ we mean

$$||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)} := \sum_{|\alpha|=j} ||e^{-\gamma t} \partial^\alpha_{\tan} u_k||^2_{L^2(Q_t)}$$

and analogously for $||\mathcal{U}^j_{k,\gamma}||^2_{L^2(\Sigma_t)}$ and $||\mathcal{U}^j_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}$. Accordingly to estimate (10), in (75), we intend that the term $\sum\limits_{j=0}^{l-2} ||\partial_t^j F_{k|\,t=0}||^2_{H^{l-1-j}_\gamma(\mathbb{R}^n_+)}$ has not to be considered for $l = 1$.
To conclude the argument we need to make a similar estimate for the tangential

derivatives of order l. Consider the problem satisfied by $\mathcal{U}_k^l$, obtained applying each tangential derivative $\partial_{\tan}^\alpha$ of order $|\alpha| = l$ to (73). Applying $\partial_{\tan}^\alpha$ to the internal equation $(73)_1$ we obtain

$$L\partial_{\tan}^\alpha u_k + [\partial_{\tan}^\alpha, L]\, u_k = \partial_{\tan}^\alpha F_k. \tag{76}$$

We use the formula (see [18])

$$[L, \partial_{\tan}^\alpha] = \sum_{|\beta|\le|\alpha|} \Gamma_{\alpha,\beta}\partial_{\tan}^\beta + \sum_{|\beta|<|\alpha|} \Psi_{\alpha,\beta}\partial_{\tan}^\beta L, \tag{77}$$

with suitable $\Gamma_{\alpha,\beta}, \Psi_{\alpha,\beta}$ smooth matrices, to restate (76) in the form

$$L\partial_{\tan}^\alpha u_k - \sum_{|\beta|=|\alpha|} \Gamma_{\alpha,\beta}\partial_{\tan}^\beta u_k = \partial_{\tan}^\alpha F_k + \sum_{|\beta|<|\alpha|} \Psi_{\alpha,\beta}\partial_{\tan}^\beta F_k + \sum_{|\beta|<|\alpha|} \Gamma_{\alpha,\beta}\partial_{\tan}^\beta u_k. \tag{78}$$

Note that the tangential derivatives of u_k in the right-hand side of (78) have already been estimated by (75), so that they can be treated as a part of the source term in the equation (78).
Applying $\partial_{\tan}^\alpha$ to the boundary and initial conditions in (73) we obtain

$$M\partial_{\tan}^\alpha u_k = \partial_{\tan}^\alpha G_k, \quad \text{on } \Sigma_T \tag{79}$$

and

$$\partial_{\tan}^\alpha u_{k\,|t=0} = \partial_{\tan}^{\alpha'} f_k^{(\alpha_{n+1})}. \tag{80}$$

Then we consider the problem satisfied by the vector $\mathcal{U}_k^l$ of all tangential derivatives $\partial_{\tan}^\alpha u$ of order $|\alpha| = l$. This problem takes the form

$$\begin{array}{ll} \mathcal{L}\mathcal{U}_k^l + \mathcal{B}\mathcal{U}_k^l = \mathcal{F}_k & \text{in } Q_T, \\ \mathcal{M}\mathcal{U}_k^l = \mathcal{G}_k & \text{on } \Sigma_T, \\ \mathcal{U}_{k\,|t=0}^l = \overline{f}_k & \text{in } \mathbb{R}_+^n, \end{array} \tag{81}$$

where

$$\mathcal{L} = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix},$$

($\mathcal{L}$ is a $q \times q$ matrix, $\mathcal{M}$ is a $q \times d$ matrix $q := \sharp\{\alpha : |\alpha| = l\}$) and with

$$\begin{aligned} \mathcal{F}_k &:= \left(\partial_{\tan}^\alpha F_k + \sum_{|\beta|<|\alpha|} \Psi_{\alpha,\beta}\partial_{\tan}^\beta F_k + \sum_{|\beta|<|\alpha|} \Gamma_{\alpha,\beta}\partial_{\tan}^\beta u_k \right)_{|\alpha|=l}, \\ \mathcal{G}_k &:= \left(\partial_{\tan}^\alpha G_k\right)_{|\alpha|=l}, \\ \overline{f}_k &:= \left(\partial_{x'}^{\alpha'} f_k^{(\alpha_{n+1})} \right)_{|\alpha|=l}. \end{aligned} \tag{82}$$

By Theorem 1 we derive that the solution $\mathcal{U}_k^l$ of system (81) is such that

$$\mathcal{U}_k^l \in C([0,T]; L^2(\mathbb{R}_+^n)), \quad \mathcal{U}_{k\,|\Sigma_T}^l \in L^2(\Sigma_T)$$

and obeys the a priori estimate (10), i.e.

$$\gamma||\mathcal{U}^l_{k,\gamma}||^2_{L^2(Q_t)} + ||\mathcal{U}^l_{k,\gamma\,|\Sigma_T}||^2_{L^2(\Sigma_t)} + ||\mathcal{U}^l_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}$$
$$\leq C\left(\frac{1}{\gamma^3}||\mathcal{F}_{k,\gamma}||^2_{L^2(H^1_\gamma(\Sigma_t))} + \frac{1}{\gamma^2}||\mathcal{G}_{k,\gamma}||^2_{H^1_\gamma(\Sigma_t)} + \frac{1}{\gamma^2}||\overline{f}_k||^2_{H^1_\gamma(\mathbb{R}^n_+)}\right),\ \ 0 < t \leq T. \tag{83}$$

