THE SEIBERG-WITTEN EQUATIONS AND FOUR-MANIFOLDS WITH BOUNDARY

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0. Introduction

Let $Z$ be a smooth, compact, oriented 4-manifold. The intersection form of $Z$ is a symmetric bilinear pairing

$$J_Z = H_2(Z; \mathbb{Z})/\text{Torsion} \to \mathbb{Z}.$$  

An early result by Donaldson says that if $Z$ is closed and $J_Z$ is negative definite then $J_Z$ is isomorphic to some diagonal form $(-1) \oplus \cdots \oplus (-1)$. More generally one may ask which negative definite forms can occur if $Z$ is allowed to have some fixed oriented rational homology sphere $Y$ as boundary. The main purpose of the present paper is to apply the equations recently introduced by Seiberg and Witten [W] to prove a finiteness result about the definite forms associated to an arbitrary $Y$.

It is useful to consider the more general situation where the boundary of $Z$ is a disjoint union of rational homology spheres: $\partial Z = Y_1 \cup \cdots \cup Y_l$. (Of course, $\cup_j Y_j$ and $\#_j Y_j$ bound the same intersection forms, since the standard cobordism connecting them has no rational homology in dimension 2.) Let

$$J_Z = m(-1) \oplus \tilde{J}_Z,$$

where $\tilde{J}_Z$ has no elements of square $-1$. Note that

$$|\det(\tilde{J}_Z)| = |\det(J_Z)| \leq \text{order}(H_1(\partial Z; \mathbb{Z})),  \tag{1}$$  

with equality if $H_1(Z; \mathbb{Z})$ is free.

Let $J_Z^\# = H_2(Z, \partial Z)/\text{Torsion}$ be the lattice dual to $J_Z$. A characteristic vector for $J_Z$ is an element $\xi \in J_Z^\#$ such that $\xi \cdot x \equiv x \cdot x \mod 2$ for all $x \in J_Z$. If $J_Z$ is unimodular then $|\xi|^2 \equiv \text{rk}(J_Z) \mod 8$ for any characteristic vector.
vector $\xi$. If $J_Z$ is unimodular and not diagonal then by a theorem of Elkies [E1] it has a characteristic vector $\xi$ satisfying
\[ \text{rk}(J_Z) - |\xi|^2 \geq 8. \]

Moreover, according to [E2] it has a characteristic vector $\xi$ satisfying
\[ \text{rk}(J_Z) - |\xi|^2 \geq 16 \text{ unless } \tilde{J}_Z \text{ is among the finite number of forms listed in [E2].} \]

Our main result is the following theorem:

**Theorem 1.** Let $Z$ be a smooth, compact, oriented 4-manifold whose boundary is a disjoint union of rational homology spheres: $\partial Z = Y_1 \cup \cdots \cup Y_l$. Suppose the intersection form $J_Z$ is negative definite. Then for any characteristic vector $\xi \in J^\#_Z$ we have
\[ \text{rk}(J_Z) - |\xi|^2 \leq \sum_j \gamma(Y_j), \]
where $\gamma(\cdot) \in \mathbb{Q}$ is a certain invariant of oriented rational homology 3-spheres. In particular, if we assume $\tilde{J}_Z$ is even then for a fixed boundary $\cup Y_j$ there are only finitely many possibilities for $\tilde{J}_Z$.

The author does not know whether the finiteness assertion is true in general for odd $\tilde{J}_Z$ (compare the concluding remarks in [E2]).

We will derive Theorem 1 from a slightly stronger result involving the spin$^c$-structures on $Z$; this is given in section 3.

For a closed 4-manifold $Z$ Theorem 1 simply says that $\text{rk}(J_Z) - |\xi|^2 \leq 0$ for any characteristic vector. The proof of this inequality by means of the Seiberg-Witten equations was first found by Kronheimer. When combined with Elkies’ result it provides a new proof of Donaldson’s theorem.

The invariant $\gamma$ is defined in section 3. For the standard 3-sphere we have $\gamma(S^3) = 0$. In section 4 we use Kronheimer’s construction of ALE-spaces as hyperkähler quotients to show that for the Poincaré homology sphere $P$, oriented as the link of the $E_8$ singularity, $\gamma(P) = -\gamma(\bar{P}) = 8$. Since $E_8$ is the only unimodular even positive definite form of rank $\leq 8$ we deduce

**Proposition 2.** Let $Z$ be a smooth, compact, oriented 4-manifold whose boundary is the Poincaré sphere oriented as above. If the intersection form $J_Z$ is negative definite and $\tilde{J}_Z$ is even then either $\tilde{J}_Z = 0$ or $\tilde{J}_Z = -E_8$.

In [F2] we will show that in Proposition 2, $\tilde{J}_Z$ cannot be odd if $Z$ is simply-connected.
1. The Seiberg-Witten equations on a cylinder

Let \( X \) be a smooth, oriented, Riemannian 4-manifold. Choose a spin\(^c\)-structure for \( X \), and let \( W^+ \) and \( W^- \) be the associated spin\(^c\)-bundles. Clifford multiplication defines a (fibrewise) map

\[
T^*X \times W^\pm \to W^\mp
\]

and a linear map

\[
\rho : i\Lambda^2 \to \text{End}_\mathbb{C}(W^+),
\]

whose kernel is \( i\Lambda^- \). Here \( \Lambda^- \subset \Lambda^2 \) is the subbundle of anti-selfdual 2-forms and \( i = \sqrt{-1} \). For \( x \in X \) the image \( \rho(i\Lambda^2_x) \) is the subspace of trace-free Hermitian endomorphisms of \( W^+ \). We write \( L = \det(W^+) \) and let \( \mathcal{A}(L) \) denote the affine space of connections in \( L \).

Let \( \mu \in \Omega^2_X \) be a parameter. We will study the perturbed Seiberg-Witten equations for a pair \( (A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+) \) given by the following pair of equations:

\[
\begin{align*}
D_A \phi &= 0 \\
\rho(F_A + i\mu) &= \frac{1}{2} \phi \otimes \phi^* - \frac{1}{4} |\phi|^2 \cdot 1
\end{align*}
\]

where \( D_A : \Gamma(W^+) \to \Gamma(W^-) \) is the Dirac operator associated to \( A \). The group \( G = \text{Map}(X, U(1)) \) acts on \( \mathcal{A}(L) \times \Gamma(W^+) \) by

\[
u(A, \phi) = (u^2(A), u\phi).
\]

This action preserves the set of solutions to (1.1).

