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Affine cellularity of $S_\vartriangle(2,2)$

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In this paper we prove that the affine Schur algebra $S_\vartriangle(2,2)$ is affine cellular over $\mathbb{Q}$. As an application, we prove that the global dimension of $S_\vartriangle(2,2)$ is finite over $\mathbb{Q}$.

1. Introduction

Affine Schur algebras have several equivalent definitions given by [8, 11, 18, 20, 21] respectively. There are clear correspondences between these definitions on basis elements. Affine Schur algebras play a central role in linking the representations of affine quantum groups and affine Hecke algebras.

Affine cellular algebras are introduced by Koenig and Xi in [17]. They extend the framework of cellular algebras due to Graham and Lehrer in [9] to affine cellular algebras which are not necessarily finite dimensional over a field. Many examples such as affine Temperley-Lieb algebras and affine Hecke algebras of type A when the parameter q is not a root of Poincaré polynomial are proved to be affine cellular in [17]. Recently A. S. Kleshchev, J. W. Loubert, V. Miemietz and J. Guilhot prove that KLR algebras of type A and affine Hecke algebras of rank two are affine cellular in [15] and [12] respectively.

The aim of this paper is to prove that the affine Schur algebra $S_\vartriangle(2,2)$ in case $q = 1$ is an affine cellular algebra over $\mathbb{Q}$. We use the equivalent definitions of $S_\vartriangle(n,r)$ given by [8, 18, 21] respectively. By using the multiplication formulas given in [3, 18, 22], we investigate the ideal generated by the idempotent corresponding to a particular diagonal matrix and construct a chain of idempotent ideals in $S_\vartriangle(2,2)$. Then we prove that this chain

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affords an affine cellular structure for $S_\vartriangle(2,2)$. As an application, we prove that $S_\vartriangle(2,2)$ over $\mathbb{Q}$ is of finite global dimension.

The paper is organized as follows. We recall equivalent definitions of affine Schur algebras and some multiplication formulas in Section 2. In Section 3 we recall affine cellular algebras defined by Koenig and Xi. In Section 4 we prove that $S_\vartriangle(2,2)$ has an affine cellular structure and is of finite global dimension.

2. Affine Schur algebras

In this section we recall equivalent definitions of affine Schur algebras given by [8, 18, 21]. We give correspondences of the basis elements between these definitions. At the end of this section, we recall some multiplication formulas given by [3, 7, 18, 21] respectively.

First we recall a geometric definition of affine quantum Schur algebras introduced by Ginzburg–Vasserot [8] and Lusztig [18]. Let $\mathbb{F}$ be a field and let $\mathbb{F}[x,x^{-1}]$ be the Laurent polynomial ring in indeterminate x. Fix an $\mathbb{F}[x,x^{-1}]$-free module V of rank $r \geq 1$. A lattice in V is a free $\mathbb{F}[x]$-submodule L of V such that $V = L \otimes_{\mathbb{F}[x]} \mathbb{F}[x,x^{-1}]$.

Let $\mathfrak{F}_\vartriangle = \mathfrak{F}_{\vartriangle,n}$ be the set of all cyclic flags $L = (L_i)_{i\in\mathbb{Z}}$ of lattices, where each L_i is a lattice in V such that $L_{i-1} \subseteq L_i$ and $L_{i-n} = xL_i$ for all $i \in \mathbb{Z}$. The group G of automorphisms of the $\mathbb{F}[x,x^{-1}]$-module V acts on $\mathfrak{F}_\vartriangle$ by $g \cdot L = (g(L_i))_{i\in\mathbb{Z}}$ for $g \in G$ and $L \in \mathfrak{F}_\vartriangle$. Thus, the map

$$\phi : \mathfrak{F}_\vartriangle \to \Lambda_\vartriangle(n,r), \quad L \to \underline{\dim}\, L = (\dim_{\mathbb{F}} L_i/L_{i-1})_{i\in\mathbb{Z}}$$

induces a bijection between the set of G-orbits in $\mathfrak{F}_\vartriangle$ and $\Lambda_\vartriangle(n,r)$, where

$$\Lambda_\vartriangle(n,r) := \left\{ (\lambda_i)_{i\in\mathbb{Z}} \,\middle|\, \lambda_i \in \mathbb{N},\ \sum_{i=1}^{n} \lambda_i = r \text{ and } \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z} \right\}.$$

Let

$$\Lambda(n,r) := \left\{ (\lambda_1,\ldots,\lambda_n) \,\middle|\, \lambda_i \in \mathbb{N},\ \sum_{1\leq i\leq n} \lambda_i = r \right\}.$$

We usually identify $\Lambda(n,r)$ with $\Lambda_\vartriangle(n,r)$ via the following bijection:

$$b : \Lambda_\vartriangle(n,r) \longrightarrow \Lambda(n,r), \quad \lambda \longrightarrow (\lambda_1,\ldots,\lambda_n).$$

The group G also acts diagonally on $\mathfrak{F}_\vartriangle \times \mathfrak{F}_\vartriangle$ by $g(L,L') = (gL, gL')$, where $g \in G$ and $L, L' \in \mathfrak{F}_\vartriangle$. By [18, 1.5], there is a bijection between the set of G-orbits in $\mathfrak{F}_\triangle \times \mathfrak{F}_\triangle$ and the set $\Theta_\triangle(n,r)$ by sending (L,L') to $A = (a_{i,j})_{i,j\in\mathbb{Z}}$, where

$$a_{i,j} = \dim_{\mathbb{F}} \frac{L_i \cap L'_j}{L_{i-1}\cap L'_j + L_i \cap L'_{j-1}} \quad \text{for } i,j \in \mathbb{Z}, \tag{2.1}$$

$$\Theta_\triangle(n,r) := \left\{ A = (a_{i,j})_{i,j\in\mathbb{Z}} \in M_{\triangle,n}(\mathbb{N}) \;\middle|\; \sum_{\substack{1\le i\le n\\ j\in\mathbb{Z}}} a_{i,j} = \sum_{\substack{1\le j\le n\\ i\in\mathbb{Z}}} a_{i,j} = r \right\} \tag{2.2}$$

and $M_{\triangle,n}(\mathbb{N})$ is the set of all $\mathbb{Z}\times\mathbb{Z}$ matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in \mathbb{N}$ such that

(a) $a_{i,j} = a_{i+n,j+n}$ for $i,j \in \mathbb{Z}$;

(b) for every $i \in \mathbb{Z}$, the set $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ is finite.

Let $\mathcal{O}_A$ denote the orbit in $\mathfrak{F}_\triangle \times \mathfrak{F}_\triangle$ corresponding to A. If $(L,L') \in \mathcal{O}_A$, then $\mathrm{row}(A) = \underline{\dim}\, L$ and $\mathrm{col}(A) = \underline{\dim}\, L'$, where

$$\mathrm{row}(A) = \left(\sum_{j\in\mathbb{Z}} a_{i,j}\right)_{i\in\mathbb{Z}}, \quad \mathrm{col}(A) = \left(\sum_{i\in\mathbb{Z}} a_{i,j}\right)_{j\in\mathbb{Z}}.$$

Assume now that $\mathbb{F} = \mathbb{F}_q$ is a finite field of q elements and write $\mathfrak{F}_\triangle(q)$ for $\mathfrak{F}_\triangle$. For any fixed $(L,L'') \in \mathcal{O}_{A''}$, let

$$c_{A,A',A'';q} = \left|\{L' \in \mathfrak{F}_\triangle(q) \mid (L,L') \in \mathcal{O}_A,\ (L',L'') \in \mathcal{O}_{A'}\}\right|.$$

Clearly, $c_{A,A',A'';q}$ is independent of the choice of (L,L''), and a necessary condition for $c_{A,A',A'';q} \neq 0$ is that

$$\mathrm{col}(A) = \mathrm{row}(A'), \quad \mathrm{row}(A) = \mathrm{row}(A'') \quad \text{and} \quad \mathrm{col}(A') = \mathrm{col}(A''). \tag{2.3}$$

Let $\mathscr{A} = \mathbb{Z}[\boldsymbol{q}]$ be the polynomial ring with indeterminate $\boldsymbol{q}$. By [18, 1.8], there is a polynomial $p_{A,A',A''} \in \mathbb{Z}[\boldsymbol{q}]$ in $\boldsymbol{q}$ such that for each finite field $\mathbb{F}$ with q elements, $c_{A,A',A'';q} = p_{A,A',A''}(q)$.

