Advanced Lectures in Mathematics
Volume 31

Handbook of Group Actions

Volume I

Companion to the volume
Handbook of Group Actions, Volume II

edited by

Lizhen Ji
Athanase Papadopoulos
Shing-Tung Yau

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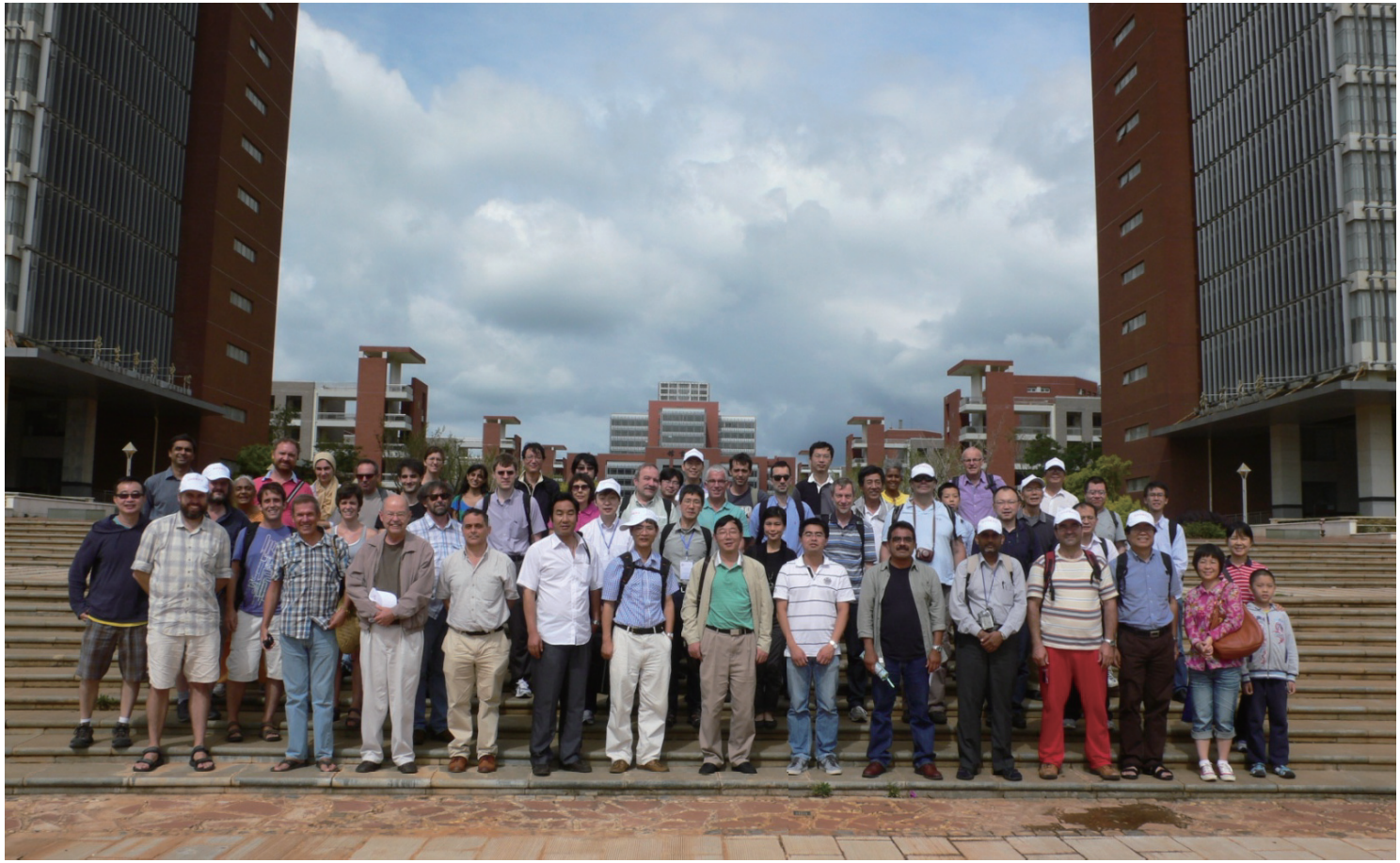
A group picture in front of the library of the original campus of Kunming University of Science and Technology