Now consider the special case where \( X = \mathbb{R} \times Y \) for some oriented, closed Riemannian 3-manifold \( Y \), and assume \( \mu \) is the pull-back of a closed form on \( Y \), also denoted \( \mu \). Given a connection \( A \) in \( L \) we can use holonomy in the \( \mathbb{R} \)-direction to identify \( L = \mathbb{R} \times L_0 \), where \( L_0 \) is a complex line bundle over \( Y \). Similarly, \( W^+ = \mathbb{R} \times W_0 \), where \( W_0 \) is the spin\(^c\)-bundle over \( Y \) with respect to the spin\(^c\)-structure inherited from \( \mathbb{R} \times Y \). Clifford multiplication with \( dt \) gives an isomorphism \( W^+ \approx W^- \) and an action of the Clifford bundle of \( Y \) on \( W_0 \). As noted in [KM] the equations (1.1) now become the downward gradient flow equation for the functional \( C : \mathcal{A}(L_0) \times \Gamma(W_0) \to \mathbb{R} \) determined up to a constant by the formula

\[
C(A + a, \phi) = C(A, 0) - \int_Y (F_\mu(A) + \frac{1}{2} da) \wedge a - \frac{1}{2} \int_Y (D_{A+a}(\phi), \phi),
\]
where \( a \in i\Omega^1_Y \) and \( F_\mu(A) = F(A) + i\mu \). Explicitly,

\[
\text{grad}(C)_{(A,\phi)} = (\ast F_\mu(A) - \frac{1}{4}\sigma(\phi,\phi), -D_A\phi),
\]

where \( \sigma : W_0 \times W_0 \to iT^*Y \) is the symmetric bilinear pairing satisfying

\[
\langle a, \sigma(\phi, \psi) \rangle = \langle a\phi, \psi \rangle
\]

for all \( a \in i\Omega^1_Y, \phi, \psi \in \Gamma(W_0) \). In general we write \( \langle , \rangle \) for Euclidean inner products and \( \langle , \rangle_C \) for Hermitian inner products. If \( u : Y \to U(1) \) is a map and \( z = u^*[U(1)] \in H^1(Y;\mathbb{Z}) \) then we find

\[
C(u(A,\phi)) - C(A,\phi) = -(2\pi c_1(L_0) + [\mu]) \cup 4\pi z.
\]

In the remainder of this section let \( Y \) be a rational homology sphere. In this case \( C \) descends to a map \( B = (A(L_0) \times \Gamma(W_0))/G \to \mathbb{R} \). Also, we can take \( \mu = d\nu \) for some \( \nu \in \Omega^1 \) and we write \( C_\nu \) for the corresponding functional on \( B \). Let \( \mathcal{R} \subset B \) be the set of (equivalence classes of) critical points of \( C \), which is a compact set (this follows for instance from Corollary 3 in [KM]). Let \( \theta \in \mathcal{R} \) be the unique critical point with zero spinor field, and set \( \mathcal{R}^* = \mathcal{R}\setminus\{\theta\} \). When convenient we will consider the completions of \( A(L_0) \times \Gamma(W_0) \) and \( G \) in the \( L^2_1 \) and \( L^2_2 \)-metrics, respectively. Then \( \mathcal{B}^* \subset \mathcal{B} \), the subset of elements with spinor field not identically zero, is a Hilbert manifold.

For any critical point \((A,\phi)\) of \( C \) let \( H_{(A,\phi)} : i\Omega^1 \times \Gamma(W_0) \to i\Omega^1 \times \Gamma(W_0) \) be the derivative of \( \text{grad}(C) \) at \((A,\phi)\), i.e.

\[
H_{(A,\phi)}(a,\psi) = (\ast da - \frac{1}{2}\sigma(\phi,\psi), -\frac{1}{2}a\phi - D_A\psi).
\]

Also, let \( \lambda_\phi : i\Omega^0 \to i\Omega^1 \times \Gamma(W_0) \) be the infinitesimal action of \( G \) on \((A,\phi)\), namely

\[
\lambda_\phi(\zeta) = (-2d\zeta, \zeta\phi).
\]

We say \((A,\phi)\) is a non-degenerate critical point if the middle cohomology group of the following complex is zero:

\[
(1.2) \quad i\Omega^0 \xrightarrow{\lambda_\phi} i\Omega^1 \times \Gamma(W_0) \xrightarrow{H_{(A,\phi)}} i\Omega^1 \times \Gamma(W_0).
\]

A standard elliptic argument shows that a non-degenerate critical point is isolated in \( \mathcal{R} \).
Proposition 3. There is a Baire set of forms $\nu \in \Omega_{C_k+1}^1$ for which all critical points of $C_\nu$ are non-degenerate.

Proof. Let

$$\tilde{E} = \{(A, \phi, a, \psi) \in \mathcal{A}(L_0) \times \Gamma(W_0) \times i\Omega^1 \times \Gamma(W_0) \mid \lambda_a^*(a, \psi) = 0\}. $$

Then $E = \tilde{E}/G$ is a vector-bundle over $\mathcal{B}^*$ and $\text{grad} C_\nu$ defines a smooth section $s_\nu$. As we will see in a moment, a critical point of $C_\nu$ is non-degenerate precisely when it is a regular zero of $s_\nu$.

We can regard the family of sections $s = \{s_\nu\}$ as a section in the bundle $\Omega^1 \times E \to \Omega^1 \times \mathcal{B}^*$. Writing $V = \ker(\lambda_a^*) \subset i\Omega^1 \times \Gamma(W_0)$ the intrinsic derivative of $s$ at a zero-point $[\nu, A, \phi]$ can be identified with

$$Ds : \Omega^1 \times V \to V, \quad (\nu, a, \psi) \mapsto (i * d\nu, 0) + H_{(A, \phi)}(a, \psi).$$

We will show that $Ds$ is surjective. Suppose $(a, \psi) \in V$ is orthogonal to $\text{im}(Ds)$. This means the following four equations are satisfied:

$$(i) \ da = 0; \quad (ii) \ \sigma(\phi, \psi) = 0$$

$$(iii) \ -2d^*a + \langle i\phi, \psi \rangle i = 0; \quad (iv) \ \frac{1}{2}a\phi + D_A\psi = 0.$$ 

Note first that these equations imply that $a$ and $\psi$ are smooth. From (ii) we see that on the complement of $\phi^{-1}(0)$ we have

$$\psi = ir\phi$$

for some smooth function $r : Y \setminus \phi^{-1}(0) \to \mathbb{R}$. Combining this with (iv) we deduce

$$-\frac{1}{2}a\phi = D_A(\psi) = idr \cdot \phi,$$

which implies $-\frac{1}{2}a = idr$ where $\phi \neq 0$. Now, since $\phi$ satisfies the Dirac equation we can argue as in [FU, pp. 57-58] to show that $Y \setminus \phi^{-1}(0)$ is connected. By (i) and the assumption $H^1(Y; \mathbb{R}) = 0$ it then follows that $r$ can be smoothly extended to all of $Y$. Inserting (1.3) into (iii) we get

$$(1.4) \quad 4d^*dr + r|\phi|^2 = 0,$$

so

$$0 \geq \int_Y \langle d^*dr, r \rangle = \int_Y |dr|^2,$$
whence $a = -2idr = 0$. (1.4) and (1.3) then gives $\psi = 0$. Therefore the intrinsic derivative $D_s$ is surjective at any zero.

