Acta Math., 223 (2019), 113–149
DOI: 10.4310/ACTA.2019.v223.n1.a2
© 2019 by Institut Mittag-Leffler. All rights reserved

Bounds on the topology and index of minimal surfaces

by

William H. Meeks III

University of Massachusetts
Amherst, MA, U.S.A.

Joaquín Pérez

University of Granada
Granada, Spain

Antonio Ros

University of Granada
Granada, Spain

1. Introduction

Let $\mathcal{M}$ be the space of connected, properly embedded minimal surfaces in $\mathbb{R}^3$. The focus of this paper is to prove the existence of an upper bound on the number of ends for a surface $M\in\mathcal{M}$ having finite topology, solely in terms of the genus of M. In the case that M has more than one end, this topological bound also produces an upper bound for the index of stability of M.

There are three classical conjectures that attempt to describe the topological types of the surfaces occurring in $\mathcal{M}$.

Conjecture 1.1. (Finite topology conjecture I; Hoffman–Meeks) A non-compact orientable surface with finite genus g and a finite number of ends $k>2$ occurs as the topological type of an example in $\mathcal{M}$ if and only if $k\leqslant g+2$. A minimal surface in $\mathcal{M}$ with finite genus and two ends has genus zero and is a catenoid.

Conjecture 1.2. (Finite topology conjecture II; Meeks–Rosenberg) For every positive integer g, there exists a $\Sigma_g\in\mathcal{M}$ with one end and genus g, which is unique up to congruences and homotheties. Furthermore, if $g=0$, such a Σ_g is a plane or a helicoid.

The first author was supported by the National Science Foundation under Award No. DMS-1309236 (any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF). Research of the second and third authors was partially supported by MINECO/FEDER grants no. MTM2014-52368-P and MTM2017-89677-P.

Conjecture 1.3. (Infinite topology conjecture; Meeks) A non-compact orientable surface of infinite topology occurs as the topological type of an example in $\mathcal{M}$ if and only if it has at most two limit ends, and when it has one limit end, then its limit end has infinite genus.

For a detailed discussion of these conjectures and related results, we refer the reader to the survey by Meeks and Pérez [26]. However, we make here a few brief comments on what is known concerning these conjectures and which will be used in the proof of the main theorem of this paper.

Regarding Conjecture 1.1, a theorem by Collin [10] states that if $M \in \mathcal{M}$ has finite topology and at least two ends, then M has finite total Gaussian curvature. This result implies that such a surface is conformally a compact Riemann surface $\overline{M}$ punctured in a finite number of points and M can be defined in terms of meromorphic data on its conformal compactification $\overline{M}$ (Osserman [40]). Collin's theorem reduces the question of finding topological obstructions for surfaces in $\mathcal{M}$ of finite topology and more than one end to the question of finding topological obstructions for complete, embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$. For example, if M is a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature, genus g and k ends, then M is properly embedded in $\mathbb{R}^3$ and the Jorge–Meeks formula [22] calculates its total curvature to be $-4\pi(g+k-1)$. The first topological obstructions for complete, embedded minimal surfaces M of finite total curvature were given by Jorge and Meeks [22], who proved that if M has genus zero, then M does not have $3, 4$ or 5 ends. Later this result was generalized by López and Ros [25], who proved that the plane and the catenoid are the only genus-zero minimal surfaces of finite total curvature in $\mathcal{M}$. At about the same time, Schoen [44] proved that a complete, embedded minimal surface of finite total curvature and two ends must be a catenoid.

The existence theory for properly embedded minimal surfaces with finite total curvature was begun by Costa [12] and by Hoffman and Meeks [18], with important theoretical advances by Kapouleas [23] and Traizet [45]. A paper by Weber and Wolf [48] makes the existence assertion in Conjecture 1.1 seem likely to hold, although their results actually fall short of giving a proof of embeddedness for their examples.

Concerning Conjecture 1.2, a theorem by Meeks and Rosenberg [34] states that the plane and the helicoid are the only properly embedded, simply connected minimal surfaces in $\mathbb{R}^3$. They also claimed that if $M \in \mathcal{M}$ has finite positive genus and just one end, then it is asymptotic to the end of the helicoid and can be defined analytically in terms of meromorphic data on its conformal completion, which is a closed Riemann surface; the proof of this result was given by Bernstein and Breiner [1] and a related more general result was proved by Meeks and Pérez [27]. These theoretical results together

with theorems developed by Weber and Traizet [46], by Hoffman and White [21] and by Hoffman, Weber and Wolf [20] provide the theory on which a proof of the uniqueness part of Conjecture 1.2 might be based. The existence part of Conjecture 1.2 has been recently solved by Hoffman, Traizet and White [19].

In relation to Conjecture 1.3, there are two important topological obstructions for surfaces in $\mathcal{M}$ with infinite topology: Collin, Kusner, Meeks and Rosenberg [11] proved that an example in $\mathcal{M}$ cannot have more than two limit ends, and in [29] we proved that an example in $\mathcal{M}$ with one limit end cannot have finite genus. This last result depends on Colding–Minicozzi theory, as well as on our previous results in [28] where we presented a descriptive theorem for minimal surfaces in $\mathcal{M}$ with two limit ends and finite genus. The study of two-limit-end minimal surfaces is motivated by a 1-parameter family of periodic examples of genus zero discovered by Riemann [41], which are defined in terms of elliptic functions on rectangular elliptic curves. In [30] we showed that if $M\in\mathcal{M}$ has genus zero and infinite topology, then M is one of the Riemann minimal examples, and if M has finite genus and infinite topology, then each of its two limit ends are in a natural sense asymptotic to the end of a Riemann minimal example. It is worth mentioning that Hauswirth and Pacard [16] have produced examples in $\mathcal{M}$ with finite genus and infinite topology.

A priori, one procedure to obtain surfaces in $\mathcal{M}$ with finite genus and infinite topology might be to take limits of sequences of finite total curvature examples in $\mathcal{M}$ with a bound on their genus but with a strictly increasing number of ends. Our results in [28], [29], [32], [33] are crucial in understanding that such sequences cannot exist, and they will lead us to a proof of the following main theorem of this manuscript.

THEOREM 1.4. *A properly embedded minimal surface in $\mathbb{R}^3$ with finite topology has a bound on the number of its ends that only depends on its genus.*

Colding and Minicozzi [8] proved that a complete, embedded minimal surface of finite topology in $\mathbb{R}^3$ is properly embedded. In particular, the conclusion of Theorem 1.4 remains valid if we weaken the hypothesis of properness to the hypothesis of completeness.

By a theorem of Fischer–Colbrie [14], a complete, immersed, orientable minimal surface M in $\mathbb{R}^3$ has finite index of stability if and only if it has finite total curvature. The index of such an M is equal to the index of the Schrödinger operator $L=\Delta+\|\nabla N\|^2$ associated with the meromorphic extension of the Gauss map N of M to the compactification of M by attaching its ends. Grigor′yan, Netrusov and Yau [15] made an in-depth study of the relation between the degree of the Gauss map and the index of a complete minimal surface of finite total curvature. In particular, they proved that the index of a complete, embedded minimal surface with k ends is bounded from below by $k-1$. On the

other hand, Tysk [47] proved that the stability index of L can be explicitly bounded from above in terms of the degree of N. By the Jorge–Meeks formula for such an embedded M, the degree of N equals $g+k-1$, where g is the genus and k is the number of ends. Hence, by Theorem 1.4, if g is fixed, then k is bounded for an embedded M. Thus, one obtains the following consequence to Theorem 1.4.

THEOREM 1.5. *If $M\subset\mathbb{R}^3$ is a complete, connected, embedded minimal surface with finite index of stability, then the index of M can be bounded by a constant that only depends on its (finite) genus. In the case of genus zero, the surface M is a plane or catenoid, and so this index upper bound is* 1.

For any integer $k\geqslant 2$, the k-noid defined by Jorge and Meeks [22] has genus zero, k catenoid-type ends and index $2k-3$ (Montiel–Ros [39] and Ejiri–Kotani [13]). Also, there exist examples of complete, immersed minimal surfaces of genus zero with a finite number of parallel catenoidal ends but which have arbitrarily large index of stability. These examples demonstrate the necessity of the embeddedness hypothesis in Theorem 1.5.

The proof of Theorem 1.4 depends heavily on results developed in our previous papers [28], [29], [32], [33]. These papers, as well as the present one, rely on a series of deep works by Colding and Minicozzi [4], [5], [6], [7], [9], in which they describe the local geometry of a complete, embedded minimal surface in a Riemannian 3-manifold, where there is a local bound on the genus of the surface.

2. Preliminaries

Throughout the paper, we will denote by $\mathbb{B}(x,r)$ the open ball in $\mathbb{R}^3$ with center $x\in\mathbb{R}^3$ and radius $r>0$, and by $\overline{\mathbb{B}}(x,r)$ its closure. When x is the origin, we will simply write $\mathbb{B}(r)$ and $\overline{\mathbb{B}}(r)$, respectively, for these particular open and closed balls. We let $\mathbb{D}(r)$ denote the open disk of radius r centered at the origin in $\mathbb{R}^2$. For a surface $\Sigma\subset\mathbb{R}^3$, K_Σ will denote its Gaussian curvature function.

Definition 2.1. Let A be an open subset of $\mathbb{R}^3$. A sequence of surfaces $\{\Sigma_n\}_n$ in A is said to have *locally bounded curvature in* A, if for every compact ball $B\subset A$, the sequence of functions $\{K_{\Sigma_n\cap B}\}_n$ is uniformly bounded. A sequence $\{\Sigma_n\}_n$ of properly embedded surfaces in A is called *locally simply connected*, if for every $q\in A$, there exist $\varepsilon_q>0$ and $n_q\in\mathbb{N}$ such that for $n>n_q$, the components of $\Sigma_n\cap\mathbb{B}(q,\varepsilon_q)$ are disks with their boundaries in the boundary of $\mathbb{B}(q,\varepsilon_q)$.

Proposition 1.1 in Colding and Minicozzi [8] ensures that the property that a sequence of embedded minimal surfaces $\{M(n)\}_n$ is locally simply connected in A is equivalent to the property that $\{M(n)\}_n$ has *locally positive injectivity radius in* A, in the

sense that for every $q\in A$, there exist $\varepsilon_q>0$ and $n_q\in\mathbb{N}$ such that for $n>n_q$, the restriction to $M(n)\cap\mathbb{B}(q,\varepsilon_q)$ of the injectivity radius function $I_{M(n)}$ of $M(n)$ is greater than a positive constant independent of n. In particular, if the $M(n)$ also have boundary, then for any $p\in A$ there exist $\delta_p>0$ and $n_p\in\mathbb{N}$ such that $\partial M(n)\cap\mathbb{B}(p,\delta_p)=\varnothing$ for all $n>n_p$, i.e., points in the boundary of $M(n)$ must eventually diverge in space or converge to a subset of $\mathbb{R}^3\setminus A$.

We will call a sequence of surfaces $\{\Sigma_n\}_n$ in $\mathbb{R}^3$ *uniformly locally simply connected*, if there exists $\varepsilon>0$ such that for every $x\in\mathbb{R}^3$, $\Sigma_n\cap\mathbb{B}(x,\varepsilon)$ consists of disks with boundary in $\partial\mathbb{B}(x,\varepsilon)$ for all n sufficiently large (depending on x).

In the proof of our main Theorem 1.4 we will use repeatedly the following statement, which is an application of Theorem 1.6 in [33].

THEOREM 2.2. *Suppose W is a closed countable subset of $\mathbb{R}^3$. Let $\{M_n\}_n$ be a sequence of smooth, connected, properly embedded minimal surfaces in $A=\mathbb{R}^3\setminus W$ such that the following conditions hold*:

(i) *$\{M_n\}_n$ has locally positive injectivity radius in A*;

(ii) *for each $n\in\mathbb{N}$, M_n has compact boundary (possibly empty) and genus at most $g\in\mathbb{N}\cup\{0\}$, for some g independent of n.*

Then, after replacing $\{M_n\}_n$ by a subsequence and composing the surfaces with a fixed rotation, there exists a minimal lamination $\mathcal{L}$ of A and a closed subset $\mathcal{S}(\mathcal{L})\subset\mathcal{L}$ such that $\{M_n\}_n$ converges C^α, for all $\alpha\in(0,1)$, on compact subsets of $A\setminus S(\mathcal{L})$ to $\mathcal{L}$; here, $\mathcal{S}(\mathcal{L})$ is the singular set of convergence(1) *of the M_n to $\mathcal{L}$.*

Furthermore, we have the following:

(1) *The closure $\bar{\mathcal{L}}$ of $\mathcal{L}$ in $\mathbb{R}^3$ has the structure of a minimal lamination of $\mathbb{R}^3$.*

(2) *If $S(\mathcal{L})\neq\varnothing$, then the convergence of $\{M_n\}_n$ to $\bar{\mathcal{L}}$ has the structure of a horizontal limiting parking garage structure in the following sense*:

(2.1) *$\bar{\mathcal{L}}$ is a foliation of $\mathbb{R}^3$ by horizontal planes and $\overline{S(\mathcal{L})}$ consists of one or two vertical straight lines (called* columns of the limiting parking garage structure*).*

(2.2) *As $n\to\infty$, a pair of highly sheeted multivalued graphs are forming inside M_n around each of the lines in $\overline{S(\mathcal{L})}$, and if $\overline{S(\mathcal{L})}$ consists of two lines, then these pairs of multivalued graphs inside the M_n around different lines are oppositely handed.*

(3) *If $\bar{\mathcal{L}}$ contains a non-flat leaf, then $S(\mathcal{L})=\varnothing$ and $\bar{\mathcal{L}}$ consists of a single leaf L_1, which is properly embedded in $\mathbb{R}^3$ and the genus of L_1 is at most g. Furthermore, $\{M_n\}_n$ converges smoothly on compact sets in $\mathbb{R}^3$ to L_1 with multiplicity 1 and one of the following three cases holds for L_1*:

(3.a) *L_1 has one end and it is asymptotic to a helicoid*;

(1) This means that for every $x\in S(\mathcal{L})$ and for all $r>0$, we have $\limsup|K_{M_n\cap\mathbb{B}(x,r)}|=\infty$.

(3.b) *L_1 has non-zero finite total curvature*;

(3.c) *L_1 has two limit ends.*

Remark 2.3. Before giving the proof of Theorem 2.2, we will illustrate its statement with some examples.

(1) The limit of homothetic shrinkings $M_n = C/n$ of a vertical catenoid

$$C = \{(x,y,z) \in \mathbb{R}^3 : \cosh^2(x^2+y^2) = z^2\}$$

is a particular case of Theorem 2.2 with $W = \{\vec{0}\}$ and $g=0$. In this example, $\mathcal{L}$ is the punctured plane $\{z=0\} \setminus \{\vec{0}\}$ and $S(\mathcal{L}) = \varnothing$.

(2) The classical Riemann minimal examples R_t, $t>0$, form a 1-parameter family of properly embedded, singly-periodic minimal surfaces with genus zero and infinitely many planar ends asymptotic to horizontal planes. As a particular case of Theorem 2.2, one can consider the limit of the R_t when the flux vector of R_t along a compact horizontal section converges to $(2,0,0)$. In this example, $W = \varnothing$, $g=0$ and item (2.2) of Theorem 2.2 holds.

(3) Theorem 0.9 in Colding–Minicozzi [9] can be viewed as a particular case of Theorem 2.2 when $\{M_n\}_n$ is a locally simply connected sequence in $\mathbb{R}^3$ of compact planar domains with $\partial M_n \subset \partial \mathbb{B}(R_n)$ and $R_n \to \infty$ (hence $W = \varnothing$ and $g=0$), with the additional assumptions:

- $\sup |K_{M_n \cap \mathbb{B}(y,r)}| \to \infty$ for some $y \in \mathbb{R}^3$ and for all $r>0$ (hypothesis (0.2) in [9]); this implies the $S(\mathcal{L}) \neq \varnothing$, and thus, item (2) of Theorem 2.2 holds.
- There exists $R>0$ such that each M_n intersects $\mathbb{B}(R)$ in a component that is not a disk (hypothesis (0.5) in [9]). This allows us to discard the case of a single column for the limiting parking garage structure in item (2.1) of Theorem 2.2, and item (2.2) holds.