A group picture on the new campus of Kunming University of Science and Technology

Foreword to Volumes I and II

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Kunming is a wonderful place for meetings and for mathematical discussions, especially in the summer, when the weather is most favorable. The conference was a success, from the mathematical and the human point of view. The city is warm, and the landscape is beautiful. There is a big lake, and a mountain behind the lake. Mathematicians like beauty. Hermann Weyl said: "My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful." (Quoted in *Hermann Weyl's Legacy*, Institute for Advanced Study.)

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This Handbook will serve as an introduction and a reference to both beginners, non-experts, experts and users of group actions.

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L. Ji, A. Papadopoulos, and S.-T. Yau
Ann Arbor, Strasbourg, and Cambridge MA
November, 2014

Introduction

The subject of this Handbook is groups and group actions. Although groups are omnipresent in mathematics, the notion of group was singled out relatively recently. We recall that the first definition of a (finite) group was formulated by Cayley around the middle of the nineteenth century. But the concept itself is inherent in the work of Galois (as a group of permutations of solutions of polynomial equations), and it is also contained — at least implicitly — in works of Ruffini, Lagrange and Gauss.

On the other hand, the idea of group is closely related to that of symmetry, or rather, to the mathematics behind symmetry, and the use of groups, seen as symmetries, can be traced back to antiquity. In fact, the notion of symmetry reflects a group action, not only in mathematics, but also in other sciences, including chemistry, biological physics, and the humanities. Symmetry is also one of the most fundamental concepts in art.

In mathematics, the notion of abstract group is at the heart of the formulation of many problems. Still, it is usually the concept of transformation group, or of a group acting on a space, rather than that of group alone, which is of fundamental importance. A group action brings in an additional notion to the group at hand, coming from the space on which the group acts. It is also the group action which makes groups interesting, useful and understandable. The precise identification of a group with a group of symmetries of a space is made through the action of the group on that space. But as the same group can act on different spaces, this group can be realized in several different ways as a group of symmetries.

The notion of transformation group was inherent in eighteenth-century geometry, in particular in projective geometry. But it was Klein, in his *Erlangen program* manifesto, and mathematicians like Lie, Poincaré and others who worked in the spirit of this program (some of them without being aware of the program) who highlighted the importance of a transformation group as a basic concept associated to a geometry, with the view that a geometry is characterized (and, in a certain way, it is defined) by a transformation group rather than by a space.

Central to contemporary research is the study of discrete group actions on homogeneous spaces, in particular on manifolds of constant curvature and locally symmetric spaces of finite volume. The most famous of such groups are probably the Fuchsian groups, the Kleinian groups and the arithmetic subgroups of semisimple Lie groups, where not only the groups are studied individually, but their deformation theory is also very rich.

Other interesting classes of examples of infinite groups are, on the one hand, the automorphism groups $\mathrm{Aut}(F_n)$ and the outer automorphism groups $\mathrm{Out}(F_n)$

of a free group F_n on n generators ($n \geq 2$), and on the other hand, the automorphism groups $\mathrm{Aut}(\pi_1(S_g))$ and the outer automorphism groups $\mathrm{Out}(\pi_1(S_g))$ of fundamental group of (say, closed) surfaces S_g of genus g ($g \geq 2$), i.e., the mapping class groups of S_g. One can also mention the Coxeter groups.