Since $Ds_\nu$ is Fredholm at every zero of $s_\nu$ it follows from the Sard-Smale theorem that for a Baire set of parameters $\nu \in \Omega_{C^k+1}^1$, the zero-set of $s_\nu$ is regular.

Finally, we must show that the critical point with zero spinor field is generically non-degenerate. Let $A$ be a smooth flat connection in $L_0$. Then $(A - i\nu, 0)$ is a critical point of $C_\nu$ which is non-degenerate if and only if ker$(D_{A - i\nu}) = 0$. Consider the vector bundle $E \to \Omega^1 \times \Gamma(W_0) \setminus 0$ whose fibre over $(\nu, \phi)$ is the (real) $L^2$-orthogonal complement $(i\phi)^\perp \subset \Gamma(W_0)$. Then $s(\nu, \phi) = D_{A - i\nu}(\phi)$ is a smooth section $C^{k+1} \times L_1^2 \to L^2$. Arguing as above one finds that the zero-set of $s$ is regular. Hence for a Baire set of parameters $\nu$ the real dimension of ker$(D_{A - i\nu}) \setminus 0$ is equal to

$$\text{ind}_R(p_\phi \circ D_{A - i\nu}) = 1,$$

where $p_\phi : L^2(W_0) \to L^2(W_0)$ is projection onto $(i\phi)^\perp$. Since $D_{A - i\nu}$ is complex linear we must have ker$(D_{A - i\nu}) = 0$ for such $\nu$. \hfill \Box

For any pair $(A, \phi)$ consider the self-adjoint elliptic operator

$$P_{(A, \phi)} = \begin{pmatrix} 0 & \frac{1}{2} \lambda \phi^* \\ \frac{1}{2} \lambda \phi & H_{(A, \phi)} \end{pmatrix},$$

acting on sections of $E_0 = i\Lambda^0 \oplus (i\Lambda^1 \oplus W_0)$. If $D_A \phi = 0$ then (1.2) is a complex, with cohomology group $\mathcal{H}^*_0(A, \phi)$ say, and

$$\text{ker}(P_{(A, \phi)}) = \mathcal{H}^0_{(A, \phi)} \oplus \mathcal{H}^1_{(A, \phi)}.$$

For any non-degenerate critical points $\alpha, \beta$ we define a relative index $i(\alpha, \beta)$ as follows. Let $(A_t, \phi_t) \subset A(L_0) \times \Gamma(W_0)$ be a smooth path which is constant in $t$ outside some interval $(t_-, t_+)$, and represents $\alpha$ for $t \leq t_-$ and $\beta$ for $t \geq t_+$. Choose $\lambda > 0$ such that neither of the operators $P_\alpha$ and $P_\beta$ has any eigenvalue in the interval $(0, \lambda)$. Let $E = \pi_2^*(E_0)$, a bundle over $\mathbb{R} \times Y$, and define $i(\alpha, \beta)$ to be the index of the Fredholm operator

$$\frac{\partial}{\partial t} + P_{(A_t, \phi_t)} + \lambda : L^2_1(E) \to L^2(E).$$

This index is the same as the spectral flow of the family of operators $\{P_{(A_t, \phi_t)} + \lambda\}$, see [APS].
2. Moduli spaces

Consider again the set-up of the previous section, with $X = \mathbb{R} \times Y$ and $Y$ an oriented rational homology 3-sphere. Fix $\nu \in \Omega^1(Y)$ such that all critical points of $C = C_\nu$ are non-degenerate. For each pair $\alpha, \beta$ of critical points we shall define a moduli space $M(\alpha, \beta) = M(\mathbb{R} \times Y; \alpha, \beta)$ which will be a perturbed version of the space of gradient lines connecting $\alpha$ and $\beta$. In the language of finite-dimensional Morse theory our approach is somewhat analogous to perturbing the gradient vector field away from the critical points.

To define these perturbations let $\eta_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth function supported in $[-1, 1]$ and satisfying $\int \eta_1 = 1$, and let $\eta_2 : \mathbb{R} \to \mathbb{R}$ be a smooth, compactly supported function such that $\eta_2(t) = t$ on some interval containing all critical values of $C$. If $A$ is any connection in $L$ and $\phi$ a section of $W^+$ let $S = (A, \phi)$ and define a smooth function $h_S : \mathbb{R} \to \mathbb{R}$ by

$$h_S(T) = \int_{\mathbb{R}} \eta_1(t_1 - T) \eta_2(\int_{\mathbb{R}} \eta_1(t_2 - t_1)C(S_{t_2}) dt_2) dt_1,$$

where $S_t = S(t)$ is the restriction of $S$ to $\{t\} \times Y$. Note that $h_S$ is bounded with all its derivatives independently of $S$, and the association $S \mapsto h_S$ is a $C^\infty$-map $L^p_1 \to C^k$ for all $k$ (and $p \geq 2$, say).

Let the parameter $\omega \in \Omega^2(\mathbb{R} \times Y)$ have compact support contained in a set $\Xi \times Y$, where $\Xi$ is the result of removing from $\mathbb{R}$ a small open interval around each critical value of $C$. Let $h_S^\omega(\omega)$ denote the pull-back of $\omega$ by the smooth map $h_S \times \mathrm{id}_Y : \mathbb{R} \times Y \to \mathbb{R} \times Y$. We will use the following translationary invariant equations for $S = (A, \phi)$:

$$D_A\phi = 0$$

$$\rho(F_A + i\pi_2^*(d\nu) + i h_{(A, \phi)}^*(\omega)) = \frac{1}{2} \phi \otimes \phi^* - \frac{1}{4} |\phi|^2 \cdot 1,$$

where $\pi_2 : \mathbb{R} \times Y \to Y$ is the projection. If $S = (A, \phi)$ is in temporal gauge then the equations take the form

$$\frac{\partial S_t}{\partial t} = -\mathrm{grad}(C_\nu)_{S_t} + E_S(t),$$

where the perturbation term $E_S(t)$ depends on $S_{|(t-1,t+1) \times Y}$ and vanishes if $h_S(t) \not\in \Xi$.