(4) Take a properly embedded minimal surface $M \subset \mathbb{R}^3$ with infinite total curvature. By the dynamics theorem (Theorem 2 in [31]), the set $D_1(M)$ of non-flat properly embedded minimal surfaces $\Sigma \subset \mathbb{R}^3$ which are obtained as C^2-limits of a divergent sequence $\{\lambda_n(M-p_n)\}_n$ of dilations of M (i.e., the translational part $p_n \in \mathbb{R}^3$ of the dilations diverges) such that $\vec{0} \in \Sigma$, $|K_\Sigma| \leqslant 1$ on Σ and $|K_\Sigma|(\vec{0}) = 1$, is non-empty. Given $\Sigma \in D_1(M)$ obtained as the limit of $M_n = \lambda_n(M-p_n)$, the fact that Σ has locally positive injectivity radius in $\mathbb{R}^3$ implies that $\{M_n\}_n$ also has locally positive injectivity radius in $\mathbb{R}^3$ (i.e., $W = \varnothing$ in Theorem 2.2). Now, assume that Σ has finite genus g. After possibly replacing M_n by $[\lambda_n(M-p_n)] \cap \mathbb{B}(R_n)$ for an appropriate sequence of radii $R_n \to \infty$, we may assume that hypothesis (ii) of Theorem 2.2 holds for $\{M_n\}_n$. In this example, $\mathcal{L} = \Sigma$, $S(\mathcal{L}) = \varnothing$ and item (3) of Theorem 2.2 holds.

Proof of Theorem 2.2. The existence of the minimal lamination $\mathcal{L}$ of A such that the M_n converge to $\mathcal{L}$ (after passing to a subsequence) in $A \setminus S(\mathcal{L})$ follows directly from

the main statement of Theorem 1.5 in [33]; note that singularities of $\mathcal{L}$ are ruled out in our setting by item (7.1) of Theorem 1.5 in [33] because the M_n have uniformly bounded genus. The same argument using item (7.1) of Theorem 1.5 in [33] ensures that item (1) of Theorem 2.2 holds.

Now, assume that $S(\mathcal{L})\neq\varnothing$ and we will prove that item (2) holds. To accomplish this, we first define a sequence of auxiliary compact minimal surfaces. For each $k\in\mathbb{N}$, choose an $R_k\in(k,k+1)$ such that the sphere $\partial\mathbb{B}(R_k)$ is disjoint from W, which is possible since W is a countable set. Since $W\cap\bar{\mathbb{B}}(k+1)$ is a compact set, then $\partial\mathbb{B}(R_k)$ is at a positive distance $2d_k$ from W; define $W_k=W\cap\mathbb{B}(R_k)$ and for any $\varepsilon>0$ let $W_k(\varepsilon)$ be the open ε-neighborhood of W_k in $\mathbb{R}^3$. As the sequence $\{M_n\}_n$ is locally simply connected in A and W is at a positive distance from $\partial\mathbb{B}(R_k)$, then for k fixed and n sufficiently large, ∂M_n is disjoint from $\partial\mathbb{B}(R_k)$ and, by Sard's theorem, we may also assume that R_k is chosen so that $\partial\mathbb{B}(R_k)$ is transverse to all of the surfaces M_n. Also, by Sard's theorem, there exist smooth, compact, possibly disconnected 3-dimensional domains $\Delta_{n,k}\subset\mathbb{B}(R_k)$ satisfying $W_k\subset\mathrm{Int}(\Delta_{n,k})\subset W_k(d_k/n)$ and such that $\partial\Delta_{n,k}$ is transverse to M_n for all k and for n sufficiently large. It follows that $M_{n,k}=(M_n\setminus\mathrm{Int}(\Delta_{n,k}))\cap\bar{\mathbb{B}}(R_k)$ is a doubly indexed sequence of smooth compact surfaces such that, after replacing M_n by a subsequence, $\{M_{k,k}\}_k$ is a sequence of smooth, compact embedded minimal surfaces that satisfy the hypotheses of Theorem 2.2. By construction, the sequence $\{M_{k,k}\}_k$ also converges to $\mathcal{L}$ with the same singular set of convergence $S(\mathcal{L})$. However, since the minimal surfaces in the sequence $\{M_{k,k}\}_k$ are compact with genus at most g, then item (7.3) in Theorem 1.5 of [33] implies that item (2) of Theorem 2.2 holds.

Finally, we demonstrate item (3) of Theorem 2.2. Suppose that the non-singular minimal lamination $\bar{\mathcal{L}}$ contains a non-flat leaf L_1. By item (6) of Theorem 1.5 in [33], L_1 is properly embedded in a simply connected domain in $\mathbb{R}^3$, which implies that it is two-sided. In particular, L_1 is not stable and thus, the convergence of portions of the $\{M_{k,k}\}_k$ to L_1 must have multiplicity 1 (see e.g., Lemma 3 in [31]). This last property and a standard curve lifting argument ensure that the genus of L_1 is at most g. By item (6) of Theorem 1.5 in [33], L_1 is proper in $\mathbb{R}^3$ and it is the unique leaf of $\bar{\mathcal{L}}$ (that is, possibility (6.1) in that theorem holds, because case (6.2) in the same theorem cannot occur, due to the finiteness of the genus of L_1). In particular, $S(\mathcal{L})=\varnothing$, since through every point in $S(\mathcal{L})$ there passes a planar leaf of $\bar{\mathcal{L}}$ (by item (4) of Theorem 1.5 in [33]). This proves the main statement in item (3) of Theorem 2.2. Item (3.a) follows from work of Bernstein and Breiner [1] or Meeks and Pérez [27]. Item (3.b) occurs when the number k of ends of L_1 satisfies $2\leqslant k<\infty$, as follows from Collin [10]. Since L_1 has finite genus, then item (3.c) occurs when $k=\infty$ by Theorem 1 in [29]. Now the proof is complete. $\square$

3. The proof of Theorem 1.4

By Collin [10] and López–Ros [25], the catenoid is the only properly embedded, connected genus-zero minimal surface with at least two ends and finite topology, and so Theorem 1.4 holds for genus-zero surfaces. Arguing by contradiction, suppose that for some positive integer g, there exists an infinite sequence $\{M(n)\}_{n\in\mathbb{N}}$ of properly embedded minimal surfaces in $\mathbb{R}^3$ of genus g such that, for every n, the number of ends of $M(n)$ is finite and strictly less than the number of ends of $M(n+1)$, and the number of ends of $M(1)$ is at least 3. By Collin's theorem [10], each of these surfaces has finite total curvature with planar and catenoidal ends, where all ends can be assumed to be horizontal for all n after a suitable rotation.

3.1. Sketch of the argument

The argument to find the desired contradiction to prove Theorem 1.4 is based on an inductive procedure, each of whose stages starts by finding a scale of smallest non-trivial topology for the sequence of surfaces; next we will identify the limit L of the sequence in this scale (after passing to a subsequence) as being a properly embedded minimal surface in $\mathbb{R}^3$ with controlled geometry; this control will allow us to perform a surgery on the surfaces of the sequence (for n sufficiently large) that simplifies their topology. This simplification of the topology will allow us to find another smallest scale of non-trivial topology in the subsequent stage of the process and then to repeat the arguments. The fact that all surfaces in the original sequence have fixed genus insures that after finitely many stages in the procedure, the simplification of the topology of the surfaces cannot be by lowering their genus. This fact will be used to prove that after some stage in the process, all limit surfaces L that we obtain with this procedure will be catenoids. In turn, this fact will allow us to find compact subdomains $\Lambda(n)$ with boundary inside the $M(n)$ (for n sufficiently large), that reproduce almost perfectly formed, large compact pieces of the limit catenoids suitably rescaled, and we will show that we can find as many of these subdomains $\Lambda(n)$ as we like and whose associated almost waist circles $\Gamma_n\subset\Lambda(n)$ separate $M(n)$ and such that these domains form a pairwise disjoint collection. This separation property of the waist circles will let us control the flux vector of $M(n)$ along Γ_n. The final contradiction will follow from an application the López–Ros deformation to a suitable non-compact, genus-zero subdomain in $M(n)$ bounded by two of these separating curves Γ_n.

3.2. Rescaling non-trivial topology in the first stage

The asymptotic behavior of $M(n)$ implies that, for each $n\in\mathbb{N}$, there exists a positive number $r_{1,n}$ such that every open ball in $\mathbb{R}^3$ of radius $r_{1,n}$ intersects the surface $M(n)$ in simply connected components, and there is some point $T_{1,n}\in\mathbb{R}^3$ such that $\bar{\mathbb{B}}(T_{1,n},r_{1,n})$ intersects $M(n)$ in at least one component that is not simply connected. Then, the rescaled and translated minimal surfaces

$$M_{1,n}=\frac{1}{r_{1,n}}(M(n)-T_{1,n})$$

of finite total curvature have horizontal ends and satisfy the following uniformly locally simply connected property for all $n\in\mathbb{N}$:

($\star$) Every open ball of radius 1 intersects $M_{1,n}$ in disk components, and the closed unit ball $\bar{\mathbb{B}}(1)$ intersects $M_{1,n}$ in a component Ω_n that is not simply connected.

By property ($\star$), the compact semi-analytic set $M_{1,n}\cap\bar{\mathbb{B}}(1)$ with $M_{1,n}\cap\partial\mathbb{B}(1)$ being analytic, which admits a triangulation by [24], contains a non-simply connected component Ω_n. It is straightforward to prove that $\partial\Omega_n$ contains a piecewise-smooth simple closed curve Γ_n that does not bound a disk in $M_{1,n}\cap\bar{\mathbb{B}}(1)$; for example, see [37] for similar constructions. Thus, property ($\star$) implies the next one:

($\star\star$) There exists a piecewise-smooth simple closed curve $\beta(n)\subset\partial\Omega_n\subset M_{1,n}\cap\partial\mathbb{B}(1)$ that is not the boundary of a disk in $M_{1,n}$, where Ω_n is defined in property ($\star$).

3.3. Controlling the limit of the rescaled surfaces in the first stage

By the main statement in Theorem 2.2 applied to the $M_{1,n}$ with $W=\varnothing$, after passing to a subsequence, the surfaces in the sequence $\{M_{1,n}\}_n$ converge to a minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3$ with related singular set of convergence $S(\mathcal{L}_1)$. Assume for the moment that $S(\mathcal{L}_1)=\varnothing$ (this property will be proven in Lemma 3.4 below).

Lemma 3.1. *If $S(\mathcal{L}_1)=\varnothing$, then $\mathcal{L}_1$ consists of a single leaf L_1 that is not simply connected, has genus at most g and is properly embedded in $\mathbb{R}^3$.*

Proof. By item (3) of Theorem 2.2, it suffices to prove that $\mathcal{L}_1$ contains a non-simply connected leaf.

Reasoning by contradiction, suppose that $\mathcal{L}_1$ consists entirely of leaves that are complete minimal surfaces that are simply connected. Theorem 2.2 implies that either $\mathcal{L}_1$ has a single leaf that is non-flat and properly embedded in $\mathbb{R}^3$, or else $\mathcal{L}_1$ consists of planar leaves. The uniqueness of the helicoid [35] demonstrates that if $\mathcal{L}_1$ has a single

leaf L_1, then this leaf is a plane or a helicoid. By Sard's theorem, we can take $\delta\in(1,2)$ such that each $M_{1,n}$ intersects the sphere $\partial\mathbb{B}(\delta)$ transversely. Let $\widetilde{\Omega}_n$ be the component of $M_{1,n}\cap\overline{\mathbb{B}}(\delta)$ that contains the homotopically non-trivial simple closed curve $\beta(n)\subset\partial\Omega_n$, where $\beta(n)$ and Ω_n are given in property $(\star\star)$.

By the convex hull property for minimal surfaces, $\widetilde{\Omega}_n$ is a compact minimal surface whose (smooth) boundary lies in the sphere $\partial\mathbb{B}(\delta)$. As the (smooth) limit $\mathcal{L}_1$ of $\{M_{1,n}\}_n$ is a collection of planes or it is a helicoid $\mathcal{H}$ that is a smooth, multiplicity-1 limit of $\{M_{1,n}\}_n$, then, for n large, $\widetilde{\Omega}_n$ is an almost-flat disk (in the case that $\mathcal{L}_1$ is a collection of planes) or $\widetilde{\Omega}_n$ is a disk that is a small normal graph over its orthogonal projection to the limit helicoid $\mathcal{H}$. By the convex hull property for minimal surfaces, the simple closed curve $\beta(n)$ is the boundary of a disk in $\Omega_n\subset M_{1,n}$, which is a contradiction that proves the lemma. □

By item (3) of Theorem 2.2, then $\mathcal{L}_1$ consists of a single leaf L_1 which is a properly embedded minimal surface in $\mathbb{R}^3$ with genus at most g, the convergence of $\{M_{1,n}\}_n\to L_1$ has multiplicity 1 and exactly one of the cases (3.a), (3.b) and (3.c) hold.

Lemma 3.2. *If $S(\mathcal{L}_1)=\varnothing$, then case* (3.c) *of Theorem 2.2 cannot hold.*

Proof. Assume that L_1 has two limit ends and we will find a contradiction. In this proof we will use the general description of such a minimal surface that is given in Theorem 1 in [28] and Theorem 8.1 in [30], as well as the terminology in those theorems. After a fixed rigid motion $A\colon\mathbb{R}^3\to\mathbb{R}^3$, we may assume that

$$\Sigma=A(L_1) \tag{1}$$

has an infinite number of middle ends that are horizontal planar ends. Furthermore, there is a representative $E\subset\Sigma$ for the top limit end of Σ that is conformally $\mathbb{S}^1\times[t_0,\infty)$ punctured in an infinite set of points $\{e_1,e_2,\dots,e_k,\dots\}$ that correspond to the planar ends of E; here, $\mathbb{S}^1$ is a circle of circumference equal to the vertical component of the flux vector of E, which is the integral of the inward pointing conormal of E along its boundary. In this conformal representation of E, we also have

$$x_3(\theta,t)=t,\quad x_3(e_k)<x_3(e_{k+1})\ \text{ for all } k\in\mathbb{N}\quad\text{and}\quad \lim_{k\to\infty}x_3(e_k)=\infty.$$

Given $i\in\mathbb{N}$, let $t_i=\frac{1}{2}(x_3(e_i)+x_3(e_{i+1}))$. Consider the simple closed curves

$$\gamma(i)={x_3}^{-1}(\{t_i\})\subset E\quad\text{and}\quad\gamma(0)=x_3^{-1}(\{t_0\})=\partial E.$$

Let S be the closed horizontal slab in $\mathbb{R}^3$ between the heights t_0 and t_{2g+2}, where g is the genus of $M_{1,n}$. Observe that $\Sigma\cap S\subset E$ is a connected minimal surface whose boundary

consists of the two simple closed planar curves $\gamma(0)$ and $\gamma(2g+2)$, and that $\Sigma\cap S$ has $2g+2$ horizontal planar ends.

Given $R>0$, let $\mathcal{C}(R)=\{(x_1,x_2,x_3):x_1^2+x_2^2\leqslant R^2\}$ be the solid cylinder of radius R. For any fixed $\varepsilon>0$ small, there exists an R_1 large such that $\gamma(i)$ is contained in $\mathcal{C}(R_1)$ for $0\leqslant i\leqslant 2g+2$, and $(\Sigma\cap S)\backslash\mathcal{C}(R_1)$ consists of $2g+2$ annular graphs which are $\left(\frac{1}{2}\varepsilon\right)$-close in the C^2-norm to the set of planes

$$\mathcal{P}=\{x_3^{-1}(\{x_3(e_1)\}),\dots,x_3^{-1}(\{x_3(e_{2g+2})\})\}.$$

We next transport the above structure from the limit surface Σ to the sequence of surfaces

$$\Sigma(n)=A(M_{1,n}),\quad n\in\mathbb{N}, \tag{2}$$

that converge smoothly to Σ as $n\to\infty$ (observe that, a priori, the $\Sigma(n)$ might fail to have horizontal ends).