It can easily be argued that the free group is much simpler to apprehend than a surface group; for instance, one can easily visualize the Cayley graph of the free group F_n, a regular tree with vertices of order $2n$, and hence, one can have a good picture of the geometry and combinatorics of that group, whereas the Cayley graph of the fundamental group $\pi_1(S_g)$ is more complex. However, it turns out that the theory of automorphism and of outer automorphisms of the free group F_n is much less understood than that of the automorphism (and the outer automorphism) group of the surface group S_g. The reason is that many actions of $\mathrm{Aut}(\pi_1(S_g))$ and $\mathrm{Out}(\pi_1(S_g))$ arising naturally from the geometry and the topology of surfaces have been studied, whereas for the groups $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$, there are not as many actions on geometric or topological spaces. The Coxeter groups are understood via to their action on Coxeter complexes.

The reader can refer to the beginning of the article by L. Ji in this volume, where many group actions are listed.

The present volume of the *Handbook of Group Actions* is more especially concerned with discrete group actions. It consists of 12 chapters, and it is divided into four parts. Each part emphasizes special discrete groups and their actions.

Part I: Geometries and General Group Actions

This part contains 2 chapters.

Chapter 1 is by S.-T. Yau. It is a record of the talk that the author gave at the Kunming conference, whose main theme was a view of a generalized geometry based on the notion of operators rather than on that of space. The relation with physics is also discussed. The concept of group is essential here, as a group of operators and as a gauge group. Several constructions of Riemannian geometry can be done in this setting, including the definitions of the Dirac and the Laplace operators, the differential topology of operator geometry, Hodge theory, Yang-Mills theory and conformal field theory. There is also a version of that theory for discrete spaces.

Chapter 2 is by L. Ji, and it is a summary of group actions that arise in mathematics. It attempts to cover all the major fields where group actions play an important role and to convey a sense of how broad group actions are in mathematics and other sciences. Hopefully it will give some content to the statement that group actions and symmetry, which are the same thing, are everywhere.

Part II: Mapping Class Groups and Teichmüller Spaces

This part concerns mapping class groups and Teichmüller spaces. The two topics are related, because the action of the mapping class group of a surface on the Teichmüller space of that surface constitutes one of the most interesting (and may be the most interesting) action of that group, in terms of the richness and the developments of the underlying theory, and also in terms of applications. Furthermore, Teichmüller spaces equipped with actions of mapping class groups are the primary source of holomorphic group actions in high dimensions, including

infinite dimensions. Teichmüller spaces are also related in other ways to the subject of group actions; for instance, an element of a Teichmüller space can be seen as a Fuchsian group acting on hyperbolic 2-space. This makes a relation between the present section and the section in Volume II of this handbook which deals with representations and deformations of subgroups of Lie groups.

In Chapter 3, A. Papadopoulos surveys some actions of mapping class groups. The latter admit actions which are of very different natures on spaces associated to surface: group-theoretic, holomorphic, combinatorial, topological, metric, piecewise-linear, etc. The author reviews in more detail actions on spaces of foliations and laminations, namely, measured foliations, unmeasured foliations, general geodesic laminations and the reduced Bers boundary. The chapter also contains a section on perspectives and open questions on actions of mapping class groups.

In Chapter 4, W. Su surveys two horofunction compactifications of Teichmüller space which are also spaces on which the mapping class group naturally acts. The horofunction boundary of a space is defined relatively to a certain metric. The two horofunction spaces that are studied in this chapter are associated to the Teichmüller metric and to the Thurston metric. The relation between these compactifications with Thurston and Gardiner-Masur's compactifications is reviewed (results of Walsh and of Lui and Su), and the isometry groups of Teichmüller space equipped with the two metrics are considered.