Definition 2.1 ([18, 1.9]). The affine quantum Schur algebra $S_\triangle(n,r)$ is the free $\mathbb{Z}[\boldsymbol{q}]$-module with basis $\{e_A \mid A \in \Theta_\triangle(n,r)\}$, and multiplication defined by

$$e_A \cdot e_{A'} = \begin{cases} \sum_{A''\in\Theta_\triangle(n,r)} p_{A,A',A''} e_{A''}, & \text{if } \mathrm{col}(A) = \mathrm{row}(A'),\\ 0, & \text{otherwise.}\end{cases}$$

As in the finite case, for each $\lambda \in \Lambda_\vartriangle(n,r)$, define $\mathrm{diag}(\lambda) = (\delta_{i,j}\lambda_i)_{i,j\in\mathbb{Z}} \in \Theta_\vartriangle(n,r)$, and $e_\lambda = e_{\mathrm{diag}(\lambda)}$. It is easy to see that for each $A \in \Theta_\vartriangle(n,r)$,

$$(2.4) \quad e_\lambda e_A = \begin{cases} e_A, & \text{if } \lambda = \mathrm{row}(A); \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad e_A e_\lambda = \begin{cases} e_A, & \text{if } \lambda = \mathrm{col}(A); \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\sum_{\lambda\in\Lambda_\vartriangle(n,r)} e_\lambda$ is the unity of $S_\vartriangle(n,r)$. By specializing $\boldsymbol{q}$ to 1 in Definition 2.1, we get the affine Schur algebra $S_\vartriangle(n,r)_\mathbb{Z}$ over $\mathbb{Z}$.

Now we introduce an algebraic definition of affine Schur algebras given by [21]. Let $\mathfrak{S} = \mathfrak{S}_r$ denote the symmetric group on r letters and let $\mathfrak{S}_\vartriangle = \mathfrak{S} \ltimes \mathbb{Z}^r$ denote the extended affine Weyl group of type A_{r-1}. For a set S, we denote by $I(S,r)$ the set $\{\underline{i} = (i_1,\ldots,i_r) \mid i_t \in S, t = 1,2,\ldots,r\}$. We will denote the set $I(S,r)$ by $I(n,r)$ if $S = \{1,2,\ldots,n\}$. Now $\mathfrak{S}$ acts on $I(n,r)$ by place permutation. $\mathfrak{S}_\vartriangle$ acts on $I(\mathbb{Z},r)$ on the right with $\mathfrak{S}$ acting by place permutation and $\mathbb{Z}^r$ acting by shifting, i.e. $\underline{i}(\sigma,\varepsilon) = \underline{i} + n\varepsilon$ for $\underline{i} \in I(\mathbb{Z},r)$, $\sigma \in \mathfrak{S}$ and $\varepsilon \in \mathbb{Z}^r$. Note that this action depends on the number n and $\mathfrak{S}_\vartriangle$ acts diagonally on $I(\mathbb{Z},r)\times I(\mathbb{Z},r)$. For each pair $(\underline{i},\underline{j}) \in I(\mathbb{Z},r)\times I(\mathbb{Z},r)$, we associate an element $\xi_{\underline{i},\underline{j}}$ such that $\xi_{\underline{i},\underline{j}} = \xi_{\underline{k},\underline{l}}$ if and only if $(\underline{i},\underline{j}) \sim_{\mathfrak{S}_\vartriangle} (\underline{k},\underline{l})$, i.e $(\underline{i},\underline{j})$ and $(\underline{k},\underline{l})$ are in the same $\mathfrak{S}_\vartriangle$-orbit.

Definition 2.2 ([21, Sec. 4]). The affine Schur algebra $\widetilde{S}_\vartriangle(n,r)$ is defined to be the $\mathbb{Z}$-algebra with basis $\{\xi_{\underline{i},\underline{j}} \mid \underline{i},\underline{j} \in I(\mathbb{Z},r)\}$ and multiplication given by the following rule:

$$\xi_{\underline{i},\underline{j}}\xi_{\underline{k},\underline{l}} = \sum_{(\underline{p},\underline{q})\in I(\mathbb{Z},r)\times I(\mathbb{Z},r)/\mathfrak{S}_\vartriangle} C(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q})\xi_{\underline{p},\underline{q}},$$

where $C(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q}) = \left|\{\underline{s} \in I(\mathbb{Z},r) \mid (\underline{i},\underline{j}) \sim_{\mathfrak{S}_\vartriangle} (\underline{p},\underline{s}),\ (\underline{s},\underline{q}) \sim_{\mathfrak{S}_\vartriangle} (\underline{k},\underline{l})\}\right|$.

Remark 2.3. There is a $\mathbb{Z}$-algebra isomorphism between $S_\vartriangle(n,r)_\mathbb{Z}$ and $\widetilde{S}_\vartriangle(n,r)$. The correspondence φ between the basis elements is given by

$$\varphi(\xi_{\underline{i},\underline{j}}) = e_A,$$

where $A = (a_{x,y})_{x,y\in\mathbb{Z}} \in \Theta_\vartriangle(n,r)$, $\underline{i},\underline{j} \in I(\mathbb{Z},r)$ and

$$a_{x,y} = \left|\{s \mid i_s = x,\ j_s = y,\ 1 \le s \le r\}\right|.$$

The $\mathbb{Z}$-algebra isomorphism above can be extended to a $\mathbb{Q}$-algebra isomorphism. In this paper, we are mainly concentrated on the affine Schur algebra $S_\vartriangle(2,2)$ over $\mathbb{Q}$. For simplicity, we usually identify the basis elements occurring in the above equivalent definitions. In the following, we always consider $S_\vartriangle(n,r)$ and $S_\vartriangle(2,2)$ as algebras over $\mathbb{Q}$.

Remark 2.4. By Section 5.2 in [3], $S_\vartriangle(n,r)$ is a graded algebra over $\mathbb{Q}$, i.e.

$$S_\vartriangle(n,r) = \bigoplus_{m\in\mathbb{Z}} S_\vartriangle(n,r)_m.$$

Let $\Theta^+_\vartriangle(n,r) = \{A = (a_{i,j}) \in \Theta_\vartriangle(n,r) \mid a_{i,j} = 0 \text{ for } i > j\}$ and $\Theta^-_\vartriangle(n,r) = \{A = (a_{i,j}) \in \Theta_\vartriangle(n,r) \mid a_{i,j} = 0 \text{ for } i < j\}$. Then the degree $\mathrm{gr}(e_A)$ for $A \in \Theta^+_\vartriangle(n,r)$ is defined by

$$\mathrm{gr}(e_A) = \sum_{i<j,1\le i\le n} a_{i,j}(j-i),$$

and the degree $\mathrm{gr}(e_A)$ for $A \in \Theta^-_\vartriangle(n,r)$ is defined by

$$\mathrm{gr}(e_A) = \sum_{i>j,1\le i\le n} -a_{i,j}(i-j).$$

Now we recall some useful multiplication formulas which are given by [3, 7, 18, 21] respectively.