For every $R_2>R_1$ there exists an $N=N(R_2)\in\mathbb{N}$ such that, for $n\geqslant N$,

$$\Sigma_S(n,R_2)=\Sigma(n)\cap S\cap\mathcal{C}(R_2)$$

is ε-close to the planar domain $\Sigma_S(R_2)=\Sigma\cap S\cap\mathcal{C}(R_2)$ in the C^2-norm; in particular, $\Sigma_S(n,R_2)$ is also a connected planar domain for all $n\geqslant N$. By initially choosing ε sufficiently small, the following properties also hold for every $n\geqslant N$:

(B1) $\partial\Sigma_S(n,R_2)$ consists of $2g+4$ simple closed curves which are arbitrarily close to $\partial\Sigma_S(R_2)$ if n is sufficiently large.

(B2) We can approximate the curves $\gamma(i)$ by simple closed planar curves $\gamma(i,n)$ in $\Sigma_S(n,R_2)\cap x_3^{-1}(\{t_i\})$, for all $i=0,\dots,2g+2$.

(B3) $\partial\Sigma_S(n,R_2)$ has two components $\gamma(0,n)$ and $\gamma(2g+2,n)$ lying on ∂S and $2g+2$ simple closed curves $\alpha_1(n),\dots,\alpha_{2g+2}(n)\subset\partial\mathcal{C}(R_2)$, which are graphs over the circle

$$\partial\mathcal{C}(R_2)\cap\{x_3=0\},$$

ordered by their relative heights; see Figure 1.

An elementary argument shows that in a compact surface with genus g and empty boundary, any connected planar subdomain with $2g+4$ boundary components has at least four boundary components that separate the surface. Since $\Sigma(n)$ has genus g and $\Sigma_S(n,R_2)$ is a connected compact planar domain with $2g+4$ boundary components, there exists an $i\in\{1,\dots,2g+1\}$ such that the corresponding curve $\alpha_i(n)$ separates $\Sigma(n)$.

Assertion 3.3. *The simple closed curve $\gamma(i,n)$ separates $\Sigma(n)$.*

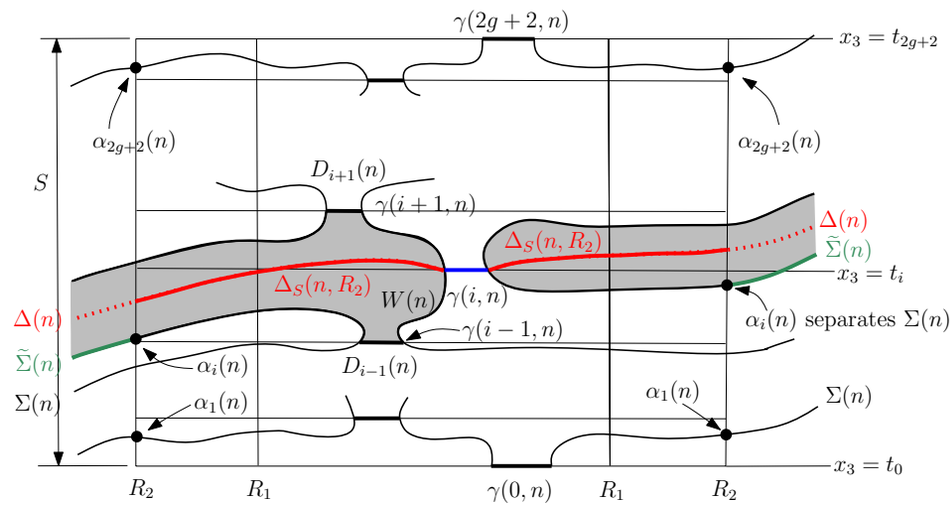

Figure 1. Producing the stable minimal surface $\Delta(n)$ in W.

Proof. Let $\widetilde{\Sigma}(n)$ be the component of $\Sigma(n)\backslash\alpha_i(n)$ that is disjoint from $\Sigma_S(n,R_2)$. Note that there is a disk in $\mathcal{C}(R_2)$ bounded by $\alpha_i(n)$ which only intersects $\Sigma(n)$ along $\alpha_i(n)\cup\gamma(i,n)$. The union of this disk with $\widetilde{\Sigma}(n)$ is a properly embedded, piecewise smooth surface in $\mathbb{R}^3$. After a slight perturbation of this surface in a small neighborhood of $\widetilde{\Sigma}(n)$, we obtain a connected properly embedded surface $\Omega(n)\subset\mathbb{R}^3$ which intersects $\Sigma(n)$ only along $\gamma(i,n)$. Since properly embedded surfaces in $\mathbb{R}^3$ separate $\mathbb{R}^3$, we deduce that $\gamma(i,n)$ separates $\Sigma(n)$, which proves the assertion.

Note that we may assume that $\Omega(n)\backslash\gamma(i,n)$ consists of two components, one of which is a planar disk contained in $S\cap\mathcal{C}(R_1)$. $\square$

Consider the surface $\Omega(n)$ defined in the proof of Assertion 3.3. Let $W(n)$ be the closed complement of $\Sigma(n)$ in $\mathbb{R}^3$ that intersects $\Omega(n)$ in a non-compact connected surface with boundary $\gamma(i,n)$. Denote by

$$D_{i-1}(n)\subset x_3^{-1}(\{t_{i-1}\}) \quad \text{and} \quad D_{i+1}(n)\subset x_3^{-1}(\{t_{i+1}\}),$$

the planar disks bounded by $\gamma(i-1,n)$ and $\gamma(i+1,n)$, respectively. Observe that

$$\partial W(n)\cup D_{i-1}(n)\cup D_{i+1}(n)$$

is a good barrier for solving Plateau problems in the abstract Riemannian piecewise smooth 3-manifold $N(n)$ obtained as the metric completion of the interior of $W(n)$ (in particular, the planar horizontal disks $D_{i-1}(n)$ and $D_{i+1}(n)$ could appear twice in the

boundary of $N(n)$); see [38]. Also note that $\gamma(i,n)$ separates the boundary of $N(n)$, since the surface $\Omega(n)\cap N(n)$ separates $N(n)$ and $\partial(\Omega(n)\cap N(n))=\gamma(i,n)$. Thus, a standard argument (see, e.g., Lemma 4 in Meeks, Simon and Yau [36]) using a compact exhaustion of the closure of one of the components of $\partial N(n)\setminus\gamma(i,n)$ implies that we can find a connected, orientable, non-compact, properly embedded least-area surface

$$\Delta(n)\subset W(n)\setminus(D_{i-1}(n)\cup D_{i+1}(n))\subset\mathbb{R}^3,\quad \text{with } \partial\Delta(n)=\gamma(i,n).$$

By the main result in Fischer–Colbrie [14], $\Delta(n)$ has finite total curvature and hence it has a positive finite number of planar and catenoidal ends that lie in $W(n)$. By the definition and the uniqueness of the limit tangent plane at infinity [2], we deduce that the limiting normal vectors to the ends of $\Delta(n)$ are parallel to the limiting normal vectors to the ends of $\Sigma(n)$.

Let $\Delta_S(n,R_2)$ be the component of $\Delta(n)\cap S\cap\mathcal{C}(R_2)$ whose boundary contains $\gamma(i,n)$. Note that the other boundary components of $\Delta_S(n,R_2)$ all lie on $\partial\mathcal{C}(R_2)\cap W(n)$. For R_1 large and $R_2\gg R_1$, curvature estimates for stable minimal surfaces [43] imply that $\Delta_S(n,R_2)\setminus\mathcal{C}(R_1)$ consists of almost-horizontal annular graphs, for n sufficiently large. By the area-minimizing property of $\Delta(n)$, we have that for R_2 much larger than R_1 and n sufficiently large, there is only one such almost-horizontal graph, which may be assumed to be oriented by the upward pointing normal.

Let $\tilde{\Delta}(n)$ be the closure of $\Delta(n)\setminus\Delta_S(n,R_2)$. Since $\Delta(n)$ has finite total curvature, then $\tilde{\Delta}(n)$ compactifies after attaching its ends to a compact Riemann surface $\overline{\tilde{\Delta}(n)}$ with boundary, and the Gauss map $G_n\colon\tilde{\Delta}(n)\to\mathbb{S}^2$ extends smoothly across the ends to $G_n\colon\overline{\tilde{\Delta}(n)}\to\mathbb{S}^2$, with values at the ends that lie in a pair of antipodal points $\pm a\in\mathbb{S}^2$. Observe that $\tilde{\Delta}(n)$ is strictly stable (because $\Delta(n)$ is stable and $\Delta(n,R_2)$ has positive area), and so $\overline{\tilde{\Delta}(n)}$ is also strictly stable. As $\tilde{\Delta}(n)$ is almost-horizontal along its boundary, $G_n(\partial\tilde{\Delta}(n))$ is contained in a small neighborhood $Q(n,R_2)$ of $(0,0,1)$ in $\mathbb{S}^2$. Since the Gaussian image of the ends of $\tilde{\Delta}(n)$ is contained in $\{\pm a\}$ and G_n is either constant or an open map, then we deduce that either $G_n(\overline{\tilde{\Delta}(n)})\subset Q(n,R_2)$, or else the interior of $G_n(\overline{\tilde{\Delta}(n)})$ contains the horizontal equator $\mathbb{S}^2\cap\{z=0\}$ for n large enough. The last possibility contradicts that $\overline{\tilde{\Delta}(n)}$ is strictly stable, as the inner product of G_n with $(0,0,1)$ is a Jacobi function on $\overline{\tilde{\Delta}(n)}$ whose zero set does not intersect the boundary $\partial\overline{\tilde{\Delta}(n)}$. Therefore, $G_n(\tilde{\Delta}(n))\subset Q(n,R_2)$. Note that, as n and R_2 approach ∞, $Q(n,R_2)$ limits to be $(0,0,1)$. Applying Lemma 1.4 in [34], it follows that $\tilde{\Delta}(n)$ is a connected graph over its projection to the (x_1,x_2)-plane and the ends of $\Sigma(n)$ are horizontal. This implies that the rigid motion A that appears in equations (1) and (2) can be taken to be the identity map; in particular, L_1 has horizontal limit tangent plane at infinity.

Since $\gamma(i,n)$ separates $\Sigma(n)$ (by Assertion 3.3) and the ends of $\Sigma(n)$ are horizontal, then $\gamma(i,n)$ has vertical flux vector, in the sense that the integral of the unit conormal

vector to $\Sigma(n)$ along $\gamma(i,n)$ is a vertical vector. As $\Sigma(n)$ converges C^2 to Σ as $n\to\infty$, and $\gamma(i,n)$ limits to $\gamma(i)$, then $\gamma(i)$ also has vertical flux vector. But Theorem 6 in [28] implies that Σ does not have vertical flux vector along such a separating curve. This contradiction finishes the proof of the lemma. □

LEMMA 3.4. $S(\mathcal{L}_1)=\varnothing$.

Proof. Reasoning by contradiction, assume that $S(\mathcal{L}_1)\neq\varnothing$. By item (2) of Theorem 2.2, $\mathcal{L}_1$ is a foliation of $\mathbb{R}^3$ by parallel planes and $S(\mathcal{L}_1)$ consists of one or two lines that are orthogonal to $\mathcal{L}_1$.

Suppose that $S(\mathcal{L}_1)$ consists of a single line. By property $(\star\star)$, $M_{1,n}\cap\overline{\mathbb{B}}(1)$ contains a piecewise-smooth simple closed curve $\beta(n)$ that does not bound a disk in $M_{1,n}$. We claim that the restriction of the injectivity radius function $I_{M_{1,n}}$ of $M_{1,n}$ to $\beta(n)$ is bounded from below by some $I_0>0$ and from above by some $I_1>0$, where both constants do not depend on n. Note that $I_0\geqslant 1$, since any open ball in $\mathbb{R}^3$ of radius 1 intersects $M_{1,n}$ in disks of non-positive Gaussian curvature with their boundaries in the boundary of the ball. If I_1 fails to exist, then, after replacing by a subsequence, there exist points $x_n\in\beta(n)$ such that $I_{M_{1,n}}(x_n)>n$. In particular, by Proposition 1.1 in Colding and Minicozzi [8] for n sufficiently large, x_n is contained in a component of $M_{1,n}\cap\mathbb{B}(x_n,3)$ that is an open disk D_n with boundary in the boundary of $\mathbb{B}(x_n,3)$. By the triangle inequality, the boundary of $\mathbb{B}(x_n,3)$ lies outside of $\overline{\mathbb{B}}(1)$. Therefore, the simple closed curve $\beta(n)\subset D_n\subset M_{1,n}$ is the boundary of a disk in $M_{1,n}$, which is a contradiction. This contradiction proves the claim that the values of $I_{M_{1,n}}$ restricted to $\beta(n)$ lie in an interval $[I_0,I_1]$.

A standard argument (see, for instance, Proposition 2.12, Chapter 13 of [3]) produces, for each $n\in\mathbb{N}$, a non-trivial geodesic loop $\Gamma_n\subset M_{1,n}$ parameterized by arc length, possibly not smooth at $\Gamma_n(0)$, such that $\Gamma_n(0)=\Gamma_n(2d_n)\in\beta(n)$ (here, $2d_n=2I_{M_{1,n}}(\Gamma_n(0))$ is the length of Γ_n), and both arcs $\Gamma_n|_{[0,d_n]}$ and $\Gamma_n|_{[d_n,2d_n]}$ minimize length among curves in $M_{1,n}$ with their extrema. Hence, the length of Γ_n lies in the interval $[2I_0,2I_1]$. We claim that the limit set $\mathrm{Lim}(\{\Gamma_n\}_n)$ satisfies $\mathrm{Lim}(\{\Gamma_n\}_n)\subset S(\mathcal{L}_1)$. If, on the contrary, there exists a point $x\in\mathrm{Lim}(\{\Gamma_n\}_n)\setminus S(\mathcal{L}_1)$, then the Γ_n converge locally around x (after extracting a subsequence) to an open straight line segment $\Gamma(x)$ passing through x and contained in one of the planes of the limit foliation $\mathcal{L}_1$. As we are assuming that $S(\mathcal{L}_1)$ consists of a single line, then the straight line that contains $\Gamma(x)$ cannot intersect $S(\mathcal{L}_1)$ at two points. This contradicts that the length of Γ_n is bounded from above by $2I_1$, and proves our claim that $\mathrm{Lim}(\{\Gamma_n\}_n)\subset S(\mathcal{L}_1)$. In particular, there exist points $a_n,b_n\in\Gamma_n$ at intrinsic distance in $M_{1,n}$ bounded away from zero, such that $|a_n-b_n|\to 0$. This contradicts the chord arc bound given by Theorem 0.5 of [8]. This contradiction proves that $S(\mathcal{L}_1)$ cannot consist of a single line.

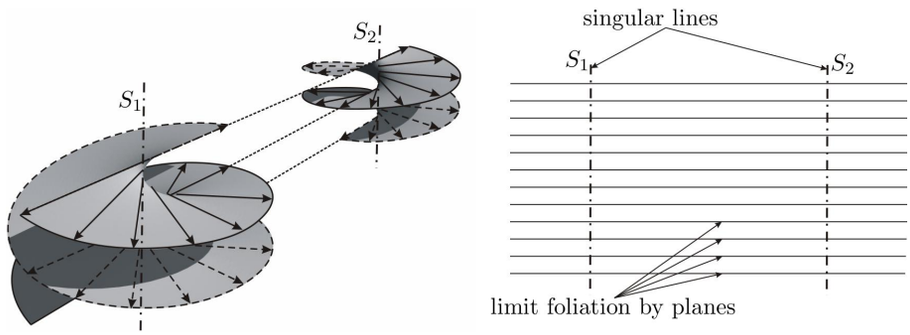

Figure 2. S_1 and S_2 are the columns of the limiting parking garage structure.

Since $S(\mathcal{L}_1)$ consists of two disjoint lines l_1 and l_2 that are orthogonal to $\mathcal{L}_1$, then item (2.2) of Theorem 2.2 implies that the pairs of multivalued graphs forming inside the $M_{1,n}$ around l_1 and l_2 are oppositely handed. From this point on, the proof of this lemma is similar to the proof of Lemma 3.2. We now outline the argument along the lines of the previous proof.