In Chapter 5, F. Herrlich studies Teichmüller disks, that is, embeddings of the hyperbolic disk in Teichmüller space that are holomorphic and isometric. More precisely, the author studies the stabilizers of these discs in the Schottky space $\mathcal{S}_g$ of a closed Riemann surface of genus g, a quotient of the Teichmüller space $\mathcal{T}_g$ by a certain (non-normal) torsion-free subgroup of the mapping class group. The Schottky space is an infinite orbifold covering of Riemann's moduli space. The stabilizer of a Teichmüller disk is sometimes a lattice in $\mathrm{PSL}(2,\mathbb{R})$. Schottky space is, like Teichmüller space, a complex manifold. The author studies in particular the stabilizers in Schottky space of the Teichmüller disks and more generally the behavior of these disks under the covering map $\mathcal{S}_g \to \mathcal{T}_g$.

Chapters 6 and 7 concern infinite-dimensional Teichmüller spaces.

Chapter 6 by E. Fujikawa concerns actions of mapping class groups of surfaces of infinite type. There are various groups which play the role of a mapping class groups, and various spaces which play the role of Teichmüller spaces, in this infinite-dimensional setting, and the author considers some of them. In particular, she considers the action of the so-called asymptotically trivial mapping class group on the asymptotic Teichmüller space, a space which was first introduced by Sullivan. The main result she describes in this context is that for surfaces satisfying a condition of bounded geometry (a quasi-isometry invariant condition which involves lower and upper bounds on certain classes of geodesics, when the Riemann surface is equipped with a hyperbolic metric), the asymptotically trivial mapping class group coincides with that of the so-called stable quasiconformal mapping class group, that is, the subgroup of conformal mapping classes which have representatives which are the identity outside a compact subset. She then introduces another Teichmüller space, which is called the intermediate Teichmüller space, which is the quotient of the classical (quasiconformal) Teichmüller space by the

asymptotically trivial mapping class group. Under the same bounded geometry condition, this space inherits a complex structure from that of the quasiconformal Teichmüller space. In general, the asymptotically trivial Teichmüller modular group is a proper subgroup of the group of holomorphic automorphisms of the asymptotic Teichmüller space. The author then studies the dynamics of the various actions that arise, and conditions under which such group actions are properly discontinuous. She also gives an asymptotic version of the Nielsen realization problem.

Chapter 7 by K. Matsuzaki is a survey of the complex analytic theory of the universal Teichmüller space and of some of its subspaces. Roughly speaking, the universal Teichmüller space is the space of equivalence classes of hyperbolic metrics on the unit disc, where two structures are considered equivalent if they differ by an isotopy which induces the identity on the boundary S^1 of the disc. This space can also be defined as a certain quotient of the group of diffeomorphisms of the unit circle. It is termed universal because it contains naturally the Teichmüller spaces of all hyperbolic surfaces. In this theory, the representation of the elements of a Teichmüller space by Fuchsian groups is useful if not essential. One of the important concepts that are studied in detail in this chapter is a natural subset of the universal Teichmüller space which is not the Teichmüller space of a surface, namely, a space of equivalence classes of diffeomorphisms of the circle with Hölder continuous derivatives. The author shows that this space is equipped with a complex structure modeled on a complex Banach space. This complex structure is described through a careful study of the Bers embedding of the space in the space of Schwarzian derivatives. The diffeomorphisms of the circle with Hölder continuous derivatives are characterized by certain properties of their quasiconformal extensions to the unit disc, and the theory bears relations with the space of asymptotically conformal maps studied by Carleson.

Chapter 8 by T. Satoh concerns mapping class groups of surfaces, and it has a more algebraic nature. It is a survey of the Johnson homomorphisms associated to mapping class groups. These are homomorphisms associated to graded quotients of a certain descending filtration of these groups. Johnson defined in the 1980s the first homomorphism in the sequence, as a tool to study the Torelli group. A similar theory for automorphisms of free groups was developed before, by Andreadakis, in his thesis in the 1960s. The Johnson homomorphism for surface mapping class groups was generalized later on to the so-called Johnson homomorphisms of higher degrees, and several people did extensive work on them, including Morita, Hain, Satoh and others, and there are recent results on the same subject by Kawazumi-Kuno and by Massuyeau-Turaev.

