ALGEBRAIC CYCLES ON SEVERI-BRAUER SCHEMES OF PRIME DEGREE OVER A CURVE

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Abstract. Let $k$ be a perfect field and let $p$ be a prime number different from the characteristic of $k$. Let $C$ be a smooth, projective and geometrically integral $k$-curve and let $X$ be a Severi-Brauer $C$-scheme of relative dimension $p-1$. In this paper we show that $CH^d(X)_{\text{tors}}$ contains a subgroup isomorphic to $CH^0(X/C)$ for every $d$ in the range $2 \leq d \leq p$. We deduce that, if $k$ is a number field, the full Chow ring $CH^*(X)$ is a finitely generated abelian group.

1. Introduction.

Let $k$ be a perfect field with algebraic closure $\overline{k}$ and let $p$ be a prime number different from the characteristic of $k$. Let $C$ be a smooth, projective and geometrically integral $k$-curve. In this paper we study a certain subgroup of $CH^d(X)_{\text{tors}}$ for a Severi-Brauer $C$-scheme $q: X \to C$ of relative dimension $p-1$ and any integer $d$ such that $2 \leq d \leq p$. Let

$$CH_0(X/C) = \text{Ker} \left[ CH_0(X) \xrightarrow{q^*} CH_0(C) \right]$$

and let $\pi^*: CH^d(X) \to CH^d(\overline{X})$ be induced by the extension-of-scalars map $\overline{X} \to X$, where $\overline{X} = X \times_{\text{Spec } k} \text{Spec } \overline{k}$. Then the following holds.

Main Theorem. For any $d$ as above, there exists a canonical isomorphism

$$\text{Ker} \left[ CH^d(X) \xrightarrow{\pi^*} CH^d(\overline{X}) \right] \simeq CH_0(X/C).$$

Corollary. Assume that $k$ is a number field. Then the Chow ring $CH^*(X)$ is a finitely generated abelian group.

The above corollary confirms a well-known conjecture of S.Bloch in a particular case. Previous work on Bloch’s conjecture include [3], where $CH^2(X)$ is shown to be finitely generated for a certain class of varieties $X$, and [4], where the same result is obtained for $CH_0(X)$ when $X \to C$ is an arbitrary (i.e., not necessarily smooth over $C$) Severi-Brauer fibration of squarefree index.

Acknowledgements.

I thank B.Kahn for some helpful comments. I also thank the referee for correcting some inaccuracies and for providing Remark 4.5.

Received by the editors January 29, 2007.

2000 Mathematics Subject Classification. Primary 14C25; Secondary 14C15.

Key words and phrases. Algebraic cycles, Chow groups, curves, Severi-Brauer schemes.

The author is partially supported by Fondecyt grant 1061209 and Universidad Andrés Bello grant DI-29-05/R.
2. Preliminaries.

Let \( k \) be a perfect field, fix an algebraic closure \( \overline{k} \) of \( k \) and let \( \Gamma = \text{Gal}(\overline{k}/k) \). Now let \( C \) be a smooth, projective and geometrically integral \( k \)-curve and let \( X \) be a Severi-Brauer scheme over \( C \) [6, §8] of dimension \( m \geq 2 \). There exists a proper and flat \( k \)-morphism \( q: X \to C \) all of whose fibers are Severi-Brauer varieties of dimension \( m - 1 \) over the appropriate residue field [loc.cit.]. We will write \( X_\eta \) for the generic fiber \( X \times_C \text{Spec}(k) \) of \( q \) and \( A \) for the central simple \( k(C) \)-algebra associated to \( X_\eta \).