Recall that the Riemann minimal examples $\{\mathcal{R}_t\}_{t>0}$ form a 1-parameter family of properly embedded, singly periodic, minimal planar domains in $\mathbb{R}^3$, each one with infinitely many horizontal planar ends (see, e.g., [30, §2] for a precise description of these classical surfaces). To identify the natural limit objects for each of the two ends of this 1-parameter family, one normalizes each $\mathcal{R}_t$ suitably: after a certain normalization, the surfaces $\mathcal{R}_t$ converge as $t\to 0$ to a vertical catenoid, while as $t\to\infty$, one can normalize the $\mathcal{R}_t$ so that under two different sequences of translations, each translated sequence converges to a vertical helicoid, with the two forming helicoids inside the $\mathcal{R}_t$, for t large, being symmetric by reflection in the vertical plane of symmetry of $\mathcal{R}_t$; hence, these helicoids have opposite handedness. This last property implies that after shrinking the $\mathcal{R}_t$ suitably and taking $t\to\infty$, the $\mathcal{R}_t$ limit to a foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence being two vertical lines. With our language in item (2) of Theorem 2.2, this limit configuration is a limiting parking garage structure with two oppositely oriented columns; see Figure 2.

In our current setting that the $\{M_{1,n}\}_n$ converge to the foliation $\mathcal{L}_1$ except at two lines l_1 and l_2 with opposite handedness, the convergence of $\{M_{1,n}\}_n$ to $\mathcal{L}_1$ has the same basic structure as a two-limit-end example. More precisely, let $\mathcal{C}(R_1)$ be a solid cylinder of radius R_1 that contains $S(\mathcal{L}_1)$ in its interior. Consider the intersection of $M_{1,n}$ with an open slab S bounded by two planes in $\mathcal{L}_1$. For $R_2>R_1$ and n sufficiently large, the part of $M_{1,n}\cap S$ in $\mathcal{C}(R_2)\backslash\mathcal{C}(R_1)$ contains an arbitrarily large number of annular graphs with

boundary curves in the cylinders $\partial\mathcal{C}(R_1)$ and $\partial\mathcal{C}(R_2)$, and these graphs are almost-parallel to the planes in $\mathcal{L}_1$; note that R_1 can be taken large enough so that the columns of the limiting parking garage structure are contained in $\mathcal{C}\big(\frac{1}{2}R_1\big)$. The remaining components of $M_{1,n}\cap S\cap(\mathcal{C}(R_2)\backslash\mathcal{C}(R_1))$ are graphical almost-horizontal disk components that are contained in arbitrarily small neighborhoods of ∂S, as n tends to infinity. Furthermore, inside $\mathcal{C}(R_1)\cap S$ we can find as many of the related curves $\gamma(i,n)\subset M_{1,n}$ from the proof of Lemma 3.2 as we desire (take the $\gamma(i,n)$ as 'connection loops' that converge to a straight line segment joining l_1 and l_2 and orthogonal to these lines). Carrying out the arguments in the proof of Lemma 3.2, we obtain a contradiction, which proves Lemma 3.4. □

So far, we have proven that, after passing to a subsequence, the surfaces $M_{1,n}$ converge smoothly with multiplicity 1 to a connected, properly embedded minimal surface L_1 in one of the following two cases:

(C1) L_1 is a one-ended surface with positive genus less than or equal to g and L_1 is asymptotic to a helicoid;

(C2) L_1 has finite total curvature, genus at most g and at least two ends.

3.4. Surgery in the first stage

We start this section with a lemma to be used later.

LEMMA 3.5. *Let $\Sigma\subset\mathbb{R}^3$ be a properly embedded minimal surface of finite genus and one end. Then, for $R>0$ sufficiently large, $\partial\mathbb{B}(R)$ intersects Σ transversely in a simple closed curve and $\Sigma\backslash\mathbb{B}(R)$ is an annular end representative of the unique end of Σ. Furthermore, given $\varepsilon>0$, R can be chosen large enough so that this end representative is ε-close in the C^2-norm to an end of a helicoid.*

Proof. Since arbitrarily small perturbations of the square of the distance function of Σ to the origin in $\mathbb{R}^3$ can be chosen to have only non-degenerate critical points of index less than 2, elementary Morse theory implies that for the first sentence of the lemma to hold it suffices to prove that, for $R>0$ sufficiently large, $\partial\mathbb{B}(R)$ intersects Σ transversely. Since Σ is a properly embedded minimal surface of finite genus and one end, then Σ is asymptotic to a helicoid (Bernstein and Breiner [1] or Meeks and Pérez [27]); in particular, after a rotation in $\mathbb{R}^3$, we may assume that for some $R>0$ large and $\delta>0$, the intersection of $\Sigma\backslash\mathbb{B}(R)$ with the region

$$C(\delta)=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2>\delta^2x_3^2\}$$

consists of two multivalued graphs over their projections to the (x_1,x_2)-plane with norm of their gradients less than 1, and given $\varepsilon>0$, R can be taken sufficiently large so that

the norm of the gradients of these multivalued graphs are less than ε. Furthermore, the analytic description in [17], [27] ensures that there exists a conformal parameterization $X: D(\infty,\varrho)\to\mathbb{R}^3$ of an end representative of the end of Σ, with associated Weierstrass data

$$g(z)=e^{iz+f(z)}, \quad dh=dz, \tag{3}$$

where $D(\infty,\varrho)=\{z\in\mathbb{C}:|z|\geqslant\varrho)$, f is a holomorphic function in $D(\infty,\varrho)$ that extends across $z=\infty$ with $f(\infty)=0$, g is the stereographically projected Gauss map of X and dh denotes its height differential. This analytic description implies that the following properties hold:

(D1) $[\Sigma\setminus\mathbb{B}(R)]\cap C(\delta)$ is transversal to $\partial\mathbb{B}(R')$ for every $R'>R$;

(D2) $\Sigma\setminus\mathbb{B}(R)$ is transverse to every horizontal plane $\{x_3=t\}$, for all $t\in\mathbb{R}$;

(D3) For $|t|$ sufficiently large, $\beta_t:=(\Sigma\setminus\mathbb{B}(R))\cap\{x_3=t\}$ consists of a smooth, proper Jordan arc at distance less than 1 to the axis of the helicoid H to which Σ is asymptotic, and the tangent lines to β_t are arbitrarily close to the constant value determined by the intersection straight line $H\cap\{x_3=t\}$.

Property (D3) ensures that circles in $\{x_3=t\}$ centered at $(0,0,t)$ and radii $r>R_0$ are transverse to β_t, for $|t|$ and R_0 sufficiently large. This implies that there exists $R_1>R_0$ large such that for R sufficiently large, $\partial\mathbb{B}(R)$ is transverse to

$$\Sigma\setminus\{(x_1,x_2,x_3): x_1^2+x_2^2\leqslant R_1^2\}.$$

Finally, transversality of $\partial\mathbb{B}(R)$ to $\Sigma\cap\{x_1^2+x_2^2\leqslant R_1^2\}$ for R large follows from the fact that under any divergent sequence of vertical translations of Σ, a subsequence converges to some vertical translation of H and, as $R\to\infty$, the unit normal vectors to $\partial\mathbb{B}(R)$ at points of $\Sigma\cap\partial\mathbb{B}(R)\cap\{x_1^2+x_2^2\leqslant R_1^2\}$ are converging to vertical unit vectors. This finishes the proof of the first sentence of the lemma. The second sentence follows from the Weierstrass representation equation (3). □

Given $R_1>0$, let $L_1(R_1)=L_1\cap\overline{\mathbb{B}}(R_1)$. By Lemma 3.5, we can take R_1 sufficiently large so that in case (C1) above, $L_1\setminus L_1(R_1)$ consists of an annular end representative of the unique end of L_1, and this annular end representative is ε-close in the C^2-norm to an end of a helicoid $H\subset\mathbb{R}^3$ for any fixed $\varepsilon>0$ small. In case (C2), the asymptotic geometry of embedded ends with finite total curvature allows us to assume that $L_1\setminus L_1(R_1)$ consists of a finite collection of annular end representatives of the at least two ends of L_1, each one ε-close in the C^2-norm to the end of a plane or of a catenoid.

We will perform the following replacement of $L_1(R_1)$:

(E1) Suppose that L_1 is in case (C1) above. Replace $L_1(R_1)$ by a smooth disk that is a small normal graph over its projection to the helicoid H, and so that the union of this

new piece with $L_1 \setminus L_1(R_1)$ produces a smooth, properly embedded surface $\tilde{L}_1 \subset \mathbb{R}^3$. For R_1 large, this replacement can be made in such a way that $\tilde{L}_1$ is ε-close to H in the C^2-norm on compact sets of $\mathbb{R}^3$ for an arbitrarily small $\varepsilon > 0$ (after choosing R_1 sufficiently large).

(E2) Now assume L_1 is in case (C2). Then, for R_1 large, $L_1 \setminus L_1(R_1)$ consists of a finite number $r \geqslant 2$ of non-compact, annular minimal graphs over the limit tangent plane at infinity of L_1, bounded by r closed curves which are almost-parallel and logarithmically close in terms of R to an equator on $\partial \mathbb{B}(R_1)$. Replace $L_1(R_1)$ by r almost-flat parallel disks contained in $\mathbb{B}(R_1)$ so that the resulting surface, after gluing these disks to $L_1 \setminus L_1(R_1)$, is a possibly disconnected, smooth, properly embedded surface $\tilde{L}_1 \subset \mathbb{R}^3$. This replacement is made so that $\tilde{L}_1 \cap \overline{\mathbb{B}}(R_1)$ has arbitrarily small second fundamental form (after choosing R_1 sufficiently large).

In either of the cases (E1) and (E2), note that the surface $\tilde{L}_1$ is no longer minimal, but it is minimal outside $\overline{\mathbb{B}}(R_1)$.

We finish this first stage by performing the surgery on the surfaces $M_{1,n}$ that converge to L_1. For n large, the surface $M_{1,n}(R_1) = M_{1,n} \cap \overline{\mathbb{B}}(R_1)$ can be assumed to be arbitrarily close to $L_1(R_1)$ in the C^2-norm. Modify $M_{1,n}$ in $\overline{\mathbb{B}}(R_1)$ as we did for L_1 to obtain a new smooth, properly embedded surface $\widetilde{M}_{1,n}$ that is C^2-close to $\tilde{L}_1$ in $\overline{\mathbb{B}}(R_1)$. Observe that if L_1 is in case (C1), then $\widetilde{M}_{1,n}$ is connected and has the same number of ends as $M_{1,n}$, while if L_1 is in case (C2), then $\widetilde{M}_{1,n}$ might fail to be connected, and the total number of ends of the connected components of $\widetilde{M}_{1,n}$ is equal to the number of ends of $M_{1,n}$ (see Figure 3 for a topological representation of this surgery procedure).

3.5. Rescaling by non-trivial topology in the second stage

Since the number of ends of $\widetilde{M}_{1,n}$ is unbounded as $n \to \infty$, then $\widetilde{M}_{1,n}$ has some component that is not simply connected for n large. As $\widetilde{M}_{1,n}$ has catenoidal or planar ends, then for n large there exists a largest positive number $r_{2,n}$ such that for every open ball B in $\mathbb{R}^3$ of radius $r_{2,n}$, every simple closed curve in $\widetilde{M}_{1,n} \cap B$ bounds a disk in $\widetilde{M}_{1,n}$ (but not necessarily inside B as $\widetilde{M}_{1,n}$ does not necessarily satisfy the convex hull property), and there exists a closed ball of radius $r_{2,n}$ centered at a point $T_{2,n} \in \mathbb{R}^3$ whose intersection with $\widetilde{M}_{1,n}$ contains a simple closed curve that is homotopically non-trivial in $\widetilde{M}_{1,n}$.

As for n large, every component of $\widetilde{M}_{1,n} \cap \mathbb{B}(2R_1)$ is simply connected and we may assume $R_1 \geqslant 2$, then every simple closed curve homotopically non-trivial on $\widetilde{M}_{1,n}$ that is contained in a ball of radius 2 is necessarily disjoint from $\mathbb{B}(R_1)$. By the last paragraph, there exists a simple closed curve $\Gamma \subset \widetilde{M}_{1,n}$ that is homotopically non-trivial in $\widetilde{M}_{1,n}$ and that is contained in a closed ball of radius $r_{2,n}$. We next check that $r_{2,n} \geqslant 1$. It clearly

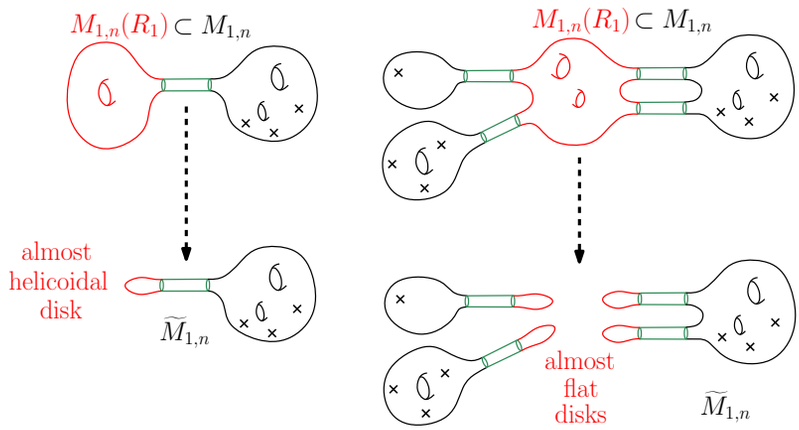

Figure 3. Topological representation of the surgery in the first stage, where the ends are represented by crosses. On the left, replacement (E1) is done on $M_{1,n}$, thus the compact surface $M_{1,n}(R_1)$ (above, red) has only one boundary component and is replaced by an almost helicoidal disk (below, red); after smoothing along a transition region (represented by a green annulus), the resulting surface $\widetilde{M}_{1,n}$ is connected and has the same number of ends as $M_{1,n}$. On the right, replacement (E2) occurs; in this example, the limit surface L_1 has genus 2 and four ends, each end producing a boundary component of $M_{1,n}(R_1)$ that is glued after the surgery to an almost flat disk. The resulting surface $\widetilde{M}_{1,n}$ disconnects into three components, with one, three and three ends, respectively. Observe that replacement (E1) strictly decreases the genus of $\widetilde{M}_{1,n}$.

suffices to show that this inequality holds assuming that $r_{2,n} \leqslant 2$; in this case, the previous argument implies that $\mathbb{B}(T_{2,n}, r_{2,n})$ does not intersect $\mathbb{B}(R_1)$, and so

$$\Gamma \subset M_{1,n} \cap \mathbb{B}(T_{2,n}, r_{2,n}).$$

Note that Γ is homotopically non-trivial in $M_{1,n}$. In particular, $r_{2,n} \geqslant 1$ by property $(\star)$, as desired.

Let

$$\widehat{M}_{2,n} = \frac{1}{r_{2,n}}(\widetilde{M}_{1,n} - T_{2,n}).$$

Thus, $\overline{\mathbb{B}}(1)$ contains a simple closed curve in $\widehat{M}_{2,n}$ that is homotopically non-trivial in $\widehat{M}_{2,n}$. Also define

$$M_{2,n} = \frac{1}{r_{2,n}}(M_{1,n} - T_{2,n}) \quad \text{and} \quad B_{1,n} = \frac{1}{r_{2,n}}(\mathbb{B}(R_1) - T_{2,n}). \tag{4}$$

Clearly, $\widehat{M}_{2,n}$ is homeomorphic to $\widetilde{M}_{1,n}$ and has simpler topology than $M_{2,n}$, in the sense that $\widetilde{M}_{1,n}$ has less generators for its first homology group than $M_{2,n}$; the simplification of the topology of $M_{2,n}$ giving $\widehat{M}_{2,n}$ as a replacement of a subdomain by disks, only occurs inside the ball $B_{1,n}$.

