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− 1 KRALL-JACOBI POLYNOMIALS*

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Abstract. We study a family of orthogonal polynomials which satisfy (apart from a 3-term recurrence relation) an eigenvalue equation involving a third order differential operator of Dunkl-type. The orthogonality measure of these polynomials consists in the continuous measure of the little -1 Jacobi polynomials to which is added an arbitrary mass located at the point $x = 0$, the middle of the orthogonality interval. This provides the first nontrivial example of Krall-type polynomials with a point mass inside the orthogonality interval. These polynomials can be obtained by a Geronimus transform of the little q-Jacobi polynomials in the limit $q = -1$.

Key words. Jacobi polynomials, little q-Jacobi polynomials, Geronimus transformation.

AMS subject classifications. 33C45, 33C47, 42C05.

1. Introduction. Significant advances have been realized in the characterization of recently discovered families of -1 orthogonal polynomials (OPs) that can be obtained as $q \to -1$ limits of q polynomials of the Askey scheme. The striking feature of these OPs is that they are classical or bispectral and that they satisfy eigenvalue equations involving Dunkl-type operators [1] in addition to the mandatory 3-term recurrence relation. They have arisen already in a number of physical problems [2]-[7] and are connected with Jordan algebras [8].

At the top of the emerging -1 scheme are the Bannai-Ito polynomials and their kernel partners ([9],Ch.1, Sect. 7), the complementary Bannai-Ito polynomials [10]. Both sets depend on 4 parameters. The Bannai-Ito polynomials are the eigensolutions of the most general operator which is of first order in Dunkl shifts, (i.e. first order in the operators T and R defined by $Tf(x) = f(x+1), Rf(x) = f(-x)$ on a function $f(x)$) and which stabilizes polynomials of given degrees. The Bannai-Ito polynomials are positive definite and orthogonal on a finite set of $N+1$ points. In the limits where $N \to \infty$, they tend to the big -1 Jacobi polynomials [11] which are orthogonal on $[-1, -c] \bigcup [c, 1]$. When $c = 0$, the little -1 Jacobi polynomials [12] arise as a special case. A Bochner-type theorem establishes [13] that the big and little -1 polynomials are the only families of OPs satisfying a differential-difference eigenvalue equation which is of first order in Dunkl-type operators.

The Bannai-Ito polynomials and their descendants all possess the Leonard duality property. (In fact, this led to their initial identification [14].) The dual -1 Hahn polynomials [15] together with the generalized Gegenbauer [16] and Hermite polynomials [17, 18] are also bispectral but, obeying eigenvalue equations of second order in Dunkl operators, they fall beyond the scope of Leonard duality.

We continue here the exploration of -1 orthogonal polynomials in that vein and look for another class of -1 OPs verifying a higher differential-difference equation.

In the wake of Krall's classification of OPs satisfying a fourth order differential equations [19], it is appreciated that the addition of discrete masses to the measure leads to OPs verifying higher order equations [20, 21, 22].

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In this connection, we present here a generalization of the little -1 Jacobi polynomials with the following features: these orthogonal polynomials obey a differential-difference equation of third order in Dunkl operators and a mass is located at the middle of the orthogonality interval.

The outline of the paper is the following. In section 2, we offer a brief review of useful results on little q-Jacobi polynomials. In section 3, we introduce the generalized little q-Jacobi polynomials [23] that are eigensolutions of higher order q-difference equations. In section 4, focusing for definiteness on one of the simpler cases, we obtain and characterize a set of generalized little -1 Jacobi polynomials by taking an appropriate q $\to -1$ limit of certain polynomials of the preceding section. The paper ends with concluding remarks.

2. Little q-Jacobi polynomials. The monic little q-Jacobi polynomials are defined as

$$P_n(x;a,b) = (-1)^n \frac{q^{n(n-1)/2}(aq;q)_n}{(abq^{n+1};q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} | qx \right) \tag{2.1}$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the q-shifted factorial and ${}_2\phi_1$ denotes the q-hypergeometric function.