Part III: Hyperbolic Manifolds and Locally Symmetric Spaces

Chapter 9 by G. J. Martin is a survey on the various aspects of the geometry and arithmetic of Kleinian groups. The author examines the geometry of Kleinian groups and he gives geometric conditions on isometry groups of hyperbolic 3-space in order to be discrete. He studies in detail the two-generator groups, giving several generalizations of Jørgensen's inequality for discreteness, and he discusses the classification of arithmetic generalized triangle groups. One motivation for this study is a problem which Siegel posed in 1943, namely, to identify the minimal co-

volume lattices of isometries of hyperbolic n-space, and more generally of rank-one symmetric spaces.

Chapter 10 by G. Prasad and A. S. Rapinchuk is a survey on several results related to the basic question: *Can you hear the shape of a drum*? They concern locally symmetric spaces of finite volume. The problem asks whether two Riemannian manifolds having the same spectrum, i.e., the same set of eigenvalues, are isometric. A closely related question is the so-called iso-length spectrum problem for locally symmetric spaces: if two Riemannian manifolds have the same length spectrum, i.e., the same set of lengths of closed geodesics, are they isometric or at least commensurable? The major portion of this paper deals with this latter question and with related problems on algebraic groups and their maximal algebraic tori and the authors give a fairly complete and detailed survey of results in this direction.

Part IV: Knot Groups

This part contains two chapters on representations of knot groups and twisted Alexander polynomials. The twisted Alexander polynomial is defined as a pair consisting of a group and a representation of that group. It generalizes the classical Alexander polynomial. The twisted Alexander polynomial is naturally defined for links in S^3 and more generally for finitely presentable groups. In some instances it can easily be calculated.

Chapter 11 by T. Morifuji is a survey on representations of knot groups and twisted Alexander polynomials, with a special focus on the twisted Alexander polynomial for finitely presentable groups introduced by Wada. This polynomial is associated to a representation into $\mathrm{SL}(2,\mathbb{C})$. There are applications to fibering and genus detecting problems of knots in S^3. The twisted Alexander polynomial of a knot is seen as a $\mathbb{C}$-valued rational function on the character variety of the knot group, and it is also expressed in terms of Reidemeister torsion. The chapter also contains a comprehensive introduction to the classical Alexander polynomials and to the algebraic theory which is behind it (presentations of knot groups, Wirtinger presentations, Tietze transformations and Fox derivatives), as well as on representations of knot groups into $\mathrm{SL}(2,\mathbb{C})$, their character varieties and their deformations. The author focuses on the deformation of an abelian representation to a nonabelian one and of a reducible representation to an nonreducible one.

Chapter 12 by M. Suzuki is devoted to the study of the existence of epimorphisms between knot groups. The author indicates by some examples how to detect the existence of a meridional epimorphism (that is, an epimorphism that preserves meridians) between knot groups and he gives explicit descriptions of some non-meridional epimorphisms. He shows that the existence of an epimorphism between finitely presentable groups implies that their twisted Alexander polynomials are divisible. He makes connections with other works on the subject, and in particular with the so-called Simon conjecture (a problem in Kirby's list) whose general case was settled recently by Agol and Liu. The result says that every knot group admits an epimorphism onto at most finitely many knot groups.

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Part IV: Knot Groups

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Introduction

This is the second volume of the Handbook of Group Actions, which will contain several volumes.

The present volume is divided into five parts, where the chapters are organized according to the nature of the groups concerned or to the applications of the actions involved.

Part I: Geometric Topology

This part consists of 5 chapters.

Discrete groups such as fundamental groups of manifolds and their actions on topological spaces are essential in topology. In geometric topology, interactions between topology and the fundamental group (or its group ring) are particularly important, and this is reflected in various far-reaching conjectures and problems such as the Borel conjecture for aspherical manifolds, the Novikov conjecture, the Baum-Connes conjecture, the Farrell-Jones conjecture and the space form problem.

