$p$-adic Banach Modules of Arithmetical Modular Forms and Triple Products of Coleman’s Families

A. A. Panchishkin

To dear Jean-Pierre Serre for his eightieth birthday with admiration

Abstract: For a prime number $p \geq 5$, we consider three classical cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j}e(nz) \in S_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3)$$

of weights $k_1, k_2, k_3$, of conductors $N_1, N_2, N_3$, and of nebentypus characters $\psi_j \mod N_j$. According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each $f_j$ ($j = 1, 2, 3$) (under suitable assumptions on $p$ and on $f_j$)

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j})q^n\}$$

into a $p$-adic analytic family of cusp eigenforms $f_{j,k_j}$ of weights $k_j$ in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain $p$-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.

The conductor of the members of the families divides the prime-to-$p$ part of the level $N = \text{LCM}(N_1,N_2,N_3)$.

The purpose of this paper is to describe a four variable $p$-adic $L$ function attached to Garrett’s triple product of three Coleman’s families

$$k_j \mapsto \left\{f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k)q^n\right\}$$

Received July 26, 2006.
Based on a talk for the French-German Seminar in Lille on November 9, 2005
of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha_{p,j}^{(1)}(k_j)) \geq 0$ where $\alpha_{p,j}^{(1)} = \alpha_{p,j}^{(1)}(k_j)$ is an eigenvalue (which depends on $k_j$) of Atkin’s operator $U = U_p$ acting on Fourier expansions by $U(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} a_{np} q^n$.

Let us consider the product of three eigenvalues:

$$\lambda = \lambda(k_1, k_2, k_3) = \alpha_{p,1}^{(1)}(k_1)\alpha_{p,2}^{(1)}(k_2)\alpha_{p,3}^{(1)}(k_3)$$

and assume that the slope of this product

$$\sigma = v_p(\lambda(k_1, k_2, k_3)) = \sigma(k_1, k_2, k_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets $(k_1, k_2, k_3)$ in an appropriate $p$-adic neighbourhood of the fixed triplet of weights $(k_1, k_2, k_3)$. The each value $\sigma_j$ is fixed.

We consider the $p$-adic weight space $X$ containing all $(k_j, \psi_j)$. Our $p$-adic $L$-functions are Mellin transforms of certain measures with values in $A$, where $A = A(B)$ denotes an affinoid algebra associated with an affinoid space $B$ as in [CoPB], where $B = B_1 \times B_2 \times B_3$, is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integers $k_j$ and fixed Dirichlet characters $\psi_j$ mod $N$).

We construct such a measure from higher twists of classical Siegel-Eisenstein series, which produce distributions with values in certain Banach $A$-modules $M = M(N; A)$ of triple modular forms with coefficients in the algebra $A$.

Acknowledgement. A part of these results was exposed in a talk at the French-German Seminar in Lille and the author is grateful to V.Gritsenko for the invitation.

It is a great pleasure for me to thank S.Boecherer, C.-G.Schmidt and R.Schulze-Pillot for valuable discussions.

Details of computations and proofs will appear elsewhere (a joint article with S.Boecherer in preparation for a special volume in the Contemporary Math. series of the AMS).

Contents

1. Introduction 1135

Why study $L$-values attached to modular forms? 1135

2. Generalities on triple products 1138

3. Statement of the problem 1139

4. Arithmetical nearly holomorphic modular forms 1142
1. Introduction

Why study $L$-values attached to modular forms? A popular procedure in Number Theory is the following:
Construct a generating function \( f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]] \) of an arithmetical function \( n \mapsto a_n \), for example \( a_n = p(n) \)

\[ \sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24} \]

**Example 1 [Chand70]:**

(Hardy-Ramanujan)

\[ p(n) = e^{2\sqrt[3]{3(n-1/24)}} + O(e^{2\sqrt[3]{3(n-1/24)}/x_n}), \quad x_n = \sqrt{n-1/24}. \]

Good bases, \( F \)-adic, finite dimensions, values of \( L \)-functions, periods, many relations, identities, congruences, . . .

Other examples: Birch and Swinnerton-Dyer conjecture, \( \ldots L \)-values attached to modular forms

**Our data: three primitive cusp eigenforms.**

\[ f_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n \in S_{k_j}(N_j, \psi_j), \quad (j = 1, 2, 3) \]

of weights \( k_1, k_2, k_3 \), of conductors \( N_1, N_2, N_3 \), and of nebentypus characters \( \psi_j \) mod \( N_j \), \( N = \text{LCM}(N_1, N_2, N_3) \).

Let \( p \) be a prime, \( p \nmid N \). We view \( f_j \in \overline{\mathbb{Q}}[[q]] \hookrightarrow \mathbb{C}_p[[q]] \) via a fixed embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \), \( \mathbb{C}_p = \hat{\mathbb{Q}}_p \) is Tate’s field.

Let \( \chi \) denote a variable Dirichlet character mod \( Np^v, v \geq 0 \).

We view \( k_j \) as a variable weight in the weight space \( X = X_{Np^v} = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^*), \)

\( Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^* \ni (y_0, y_p) \).

The space \( X \) is a \( p \)-adic analytic space first used in Serre’s [Se73] “Formes modulaires et fonctions zêta \( p \)-adiques”. Denote by \( (k, \chi) \in X \) the homomorphism \( (y_0, y_p) \mapsto \chi(y_0)\chi(y_p \mod p^v) y_p^k \). We write simply \( k_j \) for the couple \((k_j, \psi_j) \in X\).

The purpose of this paper is to describe a four variable \( p \)-adic \( L \) function attached to Garrett’s triple product of three Coleman’s families.

\[ k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k_j) q^n \right\} \]

of cusp eigenforms of three constant slopes \( \sigma_j = \text{ord}_p(\alpha_{p,j}^{(1)}(k_j)) \geq 0 \) where \( \alpha_{p,j}^{(1)}(k_j), \alpha_{p,j}^{(2)}(k_j) \) are the Satake parameters given as inverse roots of the Hecke \( p \)-polynomial \( 1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha_{p,j}^{(1)}(p)X)(1 - \alpha_{p,j}^{(2)}(p)X) \). Each
family of the associated primitive cusp eigenforms have conductors $C_{kj}$ which divide $N$ if $k_j > 2\sigma_j + 2$ (see [CoM]).

We assume that $\text{ord}_p(\alpha^{(1)}_{p,j}(k_j)) \leq \text{ord}_p(\alpha^{(2)}_{p,j}(k_j))$ and $\alpha^{(1)}_{p,j}(k_j) \neq \alpha^{(2)}_{p,j}(k_j)$

**Remark of the referee:** In the elliptic modular case, an equality is expected to never happen if the weight $> 1$ (and this fact is proven when the weight $k$ is 2 and 3 by Coleman-Edixhoven and Ulmer under different conditions). However, it happens in the Hilbert modular case, even when $k = 2$. Anyway once we assume that $\alpha^{(1)}_{p,j} \neq \alpha^{(2)}_{p,j}$ for the initial weight, this holds in an open neighborhood of the weight.

The present paper extends a previous result: (see [PaTV]) where a two variable $p$-adic $L$-function was constructed interpolating on all $k$ a function $(k, s) \mapsto L^*(f_k, s, \chi)$ ($s = 1, \cdots, k - 1$) for such a family.

We use the theory of $p$-adic integration with values in spaces of nearly holomorphic modular forms (in the sense of Shimura, see [ShiAr]).

A family of slope $\sigma > 0$ of cusp eigenforms $f_k$ of weight $k \geq 2$:

$$
\begin{align*}
&k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k)q^n \\
&\in \mathbb{Q}[q] \subset \mathbb{C}_p[[q]] \\
&\text{A model example of a p-adic family (not cusp and } \sigma = 0): \\
&E_{k} = \text{Eisenstein series} \\
&a_n(k) = \sum_{d|n} d^{k-1}, f_k = E_k \\
&1) \text{the Fourier coefficients } a_n(k) \text{ of } f_k \\
&\text{and one of the Satake } \\
&p\text{-parameters } \alpha(k) := \alpha^{(1)}_p(k) \\
&\text{are given by certain } p\text{-adic analytic} \\
&\text{functions } k \mapsto a_n(k) \text{ for } (n, p) = 1 \\
&2) \text{the slope is constant and positive:} \\
&\text{ord}(\alpha(k)) = \sigma > 0
\end{align*}
$$

The existence of families of slope $\sigma > 0$ was established in [CoPB].

R. Coleman gave an example with $p = 7, f = \Delta, k = 12$ \\
$a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1$. \\
A program in PARI for computing such families is discussed in [CST98] (see also the Web-page of W. Stein, \\
http://modular.fas.harvard.edu/ )

It was established by Coleman that:
A. A. Panchishkin

- The operator $U$ acts as a completely continuous operator on each $A$-submodule $M^u(Np^v; A) \subset A[[q]]$ (i.e. $U$ is a limit of finite-dimensional operators)

$\Rightarrow$ there exists the Fredholm determinant $P_U(T) = \det(\text{Id} - T \cdot U) \in A[[T]]$

- there is a version of the Riesz theory:
for any inverse root $\alpha \in A^*$ of $P_U(T)$ there exists an eigenfunction $g$, $Ug = \alpha g$ such that $ev_k(g) \in \mathbb{C}_p[[q]]$ are classical cusp eigenforms for all $k$ in a neighbourhood $B \subset X$ (see in [CoPB])

We refer to [CoPB], part B and [PaTV], section 1, for generalities on rigid analytic $p$-adic families of modular forms.