Observing that the forcing term $\mathcal{F}_k$ contains only tangential derivatives of u_k up to the order $l-1$, already estimated by the assumption (see estimate (75)), from (82) we can treat the right-hand side of (83) in this way:

$$\begin{aligned}
&||\mathcal{F}_{k,\gamma}||^2_{L^2(H^1_\gamma(\Sigma_t))}\\
&\leq C\left(||e^{-\gamma t}\partial^\alpha_{\tan}F_k||^2_{L^2(H^1_\gamma(\Sigma_t))} + \sum_{|\beta|<|\alpha|}||e^{-\gamma t}\partial^\beta_{\tan}F_k||^2_{L^2(H^1_\gamma(\Sigma_t))}\right.\\
&\qquad\left. + \sum_{|\beta|<|\alpha|}||e^{-\gamma t}\partial^\beta_{\tan}u_k||^2_{L^2(H^1_\gamma(\Sigma_t))}\right)\\
&\leq C\left(||F_{k,\gamma}||^2_{L^2(H^{l+1}_\gamma(\Sigma_t))} + \sum_{|\beta|<|\alpha|}||u_{k,\gamma}||^2_{L^2(H^{1+|\beta|}_\gamma(\Sigma_t))}\right).
\end{aligned} \tag{84}$$

Observing that

$$\begin{aligned}
&\sum_{|\beta|<|\alpha|}||u_{k,\gamma}||^2_{L^2(H^{1+|\beta|}_\gamma(\Sigma_t))} = \sum_{|\beta|\leq l-1}||u_{k,\gamma}||^2_{L^2(H^{1+|\beta|}_\gamma(\Sigma_t))}\\
&= \sum_{|\beta|\leq l-1}\ \sum_{|\delta|\leq|\beta|+1}\gamma^{2(|\beta|+1-|\delta|)}||\partial^\delta_{\tan}u_{k,\gamma}||^2_{L^2(Q_t)}\\
&\leq C\sum_{|\nu|\leq l}\gamma^{2(l-|\nu|)}||\partial^\nu_{\tan}u_{k,\gamma}||^2_{L^2(Q_t)} = C\sum_{j=0}^{l}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)},
\end{aligned}$$

from (84) we get

$$||\mathcal{F}_{k,\gamma}||^2_{L^2(H^1_\gamma(\Sigma_t))} \leq C\left(||F_{k,\gamma}||^2_{L^2(H^{l+1}_\gamma(\Sigma_t))} + \sum_{j=0}^{l}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)}\right), \tag{85}$$

where C denotes a $\gamma-$independent positive constant (possibly different from line to line).

Applying similar arguments to the boundary datum $\mathcal{G}_k$ gives

$$||\mathcal{G}_{k,\gamma}||^2_{H^1_\gamma(\Sigma_t)} \leq C||G_{k,\gamma}||^2_{H^{l+1}_\gamma(\Sigma_t)}. \tag{86}$$

As for the initial datum $\overline{f}_k$, from $(82)_3$ one gets immediately

$$||\overline{f}_k||^2_{H^1_\gamma(\mathbb{R}^n_+)} = \sum_{|\alpha'|+\alpha_{n+1}=l}||\partial^{\alpha'}_{x'}f_k^{(\alpha_{n+1})}||^2_{H^1_\gamma(\mathbb{R}^n_+)} \leq C\sum_{|\alpha'|+\alpha_{n+1}=l}||f_k^{(\alpha_{n+1})}||^2_{H^{1+|\alpha'|}_\gamma(\mathbb{R}^n_+)}. \tag{87}$$

On the other hand, an induction argument gives for the time-derivatives of f_k

$$
\begin{aligned}
&||f_k^{(p)}||^2_{H^m_\gamma(\mathbb{R}^n_+)}\\
&\le C_{m,p}\left(||f_k||^2_{H^{m+p}_\gamma(\mathbb{R}^n_+)}+\sum_{q=0}^{p-1}||\partial_t^q F_{k|\,t=0}||^2_{H^{m+p-1-q}_\gamma(\mathbb{R}^n_+)}\right)\quad \forall p\ge 1, \forall m\in\mathbb{N}.
\end{aligned}\tag{88}
$$

Combinig (88) (with $p=\alpha_{n+1}$ and $m=l+1-\alpha_{n+1}$, for arbitrary $1\le\alpha_{n+1}\le l$) and (87) we get

$$
||\overline{f}_k||^2_{H^1_\gamma(\mathbb{R}^n_+)}\le C\left(||f_k||^2_{H^{l+1}_\gamma(\mathbb{R}^n_+)}+\sum_{j=0}^{l-1}||\partial_t^j F_{k|\,t=0}||^2_{H^{l-j}_\gamma(\mathbb{R}^n_+)}\right)\tag{89}
$$

for a suitable $\gamma-$independent constant C.
Using (85), (86) and (89) to estimate the right-hand side of (83) gives

$$
\begin{aligned}
&\gamma||\mathcal{U}^l_{k,\gamma}||^2_{L^2(Q_t)}+||\mathcal{U}^l_{k,\gamma\,|\Sigma_T}||^2_{L^2(\Sigma_t)}+||\mathcal{U}^l_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}\\
&\le C\left(\frac{1}{\gamma^3}||F_{k,\gamma}||^2_{L^2(H^{l+1}_\gamma(\Sigma_t))}+\frac{1}{\gamma^3}\sum_{j=0}^{l}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)}+\frac{1}{\gamma^2}||G_{k,\gamma}||^2_{H^{l+1}_\gamma(\Sigma_t)}\right.\\
&\qquad\left.+\frac{1}{\gamma^2}||f_k||^2_{H^{l+1}_\gamma(\mathbb{R}^n_+)}+\frac{1}{\gamma^2}\sum_{j=0}^{l-1}||\partial_t^j F_{k|\,t=0}||^2_{H^{l-j}_\gamma(\mathbb{R}^n_+)}\right).
\end{aligned}\tag{90}
$$

