Formal Meromorphic Functions on Manifolds of Finite Type

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Dedicated to Professor J.J. Kohn on the occasion of his 75th birthday

Abstract: It is shown that a real-valued formal meromorphic function on a formal generic submanifold of finite Kohn-Bloom-Graham type is necessarily constant.

1. Introduction

It is easy to see (and known, see [1]) that if $M \subset \mathbb{C}^N$ is a connected generic real-analytic CR manifold which is of finite type in the sense of Kohn [5] and Bloom-Graham [4] at some point $p \in M$, then any meromorphic map $H: U \to \mathbb{C}^m$ defined on a connected neighbourhood of $M$ which satisfies $H(M) \subset E$, where $E \subset \mathbb{C}^m$ is a totally real real-analytic submanifold, is necessarily constant.

Let us give a short proof of this fact. First, we recall the definition of the Segre sets $S_p^j$. These are defined inductively. First, we define the Segre variety $S_p^1 = S_p$
for \( p \in M \). Let \( \rho(Z, \bar{Z}) = (\rho_1(Z, \bar{Z}), \ldots, \rho_d(Z, \bar{Z})) \) be a (vector-valued) defining function for \( M \) defined in a neighbourhood \( U \times \bar{U} \) of \((p, \bar{p})\), i.e.

\[
M \cap U = \{ Z \in U : \rho(Z, \bar{Z}) = 0 \}, \quad dp_1 \wedge \cdots \wedge dp_d \neq 0 \text{ on } U, \quad \rho(Z, \bar{Z}) = \bar{\rho}(\bar{Z}, Z).
\]

Then if \( S^1_q \) is defined by

\[
S^1_q = \{ Z \in U : \rho(Z, \bar{q}) = 0 \}, \quad q \in U,
\]

the \( j \)-th Segre set \( S^j_p \), \( j \in \mathbb{N} \), is defined inductively by

\[
S^j_p = \bigcup_{q \in S^{j-1}_p} S^1_q.
\]

We are using the following Theorem, which characterizes finite type in terms of properties of the Segre sets:

**Theorem 1** (Baouendi, Ebenfelt and Rothschild [1]). *Let \( M \subset \mathbb{C}^N \) be a generic real-analytic CR manifold. Then \( M \) is of finite type at \( p \in M \) if and only if there exists an open set \( V \subset \mathbb{C}^N \) with \( V \subset S^d_{p+1} \).*

Now assume that \( H : U \to \mathbb{C}^m \) is a meromorphic map which satisfies \( H(M) \subset E \), where \( E \) is totally real. First note that since \( M \) is of finite type at some point \( p \), it is of finite type on the complement of a proper real-analytic subvariety \( F \subset M \). So there exists a point \( p \in M \) with the property that \( M \) is of finite type at \( p \) and \( H \) is holomorphic in some neighbourhood of \( p \) (because \( M \) is generic, it is a set of uniqueness for holomorphic functions). We shall prove that in this situation, \( H \) is constant on an open set in \( \mathbb{C}^N \), and thus constant.

We can find coordinates \( \eta \) in \( \mathbb{C}^m \) such that near \( H(p) \), \( E \) is given by an equation of the form \( \eta = \varphi(\bar{\eta}) \). Thus, \( H(Z) = \varphi(\bar{H}(Z)) \), whenever \( Z \in M \), and from this we have that \( H(Z) = \varphi(\bar{H}(\zeta)) \) whenever \( Z \in S^1_{\zeta} \) (restricting to a suitable neighbourhood \( U \) of \( p \)). Thus, \( H(Z) = \varphi(\bar{H}(p)) \) for \( Z \in S^1_p \); since \( p \in S^1_p \), \( H(Z) = H(p) \) for \( Z \in S^2_p \). Now we consider \( Z \in S^2_p \). For each such \( Z \), there is \( \zeta \in S^1_p \) with \( Z \in S^1_{\zeta} \). Our equation tells us that \( H(Z) = \varphi(\bar{H}(\zeta)) = \varphi(\bar{H}(p)) \), and again, since \( p \in S^2_p \), \( H(Z) = H(p) \) for \( Z \in S^2_p \).

Continuing the iteration process like this, we see that \( H(Z) = H(p) \) for \( Z \in S^j_p \) for \( j \in \mathbb{N} \). Since \( S^{d+1}_p \) contains an open subset of \( \mathbb{C}^N \) by Theorem 1, the identity principle implies that \( H(Z) = H(p) \) on \( U \). This proves the constancy of such an \( H \).
Our main point in this paper is the extension of this result to the formal category. Here we cannot “move to a good point”. In this setting, a formal meromorphic map is given by $H = N / D$, where $D$ is a (nonvanishing) formal power series and $N: (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^m, 0)$ is a formal holomorphic map. Note that if $E \subset \mathbb{C}^m$ is a formal totally real manifold, then in suitable coordinates $\eta \in \mathbb{C}^m$, $E$ is given by $\text{Im} \eta = 0$. We say that $H = N / D$ maps $M$ into $E$ if for any formal map $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ satisfying $\rho(\gamma_1(t), \gamma_2(t)) = 0$ for every defining function $\rho$ of $M$ we have

$$N_j(\gamma_1(t))\bar{D}(\gamma_2(t)) - \bar{N}_j(\gamma_2(t))D(\gamma_1(t)) = 0$$

for every $j = 1, \ldots, m$. We shall freely use the terminology of formal real submanifolds as explained in e.g. [2]. We show the following:

