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Sharp well-posedness results of the Benjamin–Ono equation in $H^s(\mathbb{T}, \mathbb{R})$ and qualitative properties of its solutions

by

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1. Introduction

In this paper we consider the Benjamin–Ono (BO) equation on the torus,

$$\partial_t v = H\partial_x^2 v - \partial_x(v^2), \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \ t \in \mathbb{R}, \tag{1.1}$$

Thomas Kappeler sadly passed away in May 2022, after this paper was accepted.

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where $v \equiv v(t,x)$ is real valued and H denotes the Hilbert transform, defined for

$$f = \sum_{n\in\mathbb{Z}} \hat f(n) e^{inx}, \quad \hat f(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx,$$

by

$$Hf(x) := \sum_{n\in\mathbb{Z}} -i\,\mathrm{sign}(n)\hat f(n)\; e^{inx}$$

with $\mathrm{sign}(\pm n) := \pm 1$ for any $n \geqslant 1$, whereas $\mathrm{sign}(0) := 0$. This pseudo-differential equation (ΨDE) in one space dimension has been introduced by Benjamin [7] and Davis and Acrivos [13] (cf. also [32]) to model long, uni-directional internal waves in a 2-layer fluid. It has been extensively studied, both on the real line $\mathbb{R}$ and on the torus $\mathbb{T}$. For an excellent survey, including the derivation of (1.1), we refer to the recent article by Saut [34].

Our aim is to study low regularity solutions of the BO equation on $\mathbb{T}$. To state our results, we first need to review some classical results on the well-posedness problem of (1.1). Based on work of Saut [33], Abdelouhab, Bona, Felland, and Saut proved in [1] that, for any $s > \frac{3}{2}$, equation (1.1) is globally in time well-posed on the Sobolev space $H^s_r \equiv H^s(\mathbb{T},\mathbb{R})$ (endowed with the standard norm $\|\cdot\|_s$, defined by (1.5) below), meaning the following:

(S1) (*Existence and uniqueness of classical solutions*) For any initial data $v_0 \in H^s_r$, there exists a unique curve $v\colon \mathbb{R} \to H^s_r$ in $C(\mathbb{R}, H^s_r) \cap C^1(\mathbb{R}, H^{s-2}_r)$ such that $v(0) = v_0$ and, for any $t \in \mathbb{R}$, equation (1.1) is satisfied in H^{s-2}_r. (Since $s > \frac{3}{2}$, H^s_r is an algebra, and hence $\partial_x v(t)^2 \in H^{s-1}_r$ for any time $t \in \mathbb{R}$.)

(S2) (*Continuity of solution map*) The solution map $\mathcal{S}\colon H^s_r \to C(\mathbb{R}, H^s_r)$ is continuous, meaning that, for any $v_0 \in H^s_r$, $T > 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $w_0 \in H^s_r$ with $\|w_0 - v_0\|_s < \delta$, the solutions $w(t) = \mathcal{S}(t, w_0)$ and $v(t) = \mathcal{S}(t, v_0)$ of (1.1) with initial data $w(0) = w_0$ and $v(0) = v_0$, respectively, satisfy $\sup_{|t|\leqslant T} \|w(t) - v(t)\|_s \leqslant \varepsilon$.

In a straightforward way one verifies that

$$\mathcal{H}^{(-1)}(v) := \langle v \mid 1\rangle \quad \text{and} \quad \mathcal{H}^{(0)}(v) := \tfrac{1}{2}\langle v \mid v\rangle \tag{1.2}$$

are integrals of the above solutions of (1.1). Here $\langle\cdot\mid\cdot\rangle$ denotes the L^2-inner product:

$$\langle f \mid g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\bar g\,dx. \tag{1.3}$$

In particular, it follows that, for any $c \in \mathbb{R}$ and any $s > \frac{3}{2}$, the affine space $H^s_{r,c}$ is left-invariant by $\mathcal{S}$ where, for any $\sigma \in \mathbb{R}$,

$$H^\sigma_{r,c} := \{w \in H^\sigma_r : \langle w \mid 1\rangle = c\}. \tag{1.4}$$