The orthogonality relation is

$$\sum_{k=0}^{\infty} w_k P_n(q^k;a,b) P_m(q^k;a,b) = h_n \delta_{nm}, \tag{2.2}$$

where h_n are appropriate normalization constants, and the normalized weight function is

$$w_k = \frac{(aq;q)_\infty}{(abq^2;q)_\infty} \frac{(bq;q)_k (aq)^k}{(q;q)_k}. \tag{2.3}$$

It is assumed that $|q|<1$, $|aq|<1$, $|bq|<1$. The expansion coefficients of the little q-Jacobi polynomials in $P_n(x;a,b) = \sum_{s=0}^{n} B_n^{(s)} x^{n-s}$ are

$$B_n^{(s)} = b^{-s} \frac{(q^{-n}, a^{-1}q^{-n};q)_s}{(q, a^{-1}b^{-1}q^{-2n};q)_s}, \tag{2.4}$$

where $(a_1, a_2, \ldots, a_k; q)_n = (a_1,q)_n (a_2,q)_n \ldots (a_k,q)_n$.

It is known that the little q-Jacobi polynomials satisfy a second-order difference equation [24].

Introduce the functions of the second kind,

$$Q_n(z) = \int_a^b \frac{P_n(x)w(x)}{z-x} dx, \tag{2.5}$$

where $w(x)$ is assumed to be the normalized weight function, i.e.

$$\int_a^b w(x)dx = 1.$$

The values of the functions $Q_n(z)$, at $z=0$ (an accumulation point of the orthogonality measure), are :

$$Q_n(0;a,b) = -\sum_{k=0}^{\infty} \frac{P_n(q^k;a,b)w_k}{q^k}. \tag{2.6}$$

Using the $q-$ binomial theorem and the $q-$Saalschütz formula (see, e.g.,[25]) we have

$$Q_n(0;a,b) = (-1)^{n+1}a^n q^{n(n-1)/2}\frac{1-abq}{1-a}\frac{(q;q)_n(bq;q)_n}{(abq;q)_n(abq^{n+1};q)_n}. \tag{2.7}$$

3. Transformed $q-$Jacobi polynomials. Let $P_n(x)$ be orthogonal polynomials with measure localized on the interval $[a,b]$. Let $w(x)$ be the corresponding normalized unit weight function and $Q_n(z)$ be defined by (2.5). Let finally c be a point beyond the orthogonality interval $[a,b]$ such that $Q_n(c)$ exists.

Consider the Geronimus transformation [26, 27] of the polynomials $P_n(x)$ at the point $x=c$

$$\tilde{P}_n(x) = \mathcal{G}(c)P_n(x) = P_n(x) - \frac{\Phi_n}{\Phi_{n-1}}P_{n-1}(x), n=1,2,\cdots, \tilde{P}_0(x)=1, \tag{3.1}$$

where

$$\Phi_n = Q_n(c) + MP_n(c), \tag{3.2}$$

with M some arbitrary real parameter.

We note that if $a=q^j, j=1,2,3,\cdots$, then

$$\begin{aligned}\Phi_n &= Q_n(0;q^j,b) + MP_n(0;q^j,b)\\ &= (-1)^n q^{n(n-1)/2}\frac{(q^{j+1};q)_n}{(bq^{n+j+1};q)_n}\left(M - q^{nj}\frac{(1-bq^{j+1})(bq;q)_j(q;q)_j}{(1-q^j)(q^{n+1};q)_j(bq^{n+1};q)_j}\right).\end{aligned} \tag{3.3}$$

The weight function $\tilde{w}(x)$ of the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$\tilde{w}(x) = \kappa\left(\frac{w(x)}{x-c} - M\delta(x-c)\right), \tag{3.4}$$

where κ is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point $x=c$. The value of this mass depends on the parameter M. Now take for $P_n(x)$ the little q-Jacobi polynomials with $a = q^j, j=1,2,3,\cdots$, and perform the Geronimus transformation (3.1) with Φ_n given by (3.3). (In this case $c=0$.)