Lemma 4. If $b$ is any constant then for $\|\omega\|_{C^k}$ sufficiently small the following holds. Let $S = (A, \phi)$ be any smooth solution to the equations (2.1) satisfying a pointwise bound $|\phi| \leq b$. Then

1. Either $\frac{\partial}{\partial t} C(S_t) < 0$ for all $t$, or $[S_t] \equiv \alpha$ for some critical point $\alpha$. 
(2) If \( C(S_t) \) is bounded in \( t \) then there are critical points \( \alpha_+, \alpha_- \) of \( C \) such that the gauge equivalence class \([S_t]\) converges in \( C^k \) to \( \alpha_\pm \) as \( t \to \pm \infty \).

The proof is a compactness and unique continuation argument and is deferred to an appendix.

For any critical points \( \alpha, \beta \) of \( C \) we define, as a set,

\[
M(\alpha, \beta) = \{ S = (A, \phi) \text{ satisfying (2.1)} \mid \lim_{t \to -\infty} [S_t] = \alpha; \lim_{t \to \infty} [S_t] = \beta \}/\text{Gauge},
\]

where the limits refer to the \( C^k \)-topology. Note that the elements of \( M(\alpha, \beta) \) satisfy the gradient flow equation for \( C \) outside a compact subset of \( \mathbb{R} \times Y \).

It was proved in [KM] that if \((A, \phi)\) is a solution to (1.1) then at points where \(|\phi|\) has a local maximum one has

\[
|\phi|^2 \leq \max(0, 4|\rho(i\mu)| - 2s),
\]

where \( s \) is the scalar curvature of the underlying 4-manifold. It follows that if \( \|\omega\|_{C^0} \leq 1 \), say, there is a constant \( b \) such that for any critical points \( \alpha, \beta \) and any \((A, \phi) \in M(\alpha, \beta)\) we have a pointwise bound \(|\phi| \leq b\). In the following we will assume \( \omega \) chosen so that the conclusions of Lemma 4 hold for this constant \( b \).

It is convenient at this stage to introduce suitable function spaces. Let \( \alpha, \beta \) be critical points of \( C \). Let \( v > 0 \) be smaller than the first positive eigenvalue of \( P_\alpha \), and let \( w > 0 \) be smaller than the first positive eigenvalue of \( P_\beta \). Choose a connection \( A_0 \) in \( L \) and a section \( \phi_0 \) of \( W^+ \) such that \( S_0 = (A_0, \phi_0) \) is in temporal gauge outside a compact set and satisfies \([S_t] = \alpha\) for \( t \ll 0 \) and \([S_t] = \beta\) for \( t \gg 0 \). Let \( 2 < p < 4 \) and set

\[
\mathcal{C} = \mathcal{C}(\alpha, \beta) = S_0 + L^p_{m}(i\Lambda^1 \times W^+).
\]

Here \( L^p_{m} \) Sobolev space defined using a weight \( \exp(-vt) \) on the negative end and \( \exp(\omega t) \) on the positive end.

For the group of gauge transformations we take

\[
\mathcal{G} = \{ u : \mathbb{R} \times Y \to U(1) \mid u \in L^p_{2, \text{loc}}; du \cdot u^{-1} \in L^p_{1, \text{conv}} \}. \]

Then \( \mathcal{G} \) acts on \( \mathcal{C} \), and if \( \mathcal{C}^* \subset \mathcal{C} \) is the subset of elements with spinor field not identically zero then \( \mathcal{C}^*/\mathcal{G} \) is a Banach manifold.

There is now a natural identification

\[
(2.3) \quad M(\alpha, \beta) = \{ (A, \phi) \in \mathcal{C}(\alpha, \beta) \text{ satisfying (2.1)} \}/\mathcal{G}.
\]
The existence of a natural injective map from right to left in (2.3) follows from the usual elliptic techniques. To see that this map is onto one can use the non-degeneracy of the critical points of $C$ to prove an exponential decay result for solutions in temporal gauge.

Let $M(\alpha, \beta) \subset \mathcal{C}/\mathcal{G}$ have the subspace topology.

The moduli spaces $M(\alpha, \beta)$ have compactness properties familiar from finite-dimensional Morse theory. If $S_n = (A_n, \phi_n) \in M(\alpha, \beta)$ is any sequence then we can fix gauge by requiring $d^*(A_n - A_1) = 0$. Since $|\phi_n|$ is uniformly bounded we find as in [KM] that a subsequence of $\{S_n\}$ converges in $C^k$ over compact subsets to some solution $S_\infty$ of the equations (2.1). By Lemma 4 this limit lies in some moduli space $M(\alpha', \beta')$ with $C(\alpha) \geq C(\alpha') \geq C(\beta') \geq C(\beta)$. (If $[S_\infty] \in M(\alpha, \beta)$ then the convergence is global.) Now let $C(\alpha) = x_0 > x_1 > \cdots > x_m = C(\beta)$ be the critical values of $C$ in the interval $[C(\beta), C(\alpha)]$ and choose $y_j \in (x_j, x_{j-1})$. Let $S_n^j$ be the translate of $S_n$ with $C(S_n^j) = y_j$. Then a subsequence of $\{S_n^j\}$ converges over compacta to an element of some moduli space $M(\alpha_j, \beta_j)$, and by continuity we must have $\beta_{j-1} = \alpha_j$.

**Proposition 5.** For generic small $\omega \in C^k$, each moduli space $M(\alpha, \beta)$ is a smooth finite-dimensional manifold of dimension

$$\dim(M(\alpha, \beta)) = -d_\alpha + i(\alpha, \beta),$$

where $d_\alpha = 1$ if $\alpha = \theta$ and $d_\alpha = 0$ otherwise.