2. Generalities on triple products

The triple product with a Dirichlet character $\chi$ is defined as the following complex $L$-function (an Euler product of degree eight):

$$L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p)p^{-s}), \quad (2.2)$$

where $L((f_1 \otimes f_2 \otimes f_3)_p, X)^{-1} = \det(1 - X \alpha(1)_{p,j} \otimes \alpha(2)_{p,j} \otimes \alpha(3)_{p,j} \cdot X)$

$$= \prod_{\eta}(1 - \alpha^{(\eta(1))}_{p,1} \alpha^{(\eta(2))}_{p,2} \alpha^{(\eta(3))}_{p,3} \cdot X)$$

$$= (1 - \alpha^{(1)}_{p,1} \alpha^{(1)}_{p,2} \alpha^{(1)}_{p,3} X)(1 - \alpha^{(1)}_{p,1} \alpha^{(2)}_{p,2} \alpha^{(2)}_{p,3} X) \cdots (1 - \alpha^{(2)}_{p,1} \alpha^{(2)}_{p,2} \alpha^{(2)}_{p,3} X)$$

$$\text{product taken over all 8 maps } \eta : \{1, 2, 3\} \to \{1, 2\}.$$ 

The Satake parameters and Hecke $p$-polynomials of forms $f_j$: Here the Satake parameters $\alpha^{(1)}_{p,j}, \alpha^{(2)}_{p,j}$ are given as inverse roots of the Hecke $p$-polynomials

$$1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha^{(1)}_{p,j}(p)X)(1 - \alpha^{(2)}_{p,j}(p)X).$$

We always assume that the weights are “balanced”:

$$k_1 \geq k_2 \geq k_3 \geq 2, \text{ and } k_1 \leq k_2 + k_3 - 2 \quad (2.4)$$

The non-balanced case (i.e. $k_1 > k_2 + k_3 - 2$) was treated in [HaTi], where $p$-adic measures for the square roots of special values of triple product $L$-functions, were constructed in the ordinary case. This construction can be probably extended to Coleman’s case by techniques similar to the present paper.
**Critical values and functional equation.** We use the corresponding normalized $L$ function (see [De79], [Co], [Co-PeRi]), which has the form:

\[(2.5)\quad \Lambda(f_1 \otimes f_2 \otimes f_3, s, \chi) = \Gamma_C(s)\Gamma_C(s - k_3 + 1)\Gamma_C(s - k_1 + 1)\Gamma_C(s - k_2 + 1)L(f_1 \otimes f_2 \otimes f_3, s, \chi),\]

where $\Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s)$.

The Gamma-factor determines the critical values $s = k_1, \ldots, k_2 + k_3 - 2$ of $\Lambda(s)$, which we explicitly evaluate (like in the classical formula $\zeta(2) = \frac{\pi^2}{6}$). A functional equation of $\Lambda(s)$ has the form:

\[s \mapsto k_1 + k_2 + k_3 - 2 - s.\]

According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each $f_j$ ($j = 1, 2, 3$) (under suitable assumptions on $p$ and on $f_j$) into a $p$-adic analytic family

\[f_j : k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j})q^n\}\]

of cusp eigenforms $f_{j,k_j}$ of weights $k_j$ in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain $p$-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.

### 3. Statement of the problem

Given three $p$-adic analytic families $f_j$ of slope $\sigma_j \geq 0$, to construct a four-variable $p$-adic $L$-function attached to Garrett’s triple product of these families. We show that this function interpolates the special values

\[\Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, s, \chi)\]

at critical points $s = k_1, \ldots, k_2 + k_3 - 2$ for balanced weights $k_1 \leq k_2 + k_3 - 2$; we prove that these values are algebraic numbers after dividing out certain “periods”.

However the construction uses directly modular forms, and not the $L$-values in question, and a comparison of special values of two functions is done after the construction.

Consider the product of the Satake parameters

\[\lambda_p = \alpha_{p,1}^{(1)}\alpha_{p,2}^{(1)}\alpha_{p,3}^{(1)} = \lambda_p(k_1, k_2, k_3)\]

We assume that $\text{ord}_p\alpha_{p,j}^{(1)} \leq \text{ord}_p\alpha_{p,j}^{(2)}$, and that the slope $\sigma = \text{ord}_p(\lambda_p(k_1, k_2, k_3))$ is constant and positive for all triplets $(k_1, k_2, k_3)$ in a $p$-adic neighbourhood $B \subset X^3$ of the fixed triplet of weights $(k_1, k_2, k_3)$. 

Our method includes: • a version of Garrett’s integral representation for the triple
$L$-functions of the form: for $r = 0, \cdots, k_2 + k_3 - k_1 - 2$,
$$
\Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - r, \chi) =
\int \int \int \frac{f_{1,k_1}(z_1)f_{2,k_2}(z_2)f_{3,k_3}(z_3)E(z_1, z_2, z_3; -r, \chi)}{y_j^3}
\prod_j \left( \frac{dy_j}{y_j^3} \right)
$$
where $f_{j,k_j} = f_{j,k_j}^0$ is an eigenfunction of $U_p^\ast$ in $M_{k_j}(Np, \psi_j)$, $f_{j,k_j,0}$ is the corresponding eigenfunction of $U_p$, and
$$
E(z_1, z_2, z_3; -r, \chi) \in M_T(N^2p^2v) = M_{k_1,r^\ast}(N^2p^2v, \psi_1) \otimes M_{k_2,r^\ast}(N^2p^2v, \psi_2) \otimes M_{k_3,r^\ast}(N^2p^2v, \psi_3)
$$
is a (classical) nearly holomorphic triple modular form of triple weight $(k_1, k_2, k_3)$
of some type $r^\ast \geq 0$ (see [ShiAr]), and of fixed triple Nebentupus character
$(\psi_1, \psi_2, \psi_3)$, obtained from a nearly holomorphic Siegel-Eisenstein series
$F_{\chi,r} = G^\ast(z, -r; k, (Np^v)^2, \psi)$, of degree 3, of weight $k = k_2 + k_3 - k_1$, and the Nebentupus
character $\psi = \chi^2\psi_1\psi_2\psi_3$ ([Sh83]).

We obtain $E(z_1, z_2, z_3; -r, \chi)$ from a Siegel-Eisenstein series by applying to $F_{\chi,r}$
Boecherer’s higher twist (see [11.22]) and Ibukiyama’s differential operator (see
[11.23]).

These operations act explicitly on the Fourier expansions. Then one uses:

• The theory of $p$-adic integration with values in Serre’s type $A$-modules $M_T(A)$
of triple arithmetical nearly holomorphic modular forms over $p$-adic Banach algebras $A$. We shall use the notation
$$
M_T(A) = M(A(B_1)) \hat{\otimes} M(A(B_2)) \hat{\otimes} M(A(B_3))
$$
for certain $A$-modules of $p$-adic families of triple modular forms.

Explicit Fourier coefficients $a_{\chi,r}(R, T) \in \overline{Q}[R, T]$ of $E(-r, \chi)$ are given by special polynomials of matrices
$T = (t_{ij})$, $R = (R_{ij})$ and of $\chi(\beta)\beta^r$ (with $\beta \in \mathbb{Z}_p^* \cap \overline{Q}$)
i.e. the coefficients of $a_{\chi,r}$ by some elementry $p$-adic measures
$$
\int_Y \chi y^r \mu_{\pi_T} \in A.
$$
Here $A = A(B)$ is a certain $p$-adic Banach algebra of functions on an open analytic
subspace $B = B_1 \times B_2 \times B_3 \subset X^3$ in the product of three copies of the weight space
$X = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^s)$.

These measures on the group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$ produce the coefficients of
$a_{\chi,r}$ of $E(-r, \chi)$ of $M_T(A)$ for all $p$-adic weights $x \in X$, given by
$$
\int_Y x(y) \mu_{\pi_T} \in A
$$
(an interpolation from $x = \chi y^r$ to all $x \in X$).

• The spectral theory of triple Atkin’s operator $U = U_{p,T}$. allows to evaluate the integral using at each weight $(k_1, k_2, k_3)$ the equality
$$
\langle f^0, E(-r, \chi) \rangle = \langle f^0, \pi_\lambda(E(-r, \chi)) \rangle
$$
with the projection $\pi_\lambda$ of $M_T(A)$ to the $\lambda$-part $M_T(A)^\lambda$, defined by:

$$\text{Ker } \pi_\lambda := \bigcap_{n \geq 1} \text{Im } (U_T - \lambda I)^n, \quad \text{Im } \pi_\lambda := \bigcup_{n \geq 1} \text{Ker } (U_T - \lambda I)^n.$$ 

Note that

$$f^0 = f_{k_1,1}^0 \otimes f_{k_2,2}^0 \otimes f_{k_3,3}^0, \quad f_{k_1,1}^0 = f_{k_1,j,0}|_{k_1} \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix} (\text{for } j = 1, 2, 3)$$

is a classical form, defined at triple weights $(k_1, k_2, k_3)$ as the Weil-involution image of $f_0 = f_{k_1,1,0} \otimes f_{k_2,2,0} \otimes f_{k_3,3,0}$ as in [PaTV], at p.555. Note that the functions $f^0$ are eigenfunctions of the adjoint triple operator $U_p^*$. We consider also the sequences of triple modular forms:

$$f_0 = \{f_{k_1,1,0} \otimes f_{k_2,2,0} \otimes f_{k_3,3,0}\}_{(k_1,k_2,k_3)}, \quad f^0 = \{f_{k_1,1}^0 \otimes f_{k_2,2}^0 \otimes f_{k_3,3}^0\}_{(k_1,k_2,k_3)}.$$ 

The functions $f_0$ form a $p$-adic family. The evaluation at triple weight $k = (k_1, k_2, k_3)$ of the $p$-adic coordinate with respect to $p$-adic family $f_0$ is expressed through the triple Petersson scalar product with $f^0$, which is algebraically orthogonal to $f_0$ in the sense of Hida [H90] for classical weights (and in [PaTV] for Coleman’s families).

We prove that $U$ is a completely continuous $A$-linear operator on a certain Coleman’s submodule $M(A)^\dagger$ of Serre’s type module $M(A)$. Then the projection $\pi_\lambda$ exists (on this submodule) due to general results of Serre and Coleman, see [CoPB], [SePB].

We show that there exists an element $\tilde{E}(-r, \chi) \in M(A)^\dagger$ such that at each weight $(k_1, k_2, k_3)$ the equality holds: $\langle f^0, \mathcal{E}(-r, \chi) \rangle = \mathcal{E}(\tilde{E}(-r, \chi))$, and the product can be expressed through certain coefficients the series $\tilde{E}(-r, \chi)$ which are the same as those of $\mathcal{E}(-r, \chi)$.