The weight function $\tilde{w}(x)$ for the polynomials $\mathcal{G}(0)\{P_n(x)\}$ is

$$\tilde{w}(x) = \kappa\left(\sum_{k=0}^{\infty}\tilde{w}_k\delta(x-q^k) - M\delta(x)\right), \tag{3.5}$$

where

$$\tilde{w}_k = \frac{(q^{j+1};q)_\infty}{(bq^{j+2};q)_\infty}\frac{(bq^k;q)_k q^{jk}}{(q;q)_k}. \tag{3.6}$$

The coefficients $B_n^{(s)}$ in the expansion

$$\tilde{P}_n(x) = \mathcal{G}(0)\{P_n(x;q^j,b)\} = \sum_{s=0}^{n}B_n^{(s)}x^{n-s} \tag{3.7}$$

have been given in [23]. In the same paper, it was shown that there exists an operator of the form

$$\mathcal{L}_q = \sum_{k=0}^{2N} a_k(q^{-N}x)T^{-N}\mathcal{D}_q^k, \tag{3.8}$$

with

$$a_k(x) = \sum_{s=0}^{k} \alpha_{ks}x^s, \quad k = 0, 1, \cdots, 2N, \tag{3.9}$$

T the $q-$shift operator and $\mathcal{D}_q$ the $q-$derivative operator, such that the orthogonal polynomials $P_n(x)$ satisfy an eigenvalue equation of the kind

$$\mathcal{L}_q P_n(x) = \lambda_n P_n(x). \tag{3.10}$$

Consider the action of the operator $\mathcal{L}_q$ upon the monomials x^n. From (3.8) and (3.9) we get

$$\mathcal{L}_q x^n = \sum_{s=0} A_n^{(s)} x^{n-s}, \tag{3.11}$$

where

$$A_n^{(s)} = q^{N(s-n)}[n][n-1]\cdots[n-s+1]\pi_s(q^n), \tag{3.12}$$

and

$$\pi_s(q^n) = \alpha_{s0} + \sum_{i=1}^{2N-s} \alpha_{s+i,i}[n-s][n-s-1]\cdots[n-s-i+1] \tag{3.13}$$

are polynomials in $z = q^n$ of degree not exceeding $2N - s$. It is clear that

$$A_n^{(s)} = 0, \quad s > 2N \tag{3.14}$$

(see [23]). The coefficients $A_n^{(s)}$ completely characterize the operator $\mathcal{L}_q$ and $A_n^{(s)}$ are called the representation coefficients of the operator $\mathcal{L}_q$.

The coefficients $A_n^{(s)}$ for the q-difference operator $\mathcal{L}_q$ that has the polynomials $\tilde{P}_n(x)$ (3.7) as eigenfunctions have been constructed in [23], they are the following

$$\begin{aligned} A_n^{(0)} &= \lambda_n \\ &= \frac{M(q-1)q^{-n(j+1)-1}(q^n;q)_{j+1}(bq^n;q)_{j+1}}{1-q^{-j-1}} \\ &\quad -(q^{-n}-1)(1-bq^{n+j})(bq;q)_{j+1}(q;q)_{j-1}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} A_n^{(1)} = (1-q^{-n})\Big(&Mq^{j(1-n)}(q^{n+1};q)_j(bq^n;q)_j(1-q^{n-1}) \\ &-(q;q)_{j-1}(bq;q)_{j+1}(1-q^{n+j-1})\Big), \end{aligned} \tag{3.16}$$

$$\begin{aligned} A_n^{(s)} = M(q-1)q^{(s-n)(j+1)-1}\ &\frac{(q^{-j};q)_{s-1}(q^{n-s};q)_{s+j+1}(bq^n;q)_{j-s+1}}{(q;q)_s} \\ &\text{if} \quad s \geq 2. \end{aligned} \tag{3.17}$$

Note that from the above formula it follows $A_n^{(s)} = 0$ if $s > j + 1$

4. Limit of the Krall-Jacobi polynomials as $q \to -1$. In this section we construct the $q \to -1$ limit of the coefficients $A_n^{(s)}$. We take $j = 2$ and put

$$q = -e^{\epsilon}, \quad b = -e^{\beta\epsilon}, \tag{4.1}$$

Note that $|qb| < 1$ implies that $\beta + 1 > 0$.