Moving $\dfrac{1}{\gamma^3}||\mathcal{U}^l_{k,\gamma}||^2_{L^2(Q_t)}$ from the right-hand side to the left-hand side of the above inequality, and using (75) of order $l-1$ to estimate $\sum\limits_{j=0}^{l-1}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)}$, gives

$$
\begin{aligned}
&\gamma||\mathcal{U}^l_{k,\gamma}||^2_{L^2(Q_t)}+||\mathcal{U}^l_{k,\gamma\,|\Sigma_T}||^2_{L^2(\Sigma_t)}+||\mathcal{U}^l_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}\\
&\le C\left(\frac{1}{\gamma^3}||F_{k,\gamma}||^2_{L^2(H^{l+1}_\gamma(\Sigma_t))}+\frac{1}{\gamma^2}||G_{k,\gamma}||^2_{H^{l+1}_\gamma(\Sigma_t)}\right.\\
&\qquad\left.+\frac{1}{\gamma^2}||f_k||^2_{H^{l+1}_\gamma(\mathbb{R}^n_+)}+\frac{1}{\gamma^2}\sum_{j=0}^{l-1}||\partial_t^j F_{k|\,t=0}||^2_{H^{l-j}_\gamma(\mathbb{R}^n_+)}\right).
\end{aligned}\tag{91}
$$

Lastly, adding (91) and (75), multiplied by γ^2, yields

$$
\begin{aligned}
&\gamma\sum_{j=0}^{l}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma}||^2_{L^2(Q_t)}+\sum_{j=0}^{l}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma\,|\Sigma_t}||^2_{L^2(\Sigma_t)}+\sum_{j=0}^{l}\gamma^{2(l-j)}||\mathcal{U}^j_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}\\
&\le C\left(\frac{1}{\gamma^3}||F_{k,\gamma}||^2_{L^2(H^{l+1}_\gamma(\Sigma_t))}+\frac{1}{\gamma^2}||G_{k,\gamma}||^2_{H^{l+1}_\gamma(\Sigma_t)}\right.\\
&\qquad\left.+\frac{1}{\gamma^2}||f_k||^2_{H^{l+1}_\gamma(\mathbb{R}^n_+)}+\frac{1}{\gamma^2}\sum_{j=0}^{l-1}||\partial_t^j F_{k|\,t=0}||^2_{H^{l-j}_\gamma(\mathbb{R}^n_+)}\right),
\end{aligned}\tag{92}
$$

for all $l \leq m$. This completes the proof that all the tangential derivatives of u_k up to the order m belong to $C([0,T];L^2(\mathbb{R}^n_+))$, obeying the a priori estimate

$$
\begin{aligned}
&\gamma||u_{k,\gamma}||^2_{L^2(H^m_\gamma(\Sigma_t))} + ||u_{k,\gamma\,|\Sigma_t}||^2_{H^m_\gamma(\Sigma_t)} + \sum_{j=0}^{m} ||\partial_t^j u_{k,\gamma}(t)||^2_{L^2(H^{m-j}_\gamma)} \\
&\leq C\left(\frac{1}{\gamma^3}||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))} + \frac{1}{\gamma^2}||G_{k,\gamma}||^2_{H^{m+1}_\gamma(\Sigma_t)}\right. \\
&\qquad \left. + \frac{1}{\gamma^2}||f_k||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2}\sum_{j=0}^{m-1}||\partial_t^j F_{k|\,t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}\right),
\end{aligned}
\tag{93}
$$

for all $0 < t \leq T$ and $\gamma \geq \gamma_m$, provided that $\gamma_m \geq 1$ is large enough.
2^{nd} *Step: full regularity.*
It remains now to control all the derivatives of u_k of order less than or equal to m including at least one normal derivative, i.e. all derivatives of the type

$$
\partial_1^h \partial^\alpha_{\tan} u_k\,, \quad h + |\alpha| \leq m\,,\ h \geq 1\,, \tag{94}
$$

where $\partial^\alpha_{\tan}$ is defined by (74).
To do that, we proceed exactly as done for the case of the BVP, see the proof of Theorem 12. From the interior equation $(73)_1$, written for $u_{k,\gamma}$, we write the normal derivative $\partial_1 u_{k,\gamma}$ as a function of tangential derivatives of $u_{k,\gamma}$ and $F_{k,\gamma}$ by

$$
\partial_1 u_{k,\gamma} = A_1^{-1}\left(F_{k,\gamma} - Bu_{k,\gamma} - \sum_{j\geq 2} A_j \partial_j u_{k,\gamma} - \partial_t u_{k,\gamma} - \gamma u_{k,\gamma}\right). \tag{95}
$$

Firstly, applying to (95) the operator $\partial^\alpha_{\tan}$ for an arbitrary α such that $|\alpha| \leq m-1$, we express all the derivatives of type (94) with $h = 1$ in terms of tangential derivatives of $F_{k,\gamma}$ of order at most $m-1$ and tangential derivatives of $u_{k,\gamma}$ of order at most m, which have already been estimated. This gives that all these derivatives belong to $C([0,T];L^2(\mathbb{R}^n_+))$, with the estimate

$$
\begin{aligned}
&\gamma \sum_{|\alpha|\leq m-1} \gamma^{2(m-1-|\alpha|)}||\partial_1\partial^\alpha_{\tan} u_{k,\gamma}||^2_{L^2(Q_t)} \\
&\qquad + \sum_{|\alpha|\leq m-1} \gamma^{2(m-1-|\alpha|)}||\partial_1\partial^\alpha_{\tan} u_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)} \\
&\leq C_m\left(\gamma||\partial^{m-1}_{\tan}F_{k,\gamma}||^2_{L^2(Q_t)} + ||\partial^{m-1}_{\tan}F_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}\right. \\
&\qquad \left. + \gamma||u_{k,\gamma}||^2_{L^2(H^m_\gamma(\Sigma_t))} + \sum_{j=0}^{m}||\partial_t^j u_{k,\gamma}(t)||^2_{L^2(H^{m-j}_\gamma)}\right),
\end{aligned}
\tag{96}
$$

where, for simplicity, we write $\partial^{m-1}_{\tan}$ for the vector of all tangential derivatives of order $\leq m-1$.
Then, the last two terms in the right-hand side of (96) are estimated by (93). Using

the imbedding inequality (16), the first term in the right-hand side of (96) is estimated by

$$||\partial_{\tan}^{m-1}F_{k,\gamma}||^2_{L^2(Q_t)} \le ||F_{k,\gamma}||^2_{L^2(H^{m-1}_\gamma(\Sigma_t))} \le \frac{1}{\gamma^4}||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))}\,. \tag{97}$$