**Theorem 2.** Let $M \subset \mathbb{C}^N$ be a formal generic manifold of finite type, $H: (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^m, 0)$ a formal meromorphic map which satisfies $H(M) \subset E$, where $E$ is a formal totally real manifold. Then $H$ is formal holomorphic, and thus, constant.

We note that the finite type assumption is necessary. Indeed, every manifold of the form $M = \tilde{M} \times E$ where $\tilde{M}$ is some CR manifold and $E$ is totally real has nonconstant CR maps onto a totally real manifold (the projection onto its second coordinate). On the other hand, here is another example, due to J. Lebl:

**Example 1.** Let $M \subset \mathbb{C}^3$ be given by

$$w_1 = \overline{w_1}e^{p|z|^2}, \quad w_2 = \overline{w_2}e^{q|z|^2},$$

for some integers $p$ and $q$. Then the function

$$H(z, w_1, w_2) = \frac{w_1^q}{w_2^p}$$

maps $M$ into $\mathbb{R}$ and is not the restriction of a holomorphic function. Also note that this function is not even continuous on $M$. Our results imply that no nonconstant holomorphic choice of projection onto $\mathbb{R}$ can be made.

2. **Reflection Identities and Consequences**

We shall first show that we can simplify our situation somewhat by choosing ”normal” coordinates. Recall that normal coordinates for a formal generic
submanifold \((M, 0) \subset (\mathbb{C}^N, 0)\) means a choice of coordinates \((z, w) \in \mathbb{C}^n \times \mathbb{C}^d\) (\(d\) being the real codimension of \((M, 0)\)) together with formal functions \(Q_j(z, \chi, \tau) \in \mathbb{C}[[z, \chi, \tau]], j = 1, \ldots, d,\) satisfying

\[Q_j(z, 0, \tau) = Q_j(0, \chi, \tau) = \tau_j, \quad j = 1, \ldots, d,\]

such that \(w_j - Q_j(z, \chi, \tau)\) generate the manifold ideal associated to \((M, 0)\) in \(\mathbb{C}[[z, w, \chi, \tau]].\) We will write \(Q = (Q_1, \ldots, Q_d),\) and abbreviate the generating set with \(w - Q(z, \chi, \tau).\)

We will show that in normal coordinates, a formal meromorphic function \(H\) which maps \((M, 0)\) into \((\mathbb{R}, 0)\) actually only depends on the transverse variables \(w.\) To do this, we first give a reflection identity which we will use.

**Proposition 1.** If \((M, 0) \subset (\mathbb{C}^N, 0)\) is a formal generic submanifold, and \((z, w)\) are normal coordinates for \((M, 0)\) with corresponding generators \(w - Q(z, \chi, \tau).\) If \(H = ND: (M, 0) \to (\mathbb{R}, 0)\) is formal meromorphic, and \(N\) and \(D\) do not have any common factors, then there exists a formal holomorphic function \(a(z, \chi, z^1, w),\) with \(a(0, 0, 0, 0) = 1,\) such that

\[N(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))) = a(z, \chi, z^1, w)N(z^1, w),\]
\[D(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))) = a(z, \chi, z^1, w)D(z^1, w).\]

**Proof.** The conclusion is clear if \(N\) is identically zero, so we assume that this is not the case. By definition, we have

\[\bar{D}(\chi, \tau)N(z, Q(z, \chi, \tau)) = \bar{N}(\chi, \tau)D(z, Q(z, \chi, \tau)).\]

Taking the complex conjugate of the series and replacing \(\chi\) by \(z^1,\) \(\tau\) by \(w,\) and \(z\) by \(\bar{\chi}\) in this equation, we also have that

\[D(z^1, w)\bar{N}(\chi, Q(\chi, z^1, w)) = N(z^1, w)\bar{D}(\chi, Q(\chi, z^1, w)).\]

We now substitute \(\tau = \bar{Q}(\chi, z^1, w)\) into (2) to obtain

\[\bar{D}(\chi, \bar{Q}(\chi, z^1, w))N(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))) = \bar{N}(\chi, \bar{Q}(\chi, z^1, w))D(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))).\]