Substituting into the $A_n^{(s)}$ as given in (3.15)-(3.17) and taking the limit $\epsilon \to 0$, we have

(4.2)
$$\frac{A_n^{(0)}}{\epsilon^3} \to E_n^{(0)} = \begin{cases} -8Mn(n+2)(n+1+\beta) + 8n(\beta+1)(\beta+3) & n \ \texttt{even} \\ 8M(n+1)(n+\beta)(n+2+\beta) - 8(n+2+\beta)(\beta+1)(\beta+3) & n \ \texttt{odd} \end{cases}$$
(4.3)
$$\frac{A_n^{(1)}}{\epsilon^3} \to E_n^{(1)} = \begin{cases} 8Mn(n+2)(n+1+\beta) - 8n(\beta+1)(\beta+3), & n \ \texttt{even} \\ 8(\beta+1)(\beta+3)(n+1) - 8M(n^2-1)(n+\beta), & n \ \texttt{odd} \end{cases}$$
(4.4)
$$\frac{A_n^{(2)}}{\epsilon^3} \to E_n^{(2)} = \begin{cases} 8Mn(n+2)(n-2), & n \ \texttt{even} \\ -8M(n+1)(n-1)(n+\beta), & n \ \texttt{odd} \end{cases}$$
(4.5)
$$\frac{A_n^{(3)}}{\epsilon^3} \to E_n^{(3)} = \begin{cases} -8Mn(n+2)(n-2), & n \ \texttt{even} \\ 8M(n+1)(n-1)(n-3), & n \ \texttt{odd} \end{cases}.$$

and $E_n^{(s)} = 0$ for $s = 4, 5, 6, \ldots$.

Consider the form of the q-difference equation (3.10) in this limit. We divide both sides of (3.10) by ϵ^3 and introduce the operator L_ϵ which acts on the polynomials $\tilde{P}_n(x)$ as

$$L_\epsilon \tilde{P}_n(x) = \epsilon^{-3}\lambda_n \tilde{P}_n(x) \tag{4.6}$$

For monomials x^n, from (3.11) and (4.3)-(4.6) we have in the limit $\epsilon \to 0$,

$$L_0 x^n = \lim_{\epsilon\to 0} \frac{L_\epsilon}{\epsilon^3} x^n = E_n^{(0)} x^n + E_n^{(1)} x^{n-1} + E_n^{(2)} x^{n-2} + E_n^{(3)} x^{n-3} \tag{4.7}$$

This allows one to present the operator $L_0 = \lim_{\epsilon\to 0} \frac{L_\epsilon}{\epsilon^3}$ in the form

$$\begin{aligned} L_0 =& (-8M + 8Mx + 8Mx^2 - 8Mx^3)\partial_x^3 R \\ &+ \left[-12M/x + 24M + 4\beta M + (36M + 8\beta M)x - (12\beta M + 48M)x^2\right] \partial_x^2 R \\ &+ (12Mx + 4\beta Mx^2 - 12M/x - 4\beta M)\partial_x^2 + \left[(24M + 16\beta M - 8\beta^2 - 32\beta - 24)\right. \\ &\left. + 24M/x^2 + (4\beta - 12)M/x + (8\beta^2 - 36\beta M - 48M - 4\beta^2 M + 32\beta + 24)x\right] \partial_x R \\ &+ \left[(4\beta^2 M + 12\beta M)x - (12 + 4\beta)M/x + 24M + 8\beta M\right] \partial_x \\ &+ \left[12M/x^3 + 4\beta M/x^2 + (12 + 4\beta^2 + 4\beta M + 16\beta)/x\right. \\ &\left. + (8\beta M + 4\beta^2 M - 44\beta - 24 - 4\beta^3 - 24\beta^2)\right] (1 - R), \end{aligned} \tag{4.8}$$

where R is the reflection operator $Rf(x) = f(-x)$. We thus have that the polynomials $\tilde{P}_n^{(-1)}(x)$ are classical and satisfy the eigenvalue equation

$$L_0 \tilde{P}_n^{(-1)}(x) = \tilde{\lambda}_n \tilde{P}_n^{(-1)}(x), \tag{4.9}$$

where

$$\tilde{\lambda}_n = E_n^{(0)}. \tag{4.10}$$

The lower degree eigensolutions of (4.9) can be obtained directly as a check and as examples. We take

$$\tilde{\lambda}_1 = (\beta+1)(\beta+3)(16M - 8(\beta+3)), \tag{4.11}$$

and find the first order polynomial solution

$$\tilde{P}_1^{(-1)}(x) = x - 1 + \frac{2\beta - 1 - \alpha}{2\beta - 3 - \alpha}. \tag{4.12}$$