In the sequel, further progress has been made on the well-posedness of (1.1) on Sobolev spaces of low regularity. The best results so far in this direction were obtained by Molinet, by using the gauge transformation introduced by Tao [36]. Molinet's results in [27] (cf. also [30]) imply that the solution map $\mathcal{S}$, introduced in (S2) above, continuously extends to any Sobolev space H^s_r with $0\leqslant s\leqslant\frac{3}{2}$. More precisely, for any such s, $\mathcal{S}: H^s_r\to C(\mathbb{R}, H^s_r)$ is continuous and, for any $v_0\in H^s_r$, $\mathcal{S}(t, v_0)$ satisfies equation (1.1) in H^{s-2}_r. The fact that $\mathcal{S}$ continuously extends to $L^2_r\equiv H^0_r$, $\mathcal{S}: L^2_r\to C(\mathbb{R}, L^2_r)$, can also be deduced by methods recently developed in [16]. Furthermore, one infers from [16] that any solution $\mathcal{S}(t, v_0)$ with initial data $v_0\in L^2_r$ can be approximated in $C(\mathbb{R}, L^2_r)$ by solutions of (1.1) which are rational functions of $\cos x$ and $\sin x$. We refer to these solutions as rational solutions.

In this paper, we show that the BO equation is well-posed in the Sobolev space H^{-s}_r for any $0<s<\frac{1}{2}$, and that this result is sharp. Since the non-linear term $\partial_x v^2$ in equation (1.1) is not well defined for elements in H^{-s}_r, we first need to define what we mean by a solution of (1.1) in such a space.

Definition 1.1. Let $s\geqslant 0$. A continuous curve $\gamma: \mathbb{R}\to H^{-s}_r$, with $\gamma(0)=v_0$ for a given $v_0\in H^{-s}_r$, is called a *global-in-time* solution of the BO equation in H^{-s}_r with initial data v_0 if, for any sequence $(v_0^{(k)})_{k\geqslant 1}$ in H^σ_r with $\sigma>\frac{3}{2}$, which converges to v_0 in H^{-s}_r, the corresponding sequence of classical solutions $\mathcal{S}(\,\cdot\,, v_0^{(k)})$ converges to γ in $C(\mathbb{R}, H^{-s}_r)$. The solution γ is denoted by $\mathcal{S}(\,\cdot\,, v_0)$.

We remark that, for any $v_0\in L^2_r$, the solution $\mathcal{S}(\,\cdot\,, v_0)$ in the sense of Definition 1.1 coincides with the solution obtained by Molinet in [27].

Definition 1.2. Let $s\geqslant 0$. Equation (1.1) is said to be *globally C^0-well-posed* in H^{-s}_r if the following holds:

(i) For any $v_0\in H^{-s}_r$, there exists a global-in-time solution of (1.1) with initial data v_0 in the sense of Definition 1.1.

(ii) The solution map $\mathcal{S}: H^{-s}_r\to C(\mathbb{R}, H^{-s}_r)$ is continuous, i.e. satisfies (S2).

Our main results are the following ones.

THEOREM 1.3. *For any $0\leqslant s<\frac{1}{2}$, the Benjamin–Ono equation is globally C^0-well-posed on H^{-s}_r in the sense of Definition 1.2. For any $c\in\mathbb{R}$ and $t\in\mathbb{R}$, the flow map $S^t=\mathcal{S}(t,\cdot)$ leaves the affine space $H^{-s}_{r,c}$, introduced in (1.4), invariant. Furthermore, there exists a conservation law $I_{-s}: H^{-s}_r\to\mathbb{R}_{\geqslant 0}$ of (1.1) satisfying*

$$\|v\|_{-s}\leqslant I_{-s}(v)\quad \text{for all } v\in H^{-s}_r.$$

In particular, one has

$$\sup_{t\in\mathbb{R}}\|\mathcal{S}(t, v_0)\|_{-s}\leqslant I_{-s}(v_0)\quad \text{for all } v_0\in H^{-s}_r.$$

Remark 1.4. (i) Theorem 1.3 continues to hold on H^s_r for any $s>0$. See Corollary A.4 in Appendix A.