We define

\[
\text{CH}_0(X/C) = \text{Ker} \left[ \text{CH}_0(X) \xrightarrow{\alpha} \text{CH}_0(C) \right].
\]

Now let \( C_0 \) be the set of closed points of \( C \). The group of divisorial norms of \( X/C \) (cf. [8]) is the group

\[
k(C)_\text{dn} = \{ f \in k(C)^*: \forall y \in C_0, \text{ord}_y(f) \in (q_y)_*(\text{CH}_0(X_y)) \}
\]

where, for each \( y \in C_0 \), \( q_y: X_y \to \text{Spec}(k) \) is the structural morphism of the fiber \( X_y \). This group is closely related to \( \text{CH}_0(X/C) \) (see [4, Proposition 3.1]). Indeed, there exists a canonical isomorphism

\[
\text{CH}_0(X/C) \simeq k(C)_\text{dn}/k_* \text{Nrd} A^*.
\]

Remark 2.1. Fix an integer \( d \) such that \( 1 \leq d \leq m \) and let

\[
\text{CH}^d(X)' = \text{Ker} \left[ \text{CH}^d(X) \xrightarrow{\pi'} \text{CH}^d(\overline{X} \overline{\Gamma}) \right],
\]

where \( \pi: \overline{X} \to X \) is the canonical map. A simple transfer argument shows that \( \text{CH}^d(X)' \) is a subgroup of \( \text{CH}^d(X)_{\text{tors}} \). Now, since \( \overline{X} \to \overline{C} \) has a section, \( \overline{X} \) is a projective bundle over \( \overline{C} \). Thus, by [5, Theorem 3.3(b), p.64], there exist isomorphisms

\[
\text{CH}^d(\overline{X}) \simeq \begin{cases} 
\mathbb{Z} \oplus \text{CH}_0(\overline{C}) & \text{if } 1 \leq d \leq m - 1 \\
\text{CH}_0(\overline{C}) & \text{if } d = m.
\end{cases}
\]

Therefore, if \( J_C(k) \) is finitely generated, where \( J_C \) is the Jacobian variety of \( C \) (e.g., \( k \) is a number field or \( C = \mathbb{P}^1_k \)), then \( \text{CH}^d(X) \) is finitely generated if and only if \( \text{CH}^d(X)' \) is finite.

3. The general method.

Let \( C \) be as above and let \( X \) be any smooth, projective and geometrically integral \( k \)-variety such that there exists a proper and flat morphism \( q: X \to C \) whose generic fiber \( X_\eta \) is geometrically integral. We have an exact sequence [9]

\[
H^{d-1}(X_\eta, K_d) \xrightarrow{\delta} \bigoplus_{y \in C_0} \text{CH}^{d-1}(X_y) \to \text{CH}^d(X) \xrightarrow{j^*} \text{CH}^d(X_\eta) \to 0,
\]

where \( j: X_\eta \to X \) is the natural map and the map which we have labeled \( \delta \) will play a role later when \( k = \overline{k} \). A similar exact sequence exists over \( \overline{k} \), and we have two
natural exact commutative diagrams:

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Ker } j^* & \rightarrow & CH^d(X) & \rightarrow & CH^d(X_\eta) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (\text{Ker } j^*)^\Gamma & \rightarrow & CH^d(\overline{X})^\Gamma & \rightarrow & CH^d(\overline{X}_\eta)^\Gamma & \rightarrow & 0
\end{array}
\]

and

\[
\begin{array}{cccccc}
0 & \rightarrow & H^{d-1}(X_\eta, K_d) & \rightarrow & \bigoplus_{y \in C_0} CH^{d-1}(X_y) & \rightarrow & \text{Ker } j^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \left( H^{d-1}(\overline{X}_\eta, K_d) \right)^\Gamma & \rightarrow & \bigoplus_{y \in C_0} CH^{d-1}(\overline{X}_\eta)^\Gamma_y & \rightarrow & (\text{Ker } j^*)^\Gamma
\end{array}
\]

where, for each \( y \in C_0 \), we have fixed a closed point \( \overline{y} \) of \( \overline{C} \) lying above \( y \) and written \( \Gamma_y = \text{Gal } (\overline{k}/k(y)) \). Set

\[ CH^d(X_\eta)' = \text{Ker } \left[ CH^d(X_\eta) \rightarrow CH^d(\overline{X}_\eta)^\Gamma \right] \]

and, for each \( y \in C_0 \),

\[ CH^{d-1}(X_y)' = \text{Ker } \left[ CH^{d-1}(X_y) \xrightarrow{\pi_\eta^*} CH^{d-1}(\overline{X}_\eta)^\Gamma_y \right] \]