Since $F_{k,\gamma} \in L^2(0,+\infty; H^{m+1}_\gamma(\Sigma_T))$ then

$$\partial_{\tan}^{m-1}F_{k,\gamma} \in L^2(0,+\infty; H^2_\gamma(\Sigma_T)) \hookrightarrow H^1_\gamma(0,T; L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1})))$$
$$\hookrightarrow C([0,T]; L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1}))) \hookrightarrow C([0,T]; L^2(\mathbb{R}^n_+)))\,.$$

Now we use the following $\gamma-$weighted imbedding inequalities

i. $C([0,T]; L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1}))) \hookrightarrow C([0,T]; L^2(\mathbb{R}^n_+))$: using (16) we obtain

$$||u||_{C([0,T];L^2(\mathbb{R}^n_+))} \le \frac{1}{\gamma}||u||_{C([0,T];L^2(H^1_\gamma))}\,.$$

ii. From Lemma 15 with $X = L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1}))$, we have $H^1_\gamma(0,T; L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1}))) \hookrightarrow C([0,T]; L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1})))$, with

$$||u||_{C([0,T];L^2(H^1_\gamma))} \le \frac{1}{\gamma^{1/2}}||u||_{H^1_\gamma(0,T;L^2(H^1_\gamma))}\,.$$

iii. We have $L^2(0,+\infty; H^2_\gamma(\Sigma_T)) \hookrightarrow H^1_\gamma(0,T; L^2(0,+\infty; H^1_\gamma(\mathbb{R}^{n-1})))$, and directly comparing the related norms gives

$$||u||_{H^1_\gamma(0,T;L^2(H^1_\gamma))} \le ||u||_{L^2(H^2_\gamma(\Sigma_T))}.$$

Applying these inequalities to estimate $\partial_{\tan}^{m-1}F_{k,\gamma}(t)$ we get

$$\begin{aligned}
&||\partial_{\tan}^{m-1}F_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)} \le ||\partial_{\tan}^{m-1}F_{k,\gamma}||^2_{C([0,t];L^2(\mathbb{R}^n_+))}\\
&\le \frac{1}{\gamma^2}||\partial_{\tan}^{m-1}F_{k,\gamma}||^2_{C([0,t];L^2(H^1_\gamma))} \le \frac{1}{\gamma^3}||\partial_{\tan}^{m-1}F_{k,\gamma}||^2_{H^1_\gamma(0,t;L^2(H^1_\gamma))}\\
&\le \frac{1}{\gamma^3}||\partial_{\tan}^{m-1}F_{k,\gamma}||^2_{L^2(H^2_\gamma(\Sigma_t))} \le \frac{1}{\gamma^3}||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))}\,.
\end{aligned} \tag{98}$$

Collecting (97) and (98), and using (93) the estimate (96) becomes

$$\begin{aligned}
&\gamma \sum_{|\alpha|\le m-1} \gamma^{2(m-1-|\alpha|)}||\partial_1\partial_{\tan}^\alpha u_{k,\gamma}||^2_{L^2(Q_t)}\\
&\qquad + \sum_{|\alpha|\le m-1} \gamma^{2(m-1-|\alpha|)}||\partial_1\partial_{\tan}^\alpha u_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)}\\
&\le C_m\left(\frac{1}{\gamma^3}||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))} + \frac{1}{\gamma^2}||G_{k,\gamma}||^2_{H^{m+1}_\gamma(\Sigma_t)}\right.\\
&\qquad \left. + \frac{1}{\gamma^2}||f_k||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2}\sum_{j=0}^{m-1}||\partial_t^j F_{k|\,t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}\right)\,.
\end{aligned} \tag{99}$$

Then, as already done in the proof of Theorem 12, we assume that for every integer $2 \leq h \leq m$, all derivatives of the type $\partial_1^q \partial_{\tan}^\beta u_{k,\gamma}$, with $q+|\beta| \leq m$ and $1 \leq q \leq h-1$, belong to $C([0,T];L^2(\mathbb{R}^n_+))$ and are estimated by

$$\begin{aligned}
&\gamma \sum_{\substack{q+|\beta|\leq m,\\ 1\leq q\leq h-1}} \gamma^{2(m-q-|\beta|)} ||\partial_1^q \partial_{\tan}^\beta u_{k,\gamma}||^2_{L^2(Q_t)} \\
&\qquad + \sum_{\substack{q+|\beta|\leq m,\\ 1\leq q\leq h-1}} \gamma^{2(m-q-|\beta|)} ||\partial_1^q \partial_{\tan}^\beta u_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)} \\
&\leq C_m \Bigg(\frac{1}{\gamma^3} ||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))\cap H^{h-2}_\gamma(Q_t)} + \frac{1}{\gamma^2} ||G_{k,\gamma}||^2_{H^{m+1}_\gamma(\Sigma_t)} \\
&\qquad + \frac{1}{\gamma^2} ||f_k||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2} \sum_{j=0}^{m-1} ||\partial_t^j F_{k|\,t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)} \Bigg), \quad 0 < t \leq T.
\end{aligned} \tag{100}$$

