A SPLITTING THEOREM ON TORIC MANIFOLDS

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Abstract. Using the Calabi flow, we prove that any extremal Kähler metric $\omega_E$ on a product toric variety $X_1 \times X_2$ is a product extremal Kähler metric.

1. Introduction

In [3], the authors considered the following problem:

**Problem 1.1.** Let $X_i, i = 1, 2$ be two Kähler manifolds with Kähler classes $[\omega_i]$. Suppose $\omega_E$ is an extremal Kähler metric in the Kähler class $[\omega_1 + \omega_2]$. Can we conclude that $\omega_E$ is a product metric, i.e., $\omega_E = \omega_{E,1} + \omega_{E,2}$ where $\omega_{E,i}$ is an extremal Kähler metric in $[\omega_i]$.

In this short paper, we solve the problem in the case of toric manifolds.

**Theorem 1.2.** If $X_i$ are toric manifolds, then $\omega_E$ is a product metric.

2. Motivations and setup

Let $X$ be a $n$-dimensional Kähler manifold with Kähler class $[\omega]$. The set of relative Kähler potentials is

$$\mathcal{H} = \{ \varphi \in C^\infty(X) \mid \omega_\varphi = \omega + i\partial \bar{\partial} \varphi > 0 \}.$$ 

An extremal Kähler metric $\omega_\varphi$ in the sense of Calabi is defined by the condition that its scalar curvature $R_\varphi$ is a potential of a Killing vector field of $(X, \omega_\varphi)$. In [5], Calabi proved that any extremal metric is invariant under a maximal compact subgroup of the reduced automorphism group of $X$. As any such groups are conjugated, one can fix the isometry group of an extremal Kähler metric. In particular, if we further assume that $X$ is a toric manifold, then without loss of generality we can assume that if an extremal metric $\omega_E$ exists then it is invariant under the (real) torus $\mathbb{T}^n$. We thus focus on the space of $\mathbb{T}^n$-invariant relative Kähler potentials:

$$\mathcal{H}_{\mathbb{T}^n} = \{ \varphi \in C^\infty(X) \mid \varphi \text{ is invariant under } \mathbb{T}^n, \omega_\varphi = \omega + i\partial \bar{\partial} \varphi > 0 \},$$

where $\omega$ is a $\mathbb{T}^n$-invariant Kähler metric on $X$. By the equivariant Moser lemma, the space of $\mathbb{T}^n$-invariant Kähler metrics $\omega_\varphi$ given by elements in $\mathcal{H}_{\mathbb{T}^n}$ can be alternatively realized as the space $\mathcal{J}_\varphi^{\omega}$ of $\mathbb{T}^n$-invariant $\omega$-compatible complex structures on the toric symplectic manifold $(\bar{X}, \omega)$, see Abreu [2], Guillemin [19, 20] and Donaldson [9]. The latter is identified, via the momentum map, to the corresponding Delzant polytope $P \subset \mathbb{R}^n$ (see Delzant [16]). Recall that $P$ is a compact convex polytope satisfying...

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the following conditions:

- For any facet $P_i$ of $P$, there exists an inward normal vector $\vec{v}_i$ corresponding to $P_i$.
- For any vertex $v$ of $P$, there are exactly $n$ facets $P_1, \ldots, P_n$ meeting at $v$ and the inward normal vectors $\vec{v}_1, \ldots, \vec{v}_n$ form a basis of $\mathbb{Z}^n$.

Suppose that $P$ has $d$-facets. For every facet $P_i$, we choose $c_i$ such that $l_i(x) = \langle x, \vec{v}_i \rangle + c_i$ vanishes on $P_i$.

**Definition 2.1.** A smooth strictly convex function $u$ on the interior of $P$ is called a symplectic potential if

- $u$ extends as a continuous function over $\partial P$ and its restriction to the interior of each face of $P$ is smooth and strictly convex;
- 

\[ u(x) = \sum_{i=1}^{d} \frac{1}{2} l_i(x) \ln l_i(x) + f(x), \]

where $f(x)$ is a smooth function up to the boundary of $P$.