3.6. Controlling the limit of the rescaled surfaces in the second stage

LEMMA 3.6. *Let $C\subset\mathbb{R}^3$ be any compact set. Then, for n large, the ball $B_{1,n}$ is disjoint from C.*

Proof. Suppose, to the contrary, that after passing to a subsequence, every $B_{1,n}$ intersects a compact set $C\subset\mathbb{R}^3$. We first show that the radii of $B_{1,n}$ go to zero as $n\to\infty$. If not, and again after taking a subsequence, we may assume that the radius of $B_{1,n}$ is larger than some $\varepsilon>0$ for every $n\in\mathbb{N}$. This condition, together with the fact that the distance from $\overline{\mathbb{B}}(1)$ to $B_{1,n}$ is bounded independently of n, imply the following two properties:

(I) The change of scale (4) between $M_{1,n}$ and $M_{2,n}$ is essentially the identity (in the sense that the ratio $1/r_{2,n}$ appearing in (4) is bounded from above and below by positive constants independently of n, and the translational part $T_{2,n}$ of (4) is bounded from above independently of n). Observe that $\widehat{M}_{1,n}$ and $\widehat{M}_{2,n}$ are related by the same change of scale as the one in (4) between $M_{1,n}$ and $M_{2,n}$.

(II) There exists $r_0>0$ such that the open ball $B'_{1,n}$ concentric with $B_{1,n}$ of radius r_0, contains $\overline{\mathbb{B}}(1)$ for every n.

Since $\{M_{1,n}\}_n$ converges smoothly on arbitrarily large compact subsets of $\mathbb{R}^3$ to L_1 and outside $\mathbb{B}(R_1)$, L_1 consists of its annular ends (here we have possibly applied Lemma 3.5), we conclude from properties (I) and (II) above that for n large, $B'_{1,n}$ intersects $\widehat{M}_{2,n}$ in disks. This last property contradicts that $\overline{\mathbb{B}}(1)$ contains a closed curve that is homotopically non-trivial in $\widehat{M}_{2,n}$. This contradiction shows that the radii of the balls $B_{1,n}$ tend to zero as $n\to\infty$, provided that these balls intersect C.

By the previous paragraph, after taking a subsequence, we may assume that the sequence of balls $\{B_{1,n}\}_n$ converges to a point $p\in C$. To proceed with the proof of Lemma 3.6, we need the following assertion.

ASSERTION 3.7. *$\{M_{2,n}\}_n$ is locally simply connected in $\mathbb{R}^3\setminus\{p\}$.*

Proof. Fix a point $q\in\mathbb{R}^3\setminus\{p\}$. Then we can write $|p-q|=d\varepsilon$ for $d\geqslant 10$, $\varepsilon>0$. Reasoning by contradiction, suppose that for ε arbitrarily small, for n large, we find a simple closed curve $\Gamma_n\subset M_{2,n}\cap\mathbb{B}(q,\varepsilon)$ that is homotopically non-trivial in $M_{2,n}$. Since ε can be assumed to be less than 1, then Γ_n must bound a disk D_n in $\widehat{M}_{2,n}$. Note that D_n cannot be contained in $M_{2,n}$, since Γ_n is homotopically non-trivial in $M_{2,n}$. Thus, D_n is not minimal and must intersect $B_{1,n}$. Note that $D_n\setminus B_{1,n}$ contains a compact, connected, minimal planar domain whose boundary intersects each of the spheres $\partial\mathbb{B}(q,\varepsilon)$ and $\partial B_{1,n}$. An elementary application of the maximum principle for minimal surfaces shows that there is no connected minimal surface whose boundary is contained in two

such spheres (pass a suitable catenoid between the closed balls $\overline{\mathbb{B}}(q,\varepsilon)$ and $\overline{B}_{1,n}$). Thus, $\{M_{2,n}\}_n$ is locally simply connected in $\mathbb{R}^3\setminus\{p\}$. □

We next continue with the proof of Lemma 3.6. Consider the sequence of compact minimal surfaces with boundary

$$\left\{M_{2,n}\cap\left(\overline{\mathbb{B}}(n)\setminus\mathbb{B}\left(p,\frac{1}{n}\right)\right)\right\}_n.$$

As we have already observed, this sequence has locally positive injectivity radius in $\mathbb{R}^3\setminus\{p\}$. By Theorem 2.2 applied to this sequence with $W=\{p\}$, there are a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3\setminus\{p\}$ and a closed subset $S(\mathcal{L})\subset\mathcal{L}$ such that

$$\left\{M_{2,n}\cap\left(\overline{\mathbb{B}}(n)\setminus\mathbb{B}\left(p,\frac{1}{n}\right)\right)\right\}_n$$

converges C^α, for all $\alpha\in(0,1)$, to $\mathcal{L}$ outside the singular set of convergence $S(\mathcal{L})$. Furthermore, the closure $\bar{\mathcal{L}}$ of $\mathcal{L}$ in $\mathbb{R}^3$ has the structure of a minimal lamination of $\mathbb{R}^3$.

Assertion 3.8. *$\bar{\mathcal{L}}$ consists entirely of planes.*

Proof. Reasoning by contradiction, assume that $\bar{\mathcal{L}}$ contains a non-flat leaf. By item (3) of Theorem 2.2, $S(\mathcal{L})=\varnothing$ and $\bar{\mathcal{L}}$ consists of a single leaf L, which is a properly embedded minimal surface in $\mathbb{R}^3$ with genus at most g. Furthermore, the convergence of portions of the surfaces $M_{2,n}\cap(\overline{\mathbb{B}}(n)\setminus\mathbb{B}(p,1/n))$ to L is of multiplicity 1.

Since for all fixed $\varepsilon>0$ and for n large the area of $M_{2,n}\cap\mathbb{B}(p,\varepsilon)$ is greater than $\frac{3}{2}\pi\varepsilon^2$ (by the monotonicity of area formula) and $S(\mathcal{L})=\varnothing$, then we deduce that $p\in L$. As L is embedded, the area of $L\cap\mathbb{B}(p,\varepsilon)$ divided by ε^2 tends to π as $\varepsilon\to0$, which contradicts that the $M_{2,n}$ converge smoothly to L in $\mathbb{B}(p,\varepsilon)\setminus\{p\}$ with multiplicity 1. This completes the proof of the assertion. □

By Assertion 3.8, $\bar{\mathcal{L}}$ consists entirely of planes, which after a rotation in $\mathbb{R}^3$ can be assumed to be horizontal.

As $\widehat{M}_{2,n}\setminus\mathbb{B}\big(p,\frac{1}{4}\big)$ is minimal for n large and every simple closed curve in $\widehat{M}_{2,n}\cap\mathbb{B}\big(p,\frac{1}{4}\big)$ is the boundary of a disk in $\widehat{M}_{2,n}$, we claim that the components of $\widehat{M}_{2,n}\cap\mathbb{B}\big(p,\frac{1}{4}\big)$ are disks in $\widehat{M}_{2,n}$, when n is large. To see that this claim holds, for n large choose a component Δ of $\widehat{M}_{2,n}\cap\mathbb{B}\big(p,\frac{1}{4}\big)$ and notice that, because $\frac{1}{4}<1$, then Δ lies in a disk $\mathcal{D}$ in $\widehat{M}_{2,n}$ such that the intersection of $\mathcal{D}$ with $\mathbb{R}^3\setminus B_{1,n}$ is minimal. Hence, by the convex hull property applied to $\mathcal{D}\cap\big(\mathbb{R}^3\setminus\mathbb{B}\big(p,\frac{1}{4}\big)\big)$, $\Delta\subset\mathcal{D}\subset\mathbb{B}\big(p,\frac{1}{4}\big)$. Since $\widehat{M}_{2,n}\cap B_{1,n}$ consists of disks, then Δ is a disk, which proves the claim.

It follows that $\widehat{M}_{2,n}\cap\overline{\mathbb{B}}(1)$ contains a homotopically non-trivial simple closed curve, such that after an isotopy of such a curve in $\widehat{M}_{2,n}$, produces another simple closed curve $\Gamma_n\subset\widehat{M}_{2,n}\cap\overline{\mathbb{B}}\big(\frac{3}{2}\big)$ which is homotopically non-trivial in $\widehat{M}_{2,n}$ and disjoint from $\mathbb{B}\big(p,\frac{1}{4}\big)$.

ASSERTION 3.9. *The singular set of convergence $S(\mathcal{L})$ of the $M_{2,n}\cap(\overline{\mathbb{B}}(n)\setminus\mathbb{B}(p,1/n))$ to $\mathcal{L}$ is non-empty and intersects $\mathbb{B}(3)$.*

Proof. Arguing by contradiction, suppose that $S(\mathcal{L})\cap\mathbb{B}(3)=\varnothing$. Note that $\mathbb{B}\big(p,\frac{1}{4}\big)$ cannot intersect both of the spheres $\partial\mathbb{B}\big(\frac{3}{2}\big)$ and $\partial\mathbb{B}(2)$; in what follows we will assume that $\mathbb{B}\big(p,\frac{1}{4}\big)\cap\partial\mathbb{B}\big(\frac{3}{2}\big)=\varnothing$, as the argument in the other case is similar. Since all leaves of $\bar{\mathcal{L}}$ are horizontal planes, $S(\mathcal{L})\cap\overline{\mathbb{B}}(2)=\varnothing$ and that the surfaces $M_{2,n}\cap\big(\overline{\mathbb{B}}\big(\frac{3}{2}\big)\setminus\mathbb{B}\big(p,\frac{1}{4}\big)\big)$ are compact, then for n large, each component of $M_{2,n}\cap\big(\overline{\mathbb{B}}\big(\frac{3}{2}\big)\setminus\mathbb{B}\big(p,\frac{1}{4}\big)\big)$ that intersects $\partial\mathbb{B}\big(\frac{5}{4}\big)$ is an almost-horizontal graph over its projection to the (x_1,x_2)-plane and is either a disk with boundary in $\partial\mathbb{B}\big(\frac{3}{2}\big)$ or a planar domain with one boundary component in $\partial\mathbb{B}\big(\frac{3}{2}\big)$ and its other boundary components in $\partial\mathbb{B}\big(p,\frac{1}{4}\big)$ (here we may assume $\partial\mathbb{B}\big(p,\frac{1}{4}\big)$ is transverse to every such minimal surface). In particular, for n large, every component of $\widehat{M}_{2,n}\cap\overline{\mathbb{B}}\big(\frac{3}{2}\big)$ is a disk, which contradicts the existence of the curve Γ_n. Thus,

$$S(\mathcal{L})\cap\mathbb{B}(3)\neq\varnothing. \qquad \square$$

By Assertion 3.9 and item (2) of Theorem 2.2, $\bar{\mathcal{L}}$ is a foliation of $\mathbb{R}^3$ by (horizontal) planes and the convergence of the $M_{2,n}\cap(\overline{\mathbb{B}}(n)\setminus\mathbb{B}(p,1/n))$ to $\bar{\mathcal{L}}$ has the structure of a horizontal limiting parking garage structure with one or two columns (vertical lines) as singular set of convergence. By Assertion 3.9, $\overline{S(\mathcal{L})}$ intersects $\mathbb{B}(3)$.

The proof of Lemma 3.4 applies to show that $\overline{S(\mathcal{L})}$ does not contain two components (observe that the presence of the point p does not affect the argument of the proof of Lemma 3.4, as this argument can be done in a horizontal slab far from p). Hence, $\overline{S(\mathcal{L})}$ consists of a single line.

We now check that the possibility that $\overline{S(\mathcal{L})}$ is a single line also leads to a contradiction, which will finish the proof of Lemma 3.6. Define $\varepsilon>0$ by

$$\varepsilon=\begin{cases}\min\big\{\frac{1}{4},\frac{1}{2}d_{\mathbb{R}^3}(p,\overline{S(\mathcal{L})})\big\}, & \text{if } p\notin\overline{S(\mathcal{L})},\\ \frac{1}{2}, & \text{if } p\in\overline{S(\mathcal{L})}.\end{cases}$$

Choose $R\geqslant 3$ sufficiently large so that $p\in\mathbb{B}\big(\frac{1}{2}R\big)$. Recall from the paragraph just before Assertion 3.9 that we found a simple closed curve $\Gamma_n\subset\widehat{M}_{2,n}\cap\overline{\mathbb{B}}\big(\frac{3}{2}\big)$ which is homotopically non-trivial in $\widehat{M}_{2,n}$ and disjoint from $\overline{\mathbb{B}}\big(p,\frac{1}{4}\big)$ (in particular, $\Gamma_n\cap\overline{B}_{1,n}=\varnothing$ for n large). For n large, each of the surfaces $M_{2,n}\cap(\overline{\mathbb{B}}(R)\setminus\mathbb{B}(p,\varepsilon))$ contains a main planar domain component C_n, which contains the curve Γ_n and a long, connected, double spiral curve on $\partial\mathbb{B}(R)$. The component C_n intersects $\partial\mathbb{B}(p,\varepsilon)$ in either a double spiral curve when $p\in\overline{S(\mathcal{L})}$, or in a large number of almost-horizontal closed curves when $p\notin S(\mathcal{L})$. It follows that $\widehat{M}_{2,n}\cap\overline{\mathbb{B}}(R)$ consists entirely of disks, which contradicts the existence of Γ_n. This contradiction completes the proof of Lemma 3.6. $\square$

LEMMA 3.10. *The sequence* $\{M_{2,n}\}_n$ *is locally simply connected in* $\mathbb{R}^3$.

Proof. The proof of this lemma is similar to the proof of Assertion 3.7. The failure of $\{M_{2,n}\}_n$ to be locally simply connected at a point $q\in\mathbb{R}^3$ implies for n large the existence of a disk $D_n\subset\widehat{M}_{2,n}$ whose boundary curve $\Gamma_n\subset M_{2,n}$ is contained in a ball $\mathbb{B}(q,\varepsilon)$ of small radius $\varepsilon\in(0,1)$ (Γ_n is homotopically non-trivial in $M_{2,n}$), and such that D_n intersects $B_{1,n}$.

Since $r_{2,n}\geqslant 1$ as proven in the paragraph just before (4), then the radius of $B_{1,n}$ is less than or equal to R_1. As $B_{1,n}$ leaves every compact set for n large, we deduce from the maximum principle that there is no connected minimal surface with boundary contained in $\mathbb{B}(q,\varepsilon)\cap B_{1,n}$ when the distance between the balls $\mathbb{B}(q,\varepsilon)$, $B_{1,n}$ is sufficiently large. This contradicts the existence of the minimal planar domain $D_n\setminus B_{1,n}$, and completes the proof of the lemma. □

Applying Theorem 2.2 to the sequence $\{M_{2,n}\}_n$ with $W=\varnothing$, we conclude that there exists a minimal lamination $\mathcal{L}_2$ of $\mathbb{R}^3$ and a closed subset $S(\mathcal{L}_2)\subset\mathcal{L}_2$ such that, after passing to a subsequence, the $M_{2,n}$ converge in $\mathbb{R}^3\setminus S(\mathcal{L}_2)$ to $\mathcal{L}_2$. Now our previous arguments in Lemmas 3.1, 3.2 and 3.4 apply without modifications and complete the proof of the following proposition.

PROPOSITION 3.11. *After passing to a subsequence, the* $M_{2,n}$ *converge smoothly with multiplicity* 1 *to a connected, properly embedded minimal surface* $L_2\subset\mathbb{R}^3$ *that lies in one of the cases* (C1) *and* (C2) *stated at the end of* §3.3.

3.7. Surgery in the second stage

With the notation of the previous proposition and given $R_2>0$, we let

$$L_2(R_2)=L_2\cap\overline{\mathbb{B}}(R_2),$$

where the radius R_2 is chosen large enough so that $L_2\setminus L_2(R_2)$ consists of annular representatives of the ends of L_2. As we did in replacements (E1) and (E2) in §3.4, we perform the corresponding replacement of $L_2(R_2)$ by disks to obtain a smooth, properly embedded (non-minimal) surface $\tilde{L}_2\subset\mathbb{R}^3$. Since $\{M_{2,n}\}_n$ converges smoothly with multiplicity 1 to L_2 and $B_{1,n}$ leaves any compact set of $\mathbb{R}^3$ for n large enough, then the sequence $\{\widehat{M}_{2,n}\}_n$ also converges smoothly with multiplicity 1 to L_2.