• **Key point: modular admissible measures.** Let us write for simplicity: $\mathcal{E}(-r, \chi)$ for $\tilde{E}(-r, \chi)$

$M_T(A)$ instead of $M_T(A)^\dagger$ (Coleman’s submodule)

One defines **admissible $p$-adic measures** $\check{\mathcal{E}}^\lambda$ with values in Banach $A$-modules $M_T^\lambda(A)$ which are locally free of finite rank, using the **test functions** $\int_y \chi y^r \check{\mathcal{E}}^\lambda = \pi_\lambda(\mathcal{E}(-r, \chi))$.

Consider the **evaluation maps** $ev_s : A \to C_p$ for any $p$-adic triple weights $s = (s_1, s_2, s_3) \in B$. 

**Triple Products of Coleman’s Families**

1141
• Passage from values in modular forms to scalar values: apply an algebraic $A$-linear form $M_{\lambda T}(A) \xrightarrow{\ell_T} A$ to the constructed measure $\tilde{\Phi}_\lambda$ (in modular forms), and the evaluation maps $A \xrightarrow{ev_s} \mathbb{C}_p$ for any $p$-adic triple weights $s \in X^3$.

The linear form $\ell_T$ is an algebraic version of the Petersson product (a geometric meaning of $\ell_T$: the first coordinate in an (orthogonal) $A$-basis of eigenfunctions of all Hecke operators $T_q$ for $q \nmid Np$, with the first basis element $f_0 \in M_{\lambda T}(A)$).

Using the evaluation map and the Mellin transform. We obtain the measure $\mu = \ell_T(\tilde{\Phi}_\lambda)$ with values in $A$ on the profinite group $Y$.

• Construct an analytic function $L_\mu : X \rightarrow A = A(\mathcal{B})$ as the $p$-adic Mellin transform $L_\mu(x) = \int_Y x(y) d\mu(y) \in A, x \in X$.

• Solution: the function in question $L_\mu(x,s)$ is given by evaluation of $L_\mu(x)$ at $s = (s_1, s_2, s_3) \in \mathcal{B}$: this is a $p$-adic analytic function in four variables

\[ (x, s) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X \]

\[ L_\mu(x, s) := ev_s(L_\mu(x)) \quad (x \in X, s \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3, L_\mu(x) \in A). \]

Final step: comparison between $\mathbb{C}$ and $\mathbb{C}_p$. • We check an equality relating the values of the constructed analytic function $L_\mu(x, s)$ at the arithmetical characters $x = y_p^r \chi \in X$, and at triple weights $s = (k_1, k_2, k_3) \in \mathcal{B}$, with the normalized critical special values

\[ L^*(f_1, f_2, f_3, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \ldots, k_2 + k_3 - k_1 - 2), \]

for certain Dirichlet characters $\chi \mod Np^v, v \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \mathbb{Q}_p$ (the Tate field of $p$-adic numbers). The normalisation of $L^*$ includes at the same time Gauss sums, Petersson scalar products, powers of $\pi$, the product $\lambda_p(k_1, k_2, k_3)$, and a certain finite Euler product.

We refer to Theorem 0.3 at p.556 of [PaTV], where the explicit form of such finite Euler product is given in the case of the $L$-function of one Coleman’s family.

4. Arithmetical nearly holomorphic modular forms

Arithmetical nearly holomorphic modular forms (the elliptic case). Let $A$ be a commutative ring (a subring of $\mathbb{C}$ or $\mathbb{C}_p$)

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr] are certain formal series

\[ g = \sum_{n=0}^{\infty} a(n; R)q^n \in A[[q]][R], \text{ with the property} \]

that for $A = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $C^\infty$-modular form on $\mathbb{H}$ of a given weight $k$ and Dirichlet character $\psi$. The coefficients $a(n; R)$ are polynomials in $A[R]$. If $\deg_R a(n; R) \leq r$ for all $n$, we call $g$ nearly holomorphic of type $r$ (it is annihilated by $(\frac{\partial}{\partial z})^{r+1}$, see [ShiAr]).

We use the notation $M_{k,r}(N, \psi, A)$ or $\tilde{M}(N, \psi, A)$ for $A$-modules of such forms (in our constructions the weight $k$ varies).

A known example (see the introduction to [ShiAr]) is given by the series

$$-12R + E_2 := -12R + 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n$$

$$= \frac{3}{\pi^2} \lim_{s \to 0} y^s \sum_{m_1, m_2 \in \mathbb{Z}} (m_1 + m_2z)^{-2}|m_1 + m_2z|^{-2s}, (R = (4\pi y)^{-1})$$

where $\sigma_1(n) = \sum_d |d| n$.

The action of the Shimura differential operator

$$\delta_k : M_{k,r}(N, \psi, A) \to M_{k+2,r+1}(N, \psi, A),$$

is given over $\mathbb{C}$ by $\delta_k(f) = \left(\frac{1}{2\pi i} \frac{\partial}{\partial z} - \frac{k}{4\pi y}\right)f$.

This operator is a correction of the Ramanujan operator

$$\theta(\sum_{n=0}^{\infty} a_nq^n) = \sum_{n=1}^{\infty} na_nq^n = \frac{1}{2\pi i} \frac{\partial}{\partial z} (\sum_{n=0}^{\infty} a_nq^n) = q \frac{\partial}{\partial q} (\sum_{n=0}^{\infty} a_nq^n),$$

which does not preserve the modularity. For example $\theta \Delta = E_2 \Delta$, where $E_2$ is a quasimodular form (in the sense of Kaneko and Zagier, see [Ka-Za]).

Notice that $\delta_k f = (\theta - kR)f$, and that Serre’s operator $f \mapsto \theta f - \frac{k}{12} E_2 f$ takes $M_k$ to $M_{k+2}$.

Note that that the arithmetical twist operator

$$\theta \chi(\sum_{n=0}^{\infty} a_nq^n) = \sum_{n=1}^{\infty} \chi(n)a_nq^n$$

is a natural analog of the Ramanujan operator.

Triple arithmetical modular forms. Let $A$ be a commutative ring. The tensor product over $A$

$$M_{k,r,T}(N, \psi, A) := M_{k_1,r}(N, \psi_1, A) \otimes M_{k_2,r}(N, \psi_2, A) \otimes M_{k_3,r}(N, \psi_3, A)$$
consists of \textbf{triple arithmetical modular forms} as certain formal series of the form
\[
g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3; R_1, R_2, R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}
\in \mathcal{A}[q_1, q_2, q_3][R_1, R_2, R_3],
\]
where \( z_j = x_j + iy_j \in \mathbb{H}, \ R_j = (4\pi y_j)^{-1}, \)
with the property that for \( \mathcal{A} = \mathbb{C} \), the series converges to a \( e^{z} \)-modular form on \( \mathbb{H}^3 \) of a given weight \( (k_1, k_2, k_3) \) and character \( (\psi_1, \psi_2, \psi_3), j = 1, 2, 3 \). The coefficients \( a(n_1,n_2,n_3; R_1, R_2, R_3) \) are polynomials in \( \mathcal{A}[R_1, R_2, R_3] \). Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.

\section{Siegel-Eisenstein series}

\textit{Siegel modular groups.} Let \( J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \). The symplectic group
\[
\text{Sp}_m(\mathbb{R}) = \{ g \in \text{GL}_{2m}(\mathbb{R}) | ^t g \cdot J_{2m} g = J_{2m} \},
\]
acts on the Siegel upper half plane
\[
\mathbb{H}_m = \{ z = ^t z \in M_m(\mathbb{C}) | \text{Im} z > 0 \}
\]
by \( g(z) = (az+b)(cz+d)^{-1} \), where we use the bloc notation \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_{2m}(\mathbb{R}) \). We use the congruence subgroup \( \Gamma_0^m(N) = \{ \gamma \in \text{Sp}_m(\mathbb{Z}) \mid \gamma \equiv (\ast \ast \ast \ast) \mod N \} \subset \text{Sp}_m(\mathbb{Z}) \).

\textit{A Siegel modular form.} \( f \in \mathcal{M}_k(\Gamma_0^m(N), \chi) \) of degree \( m > 1 \), weight \( k \) and a Dirichlet character \( \chi \mod N \) is a \textit{holomorphic function} \( f: \mathbb{H}_m \to \mathbb{C} \) such that for every \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(N) \) one has
\[
f(\gamma(z)) = \chi(\text{det} d) \text{det}(cz+d)^k f(z).
\]
The \textbf{Fourier expansion} of \( f \) uses the symbol
\[
q^\mathcal{T} = \exp(2\pi i \text{tr}(\mathcal{T} z)) = \prod_{i=1}^{m} q_i^{\mathcal{T} i} \prod_{i<j} q_{ij}^{2\mathcal{T} ij} \in \mathbb{C}[q_{11}, \cdots, q_{mm}, q_{ij}, q_{ij}^{-1}]_{1 \leq i < j \leq m}
\]
for each Fourier coefficient, where \( q_{ij} = \exp(2\pi i \sqrt{-1} z_{ij}) \), and \( \mathcal{T} \) in the semi-group \( B_m = \{ \mathcal{T} = ^t \mathcal{T} \geq 0 \mid \mathcal{T} \text{ half-integral} \} \). We may consider \( f(z) = \sum_{\mathcal{T} \in B_m} a(\mathcal{T}) q^\mathcal{T} \in \mathbb{C}[q^{B_m}] \) as a formal \( q \)-expansion in \( \mathbb{C}[q^{B_m}] \) (the subring of \( \mathbb{C}[q_{11}, \cdots, q_{mm}][q_{ij}, q_{ij}^{-1}] \)) generated by all \( q^\mathcal{T} \).
Siegel-Eisenstein series.

Example 5.1 (Nag2, p.408).

\[ E_4^{(2)}(z) = 1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^{-2} + 13440q_{12}^{-1} \]
\[ + 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \ldots \]

\[ E_6^{(2)}(z) = 1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^{-2} + 44352q_{12}^{-1} \]
\[ + 166320 + 44352q_{12} - 504q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \ldots \]

Arithmetical nearly holomorphic Siegel modular forms.