For $i,j\in\mathbb{Z}$, let $E^\vartriangle_{i,j} \in M_{\vartriangle,n}(\mathbb{N})$ be the matrix $(e^{i,j}_{k,l})_{k,l\in\mathbb{Z}}$ defined by

$$e^{i,j}_{k,l} = \begin{cases} 1, & \text{if } k = i+sn,\ l = j+sn \text{ for some } s\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Proposition 2.5 ([7, 3.1], [18, 3.5]). *Let* $1\le h\le n, A\in\Theta_\vartriangle(n,r)$ *and* $\lambda = \mathrm{row}(A)$. *Let* $B_m = \mathrm{diag}(\lambda) + mE^\vartriangle_{h,h+1} - mE^\vartriangle_{h+1,h+1}$ *and* $C_m = \mathrm{diag}(\lambda) - mE^\vartriangle_{h,h} + mE^\vartriangle_{h+1,h}$. *Then in* $S_\vartriangle(n,r)$

$$(1)\quad e_{B_m}\cdot e_A = \sum_{\substack{t\in\Lambda(\infty,m)\\ \forall u\in\mathbb{Z}, t_u\le a_{h+1,u}}} \prod_{u\in\mathbb{Z}} \binom{a_{h,u}+t_u}{t_u} e_{A+\sum_{u\in\mathbb{Z}} t_u(E^\vartriangle_{h,u}-E^\vartriangle_{h+1,u})},$$

$$(2)\quad e_{C_m}\cdot e_A = \sum_{\substack{t\in\Lambda(\infty,m)\\ \forall u\in\mathbb{Z}, t_u\le a_{h,u}}} \prod_{u\in\mathbb{Z}} \binom{a_{h+1,u}+t_u}{t_u} e_{A-\sum_{u\in\mathbb{Z}} t_u(E^\vartriangle_{h,u}-E^\vartriangle_{h+1,u})},$$

where $\Lambda(\infty,m) = \{\lambda = (\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i\in\mathbb{N},\ \sum_i \lambda_i = m\}$.

Symmetrically, we can get the following multiplication formulas.

Proposition 2.6. *Let $1 \le h \le n$, $A \in \Theta_\vartriangle(n,r)$ and $\lambda = \mathrm{col}(A)$. Let $B_m = \mathrm{diag}(\lambda) + mE^\vartriangle_{h,h+1} - mE^\vartriangle_{h,h}$ and $C_m = \mathrm{diag}(\lambda) - mE^\vartriangle_{h+1,h+1} + mE^\vartriangle_{h+1,h}$. Then in $S_\vartriangle(n,r)$*

$$(1)\quad e_A \cdot e_{B_m} = \sum_{\substack{t\in\Lambda(\infty,m)\\ \forall u\in\mathbb{Z}, t_u \le a_{u,h}}} \prod_{u\in\mathbb{Z}} \binom{a_{u,h+1}+t_u}{t_u} e_{A+\sum_{u\in\mathbb{Z}} t_u (E^\vartriangle_{u,h+1} - E^\vartriangle_{u,h})},$$

$$(2)\quad e_A \cdot e_{C_m} = \sum_{\substack{t\in\Lambda(\infty,m)\\ \forall u\in\mathbb{Z}, t_u \le a_{u,h+1}}} \prod_{u\in\mathbb{Z}} \binom{a_{u,h}+t_u}{t_u} e_{A-\sum_{u\in\mathbb{Z}} t_u (E^\vartriangle_{u,h+1} - E^\vartriangle_{u,h})},$$

where $\Lambda(\infty,m) = \{\lambda = (\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i \in \mathbb{N},\ \sum_i \lambda_i = m\}$.

Proposition 2.7 ([3, Prop. 6.2.3]). *Let $1 \le h \le n$, $A \in \Theta_\vartriangle(n,r)$ and $\lambda = \mathrm{row}(A)$. Let $D_m = \mathrm{diag}(\lambda) - E^\vartriangle_{h,h} + E^\vartriangle_{h,h+mn}$. Then in $S_\vartriangle(n,r)$*

$$e_{D_m} \cdot e_A = \sum_{\substack{u\in\mathbb{Z}\\ a_{h,u}\ge 1}} (a_{h,u+mn}+1) e_{A+(E^\vartriangle_{h,u+mn} - E^\vartriangle_{h,u})},$$

where $m \in \mathbb{Z} \setminus \{0\}$.

Proposition 2.8 ([21, 4.2]). *Let $\underline{i}, \underline{j}, \underline{k}, \underline{l} \in I(\mathbb{Z},r)$. We have the following equations in $S_\vartriangle(n,r)$.*

(1) $\xi_{\underline{i},\underline{j}}\xi_{\underline{k},\underline{l}} = 0$ *unless* $\underline{j} \sim_{\mathfrak{S}_\vartriangle} \underline{k}$.

(2) $\xi_{\underline{i},\underline{i}}\xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}}\xi_{\underline{j},\underline{j}}$.

(3) $\sum_{\underline{i}\in I(n,r)/\mathfrak{S}} \xi_{\underline{i},\underline{i}}$ *is a decomposition of unity into orthogonal idempotents.*

Proposition 2.9 ([22, Sec. 2]).

$$\begin{aligned}(1)\quad \xi_{\underline{i},\underline{j}}\xi_{\underline{j},\underline{l}} &= \sum_{\delta\in\mathfrak{S}_{\vartriangle,\underline{j},\underline{l}}\backslash\mathfrak{S}_{\vartriangle,\underline{j}}/\mathfrak{S}_{\vartriangle,\underline{i},\underline{j}}} \Big[\mathfrak{S}_{\vartriangle,\underline{i},\underline{l}\delta} : \mathfrak{S}_{\vartriangle,\underline{i},\underline{j},\underline{l}\delta}\Big]\xi_{\underline{i},\underline{l}\delta}\\ &= \sum_{\delta\in\mathfrak{S}_{\vartriangle,\underline{i},\underline{j}}\backslash\mathfrak{S}_{\vartriangle,\underline{j}}/\mathfrak{S}_{\vartriangle,\underline{j},\underline{l}}} \Big[\mathfrak{S}_{\vartriangle,\underline{i}\delta,\underline{l}} : \mathfrak{S}_{\vartriangle,\underline{i}\delta,\underline{j},\underline{l}}\Big]\xi_{\underline{i}\delta,\underline{l}},\end{aligned}$$

where $\underline{i}, \underline{j}, \underline{l}$ are in $I(\mathbb{Z},r)$, $\mathfrak{S}_{\vartriangle,\underline{i}}$ is the stabilizer subgroup of $\underline{i}$ in $\mathfrak{S}_\vartriangle$ and $\mathfrak{S}_{\vartriangle,\underline{i},\underline{j}}$ is the stabilizer of $\underline{i}$ and $\underline{j}$ in $\mathfrak{S}_\vartriangle$, i.e. $\mathfrak{S}_{\vartriangle,\underline{i},\underline{j}} = \mathfrak{S}_{\vartriangle,\underline{i}} \cap \mathfrak{S}_{\vartriangle,\underline{j}}$, etc, $\mathfrak{S}_{\vartriangle,\underline{j},\underline{l}} \backslash \mathfrak{S}_{\vartriangle,\underline{j}}/\mathfrak{S}_{\vartriangle,\underline{i},\underline{j}}$ denotes a representative set of double cosets.