Chapter 1 by F. T. Farrell, A. Gogolev and P. Ontaneda discusses how exotic differentiable structures on higher-dimensional spheres and related unexpected nontrivial topology of the stable pseudo-isotopy space led to solutions of longstanding problems on the topology of the space of negatively curved metrics on Riemannian manifolds and the topology of the space of Anosov diffeomorphisms. The space of negatively curved metrics is a generalization of the Teichmüller space of hyperbolic surfaces. The authors discuss several results showing that in dimensions higher than ten, this space is disconnected and has nontrivial homotopy groups, in sharp contrast with the Teichmüller spaces of surfaces. They also describe an analogy between metrics of negative curvature and Anosov diffeomorphisms and they present several results on the existence of these diffeomorphisms, on the rigidity of manifolds admitting them and on the nontrivial topology of spaces of Anosov diffeomorphisms.

Chapter 2 by J. Guaschi and D. Juan-Pineda constitutes a comprehensive introduction to surface braid groups and a survey of results on the Farrell-Jones isomorphism conjecture and its variants, e.g. the fibered Farrell-Jones conjecture on the lower algebraic K-theory of group rings of surface braid groups. One purpose of these isomorphism conjectures is to compute the algebraic groups explicitly, though it is very difficult to carry them out in general. This paper also contains explicit computational results for surface braid groups. The combination of these two kinds of results makes surface braid groups special.

Groups acting on trees often have particular properties, generally related to decompositions of these groups, and results about the whole groups can be as-

sembled from those of vertex and edge stabilizers. Chapter 3 by S. K. Roushon is a survey on the Farrell-Jones fibered isomorphism conjecture in both K- and L-theories and its relation with the vanishing of the Whitehead groups for groups acting on trees. The author presents both the classical material and the recent developments of the lower K-theory and the surgery L-theory of groups acting on trees. The paper ends with a list of open problems on the (fibered) isomorphism conjecture.

Motivated by Kazhdan's property T of groups acting on Hilbert spaces, some recent activity is concerned with actions of groups on Banach spaces. Affine actions of a group G on a Banach space E can be translated into properties of the first cohomology group of G with coefficients in the G-module formed by the Banach space E together with a representation of G. Fixed point properties of such group actions are equivalent to vanishing of associated cocycles. The existence of proper actions on Hilbert spaces has implications in the setting of the Baum-Connes conjecture for groups. Chapter 4 by P. W. Nowak is a survey on recent progress on group actions on Banach spaces and their fixed point properties, motivated by property T, but with a stress on actions on Banach spaces which are not Hilbert spaces. The author also thoroughly discusses the property known in the case of Hilbert spaces as a-T-menability, or the Haagerup property, that is, the existence of metrically proper affine isometric actions on Banach spaces. Applications to the dimension theory of boundaries of hyperbolic groups are mentioned.

One basic problem on spherical space forms asks which finite groups can act freely on spheres by homeomorphisms – or diffeomorphisms. In dimension three, this corresponds to the question of which finite groups can occur as fundamental groups of compact topological or smooth manifolds. In Riemannian geometry, the spherical space form problem amounts to the classification of compact Riemannian manifolds with constant sectional curvature, and it was solved quite satisfactorily. On the other hand, the topological spherical space form problem is more subtle, and there has been a lot of work on it. Chapter 5 by I. Hambleton concerns topological spherical space forms and it is an updated short survey on this problem, starting from the nineteenth century work, and ending with the work done recently based on Perelman's solution of the Poincaré conjecture. Some new directions of research in this field are also mentioned.

Part II: Representations and Deformations

This part consists of 5 chapters.