(ii) Since, by (1.2), the L^2-norm is an integral of (1.1), I_{-s} in the case $s=0$ can be chosen as $I_0(v):=\|v\|_0^2$. The definition of I_{-s} for $0<s<\frac{1}{2}$ can be found in Remark 2.5. These novel integrals are one of the key ingredients for the proof of global C^0-well-posedness of (1.1) in H^{-s}_r for $0<s<\frac{1}{2}$.

(iii) Note that global C^0-well-posedness implies the group property $S^{t_1}\circ S^{t_2}=S^{t_1+t_2}$. Consequently, S^t is a homeomorphism of H^{-s}_r.

(iv) By Rellich's compactness theorem, S^t is also weakly sequentially continuous on $H^{-s}_{r,c}$ for any $0\leqslant s<\frac{1}{2}$ and $c\in\mathbb{R}$, and hence in particular on $L^2_{r,0}$. Note that this contradicts a result stated in [28, Theorem 1.1]. Very recently, however, an error in the proof of the latter theorem has been found, leading to the withdrawal of the paper (cf. [29]). A proof of this weak continuity property was indeed the starting point of the present paper.

The next result says that the well-posedness result of Theorem 1.3 is sharp.

Theorem 1.5. *For any $c\in\mathbb{R}$, the Benjamin–Ono equation is ill-posed on $H^{-1/2}_{r,c}$. More precisely, there exists a sequence $(u^{(k)})_{k\geqslant 1}$ in $\bigcap_{n\geqslant 1} H^n_{r,0}$, converging strongly to zero in $H^{-1/2}_{r,0}$, such that, for any $c\in\mathbb{R}$, the solutions $\mathcal{S}(\cdot, u^{(k)}+c)$ of (1.1) of average c have the property that the sequence of functions*

$$t\longmapsto \langle \mathcal{S}(t,u^{(k)}+c)\,|\,e^{ix}\rangle$$

does not converge pointwise to zero on any given time interval of positive length.

Remark 1.6. It was observed in [4] that the solution map $\mathcal{S}$ does not continuously extend to H^{-s}_r with $s>\frac{1}{2}$. More precisely, for any $c\in\mathbb{R}$, the authors of [4] construct a sequence $(v_0^{(k)})_{k\geqslant 1}$ in $\bigcap_{n\geqslant 0} H^n_{r,c}$ of initial data such that, for any $s>\frac{1}{2}$, it converges to an element v_0 in $H^{-s}_{r,c}$ whereas, for any $t\neq 0$, $(\mathcal{S}(t,v_0^{(k)}))_{k\geqslant 1}$ diverges even in the sense of distributions. However, the divergence of $\mathcal{S}(t,v_0^{(k)})$ can be removed by renormalizing the flow by a translation of the space variable, $x\mapsto x+\eta_k t$. In the case $c=0$, η_k is given by $\|v_0^{(k)}\|_0^2$. We refer to [10] for a similar renormalization in the context of the non-linear Schrödinger equation. In Appendix B, we construct a sequence of initial data in $\bigcap_{n\geqslant 0} H^n_{r,c}$ with the above convergence/divergence properties, but where such a renormalization is *not* possible.

Comments on Theorems 1.3 and 1.5. (i) A straightforward computation shows that $s_c=-\frac{1}{2}$ is the critical Sobolev exponent of the Benjamin–Ono equation. Hence, Theorems 1.3 and 1.5 say that the threshold of well-posedness of (1.1) is given by the critical Sobolev exponent s_c.

(ii) In a recent, very interesting paper [35], Talbut proved, by the method of perturbation determinants, developed for the KdV and the NLS equations by Killip, Visan, and Zhang in [24], that, for any $0<s<\frac{1}{2}$, there exists a constant $C_s>0$, only depending on s, such that any sufficiently smooth solution $t\mapsto v(t)$ of (1.1) satisfies the estimate

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{-s} \leqslant C_s(1+\|v(0)\|_{-s}^{2/(1-2s)})^s \|v(0)\|_{-s}.$$