Finally, replace $\widehat{M}_{2,n}\cap\overline{\mathbb{B}}(R_2)$ in a similar way to get a new smooth properly embedded surface $\widetilde{M}_{2,n}\subset\mathbb{R}^3$ which is not minimal but is C^2-close to $\tilde{L}_2$ in $\overline{\mathbb{B}}(R_2)$. Note that $\widetilde{M}_{2,n}$ has simpler topology than $M_{2,n}$, the simplification of topology occurring as two replacements by collections of disks inside the balls $B_{1,n}$ and $\mathbb{B}(R_2)$. This finishes

the second stage in a recursive definition of properly embedded surfaces obtained as rescalings and disk replacements from the original surfaces $M(n)$.

3.8. The inductive process

We now proceed inductively to produce the kth stage. After passing to a subsequence of the original surfaces $M(n)$, we assume that for each $i<k$ the following properties hold:

(F1) There exist largest numbers $r_{i,n}\geqslant 1$ such that in every open ball $B\subset\mathbb{R}^3$ of radius $r_{i,n}$, every simple closed curve in $\widetilde{M}_{i-1,n}\cap B$ bounds a disk contained in $\widetilde{M}_{i-1,n}$ (in the case $i=1$ we let $\widetilde{M}_{0,n}$ to be $M(n)$).

(F2) There exist points $T_{i,n}\in\mathbb{R}^3$ such that $\widetilde{M}_{i-1,n}\cap\overline{\mathbb{B}}(T_{i,n},r_{i,n})$ contains a simple closed curve that is homotopically non-trivial in $\widetilde{M}_{i-1,n}$.

(F3) The sequence of surfaces

$$M_{i,n}=\frac{1}{r_{i,n}}(M_{i-1,n}-T_{i,n})$$

is locally simply connected in $\mathbb{R}^3$, all being rescaled images of the original surfaces $M(n)$.

(F4) $\{M_{i,n}\}_n$ converges smoothly with multiplicity 1 to a connected, properly embedded minimal surface $L_i\subset\mathbb{R}^3$ that lies in one of the cases (C1) and (C2) stated at the end of §3.3.

(F5) The surface

$$\widehat{M}_{i,n}=\frac{1}{r_{i,n}}(\widetilde{M}_{i-1,n}-T_{i,n})$$

has simpler topology than $M_{i,n}$: the simplification of the topology of $M_{i,n}$ giving $\widehat{M}_{i,n}$ consists of $i-1$ replacements by collections of disks and these replacements occur in $i-1$ disjoint balls that leave each compact set of $\mathbb{R}^3$ as $n\to\infty$. Furthermore, $M_{i,n}$ and $\widehat{M}_{i,n}$ coincide outside such $i-1$ balls.

(F6) There exists a large number $R_i>0$ such that $L_i\setminus L_i(R_i)$ consists of annular representatives of the ends of L_i.

(F7) There exists a smooth, properly embedded (non-minimal) surface $\tilde{L}_i\subset\mathbb{R}^3$ such that $\tilde{L}_i$ coincides with L_i in $\mathbb{R}^3\setminus\mathbb{B}(R_i)$ and $\tilde{L}_i\cap\overline{\mathbb{B}}(R_i)$ is either a disk (when L_i is in case (C1)) or a finite number of almost-flat disks (if L_i is in case (C2)).

(F8) There exist smooth, properly embedded (non-minimal) surfaces $\widetilde{M}_{i,n}\subset\mathbb{R}^3$ such that $\widetilde{M}_{i,n}$ coincides with $\widehat{M}_{i,n}$ in $\mathbb{R}^3\setminus\mathbb{B}(R_i)$ and $\widetilde{M}_{i,n}\cap\overline{\mathbb{B}}(R_i)$ is arbitrarily C^2-close to $\tilde{L}_i\cap\overline{\mathbb{B}}(R_i)$. By (F5) above, the surface $\widehat{M}_{i,n}\cap\overline{\mathbb{B}}(R_i)$ coincides with $M_{i,n}\cap\overline{\mathbb{B}}(R_i)$. As a consequence, $\widetilde{M}_{i,n}$ has simpler topology than $M_{i,n}$, with the simplification of topology consisting of i replacements by collections of disks, one of these replacements occurring

in $\overline{\mathbb{B}}(R_i)$ and the remaining ones inside $i-1$ balls that leave each compact set of $\mathbb{R}^3$ as $n\to\infty$. Outside these i balls, $\widetilde{M}_{i,n}$ and $M_{i,n}$ coincide.

We now describe how to define $r_{k,n}$, $T_{k,n}$, $M_{k,n}$, L_k, $\widehat{M}_{k,n}$, R_k, $\tilde{L}_k$ and $\widetilde{M}_{k,n}$.

We define $r_{k,n}$ to be the largest positive number such that for every open ball $B\subset\mathbb{R}^3$ of radius $r_{k,n}$, every simple closed curve in $\widetilde{M}_{k-1,n}\cap B$ bounds a disk in $\widetilde{M}_{k-1,n}$. Following the arguments in the second paragraph of §3.5, one proves that $r_{k,n}\geqslant 1$. Furthermore, there exists a closed ball of radius $r_{k,n}$ centered at a point $T_{k,n}\in\mathbb{R}^3$ whose intersection with $\widetilde{M}_{k-1,n}$ contains a simple closed curve that is homotopically non-trivial in $\widetilde{M}_{k-1,n}$. We set

$$M_{k,n}=\frac{1}{r_{k,n}}(M_{k-1,n}-T_{k,n}) \quad \text{and} \quad \widehat{M}_{k,n}=\frac{1}{r_{k,n}}(\widetilde{M}_{k-1,n}-T_{k,n}).$$

Hence, we have that $M_{k,n}$ is a rescaled and translated image of the original surface $M(n)$, and $\widehat{M}_{k,n}\cap\overline{\mathbb{B}}(1)$ contains a curve that is homotopically non-trivial in $\widehat{M}_{k,n}$. Finally, $\widehat{M}_{k,n}$ is obtained from $M_{k,n}$ after $k-1$ replacements by collections of disks, one of these replacements occurring inside the ball

$$B_{k-1,n}=\frac{1}{r_{k,n}}(\mathbb{B}(R_{k-1})-T_{k,n}),$$

and the remaining $k-2$ replacements in pairwise disjoint balls $\widetilde{B}_{1,n}(k),\dots,\widetilde{B}_{k-2,n}(k)$ that are disjoint from $B_{k-1,n}$ and where rescaled and translated images of the forming limits $L_1,\dots,L_{k-2}$ are captured (see Figure 4). Note that the radius of $B_{k-1,n}$ is

$$\frac{R_{k-1}}{r_{k,n}}\leqslant R_{k-1},$$

and repeating this argument we have that the radius of $\widetilde{B}_{j,n}(k)$ is less than or equal to R_j for each $j=1,\dots,k-2$.

Lemma 3.12. *Given any compact set $C\subset\mathbb{R}^3$,*

$$B_{k-1,n}\cup\left(\bigcup_{j=1}^{k-2}\widetilde{B}_{j,n}(k)\right)$$

is disjoint from C for n sufficiently large.

Proof. Assume that the lemma fails for some compact set C. The arguments in the first paragraph of the proof of Lemma 3.6 can be adapted to show that, if after choosing a subsequence some of the balls in the collection $\mathcal{B}=\{B_{k-1,n},\widetilde{B}_{1,n}(k),\dots,\widetilde{B}_{k-2,n}(k)\}$ stay at bounded distance from the origin as n goes to ∞, then their corresponding radii go to zero. This implies that, after extracting a subsequence,

$$\bigcup\mathcal{B}=B_{k-1,n}\cup\left(\bigcup_{j=1}^{k-2}\widetilde{B}_{j,n}(k)\right)$$

has non-empty limit set as $n\to\infty$ being a finite set of points in $\mathbb{R}^3$, denoted by $\{p_1,\dots,p_l\}$.

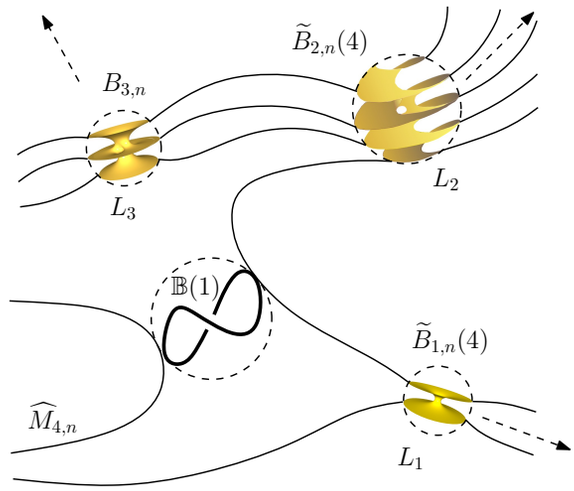

Figure 4. The inductive process in the stage $k=4$.

ASSERTION 3.13. *$\{M_{k,n}\}_n$ is locally simply connected in $\mathbb{R}^3\setminus\{p_1,...,p_l\}$.*

Proof. The proof of the similar fact in Assertion 3.7 does not work in this setting, so we will give a different proof. Arguing by contradiction, we may assume that there exists a point $p\in\mathbb{R}^3\setminus\{p_1,...,p_l\}$ such that, for any $r\in(0,1)$ fixed and for n sufficiently large, there exists a homotopically non-trivial curve $\gamma_{k,n}(r)$ in $M_{k,n}\cap\mathbb{B}(p,r)$. By our normalization, $\gamma_{k,n}(r)$ bounds a disk $\widehat{D}_{k,n}(r)$ on $\widehat{M}_{k,n}$. Since $\widehat{M}_{k,n}$ and $M_{k,n}$ coincide outside $\bigcup\mathcal{B}$, we deduce from the convex hull property that $\widehat{D}_{k,n}(r)$ must enter some of the balls in $\mathcal{B}$. Hence, $\widehat{D}_{k,n}(r)$ intersects the boundary of $\bigcup\mathcal{B}$ in a non-empty collection $\mathcal{A}$ of curves, each of which is arbitrarily close to a rescaled and translated image of the intersection of a sphere of large radius centered at the origin with either an embedded minimal end of finite total curvature or with a helicoidal end. Define

$$\Omega_{k,n}(r)=\widehat{D}_{k,n}(r)\setminus\bigcup\mathcal{B},$$

which is a planar domain in $M_{k,n}$ whose boundary consists of $\gamma_{k,n}(r)$ together with the curves in $\mathcal{A}$. Let $\widehat{\Omega}_{k,n}(r)$ be the compact subdomain on $M_{k,n}$ obtained by gluing to $\Omega_{k,n}(r)$ the forming helicoids with handles whose boundary curves lie on $\Omega_{k,n}(r)$. Since $\widehat{\Omega}_{k,n}(r)$ is a compact minimal surface with boundary and $\bigcup\mathcal{B}$ is disjoint from $\mathbb{B}(p,r)$ for n large, then the convex hull property implies that $\widehat{\Omega}_{k,n}(r)$ has at least one boundary curve outside $\overline{\mathbb{B}}(p,r)$. We will obtain the desired contradiction by an argument based on the maximum principle for minimal surfaces, applied to $\widehat{\Omega}_{k,n}(r)$ and to suitably chosen planes that leave $\widehat{\Omega}_{k,n}(r)$ at one side. To find these planes, we will first analyze the

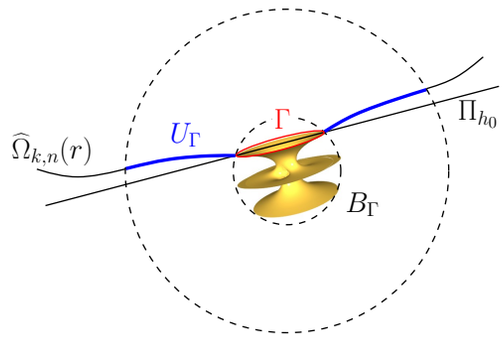

Figure 5. $\widehat{\Omega}_{k,n}(r)$ lies outside B_Γ along Γ.

behavior of $\widehat{\Omega}_{k,n}(r)$ near each of its boundary components outside $\mathbb{B}(p,r)$. For every component Γ of $\partial\widehat{\Omega}_{k,n}(r)$ outside $\overline{\mathbb{B}}(p,r)$, the following properties hold:

(G1) Γ lies in the boundary of one of the balls in $\mathcal{B}$, which we simply denote by B_Γ. Observe that Γ is the intersection of the boundary of $\Omega_{k,n}(r)$ and the boundary of a compact domain $\Omega'\subset M_{k,n}$ obtained as intersection of $M_{k,n}$ with the closure of B_Γ. Furthermore, Ω' is a compact domain in $M_{k,n}$, where a replacement of type (E2) has been made in a stage previous to the k-stage.

(G2) There exists a closed neighborhood U_Γ of Γ in $\widehat{\Omega}_{k,n}(r)$ that lies outside B_Γ, an end $\widetilde{E}$ of a vertical catenoid centered at the origin $\vec{0}\in\mathbb{R}^3$ or of the plane $\{x_3=0\}$ and a map $\phi\colon\mathbb{R}^3\to\mathbb{R}^3$, which is the composition of a homothety and a rigid motion, such that U_Γ can be taken arbitrarily close to $\phi(E)$ (by taking n large enough and the radii R_i, $i<k$, fixed but sufficiently large), where E is the intersection of $\widetilde{E}$ with the closed region between two spheres of large radii centered at $\vec{0}$, so that Γ corresponds through ϕ to the intersection of E with the inner sphere; see Figure 5.

In particular, the normal vector to $\widehat{\Omega}_{k,n}(r)$ along Γ can be assumed to lie in an arbitrarily small open disk in the unit sphere, centered at the image by the linear part of ϕ of the limit normal vector to $\widetilde{E}$. In the case $\widetilde{E}$ is the end of a catenoid, the compact subdomain E can be chosen as the intersection of $\widetilde{E}$ with a slab of the type $\{(x_1,x_2,x_3)\colon a\leqslant x_3\leqslant b\}$, for $0<a<b$ large. For a fixed and $b>a$ arbitrarily large, the sublinearity of the growth of the third coordinate function on $\widetilde{E}$ implies that, if a plane $\Pi_1\subset\mathbb{R}^3$ touches E at a point of $\{x_3=a\}$ and leaves E at one side of Π_1, then Π_1 must be arbitrarily close to the horizontal in terms of a. Therefore,

($\diamondsuit$) If for n large, a plane $\Pi\subset\mathbb{R}^3$ touches $\widehat{\Omega}_{k,n}(r)$ along Γ and leaves $\widehat{\Omega}_{k,n}(r)$ at one side of Π, then the orthogonal direction to Π must be arbitrarily close to $\pm\phi(0,0,1)$. A similar conclusion holds when $\widetilde{E}$ is the end of the plane $\{x_3=0\}$.