Then we apply the operator $\partial_1^{h-1}\partial_{\tan}^\alpha$ to (95) to write $\partial_1^h \partial_{\tan}^\alpha u_{k,\gamma}$ as a function of mixed derivatives of $F_{k,\gamma}$ and $u_{k,\gamma}$ that have already been estimated: we omit here the detailed calculations, since they are similar to those performed in the proof of Theorem 12, see (68). We thus obtain that all the derivatives $\partial_1^h \partial_{\tan}^\alpha u_{k,\gamma}$ belong to $C([0,T];L^2(\mathbb{R}^n_+))$ and are estimated by

$$\begin{aligned}
&\gamma \sum_{|\alpha|=m-h} \gamma^{2(m-h-|\alpha|)} ||\partial_1^h \partial_{\tan}^\alpha u_{k,\gamma}||^2_{L^2(Q_t)} \\
&\qquad + \sum_{|\alpha|=m-h} \gamma^{2(m-h-|\alpha|)} ||\partial_1^h \partial_{\tan}^\alpha u_{k,\gamma}(t)||^2_{L^2(\mathbb{R}^n_+)} \\
&\leq C_m \Bigg(\frac{1}{\gamma^3} ||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))\cap H^{h-1}_\gamma(Q_t)} + \frac{1}{\gamma^2} ||G_{k,\gamma}||^2_{H^{m+1}_\gamma(\Sigma_t)} \\
&\qquad + \frac{1}{\gamma^2} ||f_k||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2} \sum_{j=0}^{m-1} ||\partial_t^j F_{k|\,t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)} \Bigg), \quad 0 < t \leq T.
\end{aligned} \tag{101}$$

Adding (93), (99) and (101) over all integers $2 \leq h \leq m$ finally gives the a priori estimate of order m

$$\begin{aligned}
&\gamma ||u_{k,\gamma}||^2_{H^m_\gamma(Q_t)} + ||u_{k,\gamma\,|\Sigma_t}||^2_{H^m_\gamma(\Sigma_t)} + \sum_{j=0}^{m} ||\partial_t^j u_{k,\gamma}(t)||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)} \\
&\leq C_m \Bigg(\frac{1}{\gamma^3} ||F_{k,\gamma}||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))\cap H^{m-1}_\gamma(Q_t)} + \frac{1}{\gamma^2} ||G_{k,\gamma}||^2_{H^{m+1}_\gamma(\Sigma_t)} \\
&\qquad + \frac{1}{\gamma^2} ||f_k||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2} \sum_{j=0}^{m-1} ||\partial_t^j F_{k|\,t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)} \Bigg), \quad 0 < t \leq T,
\end{aligned} \tag{102}$$

with positive constant C_m independent of γ and k.
3^{rd} *Step:* $\{u_k\}$ *is a Cauchy sequence in* $\mathcal{C}_T(H^m_\gamma)$.
Applying the a priori estimate (102) to a difference of solutions $u_k - u_h$ of problems (73) readily gives

$$\gamma||(u_k-u_h)_\gamma||^2_{H^m_\gamma(Q_t)} + ||(u_k-u_h)_{\gamma\,|\Sigma_t}||^2_{H^m_\gamma(\Sigma_t)} + \sum_{j=0}^m ||\partial_t^j(u_k-u_h)_\gamma(t)||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}$$
$$\le C_m\left(\frac{1}{\gamma^3}||(F_k-F_h)_\gamma||^2_{L^2(H^{m+1}_\gamma(\Sigma_t))\cap H^{m-1}_\gamma(Q_t)} + \frac{1}{\gamma^2}||(G_k-G_h)_\gamma||^2_{H^{m+1}_\gamma(\Sigma_t)}\right.$$
$$\left.+\frac{1}{\gamma^2}||f_k-f_h||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2}\sum_{j=0}^{m-1}||\partial_t^j(F_k-F_h)_{|t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}\right),\ 0<t\le T. \tag{103}$$

Hence, majorizing the right-hand side of (103) by the same norms evaluated at $t = T$ and taking the supremum of $\sum_{j=0}^m ||\partial_t^j(u_k-u_h)_\gamma(t)||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}$ over $[0,T]$, we get (see (19))

$$||(u_k-u_h)_\gamma||^2_{\mathcal{C}_T(H^m_\gamma)}$$
$$\le C_m\left(\frac{1}{\gamma^3}||(F_k-F_h)_\gamma||^2_{L^2(H^{m+1}_\gamma(\Sigma_T))\cap H^{m-1}_\gamma(Q_T)} + \frac{1}{\gamma^2}||(G_k-G_h)_\gamma||^2_{H^{m+1}_\gamma(\Sigma_T)}\right.$$
$$\left.+\frac{1}{\gamma^2}||f_k-f_h||^2_{H^{m+1}_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma^2}\sum_{j=0}^{m-1}||\partial_t^j(F_k-F_h)_{|t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}\right), \tag{104}$$

from which, in view of Lemma 14, we infer that $\{u_k\}$ is a Cauchy sequence in $\mathcal{C}_T(H^m_\gamma)$. In a similar way, from (103) we obtain that $\{u_{k|\,\Sigma_T}\}$ is a Cauchy sequence in $H^m_\gamma(\Sigma_T)$. Therefore there exists a function in $\mathcal{C}_T(H^m_\gamma)$ which is the limit of $\{u_k\}$. Passing to the limit in (73) as $k\to+\infty$, we see that this function is a solution to (1)-(3). The uniqueness of the L^2 solution yields $u\in\mathcal{C}_T(H^m_\gamma)$ and $u_{|\,\Sigma_T}\in H^m_\gamma(\Sigma_T)$. Passing to the limit as $k\to+\infty$ into the estimate (102) finally gives the a priori estimate (12) for u.