Now define

\[
(3) \quad E(\overline{X}/\overline{C}) = \text{Coker } \left[ H^{d-1}(X_\eta, K_d) \xrightarrow{j^*} H^{d-1}(\overline{X}_\eta, K_d) \right] \]

Then, applying the snake lemma to the preceding diagrams, we obtain\[1\]

**Proposition 3.1.** There exists a natural exact sequence

\[
\bigoplus_{y \in C_0} CH^{d-1}(X_y)' \rightarrow \text{Ker } CH^d(X) \rightarrow CH^d(X_\eta)'
\]

\[
\rightarrow \text{Ker } E(\overline{X}/\overline{C}) \rightarrow \bigoplus_{y \in C_0} CH^{d-1}(\overline{X}_\eta)^\Gamma_y \rightarrow 0,
\]

where \( E(\overline{X}/\overline{C}) \) is the group \( (3) \).

As regards the right-hand group in the exact sequence of the proposition, the following holds. Let

\[ H^{d-1}(X_\eta, K_d)' = \text{Im } \left[ H^{d-1}(X_\eta, K_d) \rightarrow H^{d-1}(\overline{X}_\eta, K_d)^\Gamma \right] \]

\[ ^1 \text{Proposition 3.1 was inspired by [1, Proposition 1.1].} \]
and
\[ \text{Sal}_d(X/C) = \left\{ f \in H^{d-1}(X, K_d) : \forall y \in C_0, \delta_{\overline{y}}(f) \in \pi_{\overline{y}}^* \text{CH}^{d-1}(X_y) \right\}, \]
where \( \delta \) and \( \pi_{\overline{y}}^* \) are the maps of diagram (2).

**Proposition 3.2.** There exists a natural exact sequence
\[
0 \to \text{Sal}_d(X/C) \to (\gamma^* H^{d-1}(\overline{X}, K_d))^{\Gamma} \cdot H^{d-1}(X, K_d) \to \text{Ker} \left[ E(X/C) \to \bigoplus_{y \in C_0} \text{CH}^{d-1}(X_y) \right] \to H^1(\Gamma, \gamma^* H^{d-1}(\overline{X}, K_d)).
\]

**Proof.** This follows by applying the snake lemma to a diagram of the form
\[
0 \to A \to B \to B/A \to 0 \to 0 \to A^{\Gamma} \to B^{\Gamma} \to (B/A)^{\Gamma} \to H^1(\Gamma, A).
\]
with \( A = \gamma^* H^{d-1}(\overline{X}, K_d), B = H^{d-1}(X, K_d), \) etc. \( \square \)

4. Proof of the main theorem.

Let \( C \) and \( A \) be as in Section 2, let \( p \) be a prime number different from the characteristic of \( k \) and let \( X \) be a Severi-Brauer scheme over \( C \) of relative dimension \( p - 1 \).

**Lemma 4.1.** There exists a \( \Gamma \)-isomorphism
\[ \gamma^* H^{d-1}(\overline{X}, K_d) \simeq \overline{k}^*. \]

**Proof.** Clearly, \( \gamma^* H^{d-1}(\overline{X}, K_d) \) is the kernel of the map
\[ \delta : H^{d-1}(\overline{X}, K_d) \to \bigoplus_{\overline{y}} \text{CH}^{d-1}(X_{\overline{y}}) \]
appearing in the exact sequence (1) over \( \overline{k} \). Now \( \overline{X}_{\overline{y}} \simeq \mathbb{P}^{p-1}_{\overline{y}} \) and \( X_{\overline{y}} \simeq \mathbb{P}^{p-1}_{\overline{y}} \) for every \( \overline{y} \), whence we have \( \Gamma \)-isomorphisms
\[ H^{d-1}(\overline{X}_{\overline{y}}, K_d) \simeq \overline{k}(C)^{*} \]
and
\[ \text{CH}^{d-1}(\overline{X}_{\overline{y}}) \simeq \mathbb{Z} \]
for each \( \overline{y} \). Under these isomorphisms, the map \( \delta \) above corresponds to the canonical map
\[ \overline{k}(C)^{*} \to \bigoplus_{\overline{y}} \mathbb{Z}, \]
\[ f \mapsto (\text{ord}_{\overline{y}}(f))_{\overline{y}}, \]
which yields the lemma. \( \square \)
Theorem 4.2. For every $d$ such that $2 \leq d \leq p$, there exists a canonical isomorphism