Consider a commutative ring \( A \), the formal variables \( q = (q_{i,j})_{i,j=1,\ldots,m} \), \( R = (R_{i,j})_{i,j=1,\ldots,m} \), and the ring of formal Fourier series

\[
A[q^B][R_{i,j}] = \left\{ f = \sum_{\mathcal{T} \in B_m} a(\mathcal{T}, R)q^{\mathcal{T}} \bigg| a(\mathcal{T}, R) \in A[R_{i,j}] \right\}
\]

(over the complex numbers this notation corresponds to \( q^{\mathcal{T}} = \exp(2\pi i \text{tr}(\mathcal{T}z)) \), \( R = (4\pi \text{Im}(z))^{-1} \)).

The formal Fourier expansion of a nearly holomorphic Siegel modular form \( f \) with coefficients in \( A \) is a certain element of \( A[q^B][R_{i,j}] \). We call \( f \) \textit{arithmetical} in the sense of Shimura [ShiAr], if \( A = \mathbb{Q} \).

5.1. Algebraic differential operators of Maass and Shimura.

Maass differential operator. Let us consider the \textbf{Maass differential operator} (see [Maa]) \( \Delta_m \) of degree \( m \), acting on complex \( C^\infty \)-functions on \( \mathbb{H}_m \) by:

\[
\Delta_m = \det(\tilde{\partial}_{ij}), \quad \tilde{\partial}_{ij} = 2^{-1}(1 + \delta_{ij})\partial/\partial z_i,
\]

its algebraic version is the \textbf{Ramanujan operator of degree} \( m \):

\[
\Theta_m := \det\left(\frac{1}{2\pi i}\tilde{\partial}_{ij}\right) = \det(\theta_{ij}) = \frac{1}{(2\pi i)^m}\Delta_m,
\]

where \( \Theta_m(q^{\mathcal{T}}) = \det(\mathcal{T})q^{\mathcal{T}} \).

Shimura differential operator. The \textbf{Shimura differential operator} (see [Shi76, ShiAr]):

\[
\delta_k f(z) = \det(R)^{k+1-\kappa}\Theta_m \left[ \det(R)^{\kappa-1-k} f \right], \text{ where } R = (4\pi y)^{-1},
\]
acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

\[ \delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \tilde{\mathcal{M}}^m_k(N, \psi; \overline{\mathbb{Q}}) \to \tilde{\mathcal{M}}^m_{k+2rm}(N, \psi; \overline{\mathbb{Q}}), \]

where

\[ \delta_k f(z) = \left(\frac{-1}{4\pi}\right)^m \det(y)^{-1} \det(z - \bar{z})^{\kappa-k} \Delta_m \left[ \det(z - \bar{z})^{k-\kappa+1} f(z) \right]. \]

Universal polynomials \( Q(R, \mathcal{J}; k, r) \). Let \( f = \sum_{\mathcal{J} \in B_m} c(\mathcal{J})q^\mathcal{J} \in \mathcal{M}^m_k(N, \psi) \) be a formal holomorphic Fourier expansion. One shows that \( \delta_k^{(r)} f \) is given by

\[ \delta_k^{(r)} f = \sum_{\mathcal{J} \in B_m} Q(R, \mathcal{J}; k, r) c(\mathcal{J}) q^\mathcal{J}. \]

Here we use a universal polynomial (5.10) which can be defined for all \( k \in \mathbb{C} \), and it expresses the action of the Shimura operator on the exponential (of degree \( m \)):

\[ \delta_k^{(r)} (q^\mathcal{J}) = Q(R, \mathcal{J}; k, r) q^\mathcal{J}. \]

If \( m = 1, r \) arbitrary (see [Shi76]),

\[ \delta_k^{(r)} = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \frac{\Gamma(k + r)}{\Gamma(k + j)} R^{r-j} \vartheta^j, \]

\[ Q(R, n; k, r) = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} \frac{\Gamma(k + r)}{\Gamma(k + j)} R^{r-j} n^j. \]

If \( r = 1, m \) arbitrary, one has (see [Maa]):

\[ \delta_k f(z) = \sum_{\mathcal{J} \in B_m} c(\mathcal{J}) \sum_{l=0}^m (-1)^{m-l} c_{m-l}(k + 1 - \kappa) \text{tr} \left( \left( {}^t \rho_{m-l}(R) \cdot \rho^*_l(\mathcal{J}) \right) q^\mathcal{J} \right) \]

where \( R = (4\pi y)^{-1} = (R_{i,j}) \in M_m(\mathbb{R}) \), \( c_m(\alpha) = \frac{\Gamma_m(\alpha + \kappa)}{\Gamma_m(\alpha + \kappa - 1)} \).

Here we use the natural representation \( \rho_r : \text{GL}_m(\mathbb{C}) \to \text{GL}(\wedge^r \mathbb{C}^m) \) \((0 \leq r \leq m)\) of the group \( \text{GL}_m(\mathbb{C}) \) on the vector space \( \Lambda^r \mathbb{C}^m \). Thus \( \rho_r(z) \) is a matrix of size \( \binom{m}{r} \times \binom{m}{r} \) composed of the subdeterminants of \( z \) of degree \( r \). Put \( \rho^*_r(z) = \det(z) \rho_{m-r}(z)^{-1} \).

Then the representations \( \rho_r \) and \( \rho^*_r \) turn out to be polynomial representations.
In general (see [CourPa], Theorem 3.14) one has:

\[ Q(R, T) = Q(R, T; k, r) \]

\[ = \sum_{t=0}^{r} \binom{r}{t} \det(T)^{r-t} \sum_{|L| \leq mt-t} R_L(\kappa - k - r)Q_L(R, T), \]

\[ Q_L(R, T) = \text{tr} \left( (p_{m-t_1}(R)p^*_1(T)) \cdots \text{tr} (p_{m-t_t}(R)p^*_t(T)) \right). \]

In (5.10), \( L \) goes over all the multi-indices \( 0 \leq l_1 \leq \cdots \leq l_t \leq m \), such that \( |L| = l_1 + \cdots + l_t \leq mt - t \), and \( R_L(\beta) \in \mathbb{Z}[1/2][\beta] \) in (5.10) are polynomials in \( \beta \) of degree \( (mt - |L|) \) (used with \( \beta = \kappa - k - r \)).

Note the differentiation rule of degree \( m \) (see [Sh83], p.466):

\[ \Delta(fg) = \sum_{r=0}^{m} \text{tr} \left( (p_r(\partial/\partial z)f \cdot p^*_m-r(\partial/\partial z)g) \right) \]

As in (5.8), we write here \( \partial/\partial z = (\partial/\partial z_{ij}) \) for the matrix with entries \( \partial_i \partial_j = 2^{-1}(1 + \delta_{ij}) \partial/\partial_{ij} \).

Example 5.2 (Siegel-Eisenstein series of odd degree and higher level).

\[ G^*(z, s; k, \psi, N) \]

\[ = \det(y)^s \sum_{c,d} \psi(\det c) \det(cz + d)^{-k} \det(cz + d)^{-2s} \cdot \Gamma(k, s)L_N(k + 2s, \psi) \left( \prod_{i=1}^{[m/2]} L_N(2k + 4s - 2i, \psi^2) \right), \]

where \((c, d)\) runs over all "non-associated coprime symmetric pairs" with \( \det(c) \) coprime to \( N \), \( \kappa = (m + 1)/2 \), and for \( m \) odd the \( \Gamma \)-factor has the form:

\[ \Gamma(k, s) = \iota^{mk} 2^{-m(k+1)} \pi^{-m(s+k)} \Gamma_m(k + s). \]

We use this series with \( \psi = \chi^2\psi_1\psi_2\psi_3 \), \( k = k_2 + k_3 - k_1 \geq 2 \), \( m = 3 \), \( \kappa = m + 1 = 2 \), \( [m/2] = 1 \).

Theorem 5.3 (Siegel, Shimura [Sh83], P. Feit [Fei86]). Let \( m \) be an odd integer such that \( 2k > m \), and \( N > 1 \) be an integer, then:

For an integer \( s \) such that \( s = -r, 0 \leq r \leq k - \kappa \), there is the following Fourier expansion:

\[ G^*(z, -r) = G^*(z, -r; k, \psi, N) = \sum_{A_m \geq \mathcal{T} \geq 0} a(\mathcal{T}, R)q^{\mathcal{T}}, \]
where for \( s > (m + 2 - 2k)/4 \) in (5.12) the only non-zero terms occur for positive definite \( T > 0 \).

(5.13) \[ a(T, R) = M(T, \psi, k - 2r) \cdot \det(T)^{k - 2r - \kappa} Q(R, T; k - 2r, r), \]

(5.14) \[ M(T, k - 2r, \psi) = \prod_{\ell \mid \det(T)} M_\ell(T, \psi(\ell)^{2r}) \]

polynomials \( Q(R, T; k - 2r, r) \) are given by (5.10), and for all \( T > 0, T \in A_m \), is a finite Euler product, in which \( M_\ell(T, x) \in \mathbb{Z}[x] \).

\[ \square \]

6. Statement of the Main Result

Main Theorem (on \( p \)-adic analytic function in four variables). The following Main Theorem corresponds to Theorem 0.3 at p.556 of [PaTV] in the situation of the \( L \)-function attached to one Coleman’s family.