The main point of this definition is that, for appropriate choice of angular coordinates $(t_1, \ldots, t_n)$, the almost complex structure

\[ J_u = \begin{pmatrix} 0 & \cdots & -(D^2 u)^{-1} \\ \vdots & \ddots & \vdots \\ (D^2 u) & \cdots & 0 \end{pmatrix} \]

is an element of $\mathcal{J}_{\mathbb{C}^n}$ and the elements of $\mathcal{H}_{\mathbb{T}^n}$ are in one to one correspondence with symplectic potentials $u$ as above, see [2,9,19,20].

Furthermore, Abreu [1] wrote down the expression of the scalar curvature $R_u$ of the Kähler metric $g_u(\cdot, \cdot) = \omega(\cdot, J_u \cdot)$. It follows that a symplectic potential $u_E$ corresponds to an extremal Kähler metric if and only if the corresponding scalar curvature $R_E$ is an affine function. Note that in this case, $R_E$ is a priori determined by the Delzant polytope $P$, by the property that for any affine function $f$ on $P$, we have (see [9])

\[ \mathcal{L}(f) = 2 \int f \, d\sigma - \int_P f R_E \, d\mu = 0, \]

where $d\mu$ is the standard Lebesgue measure on $P$ and $d\sigma$ is the induced boundary Lebesgue measure on $\partial P$: on every facet $P_i$, we require that $dl_i \wedge d\sigma$ is $d\mu$ up to a sign.

In the case when $(X, L)$ is a compact polarized manifold, Yau [29], Tian [28] and Donaldson [9] conjectured

**Conjecture 2.2.** $(X, L)$ admits constant scalar curvature Kähler (cscK) metrics in $c_1(L)$ if and only if it is $K$-stable.

It is known by Donaldson [13], Stoppa [26], Mabuchi [23,24], Stoppa and Székelyhidi [27], Chen and Tian [8] that if $(X, L)$ admits cscK metrics in $c_1(L)$, then $(X, L)$ must be $K$-stable. In the product case $X = X_1 \times X_2$, $L = L_1 \otimes L_2$, one easily infers that each $(X_i, L_i)$ must be $K$-stable. Thus, Problem (1.1) would follow from Conjecture (2.2) by using the uniqueness of cscK metrics [8,14].
In [9], Donaldson considers the toric case and finds that the $K$-stability is related to the following condition:

**Definition 2.3.** A rational Delzant polytope is (relative) $K$-stable if for any convex continuous rational piecewise linear function $f$ one has $\mathcal{L}(f) \geq 0$. And the equality holds if and only if $f$ is an affine function.

He thus conjectures [9]:

**Conjecture 2.4 (Donaldson).** A compact toric Kähler manifold admits a compatible extremal metric if and only if $\mathcal{L}(f) \geq 0$ for any convex continuous piecewise linear function $f$ with equality if and only if $f$ is an affine function.

Once again, it is straightforward to see that if a product of two Delzant polytopes $P = P_1 \times P_2$ is $K$-stable, such is then each factor $P_i$. However, as far as Conjecture (2.4) stays open, we must find an alternative argument to establish our Theorem (1.2). To this end, we will use the Calabi flow [4], which was initially introduces as a flow on the space $\mathcal{H}$ defined by

$$\frac{\partial \varphi}{\partial t} = R_\varphi - R,$$

where $R_\varphi$ is the scalar curvature of the Kähler metric $\omega_\varphi$ and $R$ is a topological constant on $X$ defined by

$$R = \frac{2n\pi c_1(X) \wedge [\omega]^{n-1}}{[\omega]^n}.$$  

In the toric case, this flow can be rewritten in terms of symplectic potentials as ([9])

$$\frac{\partial u}{\partial t} = R - R_u.$$

We shall rather consider the modified version [21]

$$\frac{\partial u}{\partial t} = R_E - R_u.$$

Note that by Chen and He [7], the Calabi flow exists for a short time starting from any $C^{3,\alpha}$ relative Kähler potential. Thus for a smooth symplectic potential $u$, the Calabi flow starting from $u$ also exists for a short time.