The second-order polynomial solution is

$$\tilde{\lambda}_2 = (\beta+3)(16\beta + 16 - 64M), \tag{4.13}$$

$$\tilde{P}_2^{(-1)}(x) = x^2 - \frac{2(4M-\beta-1)}{(5+\beta)(2M-\beta-1)}x + \frac{2(\beta+1)}{(5+\beta)(2M-\beta-1)}. \tag{4.14}$$

and the third-order polynomial solution

$$\tilde{\lambda}_3 = (3+\beta)(5+\beta)(32M - 8 - 8\beta), \tag{4.15}$$

$$\begin{aligned} \tilde{P}_3^{(-1)}(x) &= x^3 - \frac{4(-2M+1+\beta)}{(7+\beta)(-4M+\beta+1)}x^2 - \frac{4}{7+\beta}x \\ &+ \frac{8(1+\beta)}{(7+\beta)(5+\beta)(-4M+\beta+1)}. \end{aligned} \tag{4.16}$$

Other polynomial eigensolutions can be obtained in the $q=-1$ limit of (3.1). The weight function (3.5) has the following moments

$$\tilde{c}_n = k\left(\frac{(q^2;q)_n}{(bq^3;q)_n} - \frac{1-q^2}{1-bq^3}M\delta_{n,0}\right) \tag{4.17}$$

with k a different normalization than κ in (3.5).

Using the parametrization (4.1) and taking the limit $\epsilon \to 0$, we can directly obtain from the above relation the moments corresponding to the polynomials $\tilde{P}_n^{(-1)}$.

$$\begin{aligned} \mu_{2n} = \mu_{2n-1} &= k\left[\frac{(1)_n}{(\beta/2+3/2)_n} - \frac{2}{3+\beta}M\delta_{n,0}\right] \\ &= k\left[\frac{n!}{(\beta/2+3/2)_n} + M_0\delta_{n,0}\right], \qquad n = 1,2,\cdots, \end{aligned} \tag{4.18}$$

where $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ is the ordinary Pochhammer symbol and $M_0 = -2M/(3+\beta)$. The parameter M_0 is the value of the concentrated mass inserted at the point $x=0$. It is seen that $M_0 > 0$ if $M < 0$ (recall that condition $\beta > -1$ is assumed).

Indeed, when $M_0 = 0$ we have for the moments

$$\mu_0 = 1,\ \mu_{2n} = \mu_{2n-1} = \frac{(1)_n}{(\beta/2+3/2)_n}, \quad n = 1,2,\ldots \tag{4.19}$$

They correspond to a special case of the little -1 Jacobi polynomials [12] $P_n^{(\alpha,\beta)}(x)$ when $\alpha = 1$. Hence the moments (4.18) are associated with the perturbation of the measure for the little -1 Jacobi polynomials $P_n^{(1,\beta)}(x)$ by the inserting of the concentrated mass M_0 at the point $x = 0$.

It is then easily verified that

$$w(x) = \tilde{k}\left(|x|(1-x^2)^{(\beta-1)/2}(1+x) - \frac{4}{(1+\beta)(3+\beta)}M\delta(x)\right), \tag{4.20}$$

where

$$\tilde{k} = \frac{\Gamma(\beta/2+3/2)}{\Gamma(\beta/2+1/2)}k = \frac{\beta+1}{2}k \tag{4.21}$$

is the orthogonality measure for these polynomials. Indeed we see that

$$\int_{-1}^{1} w(x)x^n\,dx = \mu_n, \qquad n = 0, 1, 2, \cdots, \tag{4.22}$$

with μ_n given by (4.18).