Appendix A. Proof of Theorem 1. Assume that all the hypotheses of Theorem 1 hold. Given the operator L with matrix A_1 as in (B) we may associate some strictly dissipative boundary conditions, namely we may find a matrix M_1 for which there exists a constant $\epsilon > 0$ such that

$$-\langle A_1(x,t)w,w\rangle \ge \epsilon|w|^2 - \frac{1}{\epsilon}|M_1w|^2 \quad \forall w\in\mathbb{R}^N,\ (x,t)\in\Sigma, \tag{105}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$. Let us consider the initial-boundary value problem

$$\begin{array}{ll} Lw = F & \text{in } Q_T,\\ M_1w = 0 & \text{on } \Sigma_T,\\ w_{|t=0} = f & \text{in } \mathbb{R}^n_+. \end{array} \tag{106}$$

It is well known, see e.g. [2, Theorem 9.15], that (106) admits a unique L^2-solution such that $w\in C([0,T];L^2(\mathbb{R}^n_+))$, $w_{|\,\Sigma_T}\in L^2(\Sigma_T)$ and which satisfies the a priori estimate ($w_\gamma := e^{-\gamma t} w$)

$$\gamma \|w_\gamma\|^2_{L^2(Q_t)} + \|w_\gamma(t)\|^2_{L^2(\mathbb{R}^n_+)} + ||w_{\gamma|\,\Sigma_t}||^2_{L^2(\Sigma_t)} \le C \left(\|f\|^2_{L^2(\mathbb{R}^n_+)} + \frac{1}{\gamma} ||F_\gamma||^2_{L^2(Q_t)} \right) \tag{107}$$

for all $\gamma \ge \gamma_0$ and $0 < t \le T$, where γ_0 is taken sufficiently large (here the hypothesis on the symmetrizability of L is essential).
Since $F \in L^2(0, +\infty; H^1_\gamma(\Sigma_T))$, $f \in H^1_\gamma(\mathbb{R}^n_+)$, from the classical theory, $w \in L^2(0, +\infty; H^1_\gamma(\Sigma_T))$ and $w_{|\,\Sigma_T} \in H^1_\gamma(\Sigma_T)$; moreover, since the boundary is non characteristic from $(106)_1$ we can write the normal derivative $\partial_1 w$ in terms of tangential derivatives to get that $\partial_1 w \in L^2(Q_T)$ and then that $w \in H^1_\gamma(Q_T)$.
Applying the operators ∂_j, for $2 \le j \le n+1$ (where $\partial_{x_{n+1}} := \partial_t$) to (106) and taking account of (77) (for $|\alpha| = 1$), we infer that $\partial_{\tan} w = (\partial_{x_2} w, \dots, \partial_{x_{n+1}} w)$ solves the problem

$$\begin{array}{ll} L\partial_j w - \sum\limits_{|\beta|=1} \Gamma_\beta \partial^\beta_{\tan} w = (\partial_j + \Psi)F + \Gamma_0 w, & \text{in } Q_T, \\ M_1 \partial_j w = -(\partial_j M_1) w, & \text{on } \Sigma_T, \\ \partial_j w_{|t=0} = f_j, & \text{in } \mathbb{R}^n_+ , \end{array} \tag{108}$$

where we have set

$$f_j := \partial_j f\,, \quad 2 \le j \le n\,, \quad f_{n+1} := f^{(1)}\,.$$

The problem (108) can be shortly restated as

$$\begin{array}{ll} \mathcal{L}\partial_{\tan} w + \mathcal{B}\partial_{\tan} w = \mathcal{F}, & \text{in } Q_T, \\ \mathcal{M}_1 \partial_{\tan} w = \mathcal{G}, & \text{on } \Sigma_T, \\ \partial_{\tan} w_{|t=0} = \widetilde{f}, & \text{in } \mathbb{R}^n_+ , \end{array} \tag{109}$$

with

$$\mathcal{L} = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}, \qquad \mathcal{M}_1 = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_1 \end{pmatrix},$$

($\mathcal{L}$ is a $n \times n$ matrix, $\mathcal{M}_1$ is a $n \times d$ matrix) and

$$\mathcal{F} := ((\partial_j + \Psi)F + \Gamma_0 w)_{2 \le j \le n+1}\,, \quad \mathcal{G} := (-(\partial_j M_1)w)_{2 \le j \le n+1}\,, \quad \widetilde{f} := (f_j)_{2 \le j \le n+1}\,.$$

The assumptions on F, G, f in Theorem 1 and $w \in L^2(Q_T)$, $w_{|\,\Sigma_T} \in H^1_\gamma(\Sigma_T)$ yield

$$\begin{array}{ll} \mathcal{F} \in L^2(Q_T) \quad \text{with} \quad ||\mathcal{F}_\gamma||_{L^2(Q_T)} \le C||F_\gamma||_{L^2(H^1_\gamma(\Sigma_T))} + C||w_\gamma||_{L^2(Q_T)}; \\ \mathcal{G} \in H^1_\gamma(\Sigma_T) \quad \text{with} \quad ||\mathcal{G}_\gamma||_{L^2(\Sigma_T)} \le C||w_{\gamma\,|\Sigma_T}||_{L^2(\Sigma_T)}\,. \end{array} \tag{110}$$

Concerning the initial data $\widetilde{f}$, we have $\widetilde{f} \in L^2(\mathbb{R}^n_+)$ with

$$||\widetilde{f}||^2_{L^2(\mathbb{R}^n_+)} = \sum_{j=2}^n ||\partial_j f||^2_{L^2(\mathbb{R}^n_+)} + ||f^{(1)}||^2_{L^2(\mathbb{R}^n_+)} \le ||f||^2_{L^2(H^1_\gamma)} + ||f^{(1)}||^2_{L^2(\mathbb{R}^n_+)}\,. \tag{111}$$

From (88) (with $m=0$, $p=1$) the L^2-norm of $f^{(1)}$ is estimated by

$$||f^{(1)}||_{L^2(\mathbb{R}^n_+)} \le C\left(||f||_{H^1_\gamma(\mathbb{R}^n_+)} + ||F_{|t=0}||_{L^2(\mathbb{R}^n_+)}\right). \tag{112}$$