Main Theorem 6.1. 1) The function \( \mathcal{L}_f : (s, k_1, k_2, k_3) \mapsto \frac{\langle f_0^0, \xi(-r, \chi) \rangle}{\langle f_0^0, f_0 \rangle} \) extends to a \( p \)-adic analytic function on four variables \((\chi \cdot y_p^r, k_1, k_2, k_3) \in X \times B_1 \times B_2 \times B_3; \)

2) Comparison of complex and \( p \)-adic values: for all \((k_1, k_2, k_3) \) in an affinoid neighborhood \( B = B_1 \times B_2 \times B_3 \subset X^3 \), satisfying \( k_1 \leq k_2 + k_3 - 2 \): the values at \( s = k_2 + k_3 - 2 - r \) coincide with the normalized critical special values

(6.15) \[ L^*(f_1, k_1 \otimes f_2, k_2 \otimes f_3, k_3, k_2 + k_3 - 2 - r, \chi) \]

\( (r = 0, \cdots, k_2 + k_3 - k_1 - 2), \)

for Dirichlet characters \( \chi \mod Np^v, v \geq 1 \), such that all three corresponding Dirichlet characters \( \chi_j \) have \( Np \)-complete conductors:

(6.16) \[ \chi_1 \mod Np^v = \chi, \chi_2 \mod Np^v = \psi_2 \bar{\psi}_3 \chi, \]

\[ \chi_3 \mod Np^v = \psi_1 \bar{\psi}_3 \chi, \psi = \chi^2 \psi_1 \bar{\psi}_2 \bar{\psi}_3. \]

The normalisation of \( L^* \) in (6.15) is the same as in Theorem C below.

3) Dependence on \( x \in X \): let \( H = [\text{ord}_p(\lambda)] + 1 \). For any fixed \((k_1, k_2, k_3) \in B \) and \( x = \chi \cdot y_p^r \) the function

\[ x \mapsto \frac{\langle f_0^0, \xi(-r, \chi) \rangle}{\langle f_0^0, f_0 \rangle} \]

extends to a \( p \)-adic analytic function of type \( o(\log^H(\cdot)) \) of the variable \( x \in X \).
Remark. The function $L_f$ depends on the variables $(s, k_1, k_2, k_3)$ in a different way: it is a mixture of the $p$-adic Mellin transform (in $s$), and of a rigid analytic function (in $k_1, k_2, k_3$).

Outline of the proof. The proof follows the lines given in Sections 5-7 in [PaTV] (the case of the $L$-function of one Coleman’s family).

1. (compare with Section 5 in [PaTV]). At each classical weight $(k_1, k_2, k_3)$ let us use the equality

$$\langle f^0, \mathcal{E}(-r, \chi) \rangle = \langle f^0, \pi_\lambda(\mathcal{E}(-r, \chi)) \rangle$$

which is deduced from the definition of the projector $\pi_\lambda$: $\text{Ker}\, \pi_\lambda := \cap_{n \geq 1} \text{Im} (U_T - \lambda I)^n$, $\text{Im}\, \pi_\lambda := \cup_{n \geq 1} \text{Ker} (U_T - \lambda I)^n$.

Notice that the coefficients of $\mathcal{E}(-r, \chi) \in \mathcal{M}(\mathcal{A})$ depend $p$-adic analytically on $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, where $\mathcal{A} = \mathcal{A}(\mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3)$ is the $p$-adic Banach algebra of rigid-analytic functions on $\mathcal{B}$.

Interpolation to all $p$-adic weights: At each classical weight $(k_1, k_2, k_3)$ the scalar product $\langle f^0, \mathcal{E}(-r, \chi) \rangle$ is given by the first coordinate of $\pi_\lambda(\mathcal{E}(-r, \chi))$ with respect to an orthogonal basis of $\mathcal{M}^\lambda(\mathcal{A})$ containing $f_0$ with respect to Hida’s algebraic Petersson product $\langle g, h \rangle_a := \left\langle g^a \left( \begin{smallmatrix} 0 & -1 \\ Np & 0 \end{smallmatrix} \right), h \right\rangle$, see [Hi90].

Let us extend the linear form $\ell(h) = \frac{\langle f^0, h \rangle}{\langle f^0, f_0 \rangle}$ (defined first only for classical weights), to Coleman’s type submodule of overconvergent families $h \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}^\lambda(\mathcal{A})$ as the first coordinate of $h$ with respect to some $\mathcal{A}$-basis of eigenfunctions of all (triple) Hecke operators $T_q$ for $q \nmid Np$, having the first basis vector $f_0 \in \mathcal{M}^\lambda(\mathcal{A})^\dagger$.

The linear form $\ell$ can be characterized as a normalized eigenfunction of the adjoint Atkin’s operator, acting on the dual $\mathcal{A}$-module of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell(f_0) = 1$.

In order to extend $\ell$ to $h = \mathcal{E}(-r, \chi)$, we need to choose a certain representative of $\mathcal{E}(-r, \chi)$ in the $\mathcal{A}$-submodule $\mathcal{M}^\lambda(\mathcal{A})^\dagger$, which is locally free of finite rank.

A representative of $\mathcal{E}(-r, \chi)$ in the (locally free of finite rank $\mathcal{A}$-submodule) $\mathcal{M}^\lambda(\mathcal{A})^\dagger$. (compare with Section 6 in [PaTV] in the case of the L-function of one Coleman’s family). Choose a (local) basis $\ell^1, \ldots, \ell^n$ given by some triple Fourier coefficients of the dual (locally free of finite rank) $\mathcal{A}$-module $\mathcal{M}^\lambda(\mathcal{A})^\dagger^*$.

Then define

$$\ell = \beta_1 \ell^1 + \cdots + \beta_n \ell^n,$$

where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and $\ell_i$ denotes the dual basis of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell^j(\ell_i) = \delta_{ij}$.
At each $p$-adic weight $(k_1, k_2, k_3) \in \mathcal{B}$ let us define
\[
\ell(E(-r, \chi)) := \beta_1 \ell_1(E(-r, \chi)) + \cdots + \beta_n \ell_n(E(-r, \chi)) \quad \text{(belongs to } A),
\]
where $\beta_i = \ell(\ell_i) \in A$, and $\ell_i(E(-r, \chi)) \in A$ are certain Fourier coefficients of the seies $E(-r, \chi)$.

**Conclusion.** There exists an element $\tilde{E}(-r, \chi) \in M_A^\lambda(\mathcal{A})^\dagger \subset M(\mathcal{A})^\dagger$ such that
\[
\ell(E(-r, \chi)) = \ell(\tilde{E}(-r, \chi)) \quad \text{(at each triple weight } (k_1, k_2, k_3)).
\]
In fact, let us define
\[
\tilde{E}(-r, \chi) := \ell_1\ell_1(E(-r, \chi)) + \cdots + \ell_n\ell_n(E(-r, \chi))
\]
\[
\Rightarrow \quad \ell(\tilde{E}(-r, \chi)) = \ell(\ell_1)\ell_1(E(-r, \chi)) + \cdots + \ell(\ell_n)\ell_n(E(-r, \chi))
\]
\[
= \beta_1 \ell_1(E(-r, \chi)) + \cdots + \beta_n \ell_n(E(-r, \chi))
\]
\[
= \ell(E(-r, \chi)) \quad \text{(at each weight } (k_1, k_2, k_3)).
\]
Thus, the dependence of $\ell(E(-r, \chi)) \in A$ on $(k_1, k_2, k_3) \in X^3$ is $p$-adic analytic.

In order to prove the remaining statements 2), 3), the dependence on $x = \chi \cdot y_p^r$ is studied in the next section.

### 7. Distributions and Admissible Measures

**Distributions and measures with values in Banach modules.** We refer to Section 4 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman’s family.

**Notation.**
- $\mathcal{A}$ (a $p$-adic Banach algebra)
- $V$ (an $\mathcal{A}$-module)
- $\mathcal{C}(Y, \mathcal{A})$ (the $\mathcal{A}$-Banach algebra of continuous functions on $Y$)
- $\mathcal{C}_{\text{loc-const}}(Y, \mathcal{A})$ (the $\mathcal{A}$-algebra of locally constant functions on $Y$)

**Definition 7.1 (Distributions and measures).** a) A **distribution** $D$ on $Y$ with values in $V$ is an $\mathcal{A}$-linear form
\[
D : \mathcal{C}_{\text{loc-const}}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto D(\varphi) = \int_Y \varphi d\mathcal{D}.
\]

b) A **measure** $\mu$ on $Y$ with values in $V$ is a continuous $\mathcal{A}$-linear form
\[
\mu : \mathcal{C}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.
\]
The integral \( \int_Y \varphi d\mu \) can be defined for any continuous function \( \varphi \), and any bounded distribution \( \mu \), using the Riemann sums.

**Admissible measures of Amice-Vélu.**

*Admissible measures.* Let \( h \) be a positive integer. A more delicate notion of an \( h \)-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

**Definition 7.2.**

a) For \( h \in \mathbb{N}, h \geq 1 \) let \( P^h(Y,A) \) denote the \( A \)-module of locally polynomial functions of degree \( < h \) of the variable \( y_p : Y \to \mathbb{Z}_p^x \hookrightarrow A^x \); in particular,

\[
P^1(Y,A) = \mathcal{C}^{\text{loc-const}}(Y,A)
\]

(the \( A \)-submodule of locally constant functions). Let also denote \( \mathcal{C}^{\text{loc-an}}(Y,A) \) the \( A \)-module of locally analytic functions, so that

\[
P^1(Y,A) \subset P^h(Y,A) \subset \mathcal{C}^{\text{loc-an}}(Y,A) \subset \mathcal{C}(Y,A).
\]

b) Let \( V \) be a normed \( A \)-module with the norm \( |\cdot|_{p,V} \). For a given positive integer \( h \) an \( h \)-admissible measure on \( Y \) with values in \( V \) is an \( A \)-module homomorphism \( \tilde{\Phi} : P^h(Y,A) \to V \) such that for fixed \( a \in Y \) and for \( v \to \infty \) the following growth condition is satisfied:

\[
(7.17) \quad \left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)})
\]

for all \( h' = 0, 1, \ldots, h-1, a_p := y_p(a) \)

The condition (7.17) allows to integrate the locally-analytic functions on \( Y \) along \( \tilde{\Phi} \) using Taylor’s expansions! This means: there exists a unique extension of \( \tilde{\Phi} \) to \( \mathcal{C}^{\text{loc-an}}(Y,A) \to V \).