We now explain how to choose the plane for which we will use property ($\diamondsuit$) to find a contradiction based on the maximum principle. Since the number of components of $\partial\widehat{\Omega}_{k,n}(r)\setminus\overline{\mathbb{B}}(p,r)$ is bounded independently of n, we deduce that the normal lines to $\widehat{\Omega}_{k,n}(r)$ along its boundary curves other than $\gamma_{k,n}(r)$ lie in a collection $\mathcal{D}_n$ of arbitrarily small (for n sufficiently large) open disks in the projective plane $\mathbb{P}^2$, the number of which is bounded independently of n. Furthermore, as $n\to\infty$, the $\mathcal{D}_n$ converge after extracting a subsequence to a finite set $\mathcal{D}_\infty\subset\mathbb{P}^2$. Given n large, consider a furthest point q_n in $\partial\widehat{\Omega}_{k,n}(r)$ to p. Let $F\in\mathbb{P}^2$ be the line obtained as a (subsequential) limit of the directions of the position vectors q_n-p. Suppose for the moment that F does not lie in $\mathcal{D}_\infty$. Consider the family of planes $\{\Pi_h\subset\mathbb{R}^3 : h\in\mathbb{R}\}$ orthogonal to F, h being the oriented distance to p. If $h_0(n)=\mathrm{dist}(p,q_n)$, then the fact that q_n is a furthest point in $\partial\widehat{\Omega}_{k,n}(r)$ to p implies that $\widehat{\Omega}_{k,n}(r)$ lies entirely at one side of the plane $\Pi_{h_0(n)}$. Taking $n\to\infty$, property ($\diamondsuit$) implies that F lies in $\mathcal{D}_\infty$, which is contrary to our assumption. Hence, $F\in\mathcal{D}_\infty$. Fix n large and consider the plane $\Pi_{h_0(n)}$ which has $\widehat{\Omega}_{k,n}(r)$ at one side, with $q_n\in\Pi_{h_0(n)}\cap\widehat{\Omega}_{k,n}(r)$. Fix $\varepsilon>0$ small and tilt slightly $\Pi_{h_0(n)+\varepsilon}$ to produce a new plane Π' so that $\widehat{\Omega}_{k,n}(r)$ still lies at one side of Π' and $\Pi'\cap\widehat{\Omega}_{k,n}(r)=\varnothing$. Observe that the normal direction to Π' can be assumed not to lie in $\mathcal{D}_\infty$. Move Π' towards $\widehat{\Omega}_{k,n}(r)$ by parallel planes until we find a first contact point (which must lie on $\partial\widehat{\Omega}_{k,n}(r)$ by the maximum principle). Now we can repeat the argument based on property ($\diamondsuit$) as before with Π' instead of $\Pi_{h_0(n)}$ to find a contradiction, if n is sufficiently large. This proves the assertion. □

Consider, for each $i=1,...,l$, a ball B_i centered at p_i, whose radius is much smaller than the minimum distance between pairs of distinct points p_j and p_h, with $j,h\in\{1,...,l\}$. To proceed with the proof of Lemma 3.12, we need the following property.

Assertion 3.14. *For each i, the components of $\widehat{M}_{k,n}$ in B_i are disks for n large.*

Proof. To see this, suppose that, for a given $i=1,...,l$, there exists a simple closed curve $\gamma_{k,n}\subset\widehat{M}_{k,n}\cap B_i$ that does not bound a disk on $\widehat{M}_{k,n}\cap B_i$. As the radius of B_i can be assumed to be less than 1, then $\gamma_{k,n}$ must bound a disk in $\widehat{M}_{k,n}$. Now the same proof of Assertion 3.13 gives a contradiction, thereby proving Assertion 3.14. □

We next continue with the proof of Lemma 3.12. For the fixed value k, define the sequence of compact minimal surfaces

$$\left\{M_{k,n}\cap\left(\overline{\mathbb{B}}(n)\setminus\bigcup_{i=1}^{l}\mathbb{B}\left(p_i,\frac{1}{n}\right)\right)\right\}_n, \tag{5}$$

which has locally positive injectivity radius in $A=\mathbb{R}^3\setminus\{p_1,...,p_l\}$, by Assertion 3.13. Applying Theorem 2.2 to this sequence with $W=\{p_1,...,p_l\}$, we deduce that there exists

a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3\setminus\{p_1,...,p_l\}$ and a closed subset $S(\mathcal{L})\subset\mathcal{L}$ such that the sequence defined by (5) converges C^α for all $\alpha\in(0,1)$ to $\mathcal{L}$, outside of the singular set of convergence $S(\mathcal{L})$. Furthermore, the closure $\bar{\mathcal{L}}$ of $\mathcal{L}$ has the structure of a minimal lamination of $\mathbb{R}^3$. In fact, the arguments in the proof of Assertions 3.8 and 3.9 can be easily adapted to demonstrate that $\bar{\mathcal{L}}$ consists of planes and $S(\mathcal{L})\neq\varnothing$. Therefore, items (2) and (3) of Theorem 2.2 imply that $\bar{\mathcal{L}}$ is a foliation of $\mathbb{R}^3$ by planes and the convergence of the sequence defined in (5) to $\bar{\mathcal{L}}$ has the structure of a limit minimal parking garage structure with $\overline{S(\mathcal{L})}$ consisting of one or two columns. Once we know this, the arguments in the proof of Assertion 3.9 ensure that $\overline{S(\mathcal{L})}$ intersects $\mathbb{B}(3)$.

Again, the arguments in the proof of Lemma 3.4 remain valid and imply that $\overline{S(\mathcal{L})}$ consists of a single line. The final step in the proof of Lemma 3.12 is to discard the case that $\overline{S(\mathcal{L})}$ is one line. Consider the positive number

$$\varepsilon=\tfrac{1}{3}\min(\{1\}\cup\{d_{\mathbb{R}^3}(p_i,p_j)\}_{i\neq j}).$$

Arguing as in the last paragraph of the proof of Assertion 3.9, we conclude that there exists a simple closed curve $\Gamma_n\subset\widehat{M}_{k,n}\cap\overline{\mathbb{B}}(2)$ which is homotopically non-trivial in $\widehat{M}_{k,n}$ and disjoint from $\bigcup_{i=1}^l\overline{\mathbb{B}}(p_i,\varepsilon)$. Furthermore, for n large, each of the surfaces

$$M_{k,n}\cap\left(\overline{\mathbb{B}}(3)\setminus\bigcup_{i=1}^{l}\mathbb{B}(p_i,\varepsilon)\right)$$

contains a main planar domain component C_n, which contains the curve Γ_n. For n large, C_n intersects $\partial\mathbb{B}(3)$ in a long, connected, double spiral curve, and it intersects $\partial\mathbb{B}(p_i,\varepsilon)$ in either a double spiral curve when $p_i\in\overline{S(\mathcal{L})}$ or in a large number of almost-horizontal closed curves when $p_i\notin\overline{S(\mathcal{L})}$, $i=1,...,l$. Thus, $\widehat{M}_{k,n}\cap\overline{\mathbb{B}}(3)$ consists of disks, which contradicts the existence of Γ_n. Now, the proof of Lemma 3.12 is complete. □

Straightforward modifications in the proof of Lemmas 3.10 and Proposition 3.11 give the following lemma, whose proof we leave to the reader.

LEMMA 3.15. *The sequence $\{M_{k,n}\}_n$ is locally simply connected in $\mathbb{R}^3$, and after passing to a subsequence, it converges with multiplicity* 1 *to a minimal lamination $\mathcal{L}_k$ of $\mathbb{R}^3$ consisting of a single leaf L_k which satisfies one of the properties* (C1) *and* (C2).

We can continue this inductive process indefinitely and using a diagonal subsequence, we will obtain an infinite sequence $\{L_k\}_{k\in\mathbb{N}}$ of non-simply connected, properly embedded minimal surfaces, each one satisfying one of the properties (C1) and (C2). For each fixed $k\in\mathbb{N}$, $L_k\cap\mathbb{B}(R_k)$ is the limit under homotheties and translations of compact domains of $M(n)$ that are contained in balls $\widehat{B}_{n,k}$. Moreover, $\widehat{B}_{n,k}$ is disjoint from $\widehat{B}_{n,k'}$ for $k\neq k'$.

3.9. The final contradiction

Since the genus of $M(n)$ is fixed and finite, for all k sufficiently large the surface L_k has genus zero (in particular, L_k cannot satisfy case (C1); see also the caption of Figure 3). By the López–Ros theorem [25], L_k is a catenoid. Fix k_0 such that, for every $k \geqslant k_0$, L_k is a catenoid. Given $k \geqslant k_0$, the convergence to L_k of suitable homotheties and translations of compact domains of the $M(n)$ contained in the balls $\widehat{B}_{n,k}$ appearing in the last paragraph ensures that there exists an integer $n(k)$ such that, for all $n \geqslant n(k)$, we may assume that $M(n) \cap \widehat{B}_{n,k_0}, \ldots, M(n) \cap \widehat{B}_{n,k}$ are close to $k - k_0 + 1$ catenoids. For these k and n and for any integer k' with $k_0 \leqslant k' \leqslant k$, let $\Gamma_{n,k'}$ be the unique simple closed geodesic in $M(n) \cap \widehat{B}_{n,k'}$ which, after scaling and translation, converges to the waist circle of $L_{k'}$ as $n \to \infty$.

Lemma 3.16. *For any $m \in \mathbb{N}$, there exists $k \geqslant k_0$ such that at least m of the simple closed curves $\Gamma_{n(k),k'} \subset M(n(k)) \cap \widehat{B}_{n(k),k'}$ separate $M(n(k))$, where $k_0 \leqslant k' \leqslant k$.*

Proof. If the lemma were to fail, then, for any $k \geqslant k_0$, there would be a bound on the number of the curves $\Gamma_{n(k),k'}$ that separate $M(n(k))$. Since the genus of $M(n(k))$ is independent of k, for k sufficiently large there exist seven of these geodesics $\Gamma_{n(k),k'}$ that bound two consecutive annuli in the conformal compactification $\overline{M}(n(k))$ of $M(n(k))$. More precisely, we find Λ_1, Γ_1, Λ_2, Γ_2, Λ_3, Γ_3 and Λ_4, seven of the non-separating curves $\Gamma_{n(k),k'}$, so that the following condition holds:

(H1) $\Lambda_1 \cup \Lambda_4$ is the boundary of a compact annulus $A(\Lambda_1, \Lambda_4) \subset \overline{M}(n(k))$ that is the union of compact annuli $A(\Lambda_1, \Gamma_1)$, $A(\Gamma_1, \Lambda_2)$, $A(\Lambda_2, \Gamma_2)$, $A(\Gamma_2, \Lambda_3)$, $A(\Lambda_3, \Gamma_3)$, $A(\Gamma_3, \Lambda_4)$ that do not intersect in their interiors. Observe that, by the convex hull property, all of these annuli contain at least one end of $M(n(k))$, as does $M(n(k)) \setminus A(\Lambda_1, \Lambda_4)$; see Figure 6.

Let $A(\Gamma_1, \Gamma_2)$ and $A(\Gamma_2, \Gamma_3)$ be the related subannuli in $\overline{M}(n(k))$ bounded by $\Gamma_1 \cup \Gamma_2$ and $\Gamma_2 \cup \Gamma_3$, respectively.

Observe that (H1) now implies the following condition:

(H2) Each of the three components of $M(n(k)) \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ contains at least two ends of $M(n(k))$.

Assertion 3.17. *In the above situation, Γ_1, Γ_2 and Γ_3 all bound disks on the same closed complement of $M(n(k))$ in $\mathbb{R}^3$.*

Proof. Suppose that the assertion fails. Then, we may assume without loss of generality that Γ_1 and Γ_2 bound disks on opposite sides of $M(n(k))$. Let W and W_1 denote the closures of the two components of $\mathbb{R}^3 \setminus M(n(k))$, so that Γ_1 bounds a disk D_1 in W_1 and Γ_2 bounds another disk D_2 in W. Observe that Γ_1 is not homologous to zero in

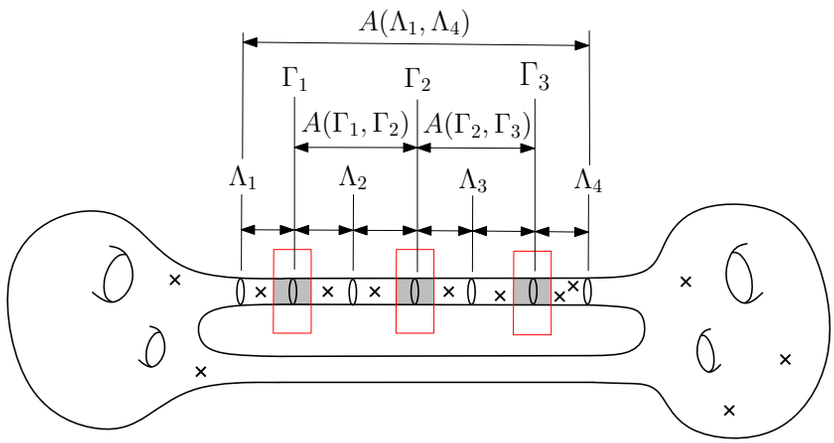

Figure 6. Properties (H1) and (H2): Topological representation of $M(n(k))$, with ends represented by crosses. When viewed in $\mathbb{R}^3$, the compact portions of $M(n(k))$ enclosed in the red rectangles represent almost perfectly formed catenoids, whose almost waist circles are the non-separating curves Γ_i.

W (otherwise Γ_1 would separate $M(n(k))$, which is contrary to our hypothesis). Let $\Omega\subset M(n(k))$ be the (non-compact) planar domain bounded by $\Gamma_1\cup\Gamma_2$. After a small perturbation of $\Omega\cup D_2$ in W that fixes Γ_1, we obtain a new surface Σ contained in W, such that $\Sigma\cap M(n(k))=\Gamma_1$. The union of Σ together with D_1 is a properly embedded surface that intersects $M(n(k))$ only along Γ_1. This implies that Γ_1 separates $M(n(k))$, which is a contradiction. □

Let W_1 be the closure of the component of $\mathbb{R}^3\backslash M(n(k))$ in which Γ_1, Γ_2 and Γ_3 all bound disks, which exists by Assertion 3.17. Since none of Γ_1, Γ_2 and Γ_3 separate $M(n(k))$, then none of the Γ_1, Γ_2 and Γ_3 bound properly embedded surfaces in the closure W of $\mathbb{R}^3\backslash W_1$. As $\Gamma_1\cup\Gamma_2\subset\partial W$ bounds a connected, non-compact orientable surface in W (which is part of $M(n(k))$) and ∂W is a good barrier for solving Plateau problems in W, a standard argument [37], [38] ensures that there exists a non-compact, connected, orientable least-area surface $\Sigma(1,2)\subset W$ with boundary $\partial\Sigma(1,2)=\Gamma_1\cup\Gamma_2$.

Assertion 3.18. *The surface $\Sigma(1,2)$ has just one end, this end has vertical limiting normal vector, and $\Sigma(1,2)\cap\partial W=\Gamma_1\cup\Gamma_2$.*

Proof. Recall that the simple closed curves Γ_1, Γ_2 and Γ_3 are the unique closed geodesics in the intersection of $M(n(k))$ with disjoint balls B_1, B_2 and B_3, and that $M(n(k))\cap B_i$ can be assumed to be arbitrarily close to a large region of a catenoid C_i centered at the center of B_i (and suitably rescaled), $i=1,2,3$. In order to check that $\Sigma(1,2)$ has exactly one end, let X be the non-simply connected region of $B_1\backslash M(n(k))$

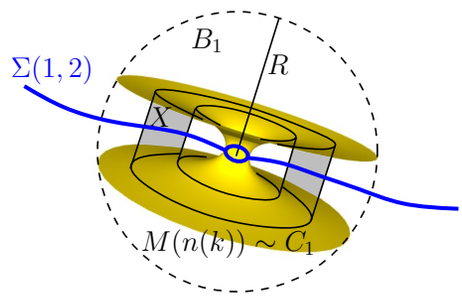

Figure 7. The normal line to $\Sigma(1,2)\cap X$ is close to the limit normal line to C_1.

that lies between two coaxial cylinders with axis the axis of C_1 and radii $\frac{1}{3}R$ and $\frac{1}{2}R$, where R denotes the radius of B_1; see Figure 7.