$$(2)\quad \xi_{\underline{i},\underline{j}+n\varepsilon}\xi_{\underline{j},\underline{l}+n\varepsilon'} = \sum_{\delta\in\mathfrak{S}_{\underline{j},\underline{l},\varepsilon'}\backslash\mathfrak{S}_{\underline{j}}/\mathfrak{S}_{\underline{i},\underline{j},\varepsilon}} \left[\mathfrak{S}_{\underline{i},\underline{l}\delta,\varepsilon'\delta+\varepsilon} : \mathfrak{S}_{\underline{i},\underline{j},\underline{l}\delta,\varepsilon'\delta,\varepsilon}\right]\xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)}$$
$$= \sum_{\delta\in\mathfrak{S}_{\underline{i},\underline{j},\varepsilon}\backslash\mathfrak{S}_{\underline{j}}/\mathfrak{S}_{\underline{j},\underline{l},\varepsilon'}} \left[\mathfrak{S}_{\underline{i}\delta,\underline{l},\varepsilon'+\varepsilon\delta} : \mathfrak{S}_{\underline{i}\delta,\underline{j},\underline{l},\varepsilon',\varepsilon\delta}\right]\xi_{\underline{i}\delta,\underline{l}+n(\varepsilon'+\varepsilon\delta)}$$

where $\underline{i},\underline{j},\underline{l}$ *are in* $I(n,r)$, $\varepsilon,\varepsilon'\in\mathbb{Z}^r$, $\mathfrak{S}_{\underline{i}}$ *is the stabilizer subgroup of* $\underline{i}$ *in* $\mathfrak{S}$ *and* $\mathfrak{S}_{\underline{i},\underline{j}}$ *is the stabilizer of* $\underline{i}$ *and* $\underline{j}$ *in* $\mathfrak{S}$, *i.e.* $\mathfrak{S}_{\underline{i},\underline{j}} = \mathfrak{S}_{\underline{i}}\cap\mathfrak{S}_{\underline{j}}$, *etc,* $\mathfrak{S}_{\underline{j},\underline{l},\varepsilon'}\backslash\mathfrak{S}_{\underline{j}}/\mathfrak{S}_{\underline{i},\underline{j},\varepsilon}$ *denotes a representative set of double cosets.*

3. Affine cellular algebras

In this section let k be a noetherian domain. A commutative k-algebra B is called affine if it is a quotient of a polynomial ring $k[x_1,\dots,x_t]$ in finitely many variables. A k-linear anti-automorphism τ of a k-algebra A with $\tau^2 = id_A$ will be called a k-involution on A.

Definition 3.1 ([17, 2.1]). Let A be a unitary k-algebra with a k-linear involution τ on A. A two-sided ideal J in A is called an affine cell ideal if the following conditions are satisfied:

(1) $\tau(J) = J$.

(2) There is a free k-module V of finite rank and an affine algebra B with a k-involution σ such that $\Delta = V\otimes_k B$ is an A-B-bimodule, where the right B-module structure is induced by that of the regular right B-module B_B .

(3) There is an A-A-bimodule isomorphism $\alpha: J\to\Delta\otimes_B\Delta'$, where $\Delta' = B\otimes_k V$ is a B-A-bimodule with the left B-structure induced by ${}_BB$ and the right A-module structure is given as $(b\otimes v)a = p(\tau(a)(v\otimes b))$, where p is the switch map:

$$p: V\otimes_k B\longrightarrow B\otimes_k V,$$
$$v\otimes b\longmapsto b\otimes v,\ \text{where } v\in V \text{ and } b\in B;$$

(4) The map α in (3) makes the the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ \downarrow \tau & & \downarrow v_1\otimes b_1 \otimes_B b_2 \otimes v_2 \to v_2 \otimes \sigma(b_2) \otimes_B \sigma(b_1) \otimes v_1 \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \end{array}$$

The algebra A is called affine cellular if and only if there is a k-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots J'_n$ with $\tau(J'_j) = J'_j$ for each j, and $J_i = \oplus_{1\le l\le i} J'_l$ gives a chain of two-sided ideals of A:

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A,$$

and for each $1 \le i \le n$, $J'_i = J_i/J_{i-1}$ is an affine cell ideal of A/J_{i-1}. We call this chain of ideals of A an affine cell chain.

Remark 3.2. Note that by the definition of affine cellular algebras, an affine k-algebra is always affine cellular with respect to the identity map as a k-involution. In particular, the affine Schur algebra $S_\vartriangle(1,r)$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Q}[x_1, x_2, \ldots, x_{r-1}, x_r, x_r^{-1}]$. So $S_\vartriangle(1,r)$ is affine cellular over $\mathbb{Q}$. In the following section we will prove that $S_\vartriangle(2,2)$ is affine cellular over $\mathbb{Q}$.

4. Affine cellularity of $S_\vartriangle(2,2)$

In this section we prove that $S_\vartriangle(2,2)$ is affine cellular and has finite global dimension over $\mathbb{Q}$. First we recall the definition of affine Hecke algebra $H_\vartriangle(r)$ over $\mathbb{Q}[\boldsymbol{q}]$. It has a set of generators $T_i (1 \le i \le r), T_\rho^{\pm 1}$ with the following relations:

$$(4.1)\quad \begin{cases} T_i^2 = (\boldsymbol{q}-1)T_i + \boldsymbol{q}, \quad \text{for } 1 \le i \le r; \\ T_iT_j = T_jT_i, \quad \text{for } 1 \le i,j \le r \text{ with } |i-j| > 1; \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad \text{for } 1 \le i \le r, \text{ with } |i-j| = 1 \text{ and } r \ge 3; \\ T_\rho T_\rho^{-1} = T_\rho^{-1}T_\rho = 1,\ T_\rho T_i T_\rho^{-1} = T_{i+1}, \quad \text{for } 1 \le i \le r. \end{cases}$$

In the relations above, we identify T_{r+1} with T_1. By specializing $\boldsymbol{q}$ to 1, $H_\vartriangle(2)$ over $\mathbb{Q}$ has a presentation as follows:

$$(4.2)\quad \begin{cases} T_1^2 = 1,\ T_2^2 = 1,\ T_\rho T_\rho^{-1} = T_\rho^{-1}T_\rho = 1; \\ T_\rho T_1 T_\rho^{-1} = T_2,\ T_\rho T_2 T_\rho^{-1} = T_1. \end{cases}$$

Let $\lambda=(2,0),\mu=(0,2),\nu=(1,1)\in\Lambda(2,2)$. Then $e_\lambda+e_\mu+e_\nu$ is the unity in $S_\vartriangle(2,2)$. We can identify $H_\vartriangle(2)$ with $e_\nu S_\vartriangle(2,2)e_\nu$. The correspondence is given as follows:

$$\varphi : H_\vartriangle(2)\longrightarrow e_\nu S_\vartriangle(2,2)e_\nu, \tag{4.3}$$
$$\varphi(1)=e_\nu,\quad \varphi(T_1)=e_{(E^\vartriangle_{1,2}+E^\vartriangle_{2,1})},\quad \varphi(T_2)=e_{(E^\vartriangle_{2,3}+E^\vartriangle_{3,2})},$$
$$\varphi(T_\rho)=e_{(E^\vartriangle_{1,2}+E^\vartriangle_{2,3})},\quad \varphi(T_{\rho^{-1}})=e_{(E^\vartriangle_{2,1}+E^\vartriangle_{3,2})}.$$

Lemma 4.1. *Let J denote the ideal generated by e_λ in $S_\vartriangle(2,2)$, i.e. $J=S_\vartriangle(2,2)e_\lambda S_\vartriangle(2,2)$. Then $\bar{S}_\vartriangle(2,2):=S_\vartriangle(2,2)/J$ is isomorphic to $\mathbb{Q}[x,x^{-1}]$, where $\mathbb{Q}[x,x^{-1}]$ is the Laurent polynomial ring in variable x.*

Proof. By Proposition 2.5, we get that $e_\mu=e_{2E^\vartriangle_{2,2}}=e_{2E^\vartriangle_{2,1}}\cdot e_{2E^\vartriangle_{1,2}}$. Since $e_{2E^\vartriangle_{2,1}}\cdot e_\lambda=e_{2E^\vartriangle_{2,1}}$ and $e_\lambda\cdot e_{2E^\vartriangle_{1,2}}=e_{2E^\vartriangle_{1,2}}$, we get that $e_\mu=e_{2E^\vartriangle_{2,1}}\cdot e_\lambda\cdot e_{2E^\vartriangle_{1,2}}\in J$.