**Proof.** By Lemma 4, for small $\omega$ all moduli spaces $M(\alpha, \alpha)$ will consist of a single point, so we may assume $\alpha \neq \beta$. Let $\Omega^2_\Xi$ be the space of parameters $\omega$, i.e. the space of $C^k$ 2-forms on $\mathbb{R} \times Y$ with compact support contained in $\Xi \times Y$. We assume $\omega$ is small in the sense spelled out in the previous section. The linearization of the equations (2.1) at a point $(\omega, A, \phi)$, taking into account the Coulomb gauge fixing condition, is the operator

$$T = T(\omega, A, \phi) : \Omega^2_\Xi \times L^p_{1}(i\Lambda^1 \oplus W^+) \rightarrow L^p_{1}(i\Lambda^0 \oplus i\Lambda^+ \oplus W^-)$$

which takes $(\kappa, a, \psi)$ to

$$(\lambda^*_\phi(a, \psi), d^+ a + i(h^*_r(\kappa)) + K(a, \psi) - \frac{1}{2}(dt \wedge \sigma(\phi, \psi))^+, \frac{1}{2}a\phi + D_A(\psi)).$$

Here $K = K(\omega, A, \phi)$ is a bounded operator $L^p_1 \rightarrow C^k$ satisfying

$$\text{supp}(K(a, \psi)) \subset h^{-1}_{(A, \phi)}(\Xi) \times Y.$$
We must first verify that if we fix \( \omega \) then the linearization at a point \((A, \phi)\) is Fredholm. For this purpose we may assume \((A, \phi)\) is smooth and in temporal gauge. Identify \( \pi^*_{\Lambda_0^0 \oplus \Lambda_1^1} = \Lambda_0^1 \Lambda_1^1 \) and \( \pi^*_{\Lambda_1^1} = \Lambda_0^1 \Lambda_1^1 \), the latter by means of the map \( \omega \mapsto dt \wedge \omega + *Y\omega \). Let

\[
E = \pi^*_{\Lambda_0^0 \oplus i\Lambda_1^1 \oplus W_0}.
\]

Then up to scalar factors the operator \( T' : (a, \psi) \mapsto T(0, a, \psi) \) can be written as

\[
T' = \frac{\partial}{\partial t} + P_t + K : L^{p,v,w}_t(E) \to L^{p,v,w}(E),
\]

where \( P_t = P(A_t, \phi_t) \) is the operator over \( Y \) defined in the previous section. Our choice of weights for the Sobolev spaces then implies that \( T' \) is Fredholm of index \(-d \alpha + i(\alpha, \beta)\).

Secondly, we must show that \( T = T_\omega(A, \phi) \) is surjective whenever \((\omega, A, \phi)\) is a solution to the equations (2.1). Let \((\zeta, a', \psi') \in L^{q,v,-w}_t \cdot \mathcal{L}^2 \cdot \mathcal{L}_v \cdot \mathcal{L}_w \cdot \mathcal{L}_t \) be \( L^2 \)-orthogonal to \( \text{im}(T) \), where \( q \) is the exponent conjugate to \( p \). Because either \( \alpha \) or \( \beta \) must have non-zero spinor field, \( \phi \) is not identically zero. Hence \( \lambda_1^\phi \lambda_1^\phi \) is an isomorphism \( L^{p,v,w}_t \to L^{p,v,w}_t \), so \( \zeta = 0 \). By Lemma 4 we have

\[
\frac{\partial}{\partial t} h(\lambda_1^\phi \lambda_1^\phi)(t) < 0
\]

for all \( t \), so by varying \( \kappa \) alone we see that \( a' \) vanishes on \( h^{-1}_1(\Xi) \times Y \).

By varying \( a \) alone we conclude as in [KM] that \( \psi' \) must also vanish on \( h^{-1}_1(\Xi) \times Y \). But then \((a', \psi')\) lies in the kernel of the operator

\[
\left( \frac{\partial}{\partial t} + P_t \right)^* = -\frac{\partial}{\partial t} + P_t,
\]

so by unique continuation, \((a', \psi')\) is identically zero. This shows \( T \) is surjective.

The discussion above carries over to 4-manifolds with tubular ends in a natural way. Let \( Y_1, \ldots, Y_l \) be oriented rational homology spheres, each with a Riemannian metric, a spin\(^c\)-structure, a good perturbation of the functional \( C_j \) (as in Proposition 3), and some form \( \omega_j \) defining a perturbation of the equations over \( \mathbb{R} \times Y_j \). Let \( X \) be an oriented Riemannian 4-manifold with an open subset isometric to \( \mathbb{R}^+ \times Y_j \) for each \( j \), such that the interior \( X_0 = X \setminus \bigcup_j (\mathbb{R}^+ \times Y_j) \) is compact. Let \( X \) have a spin\(^c\)-structure compatible with those on the ends. For any form \( \mu \in \Omega^2_c(X) \) we obtain equations on \( X \) for a pair \((A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)\) by combining the equations (1.1) on \( X_0 \) with (2.1) on the ends, using a suitable cut-off function. For any vector \( \underline{\alpha} = \{\alpha_j\} \), with \( \alpha_j \) a critical point of \( C_j \), let

\[
M(X; \underline{\alpha})
\]
be the space of solutions to these equations over $X$ which are asymptotic to $\alpha_j$ on the $j$'th end, modulo gauge. For generic $\mu$ all moduli spaces $M(X; \alpha)$ will be smooth away from points with zero spinor field.

There is also an analogue of Lemma 4 for solutions to the perturbed equations over $X$. In particular, for small $\omega_j$ we may assume that for every $[S] \in M(X; \alpha)$ and each end of $X$, either $\frac{\partial}{\partial t} C_j(S_t) < 0$ for all (non-negative) $t$, or $[S_t] \equiv \alpha_j$.

3. The main result

We first define the invariant $\gamma(Y, c)$ for an oriented rational homology 3-sphere $Y$ with spin$^c$-structure $c$. Choose a Riemannian metric $g$ on $Y$ and a perturbation $\nu$ for which the functional $C = C_\nu$ has only non-degenerate critical points. Let $A$ be a connection in $L_0$ satisfying $F_A + id\nu = 0$, i.e. $(A, 0)$ is a critical point of $C_\nu$. The non-degeneracy condition for $(A, 0)$ means that the Dirac operator $D_A$ has zero kernel. Let $V$ be any simply-connected oriented Riemannian 4-manifold with one tubular end $\mathbb{R}_+ \times Y$. Choose an extension of the spin$^c$-structure on $\mathbb{R}_+ \times Y$ to all of $V$ and let $L \to V$ be the associated $U(1)$-bundle. Let $A$ be a smooth connection in $L$ such that on the end, $A = \pi_2^2(A)$. The Dirac operator $D_A$ defines a Fredholm operator $L^2_+ (W^+) \to L^2_-(W^-)$.