Guan [18] has shown that in the toric setting, for any two symplectic potentials $u_1$ and $u_2$, the geodesic in the sense of Mabuchi [22], Semmes [25] and Donaldson [15] connecting them is given by $(1-t)u_1 + tu_2$, $t \in [0,1]$. The length of this geodesic is

$$\sqrt{\int_P (u_1 - u_2)^2 \, d\mu}.$$

Suppose that $u_1(t), u_2(t), t \in [0,1]$ are two modified Calabi flows, we want to show that the geodesic distance between $u_1(t)$ and $u_2(t)$ decreases as $t$ increases. This is essentially known by the work of Calabi and Chen [6]. In fact, we have the following lemma:

**Lemma 2.5.**

$$\frac{\partial}{\partial t} \int_P (u_1(t) - u_2(t))^2 \, d\mu \leq 0.$$
Proof.
\[
\frac{\partial}{\partial t} \int_P (u_1(t) - u_2(t))^2 \, d\mu = 2 \int_P (u_1(t) - u_2(t))(R_{u_2(t)} - R_{u_1(t)}) \, d\mu \\
= 2 \int_P (u_1(t)_{ij} - u_2(t)_{ij})(u_1(t)^{ij} - u_2(t)^{ij}) \, d\mu \quad (*)
\]
In the last step, we have used integration by parts as in Lemma 3.3.5 of [9]. For any \( x \in P \), let \( A = (D^2u_1(t))(x) \), \( B = (D^2u_2(t))(x) \), then
\[
(u_1(t)_{ij} - u_2(t)_{ij})(u_1(t)^{ij} - u_2(t)^{ij})(x) = \text{Trace}((A - B)(A^{-1} - B^{-1})).
\]
Note that \( A, B \) are positive-definite matrices, thus there exists an orthonormal matrix \( O_1 \) such that \( O_1AO_1^T \) is a diagonal matrix \( \text{diag}(\lambda_1, \ldots, \lambda_n) \). Let \( O_2 = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) \). Then \( O_2^{-1}O_1AO_1^TO_2^{-1} \) is the identity matrix and \( B = O_2^{-1}O_1BO_1^TO_2^{-1} \) is still a positive-definite matrix. Note that
\[
\text{Trace}((A - B)(A^{-1} - B^{-1})) = \text{Trace}((I_n - \tilde{B})(I_n - \tilde{B}^{-1})).
\]
We can again choose an orthonormal matrix \( O_3 \) such that \( O_3\tilde{B}O_3^T \) is a diagonal matrix \( \text{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \). Then
\[
\text{Trace}((A - B)(A^{-1} - B^{-1})) = \sum_{i=1}^n (1 - \tilde{\lambda}_i)(1 - \tilde{\lambda}_i^{-1}) \leq 0.
\]
Thus, (*) \( \leq 0 \). \( \square \)

3. Proof of Theorem (1.2)

Let \( (X_i, \omega_i) \) be toric symplectic manifolds with Delzant polytopes \( P_i \). The product manifold \( X = X_1 \times X_2 \) with symplectic form \( \omega = \omega_1 + \omega_2 \) is symplectic toric with Delzant polytope \( P = P_1 \times P_2 \). In the symplectic side, we have symplectic potentials \( u_i \) satisfying Guillemin boundary conditions of \( P_i \). We let \( x \) be the variable of \( P_1 \) and \( y \) be variable of \( P_2 \). Our assumption shows that there exists a symplectic potential \( u \) on \( P \) and
\[
u(x, y) = u_1(x) + u_2(y) + f(x, y), \quad f(x, y) \in C^\infty(\bar{P})
\]
such that the scalar curvature of \( u(x, y) \) is an affine function. Our goal is to show that \( f(x, y) \) is separable. Let
\[
f_1(x) = \frac{1}{\text{vol}(P_2)} \int_{P_2} f(x, y) \, dy, \quad f_2(y) = \frac{1}{\text{vol}(P_1)} \int_{P_1} f(x, y) \, dx.
\]
Then we have