$$CH^d(X)' \simeq CH_0(X/C).$$

Proof. By Lemma 4.1, Hilbert’s Theorem 90 and Proposition 3.2, there exists a natural isomorphism

$$\text{Ker} \left[ E(X/C) \to \bigoplus_{y \in C_0} \frac{CH^{d-1}(X_y)}{\pi_y CH^{d-1}(X_y)} \right] \simeq \frac{\text{Sal}_d(X/C)}{k^*H^{d-1}(X, K_d)'},$$

On the other hand, by [7, (8.7.2)], $H^{d-1}(X_y, K_d)' = \text{Nrd}A^*$ for every $d$ such that $2 \leq d \leq p$ and, for each $y \in C_0$, $\pi_y CH^{d-1}(X_y) \simeq \pi_y CH^{p-1}(X_y) \simeq (q_y)_* CH_0(X_y)$ ($= \mathbb{Z}$ or $p\mathbb{Z}$).

The latter implies that $\text{Sal}_d(X/C) = k(C)^*_d$, whence

$$\text{Sal}_d(X/C)/k^*H^{d-1}(X, K_d)' \simeq k(C)^*_d/k^* \text{Nrd}A^* \simeq CH_0(X/C).$$

Finally, [loc.cit.] shows that the groups $CH^d(X_y)$ and $CH^{d-1}(X_y)$ ($y \in C_0$) are torsion free, whence $CH^d(X_y)'$ and $CH^{d-1}(X_y)'$ vanish. The theorem now follows from Proposition 3.1. □

Corollary 4.3. Let $d$ be such that $2 \leq d \leq p$. Then $CH^d(X)'$ is finite if

1. $k$ is a number field, or
2. $k$ is a field of finite type over $\mathbb{Q}$, $C = \mathbb{P}^1_k$ and $X$ has a 0-cycle of degree one.

Proof. Indeed, in these cases the group $CH_0(X/C)$ is finite [4]. □

Corollary 4.4. In each of the cases listed in the previous corollary, the Chow ring $CH^*(X)$ is finitely generated as an abelian group.

Proof. The above corollary and Remark 2.1 show that $CH^d(X)$ is finitely generated for any $d$ such that $2 \leq d \leq p$. Since $CH^0(X)$ and $CH^1(X) = \text{Pic}(X)$ are well-known to be finitely generated (see [2, §1]), the proof is complete. □

Remark 4.5. The referee has suggested the following alternative approach to this paper.

Since there is only $p$-torsion in the Chow groups and dim $X = p$, it is not difficult to relate the $E_2$ and $E_\infty$ terms in the Gersten-Quillen spectral sequence (see, e.g., [7, Proposition (8.6.2), p.320]). Hence if $K_0(X)$ is finitely generated, then the Chow groups of $X$ are also finitely generated. Now let $\Lambda$ be the Azumaya algebra over $C$ corresponding to the Severi-Brauer scheme $X \to C$ (see [6]). Then, by a well-known theorem of Quillen, $K_0(X) \simeq K_0(C) \oplus K_0(\Lambda)^{p-1}$. Hence if $K_0(C)$ and $K_0(\Lambda)$ are finitely generated, then the Chow groups of $X$ are also finitely generated. Now one can construct a commutative diagram with Swan’s localization sequences for $\Lambda$ and $C$ and use it to relate the kernel of the restriction map from $K_0(\Lambda)$ to $K_0(C)$ (or $K_0(\Lambda)$) to the group $k(C)^*/k^* \text{Nrd}A^*$. This gives more transparent proofs of the finiteness results and the introduction of the Azumaya algebra $\Lambda$ provides a natural explanation for the appearance of the group $k(C)^*/k^* \text{Nrd}A^*$. 

References


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