### 7.1. \( U_p \)-Operator and the method of canonical projection.

We refer to Section 5 of [PaTV] for similar constructions in the case of the \( L \)-function of one Coleman’s family.
Using the canonical projection $\pi_\lambda$. We construct our $H$-admissible measure $\tilde{\Phi}^\lambda : P^H(Y, \mathcal{A}) \to M(\mathcal{A})$ out of a sequence of distributions $\Phi_r : P^1(Y, \mathcal{A}) \to M(\mathcal{A})$ defined on local monomials $y^r_\mu$ of each degree $r$ by the rule

$$\int_X \chi y^r_\mu \, d\tilde{\Phi}^\lambda = \pi_\lambda(\tilde{E}(-(r, \chi))),$$

where $\tilde{E}(-(r, \chi)) \in M = M(\mathcal{A})$.

Here $\tilde{E}(-(r, \chi))$ takes values in an $\mathcal{A}$-module $M = M(\mathcal{A}) \subset \mathcal{A}[\llbracket q_1, q_2, q_3 \rrbracket][R_1, R_2, R_3]$ of nearly holomorphic (overconvergent) triple modular forms over $\mathcal{A}$ (for $0 \leq r \leq H - 1$, $H = [\text{ord}_p \lambda_p] + 1$), and the formal series $\tilde{E}(-(r, \chi))$ was constructed in the proof of 1) of Main Theorem.

**Definition of the canonical projection $\pi_\lambda$.** Here $\mathcal{A}$ is a $\mathbb{C}_p$-algebra, and $\lambda \in \mathcal{A}^\times$ is a fixed non-zero eigenvalue of triple Atkin’s operator $U_T = U_{T, p}$, acting on $M(\mathcal{A})$, $\pi_\lambda : M(\mathcal{A}) \to M(\mathcal{A})^\lambda$

is the canonical projection operator onto the maximal $\mathcal{A}$-submodule $M(\mathcal{A})^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent (we call $M(\mathcal{A})^\lambda$ the $\lambda$-characteristic submodule of $M(\mathcal{A})$).

The projector $\pi_\lambda$ is defined by its kernel:

$$\text{Ker} \pi_\lambda := \bigcap_{n \geq 1} \text{Im} (U_T - \lambda I)^n, \quad \text{Im} \pi_\lambda := \bigcup_{n \geq 1} \text{Ker} (U_T - \lambda I)^n.$$

**8. Triple modular forms**

Triple modular forms are certain formal series

$$g = \sum_{n_1, n_2, n_3 = 0}^{\infty} a(n_1, n_2, n_3; R_1, R_2, R_3)q_1^{n_1}q_2^{n_2}q_3^{n_3} \in \mathcal{A}[q_1, q_2, q_3][R_1, R_2, R_3],$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $C^\infty$-modular form on $\mathbb{H}^3$ of a given weight $(k_1, k_2, k_3)$ and character $(\psi_1, \psi_2, \psi_3)$, $j = 1, 2, 3$. The coefficients $a(n_1, n_2, n_3; R_1, R_2, R_3)$ are polynomials in $\mathcal{A}[R_1, R_2, R_3]$, and the triple Atkin’s operator is given by

$$U_T(g) = \sum_{n_1, n_2, n_3 = 0}^{\infty} a(pn_1, pn_2, pn_3; pR_1, pR_2, pR_3)q_1^{n_1}q_2^{n_2}q_3^{n_3}.$$

**Eigenfunctions of $U_p$ and of $U^*_p$.**
Functions $f_{j,0}$ and $f_{j}^0$. Recall that for any primitive cusp eigenform $f_j = \sum_{n=1}^{\infty} a_n(f)q^n$, there is an eigenfunction $f_{j,0} = \sum_{n=1}^{\infty} a_n(f_{j,0})q^n \in \overline{\mathbb{Q}}[q]$ of $U = U_p$, with the eigenvalue $\lambda = \alpha_{p,j}^{(1)} \subseteq \overline{\mathbb{Q}} (U(f_{j,0}) = \alpha f_{j,0})$ given by

\begin{equation}
(8.18) \quad f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j \left( \begin{array}{c} p \\ 0 \end{array} \right) \\
\sum_{n=1}^{\infty} a_n(f_{j,0}) n^{-s} = \sum_{n=1}^{\infty} a_n(f_j) n^{-s} (1 - \alpha_{p,j}^{(1)} p^{-s})^{-1}.
\end{equation}

Moreover, there is an eigenfunction $f_{j}^0$ of $U_p^*$ given by

\begin{equation}
(8.19) \quad f_{j}^0 = f_{j,0}^p \left| \begin{array}{c} 0 \\ N_p \end{array} \right|_{k} \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right), \text{ where } f_{j,0}^p = \sum_{n=1}^{\infty} a(n, f_{0}) q^n.
\end{equation}

Therefore, $U_T(f_{1,0} \otimes f_{2,0} \otimes f_{3,0}) = \lambda(f_{1,0} \otimes f_{2,0} \otimes f_{3,0})$.

9. **Critical values of the $L$ function** $L(f_1 \otimes f_2 \otimes f_3, s, \chi)$

(compare with Section 7 in [PaTV]).

**Choice of Dirichlet characters.** For an arbitrary Dirichlet character $\chi$ mod $Np^v$ consider the following Dirichlet characters:

\begin{equation}
(9.20) \quad \chi_1 \text{ mod } Np^v = \chi, \chi_2 \text{ mod } Np^v = \psi_2 \bar{\psi}_3 \chi, \\
\chi_3 \text{ mod } Np^v = \psi_1 \bar{\psi}_3 \chi, \psi = \chi^2 \psi_1 \psi_2 \bar{\psi}_3;
\end{equation}

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_0,1, \chi_0,2, \chi_0,3$ are $Np$-completes (i.e. have the same prime divisors as resp. those of $Np$).

**Theorem A (algebraic properties of the triple product).** Assume that $k_2 + k_3 - k_1 \geq 2$, then for all pairs $(\chi, r)$ such that the corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductors, and $0 \leq r \leq k_2 + k_3 - k_1 - 2$, we have that

$$\frac{\Lambda(f_1^p \otimes f_2^p \otimes f_3^p, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)}{\langle f_1^p \otimes f_2^p \otimes f_3^p, f_1^p \otimes f_2^p \otimes f_3^p \rangle_T} \subseteq \overline{\mathbb{Q}}$$

where

$$\langle f_1^p \otimes f_2^p \otimes f_3^p, f_1^p \otimes f_2^p \otimes f_3^p \rangle_T := \langle f_1^p, f_1^p \rangle_N \langle f_2^p, f_2^p \rangle_N \langle f_3^p, f_3^p \rangle_N$$

$$= \langle f_1, f_1 \rangle_N \langle f_2, f_2 \rangle_N \langle f_3, f_3 \rangle_N.$$
10. Theorems B-D

Recall: the $p$-adic weight space and the Mellin transform. (for generalities we also refer to the introduction of [PaTV] in the case of the $L$-function of one Coleman’s family). The $p$-adic weight space is the group $X = \text{Hom}_{\text{cont}}(Y, \mathbb{C}_p^*)$ of (continuous) $p$-adic characters of the commutative profinite group $Y = \lim_{\rightarrow v}(\mathbb{Z}/Np^n\mathbb{Z})^*$

The group $X$ is isomorphic to a finite union of discs $U = \{ z \in \mathbb{C}_p \mid |z|_p < 1 \}$.

A $p$-adic $L$-function $L(p) : X \to \mathbb{C}_p$ is a certain meromorphic function on $X$. Such a function usually come from a $p$-adic measure $\mu$ on $Y$ (bounded or admissible in the sense of Amice-Vélu, see [Am-V]). The $p$-adic Mellin transform of $\mu$ is given for all $x \in X$ by

$$L(p)(x) = \int_{Y_{N,p}} x(y)d\mu(y), L(p) : X \to \mathbb{C}_p$$

Theorem B (on admissible measures attached to the triple product: fixed balanced weights case). Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde{\mu}_{f_1 \otimes f_2 \otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function

$$D(p)(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p,$$

given for all $x \in X_{N,p}$ by the integral $D(p)(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y)d\tilde{\mu}_{f_1 \otimes f_2 \otimes f_3}(y)$, and having the following properties:

(i) for all pairs $(r, \chi)$ such that $\chi \in X_{N,p}^{\text{tor}}$, and all three corresponding Dirichlet characters $\chi_j$ have $Np$-complete conductor $(j = 1, 2, 3)$, and $r \in \mathbb{Z}$ is an integer with $0 \leq r \leq k_2 + k_3 - k_1 - 2$, the following equality holds:

$$D(p)(x^{rp}, f_1 \otimes f_2 \otimes f_3) = \mu_p\left(\frac{(\psi_1 \psi_2)^2}{G(\chi_1)G(\chi_2)G(\chi_3)G(\chi_1 \psi_2 \chi_1)}\right)^{2^v}$$

where $v = \text{ord}_{p}(C_{\chi})$, $G(\chi)$ denotes the Gauss sum of a primitive Dirichlet character $\chi_0$ attached to $\chi$ (modulo the conductor of $\chi_0$),

(ii) if $\text{ord}_p \lambda_p = 0$ then the holomorphic function in (i) is a bounded $\mathbb{C}_p$-analytic function;

(iii) in the general case (but assuming that $\lambda_p \neq 0$) the holomorphic function in (i) belongs to the type $O(\log(x^{H}_p))$ with $H = [2\text{ord}_p \lambda_p] + 1$ and it can be represented as the Mellin transform of the $H$-admissible $\mathbb{C}_p$-valued measure $\tilde{\mu}_{f_1 \otimes f_2 \otimes f_3}$ (in the sense of Amice-Vélu) on $Y$.

(iv) Let $k = k_2 + k_3 - k_1 \geq 2$. If $H \leq k - 2$ then the function $D(p)$ is uniquely
determined by the above conditions (i). Let us describe now $p$-adic measures attached to Garrett’s triple product of three Coleman’s families

\[ k_j \mapsto \{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k)q^n \}(j = 1, 2, 3). \]

Consider the product of three eigenvalues:

\[ \lambda = \lambda_p(k_1, k_2, k_3) = \alpha_{p,1}^{(1)}(k_1)\alpha_{p,2}^{(1)}(k_2)\alpha_{p,3}^{(1)}(k_3) \]

and assume that the slope of this product

\[ \sigma = \text{ord}_p(\lambda(k_1, k_2, k_3)) = \sigma_1(k_1, k_2, k_3) = \sigma_1 + \sigma_2 + \sigma_3 \]

is constant and positive for all triplets \((k_1, k_2, k_3)\) in an appropriate $p$-adic neighbourhood of the fixed triplet of weights \((k_1, k_2, k_3)\).