On the other hand, applying (119) of Lemma 15 to F_γ we have

$$\begin{aligned} &||F_{|t=0}||_{L^2(\mathbb{R}^n_+)} = ||F_{\gamma\,|t=0}||_{L^2(\mathbb{R}^n_+)} \le ||F_\gamma||_{C([0,T];L^2(\mathbb{R}^n_+))} \\ &\le \tfrac{C}{\gamma^{1/2}}||F_\gamma||_{H^1_\gamma(0,T;L^2(\mathbb{R}^n_+))} \le \tfrac{C}{\gamma^{1/2}}||F_\gamma||_{L^2(H^1_\gamma(\Sigma_T))}\,. \end{aligned} \tag{113}$$

Collecting (111) - (113) finally gives

$$||\widetilde{f}||^2_{L^2(\mathbb{R}^n_+)} \le C||f||^2_{H^1_\gamma(\mathbb{R}^n_+)} + \frac{C}{\gamma}||F_\gamma||^2_{L^2(H^1_\gamma(\Sigma_T))}\,. \tag{114}$$

From [21, Theorem A.1], there exists a unique solution $\partial_{\tan}w \in C([0,T];L^2(\mathbb{R}^n_+))$ to problem (109), such that $\partial_{\tan}w_{|\,\Sigma_T} \in L^2(\Sigma_T)$, and satisfies the a priori estimate

$$\begin{aligned} &\gamma\|(\partial_{\tan}w)_\gamma\|^2_{L^2(Q_t)} + \|(\partial_{\tan}w)_\gamma(t)\|^2_{L^2(\mathbb{R}^n_+)} + ||(\partial_{\tan}w)_{\gamma|\,\Sigma_t}||^2_{L^2(\Sigma_t)} \\ &\le C\left(\|\tilde{f}\|^2_{L^2(\mathbb{R}^n_+)} + ||\mathcal{G}_\gamma||^2_{L^2(\Sigma_t)} + \frac{1}{\gamma}||\mathcal{F}_\gamma||^2_{L^2(Q_t)}\right) \\ &\le C\left(||f||^2_{H^1_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma}||F_\gamma||^2_{L^2(H^1_\gamma(\Sigma_T))} + ||w_{\gamma|\,\Sigma_t}||^2_{L^2(\Sigma_t)} + \frac{1}{\gamma}||w_\gamma||^2_{L^2(Q_t)}\right)\,, \end{aligned}$$

where we have used (110), (114).
Combining the above inequality with (107), we obtain the estimate

$$\begin{aligned} &\gamma\|w_\gamma\|^2_{L^2(H^1_\gamma(\Sigma_t))} + \sum_{|\alpha|\le 1}||\partial^\alpha_{\tan}w_\gamma(t)||^2_{L^2(\mathbb{R}^n_+)} + ||w_{\gamma|\,\Sigma_t}||^2_{H^1_\gamma(\Sigma_t)} \\ &\le C\left(||f||^2_{H^1_\gamma(\mathbb{R}^n_+)} + \frac{1}{\gamma}||F_\gamma||^2_{L^2(H^1_\gamma(\Sigma_T))}\right) \end{aligned} \tag{115}$$

for all $\gamma \ge \gamma_0$ and $0 < t \le T$, where γ_0 is taken sufficiently large.
Now we consider the initial-boundary value problem

$$\begin{array}{ll} Lv = 0 & \text{in } Q_T, \\ Mv = G - Mw & \text{on } \Sigma_T, \\ v_{|t=0} = 0 & \text{in } \mathbb{R}^n_+, \end{array} \tag{116}$$

where w is the solution of (106). Since $(G-Mw)_{|t=0} = G_{|t=0} - Mf = 0$ on $\mathbb{R}^{n-1}$, we may extend $(G-Mw)_{|\Sigma_T}$ from $[0,T]$ to $]-\infty,T]$ by setting it equal to zero for all negative times and get a function in $H^1_\gamma(\omega_T)$, that we continue to denote as $G-Mw$. Then, because of the assumption (D) applied to the BVP

$$\begin{array}{ll} Lv = 0 & \text{in } \Omega_T, \\ Mv = G - Mw & \text{on } \omega_T, \end{array} \tag{117}$$

there exists the solution of (117) $v \in L^2(\Omega_T)$ such that $v_{|\,\omega_T} \in L^2(\omega_T)$, $v_{|\,t<0} = 0$. Furthermore $v \in C([0,T];L^2(\mathbb{R}^n_+))$, and it satisfies the a priori estimate

$$\begin{aligned} &\gamma\|v_\gamma\|^2_{L^2(\Omega_t)} + \|v_\gamma(t)\|^2_{L^2(\mathbb{R}^n_+)} + ||v_{\gamma|\,\omega_t}||^2_{L^2(\omega_t)} \le \frac{C}{\gamma^2}||(G-Mw)_\gamma||^2_{H^1_\gamma(\omega_t)} \\ &\le \frac{C}{\gamma^2}\left(||G||^2_{H^1_\gamma(\Sigma_t)} + ||w_{\gamma\,|\Sigma_t}||^2_{H^1_\gamma(\Sigma_t)}\right), \end{aligned} \tag{118}$$

for all γ sufficiently large and $0 < t \le T$.
The function $v_{|[0,T]}$ is an L^2 solution to the equation $(116)_1$ and satisfies the boundary condition $(116)_2$. Moreover, $v_{|\,t=0} = 0$ (see [2, Theorem 9.12]). Then $v_{|[0,T]}$ solves the IBVP (116).
It is clear that $u = v+w$ is a solution of (1)-(3) with the required properties; combining (107), (115), (118) gives (10). Finally, we observe that the uniqueness of the solution to (1)-(3) is a consequence of (D). The proof of Theorem 1 is complete.