Note that the orthogonality measure is positive definite if $M < 0$ (i.e. the value M_0 of the concentrated mass is positive).

In the following we determine the three term recurrence relation that the polynomials $\tilde{P}_n^{(-1)}(x)$ verify. It is already known that the little q-Jacobi polynomials satisfy the relation

$$P_{n+1} + b_n P_n + u_n P_{n-1} = xP_n, \tag{4.23}$$

and that the recurrence coefficients are defined by

$$u_n = A_{n-1}C_n, \quad b_n = A_n + C_n,$$

where A_n, C_n are given as

$$A_n = q^n\frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \quad C_n = aq^n\frac{(1-q^n)(1-bq^n)}{(1-abq^{2n+1})(1-abq^{2n})}. \tag{4.24}$$

Under the Geronimus transformation

$$\tilde{P}_n(x) = P_n(x) - B_n P_{n-1}(x), \tag{4.25}$$

with $B_n = \frac{\Phi_n}{\Phi_{n-1}}$ and Φ_n defined by (3.3), one obtains the three term recurrence relation

$$\tilde{P}_{n+1} + \tilde{b}_n\tilde{P}_n + \tilde{u}_n\tilde{P}_{n-1} = x\tilde{P}_n, \tag{4.26}$$

with coefficients

$$\tilde{u}_1 = \frac{\phi_1}{\phi_0^2} \qquad \tilde{u}_n = \frac{u_{n-1}B_n}{B_{n-1}}, \quad n = 2, 3, \cdots \tag{4.27}$$

$$\tilde{b}_0 = b_0 + \frac{\phi_1}{\phi_0} \qquad \tilde{b}_n = b_n + B_{n+1} - B_n, \quad n = 1, 2, \cdots \tag{4.28}$$

When we set $a = q^2, q = -e^{\epsilon}, b = -e^{\beta\epsilon}$ and take the limit $\epsilon \to 0$, Eq. (4.26) reduces to

$$\tilde{P}_{n+1}^{(-1)} + \tilde{b}_n^{(-1)}\tilde{P}_n^{(-1)} + \tilde{u}_n^{(-1)}\tilde{P}_{n-1}^{(-1)} = x\tilde{P}_n^{(-1)}. \tag{4.29}$$

Here the coefficients are

$$\tilde{b}_n^{(-1)} = \lim_{\epsilon\to 0}(b_n + B_{n+1} - B_n) = b_n^{(-1)} + \lim_{\epsilon\to 0}(B_{n+1} - B_n) \tag{4.30}$$

$$\tilde{u}_n^{(-1)} = \lim_{\epsilon\to 0}\frac{u_{n-1}B_n}{B_{n-1}} = u_{n-1}^{(-1)}\lim_{\epsilon\to 0}\frac{B_n}{B_{n-1}}, \tag{4.31}$$

where

$$u_n^{(-1)} = -\frac{n(n+2)}{(2n+1+\beta)(2n+3+\beta)}, \quad b_n^{(-1)} = 1 \tag{4.32}$$

when n is even, and

$$u_n^{(-1)} = -\frac{(n+\beta)(n+2+\beta)}{(2n+1+\beta)(2n+3+\beta)}, \quad b_n^{(-1)} = -1 \tag{4.33}$$

when n is odd. Note that

$$\lim_{\epsilon\to 0} B_n = \lim_{q\to -1}\frac{\Phi_n}{\Phi_{n-1}} = \begin{cases} \frac{n+2}{2n+1+\beta}\frac{M-[(3+\beta)(1+\beta)]/[(n+2)(n+1+\beta)]}{M-[(3+\beta)(1+\beta)]/[n(n+1+\beta)]} & \texttt{n even} \\ -\frac{n+2+\beta}{2n+1+\beta}\frac{M-[(3+\beta)(1+\beta)]/[(n+1)(n+2+\beta)]}{M-[(3+\beta)(1+\beta)]/[(n+1)(n+\beta)]} & \texttt{n odd} \end{cases}. \tag{4.34}$$