Let \(A = A(\mathcal{B})\) denote an affinoid algebra \(A\) associated with an analytic space \(\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3\), a neighbourhood around \((k_1, k_2, k_3)\) \(\in X^3\) (with a given \(k\) and \(\psi \mod N\)).

**Theorem C (on $p$-adic measures for families of triple products).** Put \(H = [2\text{ord}_p(\lambda)] + 1\). There exists a sequence of distributions \(\Phi_r\) on \(Y\) with values in \(M = M(A)\) giving an \(H\)-admissible measure \(\Phi^\lambda\) with values in \(M^\lambda \subset M\) with the following properties:

There exists an \(A\)-linear form \(\ell = \ell_{f_1 \otimes f_2 \otimes f_3, \lambda} : M(A) \rightarrow A\) (given by (11.24), such that the composition

\[ \tilde{\mu} = \tilde{\mu}_{f_1 \otimes f_2 \otimes f_3, \lambda} := \ell_{f_1 \otimes f_2 \otimes f_3, \lambda}(\Phi^\lambda) \]

is an \(H\)-admissible measure with values in \(A\), and for all \((k_1, k_2, k_3)\) in the affinoid neighborhood \(\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3\), as above, satisfying \(k_1 \leq k_2 + k_3 - 2\) we have that the evaluated integrals

\[ \text{ev}_{(k_1, k_2, k_3)}(\ell_{f_1 \otimes f_2 \otimes f_3, \lambda}(\Phi^\lambda)(y^r_p, \chi)) \]

on the arithmetical characters \(y^r_p, \chi\) coincide with the critical special values

\[ \Lambda^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \]

for \(r = 0, \ldots, k_2 + k_3 - k_1 - 2\), and for all Dirichlet characters \(\chi \mod N p^v, v \geq 1\), with all three corresponding Dirichlet characters \(\chi_j\) given by (6.16), having \(N p\)-complete conductors. Here the normalisation of \(\Lambda^*\) includes at the same time certain Gauss sums, Petersson scalar products, powers of \(\pi\) and of \(\lambda(k_1, k_2, k_3)\), and a certain finite Euler product. The precise form of the Euler-like \(p\)-factor is given by a general motivic setting as in [Pa94], [Co], [Co-PeRi]; we also refer to Section 7 of [PaTV] in the case of the \(L\)-function of one Coleman’s family. However, our modular construction of the admissible measures of Theorem C does not use these explicit formulae. Moreover, these measures are uniquely
The $p$-adic Mellin transform and four variable $p$-adic analytic functions. Any $h$-admissible measure $\tilde{\mu}$ on $Y$ with values in a $p$-adic Banach algebra $A$ can be characterized its Mellin transform $L_{\tilde{\mu}}(x) L_{\tilde{\mu}} : X \to A$, defined by $L_{\tilde{\mu}}(x) = \int_Y x(y)d\tilde{\mu}(y)$, where $x \in X$, $L_{\tilde{\mu}}(x) \in A$.

Key property of $h$-admissible measures $\tilde{\mu}$: its Mellin transform $L_{\tilde{\mu}}$ is analytic with values in $A$.

Let $A = A(B) = A_1 \hat{\otimes} A_2 \hat{\otimes} A_3 = A(B_1) \hat{\otimes} A(B_2) \hat{\otimes} A(B_3)$ denote again the Banach algebra $A$ where $B$ is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integer $k$ and Dirichlet character $\psi \mod N$).

Theorem D (on $p$-adic analytic function in four variables). Put $H = [2\text{ord}_p(\lambda)] + 1$. There exists a $p$-adic analytic function in four variables $(x, s) \in X \times B_1 \times B_2 \times B_3 \subset X \times X \times X \times X$:

\[L_{\tilde{\mu}} : (x, s) \mapsto \text{ev}_s(L_{\tilde{\mu}}(x)) \quad (x \in X, \ L_{\tilde{\mu}}(x) \in A),\]

with values in $\mathbb{C}_p$, such that for all $(k_1, k_2, k_3)$ in the affinoid neighborhood as above $B = B_1 \times B_2 \times B_3$, satisfying $k_1 \leq k_2 + k_3 - 2$, we have that the special values $L_{\tilde{\mu}}(x, s)$ at the arithmetical characters $x = y_p^* \chi$, and $s = (k_1, k_2, k_3) \in B$ coincide with the normalized critical special values

\[L^*(f, k_1) \otimes f_2, k_2 \otimes f_3, k_3, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \cdots, k_2 + k_3 - k_1 - 2),\]

for Dirichlet characters $\chi \mod Np^v, v \geq 1$, such that all three corresponding Dirichlet characters $\chi_j$ given by (6.16), have $Np$-complete conductors where the same normalization of $L^*$ as in Theorem C.

Moreover, for any fixed $s = (k_1, k_2, k_3) \in B$ the function

\[x \mapsto L_{\tilde{\mu}}(x, s)\]

is $p$-adic analytic of type $o(\log^H(\cdot))$.

Indeed, we obtain the function in question $L_{\mu}(x, s)$ by evaluation at

\[s = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in B :\]

this is a $p$-adic analytic function in four variables $(x, s) \in X \times B_1 \times B_2 \times B_3 \subset X \times X \times X \times X$:

\[L_{\tilde{\mu}}(x, s) := \text{ev}_s(L_{\tilde{\mu}}(x)) \quad (x \in X, \ s \in B_1 \times B_2 \times B_3, \ L_{\tilde{\mu}}(x) \in A).\]

This is a joint work in progress with S. Boecherer, we use:

1) the higher twists of the Siegel-Eisenstein series, introduced in [Boe-Schm],
2) Ibukiyama’s differential operators (see [Ibu, BSY]).

11. Scheme of the Proof

11.1. Boecherer’s higher twist.

Boecherer’s Higher Twist. 1) We define the higher twist of the series \( F_{\chi,r} = \sum_{T} a_{\chi,r}(R, T)q^{T} \) by some Dirichlet characters \( \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3 \) as the following formal nearly holomorphic Fourier expansion:

\[
F_{\chi,r} = \sum_{T} \bar{\chi}_1(t_{12})\bar{\chi}_2(t_{13})\bar{\chi}_3(t_{23})a_{\chi,r}(R, T)q^{T}.
\]

(11.22)

The series (11.22) is a Siegel modular form of some higher level, but it has additional symmetries with respect to symplectic embedding \( \iota_3 : \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \rightarrow \text{Sp}_3 \): its triple Nebentypus character does not depend on \( \chi \) mod \( Np^{v} \), and is equal to \( (\psi_1, \psi_2, \psi_3) \), if we choose Dirichlet characters as in (6.16):

\[
\begin{align*}
\chi_1 \text{ mod } Np^{v} &= \chi, \quad \chi_2 \text{ mod } Np^{v} = \psi_2 \bar{\psi}_3 \chi, \\
\chi_3 \text{ mod } Np^{v} &= \psi_1 \bar{\psi}_3 \chi, \quad \psi = \chi^2 \psi_1 \psi_2 \bar{\psi}_3.
\end{align*}
\]

We use the Siegel-Eisenstein series \( F_{\chi,r} \) which depends on the character \( \chi \), but its precise nebentypus character is \( \psi = \chi^2 \psi_1 \psi_2 \bar{\psi}_3 \), and it is defined by \( F_{\chi,r} = G^*(z, -r; k, (Np^{v})^2, \psi) \), where \( z \) denotes a variable in the Siegel upper half space \( \mathbb{H}_3 \), and the normalized series \( G^*(z, s; k, (Np^{v})^2, \psi) \) is given by (5.11).

This series depends on \( s = -r \), and for the critical values at integral points \( s \in \mathbb{Z} \) such that \( 2 - k \leq s \leq 0 \), it represents a (nearly) holomorphic Siegel modular form in the sense of Shimura \cite{ShiAr}:

\[
F_{\chi,r} = \sum_{T} \det(T)^{k-2r-k}Q(R, \iota; k-2r, r)a_{\chi,r}(T)q^{T}.
\]

11.2. Ibukiyama’s differential operator.

Ibukiyama’s differential operator. 2) We use an algebraic version of Ibukiyama’s differential operator, which generalizes the algebraic “pull-back”: it transforms a nearly holomorphic Siegel modular form of weight \( k \)

to a nearly holomorphic triple modular form of weight \( (k_1, k_2, k_3) \) \( (k = k_2 + k_3 - k_1) \).
On a holomorphic Siegel modular form $F = \sum_{\mathcal{T}} a(\mathcal{T}) q^T$, this operator has the form
$$\mathcal{L}_k^{\lambda,\nu}(F) = \sum_{\mathcal{T}} \mathcal{P}(k_1, k_2, k_3, 0, \mathcal{T}) a(\mathcal{T}) q_1^{t_{11}} q_2^{t_{22}} q_3^{t_{33}},$$
where $\lambda = k_1 - k_3 \geq \mu = k_1 - k_2 \geq 0$, and $\mathcal{P}(k_1, k_2, k_3; r; \mathcal{T})$ is certain Ibukiyama's polynomial, equal to $(t_{11} t_{22} t_{33})^\lambda (t_{12} t_{13} t_{23})^\mu$, if $r = 0$.

As a result we obtain a sequence of triple modular distributions $\Phi_r(\chi)$ with values in the tensor product $\mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{A}) \otimes \mathcal{A} \mathcal{M}(\mathcal{A}) \otimes \mathcal{A} \mathcal{M}(\mathcal{A})$ of three Banach $\mathcal{A}$-modules of arithmetical nearly holomorphic modular forms (the normalizing factor $2^r$ is needed in order to prove certain congruences between $\Phi_r$). Note that $\mathcal{M}(\mathcal{T})$ is again a Banach $\mathcal{A}$-module on which $U_{\mathcal{T}}$ acts as a completely continuous operator.