Let $m$ be the smallest non-negative integer such that $C$ has no critical points $\alpha$ with $i(\alpha, \theta) = 2m + 1$, where $\theta$ is the critical point with zero spinor field. Then define

$$\gamma(Y, c, g, \nu) = 8m + c_1(L)^2 - \sigma(V) - 8 \text{ind}_c(D_A),$$

where $\sigma(V)$ is the signature of $V$. Note that on a closed spin$^c$ 4-manifold $\tilde{V}$ the index of the Dirac operator is precisely $\frac{1}{8}(c_1(\tilde{L})^2 - \sigma(\tilde{V}))$. To see that $\gamma(Y, c, g, \nu)$ depends only on $(Y, c, g, \nu)$, use the additivity of the index of the Dirac operator over $\tilde{V} = V_0 \cup Y V'_0$.

Now define

$$\gamma(Y, c) = \inf_{g, \nu} \gamma(Y, c, g, \nu); \quad \gamma(Y) = \max_c \gamma(Y, c).$$

If $Y$ is an integral homology sphere then $c_1(L)^2 \equiv \sigma(V)$ (8), whence $\gamma(Y) \in 8\mathbb{Z}$.

**Theorem 6.** Let $Z$ be any smooth, compact, oriented 4-manifold whose oriented boundary is a disjoint union of rational homology spheres: $\partial Z = Y_1 \cup \cdots \cup Y_l$. Suppose the intersection form of $Z$ is negative definite and $b_1(Z) = 0$.!
Given any spin$^c$-structure on $Z$, let $L$ be the associated $U(1)$-bundle on $Z$ and $c_j$ the induced spin$^c$-structure on $Y_j$. Then
\[c_1(L)^2 - \sigma(Z) \leq \sum_j \gamma(Y_j, c_j),\]
where $\sigma(Z)$ is the signature of $Z$.

Note that applying Theorem 6 with $Z = [0, 1] \times Y$ gives
\[(3.1) \quad 0 \leq \gamma(Y, c) + \gamma(\bar{Y}, \bar{c}),\]
so $\gamma(Y, c)$ is always finite.

**Proposition 7.** Let $Y$ be any oriented rational homology 3-sphere which admits a metric with positive scalar curvature. If $c$ is any spin$^c$-structure on $Y$ and $\bar{c}$ the corresponding spin$^c$-structure on $\bar{Y}$ then
\[\gamma(\bar{Y}, \bar{c}) = -\gamma(Y, c).\]

**Proof.** If $Y$ has a metric $g$ with positive scalar curvature then for $\nu = 0$ the functional $C_\nu$ has only one critical point, namely the one with zero spinor field, and this is non-degenerate. Choose an embedding of $Y$ in a smooth, compact, simply-connected, oriented 4-manifold: $\bar{V} = V_1 \cup_{Y} V_2$. Stretching $\bar{V}$ along $Y$ we obtain
\[0 = \gamma(Y, c, g, \nu) + \gamma(\bar{Y}, \bar{c}, g, \nu),\]
so
\[\gamma(Y, c) \leq \gamma(Y, c, g, \nu) = -\gamma(\bar{Y}, \bar{c}, g, \nu) \leq -\gamma(\bar{Y}, \bar{c}).\]
Combining this with (3.1) proves the proposition. \qed

**Proof of Theorem 1 assuming Theorem 6.** Let $Z$ be as in Theorem 1. As noticed in [FS], by performing surgery on a set of generators for the free part of $H_1(Z; \mathbb{Z})$ we obtain a new 4-manifold $Z'$ with $b^1 = 0$ and with the same intersection form. If $\xi \in J_Z^#$ is any characteristic element then we can find a spin$^c$-structure on $Z'$ with associated $U(1)$-bundle $L$ such that $c_1(L)$ is Poincaré dual to $\xi$ modulo torsion. Since $\gamma(Y) = \max_c \gamma(Y, c)$, Theorem 1 follows from Theorem 6. \qed

**Proof of Theorem 6.** Choose a metric $g_j$ and a good perturbation $\nu_j$ on $Y_j$. We will show that the assumption
\[(3.2) \quad c_1(L)^2 - \sigma(Z) > \sum_j \gamma(Y_j, c_j, g_j, \nu_j)\]
leads to a contradiction. Let $X$ be the manifold obtained by adding a half-tube $\mathbb{R}_+ \times Y_j$ to each boundary component of $Z$. Extend the product metrics on the ends to all of $X$. Let $M = M(X; \theta)$, where $\theta = \{\theta_j\}$. If $D_A$ is the Dirac operator on $X$ then $\dim M = \operatorname{ind}_R(D_A) - 1$. The assumption (3.2) is therefore equivalent to

$$\dim M + 1 > \sum 2m_j,$$

where $m_j$ is the integer entering in the definition of $\gamma(Y_j)$. Let $2m = \dim M - 1 - \sum 2m_j$, which is non-negative since $M$ has odd dimension.

If the forms $\omega_j$ defining perturbations on the tubes are zero then $M$ has a unique reducible point $P$ (i.e. one with zero spinor field), since $X$ is negative definite. Moreover, as a reducible solution, $P$ is regular. So for small non-zero $\omega_j$, $M$ will still contain a unique reducible point $P$. After perturbing $M$ in a neighbourhood of $P$ (along the lines of [D1]) if necessary, we may assume $P$ is regular as a point in $M$. $P$ will then have an open neighbourhood $U \subset M$ homeomorphic to an open cone on $\mathbb{CP}^d$ for some $d \geq 0$. Let $M^* = M \{P\}$ and $M^0 = M \setminus U$.

Let $B \subset X$ be a compact ball and let $B^*_B$ be the orbit space of pairs $(A, \phi)$ defined over $B$ with $\phi \neq 0$, modulo gauge. (Thus the $A$’s are connections in $L|_B$ and the $\phi$’s are sections in $W^+|_B$.) Suitably completed, $B^*_B$ is a Hilbert manifold. Let $L_B \to B^*_B$ be the canonical $U(1)$-bundle (i.e. the base-point fibration).

For any $\alpha$ let

$$r : M^*(X; \alpha) \to B^*_B$$

be the restriction map, where the star in $M^*$ means we remove the singular point if $\alpha = \theta$. Thus the domain of $r$ is the union of all moduli spaces $M^*(X; \alpha)$. Note that $r$ really maps into $B^*_B$: for a solution to the Dirac equation cannot vanish on an open set of $X$ unless it is identically zero.

Let $s$ be a generic section of the $m$-fold direct sum $mL$. By [DK, Lemma 5.2.9] we may assume each restriction map $r$ is transverse (on each moduli space) to $s^{-1}(0)$, so each stratum of $V = r^{-1}s^{-1}(0)$ is smooth of the correct dimension.