For a discrete subgroup Γ of a semisimple Lie group G, the most natural and obvious action is probably the action of Γ on the associated symmetric space G/K, where K is a maximal compact subgroup of G. Particular examples are the action of a Fuchsian group on the upper half-plane and the action of a Kleinian group on three-dimensional hyperbolic space. There are at least two ways of generalizing these examples. The first one is to consider representations of discrete groups into semisimple Lie groups and their actions on associated symmetric spaces, and the second one is to consider many actions of a given type, that is, families, or deformations of such actions, simultaneously.

Chapter 6 by R. D. Canary is a survey on the dynamics of the action of the

outer automorphism group $\mathrm{Out}(\Gamma)$ of a word hyperbolic group Γ on the character variety of representations of Γ in a semisimple Lie group G. The stress is on two aspects of this theory: (1) the setting started by Labourie and developed later on by Guichard, Labourie and Wienhard on the proper discontinuity of the action on spaces of Anosov representations; (2) the works of Canary, Gelander, Lee, Magid, Minsky and Storm on the special case where Γ is the fundamental group of a compact-orientable 3-manifold with boundary and $G = \mathrm{PSL}(2, \mathbb{C})$.

In Chapter 7, J. Maubon surveys the Higgs bundle theory of Hitchin and Simpson and he shows how these bundles can be used in the representation theory of complex hyperbolic lattices into Lie groups of Hermitian type. He proves in particular the rigidity of the so-called *maximal* representations. The survey covers representations of surface groups but the main part of the survey concerns representations of higher-dimensional lattices. For this reason, the main Lie group that is involved is the Hermitian Lie group $\mathrm{SU}(p; q)$, since this is the only simple Lie group of Hermitian type into which maximal representations of higher-dimensional complex hyperbolic lattices are expected to exist.

Chapter 8 by K. Ohshika concerns deformation spaces of Kleinian groups, and limits of such groups. Deformation spaces are equipped with several different topologies (this was highlighted by Thurston and others), deformation spaces have several boundary structures, with complicated structure. Ohshika studies in particular the Bers boundaries for quasi-Fuchsian groups. He presents recent results that he obtained, which provide a complete classification of the geometric limits of quasi-Fuchsian groups. Combined with work of Kerckhoff and Thurston, these results explain why the action of the mapping class group on Teichmüller space does not extend continuously to a Bers boundary, and the author describes a quotient of this boundary, the so-called *reduced Bers boundary*, which admits a continuous action of the mapping class group.

Teichmüller spaces are parameter spaces which describe deformation of (equivalence classes of) complex structures on surfaces. Several chapters of the first volume of this handbook deal with Teichmüller spaces and their associated mapping class groups. In principle, complex structures are defined abstractly by charts. But there is another point of view, namely, to consider deformation of surfaces embedded in higher-dimensional manifolds, and in particular, four-dimensional manifolds. Diffeomorphism groups and hence mapping class groups of surfaces act on the space of embeddings of the surfaces. One question is how different these embeddings are when viewed from the point of view of the diffeomorphism group of the ambient space. Chapter 9 by S. Hirose is a survey of results on deformations of smooth surfaces embedded in four-dimensional manifolds with respect to these diffeomorphism groups and to the mapping class groups of the surfaces. Given an embedding e of the surface in the 4-manifold, the main question which is investigated is to what extent a diffeomorphism ϕ of an embedded surface extends to a diffeomorphism Φ of the ambient 4-manifold respecting the embedding e, that is, satisfying $\Phi \circ e = e \circ \phi$. The author surveys some flexibility and rigidity results on this question. Knotted and unknotted embeddings of closed surfaces in the 4-sphere and algebraic curves in $\mathbb{C}P^2$ are considered. Results on the Rokhlin quadratic form and on the Arf invariant are used.