As a final observation, let us identify the matrix orthogonal polynomials that the even (or odd) part of the $\tilde{P}_n^{(-1)}(x)$ define. Split the polynomials $\tilde{P}_n^{(-1)}(x)$ into its even (E_n) and odd (O_n) parts:

$$\tilde{P}_n^{(-1)}(x) = E_n(x) + O_n(x). \tag{4.35}$$

From the recurrence relation (4.29), we have

$$\begin{aligned} x^2 E_n = E_{n+2} &+ (\tilde{b}_{n+1}^{(-1)} + \tilde{b}_n^{(-1)})E_{n+1} + (\tilde{u}_{n+1}^{(-1)} + \tilde{u}_n^{(-1)} + (\tilde{b}_n^{(-1)})^2)E_n \\ &+ (\tilde{b}_n^{(-1)} + \tilde{b}_{n-1}^{(-1)})\tilde{u}_n^{(-1)}E_{n-1} + \tilde{u}_{n-1}^{(-1)}\tilde{u}_n^{(-1)}E_{n-2}. \end{aligned} \tag{4.36}$$

With the redefinition $E_n = \sigma_n F_n$, $\sigma_n = \sqrt{\tilde{u}_1^{(-1)}\tilde{u}_2^{(-1)}\cdots\tilde{u}_n^{(-1)}}$, it is easy to see that the polynomial F_n satisfies the five-term recurrence relation,

$$x^2 F_n(x) = c_{n,0}F_n + c_{n,1}F_{n-1} + c_{n+1,1}F_{n+1} + c_{n,2}F_{n-2} + c_{n+2,2}F_{n+2} \tag{4.37}$$

where the coefficients are

$$\begin{aligned} &c_{n,0} = (\tilde{u}_{n+1}^{(-1)} + \tilde{u}_n^{(-1)} + (\tilde{b}_n^{(-1)})^2), \quad c_{n,1} = (\tilde{b}_{n-1}^{(-1)} + \tilde{b}_n^{(-1)})\sqrt{\tilde{u}_n^{(-1)}}, \\ &c_{n,2} = \sqrt{\tilde{u}_n^{(-1)}\tilde{u}_{n-1}^{(-1)}}. \end{aligned} \tag{4.38}$$

From the theorem in [28], the matrix polynomials $\{\mathsf{P}_n(x)\}$ defined by

$$\mathsf{P}_n(x) = \begin{pmatrix} R_{2,0}(F_{2n})(x) & R_{2,1}(F_{2n})(x) \\ R_{2,0}(F_{2n+1})(x) & R_{2,1}(F_{2n+1})(x) \end{pmatrix} \tag{4.39}$$

satisfy the matrix three term recurrence relation

$$xP_n(x) = D_{n+1}P_{n+1}(x) + E_nP_n(x) + D_n^*P_{n-1}(x) \tag{4.40}$$

where

$$D_n = \begin{pmatrix} c_{2n,2} & 0 \\ c_{2n,1} & c_{2n+1,2} \end{pmatrix}, \qquad E_n = \begin{pmatrix} c_{2n,0} & c_{2n+1,1} \\ c_{2n+1,1} & c_{2n+1,0} \end{pmatrix}. \tag{4.41}$$

The polynomials $R_{N,m}(p)(x)$ are defined by

$$R_{N,m}(p)(x) = \sum_{n=0}^{K} \frac{p^{(nN+m)}(0)}{(nN+m)!} x^n, \tag{4.42}$$

where K is the integer part of $(\deg(p(x)) - m)/N$

5. Conclusion. To sum up, we have added to the exploration of -1 polynomials by introducing some -1 Krall-Jacobi polynomials. We focused on the simplest positive definite case. They have been obtained from generalized $q-$Jacobi polynomials through a limiting procedure and have remarkable features. Noteworthy is the fact that they obey a third-order differential-difference eigenvalue equation involving the reflection operator. The 3-term recurrence relation has also been determined. Finally let us stress that this presents an interesting example of OPs whose measure involves a discrete mass at the center of the orthogonality interval (rather than at its boundary which is more common).

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