We now multiply the left (and right, respectively) hand sides of (3) and (4) with each other, and after cancelling the (nonvanishing) common factor \(\bar{N}(\chi, \bar{Q}(\chi, z^1, w))\bar{D}(\chi, \bar{Q}(\chi, z^1, w))\) we obtain

\[D(z^1, w)N(z, Q(z, \chi, \bar{Q}(\chi, z^1, w))) = D(z, Q(z, \chi, \bar{Q}(\chi, z^1, w)))N(z^1, w).\]
Now, using the fact that $N$ and $D$ do not have any common factors, unique factorization in the ring $\mathbb{C}[[z, \chi, z^1, w]]$ implies that there exists a unit $a(z, \chi, z^1, w)$ such that (1) holds. By evaluating (1) at $z = z^1$, and using the reality property $Q(z, \chi, \bar{Q}(\chi, z, w)) = w$, we have that $a(z, \chi, z, w) = 1$, so in particular, $a(0, 0, 0, 0) = 1$. □

Lemma 2. Let $(M, 0) \subset (\mathbb{C}^N, 0)$ be a formal generic submanifold. Assume that $H(Z) = \frac{N(Z)}{D(Z)}$ is a formal meromorphic map sending $(M, 0)$ into $(\mathbb{R}, 0)$. Then for any choice of normal coordinates $(z, w)$ for $(M, 0)$, we have that $H(z, w) = H(0, w)$; i.e., there exist formal functions $\tilde{N}(w)$ and $\tilde{D}(w)$ such that $H(z, w) = \frac{\tilde{N}(w)}{\tilde{D}(w)}$.

Proof. We use Proposition 1. Setting $\chi = z^1 = 0$, we see that $N(z, w) = a(z, 0, 0, w)N(0, w), \quad D(z, w) = a(z, 0, 0, w)D(0, w).

The Lemma follows. □

3. Prolongation of the reflection along Segre maps and proof of Theorem 2

We will denote by $v^1(z, \chi, z^1; w) = Q(z, \chi, \bar{Q}(\chi, z^1, w))$;

in the usual Segre-map terminology, $v^1(z, \chi, z^1; 0)$ is the transversal component of the second Segre map of $(M, 0)$. We define $S^{(0)} = z$, and for $j \geq 1$

$S^{(j)} = (z, \chi, z^1, \chi^1, \ldots, z^j),$

and write $S^{(j)}_k = (z^k, \chi^k, \ldots, z^j)$ for $k \leq j$. With that notation and our simplification from Lemma 2, our reflection identity (1) now reads

$$N\left(v^1(S^{(1)}; w)\right) = a(S^{(1)}, w)N\left(w\right),$$

$$D\left(v^1(S^{(1)}; w)\right) = a(S^{(1)}, w)D\left(w\right).$$

For $j \geq 2$, we define inductively

$$v^j\left(S^{(j)}; w\right) = v^1(z, \chi, z^1; v^{j-1}(S^{(j)}_1; w)).$$

We can now state the finite type criterion of Baouendi, Ebenfelt and Rothschild [3], for later reference, as follows:
Theorem 3. If \((M,0)\) is of finite type in the sense of Kohn-Bloom-Graham, then there exists a \(j \geq 1\) such that
\[
S^{(j)} \mapsto v^j \left( S^{(j)}; 0 \right), \quad (\mathbb{C}^{(2j-1)n}, 0) \to (\mathbb{C}^d, 0),
\]
is of generic full rank \(d\).

Thus, if we for \(j \geq 2\) replace \(w\) by \(v^{j-1}(S^{(j)}; w)\) in (6), we obtain
\[
N \left( v^j(S^{(j)}; w) \right) = N \left( v^1(S^{(1)}; v^{j-1}(S^{(j)}; w)) \right) = a(S^{(1)}; v^{j-1}(S^{(j)}; w)) N \left( v^{j-1}(S^{(j)}; w) \right).
\]

Applying induction, we see that the following holds:

Lemma 3. For every \(j \geq 1\), there exists a unit \(a_j(S^{(j)}, w)\) such that
\[
(7) \quad N \left( v^j(S^{(j)}; w) \right) = a_j(S^{(j)}, w) N(w), \quad D \left( v^j(S^{(j)}; w) \right) = a_j(S^{(j)}, w) D(w).
\]

We can now prove Theorem 2: By Theorem 3, there exists a \(j\) such that \(v^j(S^{(j)}; 0)\) is of generic full rank. Assuming that \(D(0) = 0\), we see that \(D(v^j(S^{(j)}; 0)) = 0\). Since \(v^j\) is of generic full rank, this implies that \(D(w) = 0\); this contradiction shows that \(D(0) \neq 0\). Hence, we can assume that \(H(w) = N(w)\) is holomorphic, and without loss of generality, \(N(0) = 0\). Now the same argument as before shows that \(N(w) = 0\), and so, \(H\) is constant.

Remark 1. More generally, if we do not assume that \((M,0)\) is of finite type, then we can define the formal variety
\[
V_j = \overline{\text{image}(v^j(S^{(j)}; 0))} \cong \{ f \in \mathbb{C}[[w]] : f \circ v^j(S^{(j)}; 0) = 0 \},
\]
and \(V = \bigcup_j V_j\) (which is again a formal variety). The same arguments as above show that \(D\), as well as \(N\), are constant on \(V\). This corresponds to the statement that a real-valued CR meromorphic function is constant along the CR-orbits of \(M\).

References


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