So $\bar{e}_\nu$ is the unity in $\bar{S}_\vartriangle(2,2)$, and $\bar{e}_\nu\bar{S}_\vartriangle(2,2)\bar{e}_\nu=\bar{S}_\vartriangle(2,2)$. Note there is the following isomorphism

$$\bar{e}_\nu\bar{S}_\vartriangle(2,2)\bar{e}_\nu\cong(e_\nu S_\vartriangle(2,2)e_\nu)/(e_\nu Je_\nu).$$

By identifying $H_\vartriangle(2)$ with $e_\nu S_\vartriangle(2,2)e_\nu$ by (4.3), we prove that J equals the ideal of $S_\vartriangle(2,2)$ generated by T_1+e_ν and T_2+e_ν.

Let J' be the ideal of $S_\vartriangle(2,2)$ generated by T_1+e_ν and T_2+e_ν. We first show that $J'\subseteq J$. By Proposition 2.5, we have

$$e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})}\cdot e_\lambda=e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})},\ e_\lambda\cdot e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}=e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}.$$

Then we get

$$e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})}\cdot e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}=e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})}\cdot e_\lambda\cdot e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}\in J.$$

Since

$$e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})}\cdot e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}=e_{(E^\vartriangle_{1,2}+E^\vartriangle_{2,1})}+e_\nu=T_1+e_\nu,$$

this proves $T_1+e_\nu\in J$. By (4.2), $T_\rho^{-1}(T_1+e_\nu)T_\rho=T_2+e_\nu$, then $T_2+e_\nu\in J$. This proves $J'\subseteq J$.

By Proposition 2.5, we have

$$e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}(T_1+e_\nu)e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})}=4e_\lambda,$$

then $e_\lambda\in J'$, which proves $J\subseteq J'$.

Note that $e_\nu J' e_\nu$ equals the ideal of $H_\vartriangle(2)$ generated by T_1+1 and T_2+1. By $J=J'$ and identifying $H_\vartriangle(2)$ with $e_\nu S_\vartriangle(2,2)e_\nu$, we have

$$(e_\nu S_\vartriangle(2,2)e_\nu)/(e_\nu J e_\nu) \cong H_\vartriangle(2)/(e_\nu J' e_\nu) \cong \mathbb{Q}[T_\rho, T_\rho^{-1}].$$

This completes the proof of the lemma. □

Remark 4.2. By Remark 2.3, $T_1 = e_{(E^\vartriangle_{1,2}+E^\vartriangle_{2,1})}$ corresponds to $\xi_{\underline{i},\underline{j}}$, where $\underline{i}=(1,2)$ and $\underline{j}=(2,1)$ are both in $I(2,2)$. Since $(\underline{i},\underline{j}) \sim_{\mathfrak{S}_\vartriangle} (\underline{j},\underline{i})$, by Proposition 2.9,

$$T_1^2 = \xi_{\underline{i},\underline{j}} \cdot \xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}} \cdot \xi_{\underline{j},\underline{i}} = \xi_{\underline{i},\underline{i}} = e_\nu.$$

In the following, we will describe J as a bimodule over $S_\vartriangle(2,2)$, and prove it is projective as both left and right $S_\vartriangle(2,2)$-module. This affords a solution to bound the global dimension of $S_\vartriangle(2,2)$.

The following lemma which is given in [22] describes the algebra $e_\lambda S_\vartriangle(2,2)e_\lambda$.

Lemma 4.3 ([22, Prop. 6]). *$e_\lambda S_\vartriangle(2,2)e_\lambda$ is isomorphic to the Laurent polynomial ring $\mathbb{Q}[x_1,x_2,x_2^{-1}]$.*

The isomorphism between $e_\lambda S_\vartriangle(2,2)e_\lambda$ and $\mathbb{Q}[x_1,x_2,x_2^{-1}]$ is given as follows:

$$\psi : e_\lambda S_\vartriangle(2,2)e_\lambda \longrightarrow \mathbb{Q}[x_1,x_2,x_2^{-1}], \tag{4.4}$$
$$\psi(e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,3})}) = x_1, \quad \psi(e_{(2E^\vartriangle_{1,3})}) = x_2, \quad \psi(e_{(2E^\vartriangle_{3,1})}) = x_2^{-1}.$$

Lemma 4.4. *Let $B = e_\lambda S_\vartriangle(2,2)e_\lambda$. Then $e_\lambda S_\vartriangle(2,2)$ is a free left B-module of rank four, $S_\vartriangle(2,2)e_\lambda$ is a free right B-module of rank four.*

Proof. $e_\lambda S_\vartriangle(2,2) = \mathbb{Q}\{e_A \mid \mathrm{row}(A)=\lambda\} = \mathbb{Q}\{e_{(E^\vartriangle_{1,i}+E^\vartriangle_{1,j})} \mid i,j\in\mathbb{Z}\}$. Identify B with $\mathbb{Q}[x_1,x_2,x_2^{-1}]$ and by Proposition 2.7, the actions of x_1 on $e_\lambda S_\vartriangle(2,2)$ is given as follows.

$$x_1 \cdot e_{(E^\vartriangle_{1,i}+E^\vartriangle_{1,j})} = \begin{cases} e_{(E^\vartriangle_{1,i+2}+E^\vartriangle_{1,j})} + e_{(E^\vartriangle_{1,i}+E^\vartriangle_{1,j+2})}, & i\neq j,\ i\neq j\pm 2; \\ e_{(E^\vartriangle_{1,i+2}+E^\vartriangle_{1,j})}, & i=j; \\ 2e_{(E^\vartriangle_{1,i+2}+E^\vartriangle_{1,j})} + e_{(E^\vartriangle_{1,i}+E^\vartriangle_{1,j+2})}, & i=j-2; \\ e_{(E^\vartriangle_{1,i+2}+E^\vartriangle_{1,j})} + 2e_{(E^\vartriangle_{1,i}+E^\vartriangle_{1,j+2})}, & i=j+2. \end{cases}$$

By Remark 2.3, identify x_2 with $\xi_{\underline{l},\underline{m}}$, where $\underline{l}=(1,1),\underline{m}=(3,3)\in I(\mathbb{Z},2)$, and identify $e_{(E^\Delta_{1,i}+E^\Delta_{1,j})}$ with $\xi_{\underline{l},\underline{u}}$, where $\underline{l}=(1,1),\underline{u}=(i,j)\in I(\mathbb{Z},2)$. By Proposition 2.9 we have

$$x_2\cdot e_{(E^\Delta_{1,i}+E^\Delta_{1,j})}=e_{(E^\Delta_{1,i+2}+E^\Delta_{1,j+2})},$$
$$x_2^{-1}\cdot e_{(E^\Delta_{1,i}+E^\Delta_{1,j})}=e_{(E^\Delta_{1,i-2}+E^\Delta_{1,j-2})}.$$

Now we prove that $e_\lambda S_\Delta(2,2)$ as a left B-module is generated by the set

$$\Pi_1=\{e_{(E^\Delta_{1,1}+E^\Delta_{1,2})},\ e_{(E^\Delta_{1,2}+E^\Delta_{1,3})},\ e_{(2E^\Delta_{1,1})},\ e_{(2E^\Delta_{1,2})}\}.$$

Let

$$\mathcal{X}=\{e_{(E^\Delta_{1,i}+E^\Delta_{1,j})}\mid i,j\in\mathbb{Z}\}.$$

We prove that each element in $\mathcal{X}$ is contained in the B-linear combinations of elements in Π_1. First if $i=j$, we have

$$e_{(2E^\Delta_{1,i})}=\begin{cases}x_2^k\cdot e_{(2E^\Delta_{1,1})}, & i=2k+1;\\ x_2^{(k-1)}\cdot e_{(2E^\Delta_{1,2})}, & i=2k.\end{cases}$$

If $i\neq j$, we have

$$e_{(E^\Delta_{1,i}+E^\Delta_{1,j})}=\begin{cases}x_2^k\cdot e_{(E^\Delta_{1,1}+E^\Delta_{1,j-i+1})}, & i=2k+1;\\ x_2^{(k-1)}\cdot e_{(E^\Delta_{1,2}+E^\Delta_{1,j-i+2})}, & i=2k.\end{cases}$$

So we only need to prove that $X_l=e_{(E^\Delta_{1,1}+E^\Delta_{1,l})}$ and $Y_l=e_{(E^\Delta_{1,2}+E^\Delta_{1,l})}$ for $l\in\mathbb{Z}$ can be generated by Π_1.