In a moduli space $M^*(X; \alpha)$ with $\alpha_j = \theta_j$ there is also another source of cohomology. To explain this, let $B^*_j$ be the orbit space of pairs $(A, \phi)$ (with $\phi \neq 0$) defined over $I_j = [0, 1] \times Y_j$. We will define a smooth map $r_j : M^*(X; \alpha) \to B^*_j$. Choose $\epsilon$ such that for each $j$, $C_j$ has no critical value in the interval $(C_j(\theta_j), \epsilon]$. Fix $T \gg 0$ and define $\bar{t}_j : M(X; \alpha) \to \mathbb{R}$ implicitly by

$$\bar{t}_j(S) = T \quad \text{if} \quad C_j(S_T) \leq \epsilon$$

$$C_j(S_{\bar{t}_j(S)}) = \epsilon \quad \text{if} \quad C_j(S_T) \geq \epsilon,$$
where $S_t = S|_{\{t\} \times Y_j}$. Let $t_j = f \circ \bar{t}_j$, where $f$ is a smooth function satisfying $f(x) = T$ for $x \leq T + 1$, $f(x) = x$ for $x \geq T + 2$, and $f' \geq 0$. Then $t_j$ is smooth. We now define $r_j$ by

$$r_j([S]) = [S|_{t_j(S), t_j(S)+1} \times Y_j].$$

Similarly, for any critical point $\alpha \neq \theta_j$ set

$$M^*(\alpha, \theta_j) = \{[S] \in M(\alpha, \theta_j) \mid C_j(S|_{\{0\} \times Y_j}) = \epsilon\}$$

and define a smooth map $r_j : M^* \to B_j^*$ by

$$r_j([S]) = [S|_{[0,1] \times Y_j}].$$

Now let $L_j \to B_j^*$ be the canonical complex line bundle and $s_j$ a generic section of $m_j L_j$. We may assume $r_j$ is transverse (on each moduli space where it is defined) to $s_j^{-1}(0)$. Let $V_j = r_j^{-1} s_j^{-1}(0)$. We can arrange that all intersections of strata in $V, V_1, \ldots, V_l$ are transverse, and similarly for the intersection $\mathbb{CP}^d \cap V \cap (\cap V_j)$ in $M^* = M^*(X; \theta)$. Now set

$$U = M^0 \cap V \cap (\cap V_j).$$

We claim $U$ is compact. For $U$ has dimension 1, so the standard compactification of $U$ consists of $U$ itself together with “broken solutions” with exactly two factors. The codimension 1 strata resulting from factorization on the $j$’th end have the form

$$(M(X; \alpha) \cap V \cap (\cap V_i)) \times (M^*(\alpha_j, \theta_j) \cap V_j),$$

with $\alpha_j \neq \theta_j$ and $\alpha_i = \theta_i$ for $i \neq j$. For reasons of dimension we must have $i(\alpha, \theta_j) = 2m_j + 1$. However, by definition of $m_j$ there is no such $\alpha_j$, hence $U$ is compact.

Now, it is easy to see that $r^*(L_j)$ and $r_j^*(L_j)$ restrict to the tautological line bundle on $\mathbb{CP}^d$. But then $\partial U = \mathbb{CP}^d \cap V \cap (\cap V_j)$ has one point counted mod 2, contradicting the fact that $U$ is a compact 1-manifold-with-boundary. Therefore (3.2) is impossible, so the theorem is proved. 

**Remark.** Our main object in this paper is to give a simple proof of Theorem 1. However, when studying concrete examples it may be of interest to examine the proof above to see if one can obtain a lower bound on $\text{rk}(J_Z) - |\xi|^2$. We will do this in [F2] and show that Theorems 1 and 6 hold for an invariant $\gamma$ defined in terms of certain distinguished elements of the (metric dependent) Seiberg-Witten-Floer cohomology groups (with coefficients in some fixed field) rather than the critical points of $C$. These distinguished elements measure interaction with the critical point $\theta$ and can be identified with the differentials in a spectral sequence for computing the equivariant Seiberg-Witten-Floer group ([AB1], [AB2], [F1]).
4. Binary polyhedral spaces

Let $\Gamma \subset SU(2)$ be a finite subgroup and $D \subset \mathbb{C}^2$ the closed unit ball. Consider the rational homology sphere $Y = \partial D / \Gamma$. Let $X$ be the smooth 4-manifold underlying the minimal resolution of the quotient singularity $\mathbb{C}^2 / \Gamma$. The portion $Z \subset X$ lying above $D / \Gamma$ is a compact 4-manifold with boundary $Y$. Recall that the intersection form $J_Z$ is a root lattice. In particular, if $\Gamma$ is the binary icosahedral group, in which case $Y$ is the Poincaré sphere, then $J_Z = -E_8$.

Proposition 8. Let $c$ be the spin-structure that $Y$ inherits from the unique spin-structure on $Z$. Then $\gamma(Y, c) = \text{rk}(J_Z)$.

Proof. Theorem 6 gives $\text{rk}(J_Z) \leq \gamma(Y, c)$. We must prove that $\text{rk}(J_Z) \geq \gamma(Y, c)$. The universal covering space $\tilde{V}$ of $V = X \setminus Z$ can be naturally identified with $\mathbb{C}^2 \setminus D$ as a smooth manifold. Let $x_i$ be euclidean coordinates on $\tilde{V}$ and $r = \sum x_i^2$. By [K1] there is a hyperkähler structure on $X$ whose metric $g$ satisfies

\begin{equation}
\partial^p (g^{ij} - \delta^{ij}) = o(r^{-4-p}), \quad p \geq 0
\end{equation}

on $V$, where $\delta$ is the euclidean metric and $\partial$ denotes differentiation with respect to the coordinates $x_i$. Extend the metric $\delta$ to all of $X$, let $u : X \to \mathbb{R}_+$ be a smooth function equal to $r^{-1}$ outside $Z$, and set $\tilde{g} = u^2 g, \tilde{\delta} = u^2 \delta$. Then $(V, \tilde{\delta}|_V)$ is isometric to $\mathbb{R}_+ \times Y$, with $t = \log(r)$ the coordinate on $\mathbb{R}_+$.

Since $X$ is simply-connected and has even intersection form it has a unique spin structure. Let $D^\delta : \Gamma(W^+) \to \Gamma(W^-)$ be the Dirac operator on $(X, \tilde{\delta})$. Recall that $Y$ has positive scalar curvature, so $D^\delta$ defines a Fredholm operator $L^2_1 \to L^2$. We will show that $\text{ind}(D^\delta) = 0$. By definition of $\gamma(Y, c)$ this will imply

$\gamma(Y, c) \leq -\sigma(Z) = \text{rk}(J_Z)$

and the proposition will be proven.