Appendix B. Technical Lemmata . For every integer $m \ge 1$ and $\gamma \ge 1$, let us define the following space

$$K^m_\gamma(Q_T) := \{F \in L^2(0,+\infty; H^{m+1}_\gamma(\Sigma_T)) \cap H^{m-1}_\gamma(Q_T)$$
$$: \partial_t^j F_{|t=0} \in H^{m-j}_\gamma(\mathbb{R}^n_+),\ j = 0,\dots,m-1\}$$

provided with the natural norm

$$||F||^2_{K^m_\gamma(Q_T)} := \gamma^4 ||F||^2_{H^{m-1}_\gamma(Q_T)} + ||F||^2_{L^2(H^{m+1}_\gamma(\Sigma_T))} + \sum_{j=0}^{m-1} ||\partial_t^j F_{|t=0}||^2_{H^{m-j}_\gamma(\mathbb{R}^n_+)}.$$

Lemma 14. *Assume that problem* (1)-(3) *obeys the assumptions* (A)–(C). *Let* $F \in K^m_\gamma(Q_T)$, $G \in H^{m+1}_\gamma(\Sigma_T)$, $f \in H^{m+1}_\gamma(\mathbb{R}^n_+)$, *such that* $Mf^{(h)} = \partial_t^h G_{|t=0}$ *in* $\mathbb{R}^{n-1}$, $h = 0,\dots,m$.

Then there exist $F_k \in L^2(0,+\infty; H^{m+2}_\gamma(\Sigma_T)) \cap H^m_\gamma(Q_T)$, $G_k \in H^{m+2}_\gamma(\Sigma_T)$, $f_k \in H^{m+2}_\gamma(\mathbb{R}^n_+)$, *such that, for every* k, $Mf_k^{(h)} = \partial_t^h G_{k|t=0}$ *on* $\mathbb{R}^{n-1}$ *for* $h = 0,\dots,m+1$, *and such that* $F_k \to F$ *in* $K^m_\gamma(Q_T)$, $G_k \to G$ *in* $H^{m+1}_\gamma(\Sigma_T)$, $f_k^{(h)} \to f^{(h)}$ *in* $H^{m+1-h}_\gamma(\mathbb{R}^n_+)$ *for* $h = 0,\dots,m+1$, *as* $k \to +\infty$.

Proof. For noncharacteristic homogeneous ($G = 0$) boundary conditions and in the case of standard Sobolev spaces H^m, a similar proposition has been proved in [19, Lemma 3.3] and in [1]. The present adaptation to the nonhomogeneous case ($G \ne 0$) follows the same lines of the proof of [20, Lemma 5.1], so we will omit the details. □

For any Banach space X and real $\gamma \ge 1$, let $H^1_\gamma(0,T;X)$ denote the Sobolev space of $X-$valued functions $u \in L^2(0,T;X)$ such that $\partial_t u \in L^2(0,T;X)$, equipped with the weighted norm

$$||u||^2_{H^1_\gamma(0,T;X)} := \gamma^2 ||u||^2_{L^2(0,T;X)} + ||\partial_t u||^2_{L^2(0,T;X)}\,.$$

For $\gamma = 1$ we set $H^1(0,T;X) := H^1_1(0,T;X)$.
We state the following weighted version of the classical Sobolev Imbedding Theorem for vector-valued functions that is used throughout the paper.

Lemma 15. *For any* $T > 0$, $H^1_\gamma(0,T;X) \hookrightarrow C([0,T];X)$ *and there exists a positive constant* C_T, *independent of* γ, *such that*

$$||u||_{C([0,T];X)} \le \frac{C_T}{\sqrt{\gamma}} ||u||_{H^1_\gamma(0,T;X)}\,, \quad \forall\, u \in H^1_\gamma(0,T;X)\,. \tag{119}$$

Proof. For an arbitrary function $u \in H^1(0,T;X)$ let $\overline{u} \in H^1(\mathbb{R};X)$ be the extension of u over $\mathbb{R}$, obtained by reflection methods (see [11]). Then we have

$$||\overline{u}||_{H^1_\gamma(\mathbb{R};X)} \le C_T ||u||_{H^1_\gamma(0,T;X)}, \quad \forall \gamma \ge 1, \tag{120}$$

where C_T is a positive constant depending only on T.
On the other hand, the Sobolev Imbedding Theorem gives that $H^1(\mathbb{R};X) \hookrightarrow L^\infty(\mathbb{R};X)\cap C^0(\mathbb{R};X)$ with

$$||v||_{L^\infty(\mathbb{R};X)} \le C||v||_{H^1(\mathbb{R};X)}, \tag{121}$$

for all $v\in H^1(\mathbb{R};X)$. Writing (121) for $v(t)=\overline{u}_\gamma(t):=\overline{u}(\frac{t}{\gamma})$ with an arbitrary $\gamma\ge 1$ and using the identity

$$||\overline{u}||^2_{H^1_\gamma(\mathbb{R};X)} = \gamma||\overline{u}_\gamma||^2_{H^1(\mathbb{R};X)}$$

from (121) we derive

$$||\overline{u}||_{L^\infty(\mathbb{R};X)} = ||\overline{u}_\gamma||_{L^\infty(\mathbb{R};X)} \le C||\overline{u}_\gamma||_{H^1(\mathbb{R};X)} = \frac{C}{\sqrt{\gamma}}||\overline{u}||_{H^1_\gamma(\mathbb{R};X)}. \tag{122}$$

From the previous results, since $u=\overline{u}_{|[0,T]}$ we get $u\in C^0([0,T];X)$ and

$$||u||_{C^0([0,T];X)} = \sup_{[0,T]}||u(t)||_X \le \sup_{\mathbb{R}}||\overline{u}(t)||_X$$
$$= ||\overline{u}||_{L^\infty(\mathbb{R};X)} \le \tfrac{C}{\sqrt{\gamma}}||\overline{u}||_{H^1_\gamma(\mathbb{R};X)} \le \tfrac{C_T}{\sqrt{\gamma}}||u||_{H^1_\gamma(\mathbb{R};X)}.$$

□

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