Note that

$$x_1\cdot X_l=\begin{cases}X_{l+2}+x_2\cdot X_{l-2}, & l\neq -1,1,3;\\ X_5+2x_2X_1, & l=3;\\ X_3, & l=1;\\ 2X_1+x_2X_{-3}, & l=-1.\end{cases}\tag{4.5}$$

This gives the equalities

$$X_{l+2}=x_1\cdot X_l-x_2\cdot X_{l-2},\quad \text{and}\quad X_{l-2}=x_2^{-1}x_1\cdot X_l-x_2^{-1}\cdot X_{l+2},\tag{4.6}$$

for $l\neq -1,3$. Since $X_1,X_2\in\Pi_1$, by (4.5),

$$X_3=x_1X_1,\quad X_4=x_1\cdot X_2-e_{(E^\Delta_{1,2}+E^\Delta_{1,3})},$$
$$X_5=x_1\cdot X_3-2x_2\cdot X_1,\quad X_{-3}=x_2^{-1}x_1\cdot X_{-1}-2x_2^{-1}\cdot X_1$$

are all generated by Π_1.

So we can prove each X_l for $l \in \mathbb{Z}$ is generated by Π_1 by induction on l for $l \in \mathbb{Z}$ using (4.6).

Similarly we have

$$x_1 \cdot Y_l = \begin{cases} Y_{l+2} + x_2 \cdot Y_{l-2}, & l \neq 0, 2, 4; \\ Y_6 + 2x_2 Y_2, & l = 4; \\ Y_4, & l = 2; \\ 2Y_2 + x_2 Y_{-2}, & l = 0. \end{cases} \tag{4.7}$$

Since $Y_1, Y_2 \in \Pi_1$, we can use a similar induction on l to prove that Y_l for $l \in \mathbb{Z}$ are all generated by Π_1.

Now we prove elements in Π_1 are $e_\lambda S_\vartriangle(2,2)e_\lambda$-linearly independent. By Remark 2.4, $S_\vartriangle(2,2)$ is a $\mathbb{Z}$-graded algebra over $\mathbb{Q}$ and the degrees of

$$e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})},\ e_{(E^\vartriangle_{1,2}+E^\vartriangle_{1,3})},\ e_{(2E^\vartriangle_{1,1})},\ e_{(2E^\vartriangle_{1,2})}$$

are $1, 3, 0, 2$ respectively, and the degrees of x_1, x_2, x_2^{-1} are $2, 4, -4$ respectively. Since x_1 and x_2 both have even degrees, we only need to prove that $e_{(2E^\vartriangle_{1,1})}$, $e_{(2E^\vartriangle_{1,2})}$ are linearly independent and $e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})}$, $e_{(E^\vartriangle_{1,2}+E^\vartriangle_{1,3})}$ are linearly independent.

Suppose there is an equation

$$f \cdot e_{(2E^\vartriangle_{1,1})} + g \cdot e_{(2E^\vartriangle_{1,2})} = 0,$$

where $f, g \in \mathbb{Q}[x_1, x_2, x_2^{-1}]$. We want to show that $f = g = 0$. Since $x_1 \cdot e_{(2E^\vartriangle_{1,1})} = x_1$ and $x_2 \cdot e_{(2E^\vartriangle_{1,1})} = x_2$, then $f \cdot e_{(2E^\vartriangle_{1,1})} = f$ and

$$f + g \cdot e_{(2E^\vartriangle_{1,2})} = 0. \tag{4.8}$$

Suppose $f = \sum_A a_A e_A$, where $a_A \in \mathbb{Q}$ and $A \in \Theta_\vartriangle(n, r)$. Then by (2.3) each term e_A in f satisfies $\mathrm{col}(A) = \mathrm{col}(2E^\vartriangle_{1,1}) = \lambda$. Suppose that

$$g \cdot e_{(2E^\vartriangle_{1,2})} = \sum_{A'} b_{A'} e_{A'},$$

where $b_{A'} \in \mathbb{Q}$ and $A' \in \Theta_\vartriangle(n, r)$. Then by (2.3) each term $e_{A'}$ in $g \cdot e_{(2E^\vartriangle_{1,2})}$ has the property that $\mathrm{col}(A') = \mathrm{col}(2E^\vartriangle_{1,2}) = \mu$. Since the set $\{e_A \mid A \in \Theta_\vartriangle(n, r)\}$ is a basis for $S_\vartriangle(2,2)$, bases corresponding to matrices of different columns are linearly independent. This implies that $f = 0$ and $g \cdot e_{(2E^\vartriangle_{1,2})} = 0$. Then $b_{A'} = 0$ for each A' occurring in $g \cdot e_{(2E^\vartriangle_{1,2})}$. By Proposition 2.6, we get that $g=\sum_{A'[1]} b_{A'}e_{A'[1]}$, where $A'[1]$ is the matrix such that $A'[1]$'s ith-column is the same as that A''s $(i+1)$th-column. This proves that $g=0$ and $e_{(2E^{\vartriangle}_{1,1})}$, $e_{(2E^{\vartriangle}_{1,2})}$ are linearly independent.

Now we prove that $e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}$, $e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}$ are linearly independent. Suppose there is an equation

$$f'\cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}+g'\cdot e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}=0, \tag{4.9}$$

where $f',g'\in\mathbb{Q}[x_1,x_2,x_2^{-1}]$ and at least one of f',g' are not zero. We want to find a contradiction. Since x_2 is invertible, we assume each term containing x_2 in f' and g' has positive degrees in x_2.

Now suppose $f'=\sum_{a,b} m_{a,b}x_1^a x_2^b$ and $g'=\sum_{c,d} n_{c,d}x_1^c x_2^d$, where $m_{a,b}$, $n_{c,d}\in\mathbb{Q}$. By Proposition 2.7 and Proposition 2.9, we get

$$x_1^a x_2^b\cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}=\sum_{0\le k\le 2a} p_k e_{(E^{\vartriangle}_{1,2b+1+k}+E^{\vartriangle}_{1,2b+2+2a-k})}, \tag{4.10}$$

$$x_1^c x_2^d\cdot e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}=\sum_{0\le l\le 2c} h_l e_{(E^{\vartriangle}_{1,2d+2+l}+E^{\vartriangle}_{1,2d+3+2c-l})}, \tag{4.11}$$

where

$$\begin{cases} p_k=1, & k=0,\text{ or } 2a;\\ p_k>1, & 1\le k\le 2a-1;\end{cases}\qquad \begin{cases} h_l=1, & l=0,\text{ or } 2c;\\ h_l>1, & 1\le l\le 2c-1.\end{cases} \tag{4.12}$$

Suppose $f'\neq 0$. Let $m_{a_0,b_0}x_1^{a_0}x_2^{b_0}$ be any nonzero term appearing in f' with b_0 minimal. If $e_{(E^{\vartriangle}_{1,2b_0+1}+E^{\vartriangle}_{1,2b_0+2+2a_0})}$ appears in $f'\cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}$ with coefficient s, then by (4.12), $s\ge m_{a_0,b_0}\neq 0$. By (4.9), there is some d_0 and some l with $0\le l\le 2c$ such that

$$e_{(E^{\vartriangle}_{1,2b_0+1}+E^{\vartriangle}_{1,2b_0+2+2a_0})}=e_{(E^{\vartriangle}_{1,2d_0+2+l}+E^{\vartriangle}_{1,2d_0+3+2c-l})}.$$

Then we get that $e_{(E^{\vartriangle}_{1,2d_0+2}+E^{\vartriangle}_{1,2d_0+3+2c})}$ appears in $g'\cdot e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}$ with nonzero coefficient by (4.12). By b_0 is minimal, we cannot find this term in $f'\cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}$. This contradicts (4.9).