**Proposition 3.1.** \( v(x, y) = u_1(x) + u_2(y) + f_1(x) + f_2(y) \) is a symplectic potential of \( P \) satisfying the Guillemin boundary conditions.

**Proof.** It is easy to see that \( f_1(x) + f_2(y) \) is a smooth function on \( \bar{P} \). Thus, we only need to show that \( (D^2v) \) is a positive matrix in order to prove that \( v \) is a symplectic potential. To show \( (D^2v) > 0 \) is equivalent to show that \( (D^2(u_1(x) + f_1(x))) > 0 \) and \( (D^2(u_2(y) + f_2(y))) > 0 \). However, \( (D^2(u_1(x) + f_1(x))) > 0 \) and \( (D^2(u_2(y) + f_2(y))) > 0 \) just follow from the fact that \( (D^2u) > 0 \). \( \square \)

Let \( S \) be the set of all symplectic potentials. We define a subset of \( S \).
Definition 3.2.
\[ \mathcal{M} = \left\{ u(x, y) \in \mathcal{S} \mid u(x, y) = u_1(x) + u_2(y) + g_1(x) + g_2(y) \text{ s.t.} \right. \\
\left. g_1(x) \in C^\infty(\bar{P}_1), \int_{P_1} f_1(x) \, dx = \int_{P_1} g_1(x) \, dx, \right. \\
\left. g_2(y) \in C^\infty(\bar{P}_2), \int_{P_2} f_2(y) \, dy = \int_{P_2} g_2(y) \, dy \right\}. \]

Then we have

Proposition 3.3. For any \( u \in \mathcal{M} \), we have
\[
\int_P (u(x, y) - v(x, y))^2 \, dx \, dy \leq \int_P (u(x, y) - u(x, y))^2 \, dx \, dy.
\]
And the equality holds if and only if \( v = u \).

Proof. (3.1) is equivalent to show that
\[
\int_P (f(x, y) - f_1(x) - f_2(y))^2 \, dx \, dy \leq \int_P (f(x, y) - g_1(x) - g_2(y))^2 \, dx \, dy.
\]
Expressing it out, we have
\[
\int_P -2f(x, y)(f_1(x) + f_2(y)) + f_1^2(x) + f_2^2(y) \, dx \, dy
\leq \int_P -2f(x, y)(g_1(x) + g_2(y)) + g_1^2(x) + g_2^2(y) \, dx \, dy
\]
which is equivalent to
\[
0 \leq \int_P (f_1(x) - g_1(x))^2 + (f_2(y) - g_2(y))^2 \, dx \, dy.
\]
The equality holds if and only if \( f_1(x) = g_1(x) \) and \( f_2(y) = g_2(y) \). \( \square \)

Proof of Theorem (1.2). We use the Calabi flow to show that \( v \) is an extremal symplectic potential. Let \( u(t) \) be a sequence of symplectic potentials satisfying the modified Calabi flow equation on \( P \) and \( u(0) = v \). By Lemma 2.5, we have
\[
\frac{d}{dt} \int_P (u(t) - u)^2 \, dx \, dy \leq 0.
\]
Since \( u(t) \in \mathcal{M} \), we obtain \( u(t) = v \). This shows that \( v \) is a separable extremal symplectic potential on \( P \). By the uniqueness of the extremal symplectic potential modulo affine functions [18], it follows that \( u \) is also separable. So, \( f \) is a separable function. \( \square \)

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