One important and effective method to deform surfaces in four-dimensional symplectic manifolds is to consider Lefschetz fibrations. This subject was motivated by the study of Lefschetz fibrations in algebraic geometry, a basic tool in that area, and in particular in the study of algebraic surfaces. The works of Donaldson and Gompf showed that these fibrations give a fairly complete description of symplectic 4-manifolds. Motivated by the question of the geography of algebraic surfaces in terms of the ratio (or the slope) of two invariants of algebraic surfaces, in Chapter 10, N. Monden gives an introduction to Lefschetz fibrations with a particular stress on the relation with mapping class groups. The author presents several methods of construction of Lefschetz fibrations and he gives a summary of some results on the geography of symplectic 4-manifolds and, in particular, on the construction of some Lefschetz fibrations over the sphere S^2 which violate the so-called slope inequality.

Part III: Geometric Groups

This part consists of 3 chapters.

Among all infinite discrete groups, the modular group $\mathrm{SL}(2,\mathbb{Z})$ is special and it plays the role of a model for several classes of groups. One reason for its importance comes from its actions on very different spaces. Chapter 11 by A. M. Uludağ is a survey on several actions of $\mathrm{SL}(2,\mathbb{Z})$, starting from the very classical ones and ending with the recently discovered ones. The examples include the actions on Farey trees, on binary quadratic forms, on planar lattices, on the unit circle considered as the boundary of the upper half plane, and on dessins d'enfants, introduced by Grothendieck in the 1980s.

Simplicial sets are basic objects in algebraic topology and they are a generalization of simplicial complexes. They are related to the notion of simplicial group. Chapter 12 by J. Wu gives an introduction to simplicial groups and simplicial structures on geometric groups such as braid groups, link groups and mapping class groups.

By definition, a Lie group is a combination of a group and a compatible smooth manifold. If we ignore the underlying smooth structure, then we obtain a discrete group, or an abstract group. One basic question in Lie group theory is to what extent the smooth structure of a Lie group can be recovered from the group structure. For example, when is a group homomorphism a Lie group homomorphism? Similar questions can be raised for rational points of linear algebraic groups over general fields. Chapter 13 by I. A. Rapinchuk is a survey on known results on these questions and on their applications such as to character varieties of elementary subgroups of Chevalley groups over finitely generated commutative rings.

Part IV: Geometric Invariants and Growth of Discrete Groups

This part consists of 2 chapters.

One important question in the field of group actions in algebraic geometry concerns the actions of algebraic groups on algebraic varieties. The behavior of orbits of such actions is a basic question. This theory is one of the basic objects of geometric invariant theory and it is closely related to the problem of constructing moduli spaces in algebraic geometry.

In Chapter 14, D. P. Băć and N. Q. Thăńg give a summary of classical geometric invariant theory over algebraically closed fields as well as a survey of geometric invariant theory over non-algebraically closed fields, i.e., actions of algebraic groups on affine varieties over non-algebraically closed fields. The authors discuss questions related to the notion of stability, of various types and closures that are akin to this algebraic geometry setting and of other topological properties of orbits.

One basic point of view in geometric group theory is to consider a finitely generated group as a metric space. One way to do this is to endow the group with a word metric (which depends on the choice of a generating set). One geometric object associated to such a group (or a word metric) is the growth series, whose coefficients are the number of group elements at fixed distance from the identity element. This gives rise to the important notion of spherical growth series. Chapter 15 by M. Fujii gives an introduction to a method of computing spherical growth series of finitely generated groups via finite-state automata. The theory is illustrated by several explicit examples of growth series related to the pure Artin group of dihedral type. The paper also contains background material on Artin groups as well as a survey on the known groups for which the growth series is rational or irrational, for some or for an arbitrary generating set.

Part V: Music and Group Actions

Chapter 16 by A. Papadopoulos, concerns applications of groups and group actions in arts, and more precisely in music theory and music composition. Several examples are presented, especially from the compositions and the theoretical work of the French composer Olivier Messiaen. Since most readers of this handbook are not familiar with the details on the relation between mathematics and music — although most of them know that such a relation exists — the chapter also contains an introduction, with a historical overview, on this subject.

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