The important property of the triple modular forms $\Phi_r(\chi)$: the nebentypus character is fixed and is equal to $(\psi_1, \psi_2, \psi_3)$ (for all $(k_1, k_2, k_3)$ and $\chi$ in question).

Using this property we compute the canonical projection $\pi_\lambda(\Phi_r(\chi))$ of the triple modular form $\Phi_r(\chi)$ onto the $\lambda$-characteristic $\mathcal{A}$-submodule $\mathcal{M}(\mathcal{T})^{\lambda}(\mathcal{A})$ of the triple Atkin's operator $U_{\mathcal{T},p}$:
$$\pi_\lambda : \mathcal{M}(\mathcal{T}) \to \mathcal{M}(\mathcal{T})^{\lambda}(\mathcal{A}).$$

We prove that the resulting sequence of modular distributions $\pi_\lambda(\Phi_r)$ on the profinite group $Y$ produces a certain $p$-adic admissible measure $\Phi^\lambda$ (in the sense of Amice-Vélu, [Am-V]) with values in a certain locally free $\mathcal{A}$-submodule of finite rank
$$\mathcal{M}(\mathcal{T})^\lambda(\mathcal{A}) \subset \mathcal{M}(\mathcal{T}) \subset \bigcup_{\nu \geq 0} \mathcal{M}(\mathcal{T}, Np^{\nu}, \psi_1, \psi_2, \psi_3; \mathcal{A})$$
of formal nearly holomorphic triple modular forms of all levels $Np^{\nu}$ and the fixed nebentypus characters $(\psi_1, \psi_2, \psi_3)$. We use congruences between triple modular forms $\Phi_r(\chi) \in \mathcal{M}(\mathcal{T})$ (they have explicit formal Fourier coefficients).

Then we use a general admissibility criterion saying that these congruences imply $H$-admissibility for their projections in $\mathcal{M}(\mathcal{T})^\lambda(\mathcal{A})$, where $H = [2\ord_p(\lambda)] + 1$.

11.3. Algebraic linear form. We refer here to Section 6 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman’s family.
3) From $M^\lambda_T(A)$ to $A$: we use a $\mathbb{Q}$-valued linear forms of type

$$\mathcal{L} : h \mapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, h \rangle}{\langle f_1^0, f_1^0 \rangle \langle f_2^0, f_2^0 \rangle \langle f_3^0, f_3^0 \rangle}$$

where $f_j^0$ is the eigenfunction (8.18) of the conjugate Atkin’s operator $U^*_p$, and $f_{j,0}$ is the eigenfunction (8.19) of $U_p$. The linear form $\mathcal{L}$ is defined on the finite dimensional $\mathbb{Q}$-vector characteristic subspace

$$h \in M_k(\mathbb{Q})^\lambda(\mathbb{Q}) \subset M_{k_1,r^*}(Np, \psi_1; \mathbb{Q}) \otimes M_{k_2,r^*}(Np, \psi_2; \mathbb{Q}) \otimes M_{k_3,r^*}(Np, \psi_3; \mathbb{Q}).$$

This map is then extended to an $A$-linear map

$$(11.24) \quad \ell = \ell_{f_1 \otimes f_2 \otimes f_3, \lambda} : M(A)^\lambda \to A;$$

on the locally free $A$-module of finite rank $M(A)^\lambda$.

This map produces a sequence of $A$-valued distributions $\mu^\lambda(\chi) \in A$ in such a way that for all suitable weights $k \in \mathcal{B}$ one has

$$ev_k(\mu^\lambda(\chi)) = \mathcal{L}(ev_k(\pi_\lambda(\Phi_p)(\chi))), \lambda \in A^x, \lambda(k) \in \mathbb{Q}^x,$$

where $k = (k_1, k_2, k_3) \in \mathcal{B}$, $ev_k : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_k : M(A) \to M_k(\mathbb{C}_p).$$

More precisely, we consider three auxiliary families of modular forms

$$(11.25) \quad \tilde{f}_{j,k_j}(z) = \sum_{n=1}^{\infty} a_{n,j,k_j} e(nz) \in S_{k_j}(\Gamma_0(Njp^{\nu_j}), \psi_j), \ (1 \leq j \leq 3, \nu_j \geq 1),$$

with the same eigenvalues as those of (10.21),

for all Hecke operators $T_q$, with $q$ prime to $Np$. In our construction we use as $\tilde{f}_{j,k_j}$ certain “easy transforms” of primitive cusp forms in (1.1). In particular, we choose as $\tilde{f}_j$ certain eigenfunctions $\tilde{f}_{j,k_j} = f_{j,k_j}^0$ of the adjoint Atkin’s operator $U_p^*$, in this case we denote by $f_{j,k_j,0}$ the corresponding eigenfunctions of $U_p$.

The $\mathbb{Q}$-linear form $\mathcal{L}$ produces a $\mathbb{C}_p$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ starting from the modular $p$-adic admissible measure $\Phi^\lambda$ of stage 3), where $\ell : M_T(\mathbb{C}_p) \to \mathbb{C}_p$ denotes a $\mathbb{C}_p$-linear form, interpolating $\mathcal{L}$.

11.4. Evaluation of $p$-adic integrals. We refer to Section 7 of [PaTV] for similar constructions in the case of the $L$-function of one Coleman’s family.
L-values and $p$-adic integrals. 4) We show that for any appropriate Dirichlet character $\chi \mod Np^v$ the integral

$$\mu_r^\lambda(\chi) = L(\pi_r(\Phi_r(\chi))) \in \mathcal{A}$$

evaluated at $(k_1, k_2, k_3) \in B = B_1 \times B_2 \times B_3$, coincides (up to a normalisation) with the special $L$-value

$$L^s(f_{1,k_1}^0 \otimes f_{2,k_2}^0 \otimes f_{3,k_3}^0, k_2 + k_3 - 2 - r, \psi_1 \psi_2 \chi)$$

under the above assumptions on $\chi$ and $r$).

A general integral representation of Garrett’s type. The basic idea how a Dirichlet character $\chi$ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $\mathcal{A}$-valued admissible measure $\tilde{\mu}^\lambda = \ell(\tilde{\Phi}^\lambda)$ established at stage 4), does not depend on this technical computation.

In order to control the denominators of the modular forms

$$\pi_\lambda(\tilde{E}(-r, \chi)) \in M^\lambda(\mathcal{A}) =: \Phi_r(\chi),$$

used in the construction (the admissibility condition) we use the following result.

12. Criterion of admissibility

Theorem 12.1 (Criterion of admissibility). Let $\alpha \in \mathcal{A}^*$, $0 < |\alpha|_p < 1$ and suppose that there exists a positive integer $\tau$ such that the following conditions are satisfied:

1) for all $r = 0, 1, \ldots, h - 1$ with $h = [\tau \text{ord}_p \alpha] + 1$, and $v \geq 1$,

\begin{equation}
\Phi_r(a + (Np^v)) \in M(Np^{\tau v}) \quad \text{(the level condition)}
\end{equation}

2) the following congruence for the coefficients holds: for all $t = 0, 1, \ldots, h - 1$

\begin{equation}
U^{\tau v} \sum_{r=0}^t \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \mod p^{vt}
\end{equation}

\begin{equation}
\text{(the divisibility condition)}
\end{equation}

Then the linear form given by $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)}y_p^r) := \pi_\alpha(\Phi_r(a + (Np^v)))$ on local monomials (for all $r = 0, 1, \ldots, h - 1$), is an $h$-admissible measure: $\tilde{\Phi}^\alpha : \mathcal{P}^h(Y, \mathbb{Q}) \to \mathcal{M}^\alpha \subset \mathcal{M}$.
Proof uses the commutative diagram:
\[
\begin{array}{ccc}
M(Np^{v+1}, \psi; \mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & M^\alpha(Np^{v+1}, \psi; \mathcal{A}) \\
U^v & & U^v
\end{array}
\]
\[M(Np, \psi; \mathcal{A}) \xrightarrow{\pi_{\alpha,0}} M^\alpha(Np, \psi; \mathcal{A}) = M^\alpha(Np^{v+1}, \psi; \mathcal{A}).\]

The existence of the projectors \(\pi_{\alpha,v}\) comes from Coleman’s Theorem A.4.3 \([\text{CoPB}]\).

On the right: \(U\) acts on the locally free \(\mathcal{A}\)-module \(M^\alpha(Np^{v+1}, \mathcal{A})\) via the matrix:
\[
\begin{pmatrix}
\alpha & \cdots & \cdots & * \\
0 & \alpha & \cdots & * \\
0 & 0 & \ddots & \cdots \\
0 & 0 & \cdots & \alpha
\end{pmatrix}
\]
where \(\alpha \in \mathcal{A}^\times\)

\[\implies U^v\] is an isomorphism over \(\mathcal{A}\),

and one controls the denominators of the modular forms of all levels \(v\) by the relation:
\[(12.28) \quad \pi_{\alpha,v}(h) = U^{-v}\pi_{\alpha,0}(U^v h) =: \pi_\alpha(h)\]

The equality \((12.28)\) can be used as the definition of \(\pi_\alpha\) at any level \(Np^v\).

The growth condition (see \((12.17)\)) for \(\pi_\alpha(\Phi_r)\) is deduced from the congruences \((12.27)\) between modular forms, using the relation \((12.28)\).
A. A. Panchishkin

REFERENCES


[Boe-Pa6] S. BÖCHERER and A.A. PANCHISHKIN, Admissible $p$-adic measures attached to triple products of elliptic cusp forms, accepted in Documenta Math. in March 2006 (a special volume dedicated to John Coates).


[Cour] COURTIER, M., Familles d’opérateurs sur les formes modulaires de Siegel et fonctions $L$ $p$-adiques,


Triple Products of Coleman’s Families


[JoH05] Jory–Hugue, F., Unicité des h–mesures admissibles p-adiques données par des valeurs de fonctions L sur les caractères, Prépublication de l’Institut Fourier (Grenoble), N°676, 1-33, 2005


A. A. Panchishkin


A. A. Panchishkin
Institut Fourier, Université Grenoble-1
E-mail: Alexei.Pantchichkine@ujf-grenoble.fr