By curvature estimates for stable surfaces, the portion of $\Sigma(1,2)$ contained in X consists of almost-flat graphs parallel to the almost-flat graphs defined by the catenoid C in the boundary of X. Since the surface $\Sigma(1,2)$ is area-minimizing in W, there is only one such an annular graph. A similar description can be made for $\Sigma(1,2)$ in the ball B_2. After removing the portion of $\Sigma(1,2)$ inside the innermost cylinder in each of these balls, we obtain a connected, non-compact, stable minimal surface $\widetilde{\Sigma}(1,2)$ whose Gauss map $\widetilde{G}\colon\widetilde{\Sigma}(1,2)\to\mathbb{S}^2$ satisfies that $\widetilde{G}(\partial\widetilde{\Sigma}(1,2))$ is contained in a small neighborhood of the limiting normal directions of the corresponding forming catenoids C_1 and C_2. Since the surface $\widetilde{\Sigma}(1,2)$ is stable and connected, it follows that $\widetilde{G}(\widetilde{\Sigma}(1,2))$ is contained in a small neighborhood U of a point in $\mathbb{S}^2$ (see a similar argument in the penultimate paragraph of the proof of Lemma 3.2). In particular, the two forming catenoids in B_1 and B_2 are almost-parallel. Since $\widetilde{\Sigma}(1,2)$ lies in the complement of $M(n(k))$, then the values of $\widetilde{G}$ at the ends of $\widetilde{\Sigma}(1,2)$ are the north or the south poles, which are contained in U. Thus, the forming catenoids inside B_1 and B_2 are approximately vertical, and $\widetilde{\Sigma}(1,2)$ is an almost-horizontal graph over its projection to the (x_1,x_2)-plane. In particular, $\Sigma(1,2)$ has exactly one end, and this end has vertical limiting normal vector.

Finally, the property that $\Sigma(1,2)\cap\partial W=\Gamma_1\cup\Gamma_2$ can be easily deduced from the maximum principle and from property (H2) above. This completes the proof of the assertion. $\square$

Note that $\Sigma(1,2)$ separates W into two regions. Let W' be the closed complement of $\Sigma(1,2)$ in W that contains Γ_3 in its boundary. Let $\Gamma_2'\subset M(n(k))\cap B_2$ be an ε-parallel curve to Γ_2 in $\partial W'$, with $\varepsilon>0$ being very small. Since $\Gamma_2'\cup\Gamma_3$ bounds a connected non-compact surface in $\partial W'$ (which is part of $M(n(k))$), then $\Gamma_2'\cup\Gamma_3$ also bounds a connected, non-compact, orientable least-area surface $\Sigma(2,3)\subset W'$. Note that the arguments in the

proof of Assertion 3.18 apply to give that $\Sigma(2,3)$ intersects $\partial W'$ only along $\Gamma_2'\cup\Gamma_3$, and that outside B_2 and B_3, the surface $\Sigma(2,3)$ is an almost-flat, almost-horizontal graph over its projection to the (x_1,x_2)-plane.

Let Σ be the connected, non-compact, piecewise smooth surface consisting of

$$\Sigma=\Sigma(1,2)\cup\Sigma(2,3)\cup D_1\cup D_3\cup A(\Gamma_2,\Gamma_2'),$$

where, for $i=1,3$, D_i is a disk in $\mathbb{R}^3\backslash W$ bounded by Γ_i and $A(\Gamma_2,\Gamma_2')\subset M(n(k))\cap B_2$ is the compact annulus bounded by $\Gamma_2\cup\Gamma_2'$. Since Σ has no boundary and is properly embedded in $\mathbb{R}^3$, then Σ separates $\mathbb{R}^3$ into two open regions. Let $\Delta(1,3)$ be the connected component of $M(n(k))\backslash(\Gamma_1\cup\Gamma_3)$ that is disjoint from Γ_2. Let W'' be the connected component of $\mathbb{R}^3\backslash\Sigma$ that contains $\Delta(1,3)$. Note that $\Delta(1,3)$ separates W'' (this follows because the piecewise smooth properly embedded surface $\Delta(1,3)\cup D_1\cup D_3$ separates $\mathbb{R}^3$, the disks D_1 and D_3 are contained in the boundary of W'' and $\Delta(1,3)$ is contained in W''). Let W''' be the closed complement of $\Delta(1,3)$ in W'' in which Γ_1 is not homologous to zero. Let $\Gamma_1',\Gamma_3'\subset M(n(k))$ be ε-parallel curves to Γ_1 and Γ_3 in $\partial W'''$, with $\varepsilon>0$ small. Since neither Γ_1' nor Γ_3' separate $M(n(k))$, then $\Gamma_1'\cup\Gamma_3'$ bounds a connected non-compact, proper surface in $\partial W'''$ which is part of $M(n(k))$. Thus, $\Gamma_1'\cup\Gamma_3'$ also bounds a connected, orientable, non-compact, properly embedded least-area surface $\Sigma(1,3)\subset W'''$. As in Assertion 3.18, $\Sigma(1,3)$ has exactly one end that is an almost-horizontal graph. The end of this graph lies between the ends of the two horizontal annular ends of Σ, since it lies in $W'''\subset W''$; see Figure 8.

We now obtain the desired contradiction. Consider the surface

$$\widetilde{\Sigma}(1,3)=\Sigma(1,3)\cup D_1'\cup D_3',$$

where D_i' is a disk in $\mathbb{R}^3\backslash W$ bounded by Γ_i', $i=1,3$. The surface $\widetilde{\Sigma}(1,3)$ is properly embedded in $\mathbb{R}^3$ and $\widetilde{\Sigma}(1,3)\cap\Sigma=\varnothing$, hence Σ must lie on one side of $\widetilde{\Sigma}(1,3)$ in $\mathbb{R}^3$. However, since the graphical end of $\widetilde{\Sigma}(1,3)$ lies between two graphical ends of Σ, we obtain a contradiction that finishes the proof of Lemma 3.16. □

We now complete the proof of Theorem 1.4. By Lemma 3.16, for any m there exists $k\geqslant k_0$ such that at least m of the $k-k_0+1$ closed geodesics of the type $\Gamma_{n(k),k'}\subset M(n(k))\cap\widehat{B}_{n(k),k'}$ separate $M(n(k))$, $k_0\leqslant k'\leqslant k$. These m separating curves $\Gamma_{n(k),k'}$ can be assumed to be arbitrarily close to the waist circles of suitable rescaled, large compact regions of m disjoint catenoids. In particular, $M(n(k))$ has non-zero flux vector along any of these curves, and the separating property implies that such flux vectors are all vertical (any separating curve in $M(n(k))$ with non-zero flux must be homologous to a finite positive number of ends of $M(n(k))$, which have vertical flux). Therefore, the m

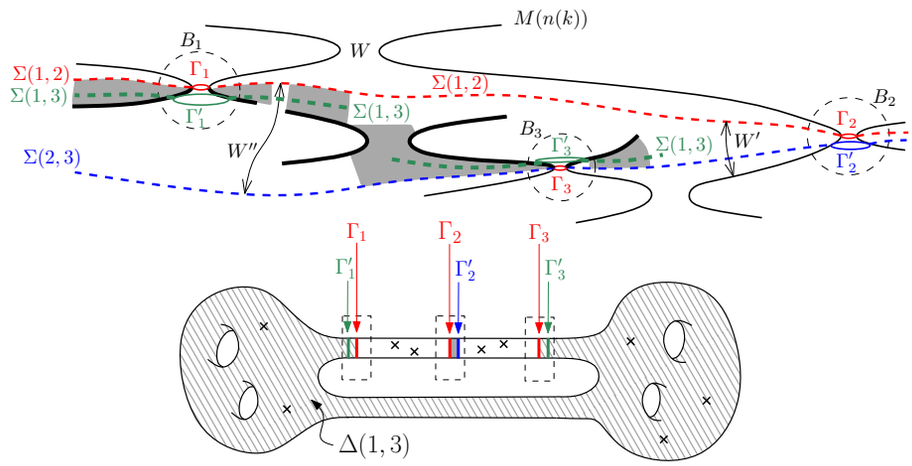

Figure 8. Top: Producing a contradiction with three non-separating curves $\Gamma_1, \Gamma_2, \Gamma_3 \subset M(n(k))$. The gray region that contains $\Sigma(1,3)$ is W'''. Bottom: A topological representation of $M(n(k))$, with the curves that appear in the top figure. The shaded component $\Delta(1,3)$ of $M(n(k)) \setminus (\Gamma_1 \cup \Gamma_3)$ in the bottom figure corresponds to the thick black curve in the top figure.

forming catenoids inside $M(n(k))$ are all vertical. Now exchange the geodesics $\Gamma_{n(k),k'}$ by planar horizontal convex curves $\widetilde{\Gamma}_{n(k),k'}$ in $M(n(k))$, which can be chosen arbitrarily close to the corresponding $\Gamma_{n(k),k'}$. Since the genus of $M(n(k))$ is fixed and finite, we can take m large enough so that at least two of these planar curves, say $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, bound a non-compact planar domain Ω inside $M(n(k))$ and bound planar horizontal disks in the same complement of $M(n(k))$ in $\mathbb{R}^3$. Since Ω has vertical catenoidal and/or planar ends, the López–Ros deformation [25], [42] applies to give the desired contradiction. This finishes the proof of the theorem.

References

[1] Bernstein, J. & Breiner, C., Conformal structure of minimal surfaces with finite topology. *Comment. Math. Helv.*, 86 (2011), 353–381.

[2] Callahan, M., Hoffman, D. & Meeks, W. H., III, The structure of singly-periodic minimal surfaces. *Invent. Math.*, 99 (1990), 455–481.

[3] do Carmo, M. P. a., *Riemannian Geometry.* Mathematics: Theory & Applications. Birkhäuser, Boston, MA, 1992.

[4] Colding, T. H. & Minicozzi, W. P., II, The space of embedded minimal surfaces of fixed genus in a 3-manifold. I. Estimates off the axis for disks. *Ann. of Math.*, 160 (2004), 27–68.

[5] — The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks. *Ann. of Math.*, 160 (2004), 69–92.

[6] — The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar domains. *Ann. of Math.*, 160 (2004), 523–572.

[7] — The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected. *Ann. of Math.*, 160 (2004), 573–615.

[8] — The Calabi–Yau conjectures for embedded surfaces. *Ann. of Math.*, 167 (2008), 211–243.

[9] — The space of embedded minimal surfaces of fixed genus in a 3-manifold V; fixed genus. *Ann. of Math.*, 181 (2015), 1–153.

[10] Collin, P., Topologie et courbure des surfaces minimales proprement plongées de $\mathbf{R}^3$. *Ann. of Math.*, 145 (1997), 1–31.

[11] Collin, P., Kusner, R., Meeks, W. H., III & Rosenberg, H., The topology, geometry and conformal structure of properly embedded minimal surfaces. *J. Differential Geom.*, 67 (2004), 377–393.

[12] Costa, C. J., Example of a complete minimal immersion in $\mathbf{R}^3$ of genus one and three embedded ends. *Bol. Soc. Brasil. Mat.*, 15 (1984), 47–54.

[13] Ejiri, N. & Kotani, M., Index and flat ends of minimal surfaces. *Tokyo J. Math.*, 16 (1993), 37–48.

[14] Fischer-Colbrie, D., On complete minimal surfaces with finite Morse index in three-manifolds. *Invent. Math.*, 82 (1985), 121–132.

[15] Grigor′yan, A., Netrusov, Y. & Yau, S. T., Eigenvalues of elliptic operators and geometric applications, in *Surveys in Differential Geometry.* Vol. 9, pp. 147–217. Int. Press, Somerville, MA, 2004.

[16] Hauswirth, L. & Pacard, F., Higher genus Riemann minimal surfaces. *Invent. Math.*, 169 (2007), 569–620.

[17] Hauswirth, L., Pérez, J. & Romon, P., Embedded minimal ends of finite type. *Trans. Amer. Math. Soc.*, 353 (2001), 1335–1370.

[18] Hoffman, D. & Meeks, W. H., III, Embedded minimal surfaces of finite topology. *Ann. of Math.*, 131 (1990), 1–34.

[19] Hoffman, D., Traizet, M. & White, B., Helicoidal minimal surfaces of prescribed genus. *Acta Math.*, 216 (2016), 217–323.

[20] Hoffman, D., Weber, M. & Wolf, M., An embedded genus-one helicoid. *Ann. of Math.*, 169 (2009), 347–448.

[21] Hoffman, D. & White, B., Genus-one helicoids from a variational point of view. *Comment. Math. Helv.*, 83 (2008), 767–813.

[22] Jorge, L. P. & Meeks, W. H., III, The topology of complete minimal surfaces of finite total Gaussian curvature. *Topology*, 22 (1983), 203–221.

[23] Kapouleas, N., Complete embedded minimal surfaces of finite total curvature. *J. Differential Geom.*, 47 (1997), 95–169.

[24] Lojasiewicz, S., Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 18 (1964), 449–474.

[25] López, F. J. & Ros, A., On embedded complete minimal surfaces of genus zero. *J. Differential Geom.*, 33 (1991), 293–300.

[26] Meeks, W. H., III & Pérez, J., The classical theory of minimal surfaces. *Bull. Amer. Math. Soc.*, 48 (2011), 325–407.

[27] Meeks. W. H., III & Pérez, J., Embedded minimal surfaces of finite topology. *J. Reine Angew. Math.*, 753 (2019), 159–191.

[28] Meeks, W. H., III, Pérez, J. & Ros, A., The geometry of minimal surfaces of finite genus. I. Curvature estimates and quasiperiodicity. *J. Differential Geom.*, 66 (2004), 1–45.

[29] — The geometry of minimal surfaces of finite genus. II. Nonexistence of one limit end examples. *Invent. Math.*, 158 (2004), 323–341.
[30] — Properly embedded minimal planar domains. *Ann. of Math.*, 181 (2015), 473–546.
[31] — The dynamics theorem for properly embedded minimal surfaces. *Math. Ann.*, 365 (2016), 1069–1089.
[32] — Local removable singularity theorems for minimal laminations. *J. Differential Geom.*, 103 (2016), 319–362.
[33] — Structure theorems for singular minimal laminations. To appear in *J. Reine Angew. Math.*
[34] Meeks, W. H., III & Rosenberg, H., The uniqueness of the helicoid. *Ann. of Math.*, 161 (2005), 727–758.
[35] — Maximum principles at infinity. *J. Differential Geom.*, 79 (2008), 141–165.
[36] Meeks, W. H., III, Simon, L. & Yau, S. T., Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math.*, 116 (1982), 621–659.
[37] Meeks, W. H., III & Yau, S. T., The classical Plateau problem and the topology of three-dimensional manifolds. The embedding of the solution given by Douglas–Morrey and an analytic proof of Dehn's lemma. *Topology*, 21 (1982), 409–442.
[38] — The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179 (1982), 151–168.
[39] Montiel, S. & Ros, A., Schrödinger operators associated to a holomorphic map, in *Global Differential Geometry and Global Analysis* (Berlin, 1990), Lecture Notes in Math., 1481, pp. 147–174. Springer, Berlin, 1991.
[40] Osserman, R., *A Survey of Minimal Surfaces*. Dover Publications, New York, 1986.
[41] Riemann, B., *Ouevres Mathématiques de Riemann*. Gauthiers-Villars, Paris, 1898.
[42] Ros, A., Embedded minimal surfaces: forces, topology and symmetries. *Calc. Var. Partial Differential Equations*, 4 (1996), 469–496.
[43] Schoen, R., Estimates for stable minimal surfaces in three-dimensional manifolds, in *Seminar on Minimal Submanifolds*, Ann. of Math. Stud., 103, pp. 111–126. Princeton Univ. Press, Princeton, NJ, 1983.
[44] Schoen, R. M., Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Differential Geom.*, 18 (1983), 791–809.
[45] Traizet, M., An embedded minimal surface with no symmetries. *J. Differential Geom.*, 60 (2002), 103–153.
[46] Traizet, M. & Weber, M., Hermite polynomials and helicoidal minimal surfaces. *Invent. Math.*, 161 (2005), 113–149.
[47] Tysk, J., Eigenvalue estimates with applications to minimal surfaces. *Pacific J. Math.*, 128 (1987), 361–366.
[48] Weber, M. & Wolf, M., Teichmüller theory and handle addition for minimal surfaces. *Ann. of Math.*, 156 (2002), 713–795.

William H. Meeks III
Mathematics Department
University of Massachusetts
Amherst, MA 01003
U.S.A.
profmeeks@gmail.com

Joaquín Pérez
Department of Geometry and Topology
and Institute of Mathematics (IEMath-GR)
University of Granada
ES-18071 Granada
Spain
jperez@ugr.es

Antonio Ros
Department of Geometry and Topology
and Institute of Mathematics (IEMath-GR)
University of Granada
ES-18071 Granada
Spain
aros@ugr.es

Received May 9, 2016