Similarly, we can find a contradiction under the assumption that $g'\neq 0$. This proves that $e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}$, $e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}$ are linearly independent.

Then Π_1 is a basis of $e_\lambda S_\vartriangle(2,2)$ over $e_\lambda S_\vartriangle(2,2)e_\lambda$. In a similar way we can prove that the set

$$\Pi_2=\left\{e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{2,1})},\ e_{(E^{\vartriangle}_{2,1}+E^{\vartriangle}_{3,1})},\ e_{(2E^{\vartriangle}_{1,1})},\ e_{(2E^{\vartriangle}_{2,1})}\right\}$$

forms a basis of $S_\vartriangle(2,2)e_\lambda$ over $e_\lambda S_\vartriangle(2,2)e_\lambda$. □

Proposition 4.5. *There is an $S_\vartriangle(2,2)$-bimodule isomorphism*

$$\alpha : S_\vartriangle(2,2)e_\lambda \otimes_B e_\lambda S_\vartriangle(2,2) \longrightarrow S_\vartriangle(2,2)e_\lambda S_\vartriangle(2,2),$$

and $J = S_\vartriangle(2,2)e_\lambda S_\vartriangle(2,2)$ is projective as both a left and a right $S_\vartriangle(2,2)$-module.

Proof. Let $B = e_\lambda S_\vartriangle(2,2)e_\lambda$ and let $\{b_i\}_{i\in I}$ be a $\mathbb{Q}$-basis of B. There is a canonical epimorphism

$$\begin{aligned}\alpha :\ S_\vartriangle(2,2)e_\lambda \otimes_B e_\lambda S_\vartriangle(2,2) &\longrightarrow S_\vartriangle(2,2)e_\lambda S_\vartriangle(2,2),\\ e_A b_i \otimes b_{i'} e_C &\longrightarrow e_A b_i b_{i'} e_C,\end{aligned}$$

where $e_A \in \Pi_2, e_C \in \Pi_1, i, i' \in I$.

The set

$$\Omega = \{e_A b_i \otimes e_C \mid e_A \in \Pi_2,\ e_C \in \Pi_1,\ i \in I\}$$

is a basis of $S_\vartriangle(2,2)e_\lambda \otimes_B e_\lambda S_\vartriangle(2,2)$ over $\mathbb{Q}$. Let

$$\Omega' = \{e_A b_i e_C \mid e_A \in \Pi_2,\ e_C \in \Pi_1,\ i \in I\}.$$

We prove α is injective by showing that elements in Ω' are $\mathbb{Q}$-linearly independent.

Recall that

$$\Pi_1 = \left\{e_{(E^\vartriangle_{1,1}+E^\vartriangle_{1,2})},\ e_{(E^\vartriangle_{1,2}+E^\vartriangle_{1,3})},\ e_{(2E^\vartriangle_{1,1})},\ e_{(2E^\vartriangle_{1,2})}\right\}$$

and

$$\Pi_2 = \left\{e_{(E^\vartriangle_{1,1}+E^\vartriangle_{2,1})},\ e_{(E^\vartriangle_{2,1}+E^\vartriangle_{3,1})},\ e_{(2E^\vartriangle_{1,1})},\ e_{(2E^\vartriangle_{2,1})}\right\}.$$

Let π_l for $1 \le l \le 4$ denote the elements in Π_1 and let π'_l for $1 \le l \le 4$ denote the elements in Π_2. Define

$$\Omega'_{l,m} = \{\pi'_l b_i \pi_m \mid i \in I\},$$

then

$$\Omega' = \cup_{1\le l,m\le 4}\Omega'_{l,m}.$$

The elements in $\Omega'_{l,m}$ are $\mathbb{Q}$ combinations of basis e_A such that

$$\mathrm{row}(A) = \mathrm{row}(C), \quad \mathrm{col}(A) = \mathrm{col}(D) \quad \text{for } 1 \le l, m \le 4,$$

where $\pi'_l = e_C$ and $\pi_m = e_D$. Since $\{e_A \mid A \in \Theta_\vartriangle(2,2)\}$ is a basis of $S_\vartriangle(2,2)$, we can divide Ω' into disjoint union of subsets according to the row and column vectors of the matrices corresponding to the basis elements, such that elements in different subsets are linearly independent. It suffices to prove elements in the following sets are linearly independent.

$$\begin{aligned} &\Omega'_{l,m}, \quad 3 \leq l, m \leq 4, \\ &\bigcup_{1\leq l,m\leq 2} \Omega'_{l,m}, \\ &\Omega'_{3,1} \cup \Omega'_{3,2}, \\ &\Omega'_{4,1} \cup \Omega'_{4,2}, \\ &\Omega'_{1,3} \cup \Omega'_{2,3}, \\ &\Omega'_{1,4} \cup \Omega'_{2,4}. \end{aligned}$$

We only prove elements in $\Omega'_{3,1} \cup \Omega'_{3,2}$ are linearly independent. Other cases can be proved similarly. Suppose

$$e_{(2E^{\vartriangle}_{1,1})} \cdot f \cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})} + e_{(2E^{\vartriangle}_{1,1})} \cdot g \cdot e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})} = 0,$$

where f, g are $\mathbb{Q}$-linear combinations of elements in $\{b_i\}_{i\in I}$. Then

$$e_{(2E^{\vartriangle}_{1,1})} \cdot \Big(f \cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})} + g \cdot e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}\Big) = 0.$$

by Proposition 2.8,

$$f \cdot e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})} + g \cdot e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})} = 0.$$

Then we can get $f = g = 0$, since $e_{(E^{\vartriangle}_{1,1}+E^{\vartriangle}_{1,2})}, e_{(E^{\vartriangle}_{1,2}+E^{\vartriangle}_{1,3})}$ belong to Π_1.

This proves that $J \cong S_{\vartriangle}(2,2)e_\lambda \otimes_B e_\lambda S_{\vartriangle}(2,2)$ as $S_{\vartriangle}(2,2)$-bimodules. Now we prove $S_{\vartriangle}(2,2)e_\lambda \otimes_B e_\lambda S_{\vartriangle}(2,2)$ is projective as both a left and a right $S_{\vartriangle}(2,2)$-module.

Since $e_\lambda S_{\vartriangle}(2,2)$ and $S_{\vartriangle}(2,2)e_\lambda$ are free modules over B, $S_{\vartriangle}(2,2)e_\lambda \otimes_B e_\lambda S_{\vartriangle}(2,2)$ is projective as both a left and a right $S_{\vartriangle}(2,2)$-module. Then J is projective as both a left and a right $S_{\vartriangle}(2,2)$-module follows from the isomorphism constructed above. □

Theorem 4.6. *$S_{\vartriangle}(2,2)$ is affine cellular over $\mathbb{Q}$.*

Proof. Let τ be the anti-involution of $S_{\vartriangle}(2,2)$ given by $\tau(e_A) = e_{A'}$, where $A \in \Theta_{\vartriangle}(2,2)$ and A' denotes the transpose of A. Let $J = S_{\vartriangle}(2,2)e_\lambda S_{\vartriangle}(2,2)$. It is obvious that $\tau(J) = J$. We first show that J is an affine cell ideal of $S_{\vartriangle}(2,2)$.