We wish to compare the Dirac operators $D^{\tilde{g}}$ and $D^\delta$. Since taking the square root of a positive real symmetric matrix is a smooth operation, there is a natural choice of an endomorphism of $TX$ which intertwines $\tilde{g}$ and $\delta$. This allows us to identify the frame bundles of $(X, \tilde{g})$ and $(X, \delta)$, hence also their spin bundles. We choose $\delta$ as reference metric and use the corresponding Levi-Civita connection $\nabla^\delta$ to define the Sobolev spaces $L^2_1(W^\pm)$. 

This being said it is standard fare to deduce from (4.1) that $D^g$ defines a Fredholm operator $L^2_1(W^+) \to L^2(W^-)$ of the same index as $D^g$. Now let $\tilde{\psi} \in L^2_1(W^+ \oplus W^-)$ be any section satisfying $D^g(\tilde{\psi}) = 0$. Then $\psi = u^{\frac{3}{2}}\tilde{\psi}$ satisfies $D^g(\psi) = 0; \quad |\psi| = o(r^{-\frac{3}{2}}); \quad |\nabla^g(\psi)| = o(r^{-\frac{5}{2}})$.

(See [H].) In particular, $\langle \nabla^g(\psi), \psi \rangle_g = o(r^{-4})$. We can now copy the argument in the proof of [K2, Lemma 2.2], using the Weitzenböck formula for the Dirac operator and the fact that $g$ has zero scalar curvature, to conclude that $\nabla^g(\psi) = 0$. Hence $\psi = 0$, so $\text{ind}(D^g) = 0$. This completes the proof of Proposition 8. □

Appendix A. Proof of Lemma 4

(0) We begin with a remark on unique continuation. Let $S = \{S_t\}$ be any smooth solution to the gradient flow equation $\frac{\partial S_t}{\partial t} = -\text{grad}(C)_{S_t}$. Then if $\frac{\partial S_t}{\partial t} = 0$ at some point $t = t_0$ we must have $S_t = S_{t_0}$ for all $t$. This follows from the unique continuation argument in [DK, section 4.3.4] since we can define another solution $\tilde{S}_t$, of class $C^1$ on $\mathbb{R} \times Y$, by

$$\tilde{S}_t = \begin{cases} S_t & \text{for } t \leq t_0 \\ S_{t_0} & \text{for } t \geq t_0. \end{cases}$$

(1) Fix $b > 0$ and let $\{\omega_n\}$ be any sequence with $\|\omega_n\|_{C^k} \to 0$. Let $\{S_n = (A_n, \phi_n)\}$ be a smooth solution to (2.1) with $\omega = \omega_n$ and suppose $\|\phi_n\|_{C^0} \leq b$. Then $\|h^*_{S_n}(\omega_n)\|_{C^k} \to 0$. If $B$ is any smooth connection in $L$ then over any bounded open subset $V$ (containing $[-1, 1]$, say) we can find gauge transformations $u_n$ such that

$$d^*(u_n^2(A_n) - B) = 0.$$ 

Arguing as in [KM] we see that there is a subsequence of $\{u_n(S_n)\}$ which converges in $C^k$ over compact subsets of $V$ to some solution $S_\infty$ of the equations (2.1) with $\omega = 0$.

Now let $C'_n(t) = \frac{\partial}{\partial t}C(S_n(t))$ and suppose $C'_n(t_n) \geq 0$ for some $t_n$. Since the equations (2.1) are translationary invariant we may assume $t_n = 0$. Equation (2.2) and the Cauchy-Schwartz inequality gives

$$\text{(A.1)} \quad \|\text{grad}(C)_{S_n(0)}\|_{L^2} \leq \|E_{S_n(0)}\|_{L^2} \leq \text{const} \cdot \|\omega_n\|_{C^0}.$$ 

Therefore $\text{grad}(C)_{S_\infty(0)} = 0$, so the argument in (0) shows that in a temporal gauge for $S_\infty$, $S_\infty(t)$ must be independent of $t$. Therefore

$$\lim_n h_{S_n}(0) = h_{S_\infty}(0) = C(S_\infty(0)) \notin \Xi.$$
But $\Xi$ is closed, so for sufficiently large $n$ we must have $h_{S_n}(0) \not\in \Xi$, whence $E_{S_n}(0) = 0$ by definition. Equation (2.2) then gives

$$C'_n(0) = -\| \text{grad}(C)_{S_n(0)} \|^2_{L^2} \leq 0,$$

so $S_n(0)$ is a critical point. By (0) above, $[S_n(t)]$ is constant in $t$ on the maximal interval containing 0 where $h_{S_n} \not\in \Xi$. Since $\Xi$ is closed and does not contain any critical points this interval must be all of $\mathbb{R}$. This proves statement (1) of the lemma.

(2) Now let $\omega$ be so small that statement (1) holds, and such that any $P \in A(L_0) \times \Gamma(W_0)$ satisfying

(A.2)  \[ \| \text{grad}(C)_P \|_{L^2} \leq \text{const} \cdot \| \omega \|_{C^0} \]

is close to some critical point of $C$, where the constant is the same as in (A.1). More precisely, we require $C(P) \not\in \Xi$ whenever (A.2) holds.

Let $S = (A, \phi)$ be a smooth solution to the equations (2.1) satisfying $\| \phi \|_{C^0} \leq b$, and let $C(S_t)$ be bounded in $t$. Then if $\{t_n\}$ is any sequence with $t_n \to \infty$ we can find gauge transformations $u_n$ such that a subsequence of $u_n(S|_{(t_n-2,t_n+2) \times Y})$ converges in $C^k$ to a point $\tilde{S}$ which is defined on $(-2,2) \times Y$, say, and provides a solution to (2.1) on $(-1,1) \times Y$. Clearly, $\frac{\partial}{\partial t} C(\tilde{S}_t) = 0$, so our assumption on $\omega$ implies that for $t \in [-1,1]$, $\tilde{S}_t$ is a critical point of $C$. Since the critical points of $C$ are isolated, the gauge equivalence class $[\tilde{S}_t]$ must be independent of $t$, and we see that as $t \to \infty$, $[S_t]$ converges in $C^k$ to some critical point $\alpha_+$. Similarly, we obtain a limit $\alpha_- \text{ as } t \to -\infty$.

**Added in proof:** At the end of Section 2, in order to obtain transversality in moduli spaces over $X$ it is useful to let the forms $\omega_j$ vary as well and argue as in the proof of Proposition 5.

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**References**


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