Let

$$\Pi_1 = \left\{e_{(E^{\triangle}_{1,1}+E^{\triangle}_{1,2})},\ e_{(E^{\triangle}_{1,2}+E^{\triangle}_{1,3})},\ e_{(2E^{\triangle}_{1,1})},\ e_{(2E^{\triangle}_{1,2})}\right\}$$

and

$$\Pi_2 = \left\{e_{(E^{\triangle}_{1,1}+E^{\triangle}_{2,1})},\ e_{(E^{\triangle}_{2,1}+E^{\triangle}_{3,1})},\ e_{(2E^{\triangle}_{1,1})},\ e_{(2E^{\triangle}_{2,1})}\right\}$$

be the same as in Lemma 4.4. It is easy to find that

$$\Pi_1 = \{\tau(e_A) \mid e_A \in \Pi_2\}. \tag{4.13}$$

Let V be the free $\mathbb{Q}$-module on the basis Π_2. Suppose that

$$\Delta = V \otimes_{\mathbb{Q}} B, \quad \Delta' = B \otimes_{\mathbb{Q}} V,$$

where $B = e_\lambda S_\triangle(2,2)e_\lambda$. Then by Lemma 4.4 we have the following isomorphism of $S_\triangle(2,2)$-B-bimodules

$$\begin{aligned}\varphi : \Delta &\longrightarrow S_\triangle(2,2)e_\lambda,\\ v\otimes b &\longrightarrow v\cdot b,\end{aligned}$$

where $v \in V, b \in B$. And by (4.13) we have the following isomorphism of B-$S_\triangle(2,2)$-bimodules

$$\begin{aligned}\psi : \Delta' &\longrightarrow e_\lambda S_\triangle(2,2),\\ b\otimes v &\longrightarrow b\cdot\tau(v),\end{aligned}$$

where $b \in B, v \in V$.

Then by Proposition 4.5, we have the following $S_\triangle(2,2)$-bimodule isomorphism:

$$\begin{aligned}\alpha : \Delta\otimes_B \Delta' &\longrightarrow J,\\ v_1\otimes b_1\otimes b_2\otimes v_2 &\longrightarrow v_1\cdot b_1\cdot b_2\cdot\tau(v_2),\end{aligned} \tag{4.14}$$

where $b_1, b_2 \in B$ and $v_1, v_2 \in V$.

Now define

$$\begin{aligned}f : \Delta\otimes_B \Delta' &\longrightarrow \Delta\otimes_B \Delta',\\ v_1\otimes b_1\otimes b_2\otimes v_2 &\longrightarrow v_2\otimes\tau(b_2)\otimes\tau(b_1)\otimes v_1,\end{aligned}$$

where $b_1, b_2 \in B$ and $v_1, v_2 \in V$.

Then we have the following commutative diagram:

$$\begin{array}{ccc} J & \xleftarrow[\alpha]{} & \Delta \otimes_B \Delta' \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle f} \\ J & \xleftarrow[\alpha]{} & \Delta \otimes_B \Delta' \end{array}$$

Let α^{-1} be the inverse of the bimodule isomorphism α given in (4.14). Since $\tau\alpha = \alpha f$, we get that $\alpha^{-1}\tau = f\alpha^{-1}$, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} J & \xrightarrow{\alpha^{-1}} & \Delta \otimes_B \Delta' \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle f} \\ J & \xrightarrow{\alpha^{-1}} & \Delta \otimes_B \Delta' \end{array}$$

This shows that J is an affine cell ideal of $S_\vartriangle(2,2)$. By Lemma 4.1, $S_\vartriangle(2,2)/J \cong \mathbb{Q}[x, x^{-1}]$. Then $S_\vartriangle(2,2)/J$ is an affine cellular algebra over $\mathbb{Q}$ by Remark 3.2. Now we have a chain of ideals

$$0 \subset J \subset S_\vartriangle(2,2)$$

and a decomposition of $S_\vartriangle(2,2)$ as a $\mathbb{Q}$-vector space(In fact, this is an decomposition of $S_\vartriangle(2,2)$-module):

$$S_\vartriangle(2,2) = J \bigoplus S_\vartriangle(2,2)/J.$$

By Lemma 4.1, $\tau(S_\vartriangle(2,2)/J) = S_\vartriangle(2,2)/J$. This implies that this chain and decomposition satisfy the conditions given in Definition 3.1. This proves that $S_\vartriangle(2,2)$ is an affine cellular algebra over $\mathbb{Q}$. □

Corollary 4.7. *The global dimension of $S_\vartriangle(2,2)$ is finite over $\mathbb{Q}$.*

Proof. By Lemma 4.5 in [17], if R is a ring and $e = e^2 \in R$ with ReR projective as a left R-module, then the global dimension of R is finite if and only if the global dimension of (R/ReR) and the global dimension of (eRe) are finite.

By Proposition 4.5, $J = S_\vartriangle(2,2)e_\lambda S_\vartriangle(2,2)$ is projective as a left $S_\vartriangle(2,2)$-module. By Lemma 4.1, $S_\vartriangle(2,2)/J \cong \mathbb{Q}[x, x^{-1}]$. By Lemma 4.3, $e_\lambda S_\vartriangle(2,2)e_\lambda \cong \mathbb{Q}[x_1, x_2, x_2^{-1}]$. Then both the global dimension of $S_\vartriangle(2,2)/J$ and the global dimension of $e_\lambda S_\vartriangle(2,2)e_\lambda$ are finite. This proves that the global dimension of $S_\vartriangle(2,2)$ is finite over $\mathbb{Q}$. □

Remark 4.8. In [2, Def. 2.4], affine cellular bases of affine cellular algebras are introduced. In fact, suppose that

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

is an affine cell chain of the affine cellular algebra A as in Definition 3.1, such that

$$J_i/J_{i-1} \cong \Delta_i \otimes_{B_i} \Delta_i$$

is an A/J_{i-1}-bimodule isomorphism for each $1 \le i \le n$. Let the set $\{c^i_{s,t} \mid s, t \in \mathcal{T}(i)\}$ be a B_i-basis of J_i/J_{i-1} which satisfies conditions in [2, Def. 2.4](analogous conditions to that of cellular basis), for $1 \le i \le n$. Then $\{c^i_{s,t} \mid s, t \in \mathcal{T}(i),\ 1 \le i \le n\}$ is called an affine cellular basis of A.

If in addition that A is a graded algebra, we can give a definition of graded affine cellular basis(algebra) which is an affine analogue of graded cellular basis(algebra) given in [13, 2.1]. An affine cellular basis(algebra) is called a graded affine cellular basis(algebra) if each element in $\{c^i_{s,t} \mid s, t \in \mathcal{T}(i),\ 1 \le i \le n\}$ is homogeneous.

By Proposition 4.5, $\{\pi_i \cdot \pi'_j \mid 1 \le i, j \le 4\}$ forms a basis of $J = S_\vartriangle(2,2)e_\lambda \cdot S_\vartriangle(2,2)$ over $e_\lambda S_\vartriangle(2,2)e_\lambda$. By Lemma 4.1, $S_\vartriangle(2,2)/J \cong e_\nu(S_\vartriangle(2,2)/J)e_\nu$. Then we find that $\{\pi_i \cdot \pi'_j, \mid 1 \le i, j \le 4\}$ and the identity element of $S_\vartriangle(2,2)$ form an affine cellular basis of $S_\vartriangle(2,2)$. It is easy to check that each basis element is homogeneous under the grading of $S_\vartriangle(2,2)$ given in Remark 2.4. Then $S_\vartriangle(2,2)$ is a graded affine cellular algebra in this sense.

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