We note that the integrals I_{-s} of Theorem 1.3 (iv) are of a different nature. Let us explain this in more detail. Our method for proving that the solution map $\mathcal{S}$ of (1.1) continuously extends to H_r^{-s} for any $0<s<\frac{1}{2}$ consists in constructing a globally defined non-linear Fourier transform Φ, also referred to as Birkhoff map (cf. §2). It means that (1.1) can be solved by quadrature, when expressed in the coordinates defined by Φ, which we refer to as Birkhoff coordinates. The integrals I_{-s} of Theorem 1.3 (iv) are Taylored to show that $\Phi: H_{r,0}^{-s} \to h_+^{1/2-s}$ is onto (cf. Theorem 2.3). Actually, the map Φ is a key ingredient not only in the proof of Theorem 1.3, but in the proof of all results, stated in §1. In particular, with regard to Theorem 1.5, we note that the standard norm inflation argument, pioneered by [11] (cf. also [17, Appendix A] and references therein), does not apply for proving ill-posedness of (1.1) in $H_{r,c}^{-1/2}$, since the mean $\langle u|1\rangle$ is an integral of (1.1). Our proof of Theorem 1.5 is based in a fundamental way on the map Φ and its properties (cf. §7).

(iii) Using a probabilistic approach developed by Tzvetkov and Visciglia [38], Deng [14] proved well-posedness result for the BO equation on the torus for almost every data with respect to a measure which is supported by $\bigcap_{\varepsilon>0} H_r^{-\varepsilon}$, and for which L_r^2 is of measure zero. Our result provides a deterministic framework for these solutions.

One of the key ingredients of our proof of Theorem 1.3 are explicit formulas for the frequencies of the Benjamin–Ono equation, defined by (2.4) below, which describe the time evolution of solutions of (1.1) when expressed in Birkhoff coordinates. They are not only used to prove the global well-posedness results for (1.1), but at the same time allow to obtain the following qualitative properties of solutions of (1.1).

THEOREM 1.7. *For any* $v_0\in H_{r,c}^{-s}$, *with* $0<s<\frac{1}{2}$ *and* $c\in\mathbb{R}$, *the solution* $\mathcal{S}(\cdot, v_0)$ *of* (1.1) *has the following properties*:

(i) *the orbit* $\{\mathcal{S}(t,v_0): t\in\mathbb{R}\}$ *is relatively compact in* $H_{r,c}^{-s}$;

(ii) *the solution* $t\mapsto\mathcal{S}(t,v_0)$ *is almost periodic in* $H_{r,c}^{-s}$.

Remark 1.8. Theorem 1.7 continues to hold for any initial data in $H_{r,c}^s$ with $s>0$ arbitrary. See Corollary A.4 in Appendix A. For $s=0$, results corresponding to the ones of Theorem 1.7 have been obtained in [16].

In [3], Amick and Toland characterized the traveling wave solutions of (1.1), originally found by Benjamin [7] . It was shown in [16, Appendix B] that they coincide with the so-called 1-gap solutions, described explicitly in [16]. Note that 1-gap potentials are rational solutions of (1.1) and evolve in $\bigcap_{n\geqslant 1} H^n_{r,0}$. In [5, §5.1], Angulo Pava and Natali proved that every traveling wave solution of (1.1) is orbitally stable in $H^{1/2}_r$. Our newly developed methods allow to complement their result as follows.

Theorem 1.9. *Every traveling wave solution of the* BO *equation is orbitally stable in* H^{-s}_r *for any* $0\leqslant s<\frac{1}{2}$.

Remark 1.10. Theorem 1.9 continues to hold on H^s_r for any $s>0$. See Corollary A.4 in Appendix A.

Method of proof. Let us explain our method for studying low regularity solutions of integrable PDEs / ΨDEs such as the Benjamin–Ono equation, in an abstract, informal way. Consider an integrable evolution equation (E) of the form $\partial_t u=X_{\mathcal{H}}(u)$, where $X_{\mathcal{H}}(u)$ denotes the Hamiltonian vector field, corresponding to the Hamiltonian $\mathcal{H}$. In a first step, we disregard equation (E) and choose instead a family of Poisson commuting Hamiltonians $\mathcal{H}_\lambda$, parameterized by $\lambda\in\Lambda$, with the property that the Hamiltonian $\mathcal{H}$ is in the Poisson algebra, generated by the family $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$, i.e., $\{\mathcal{H},\mathcal{H}_\lambda\}=0$ for any $\lambda\in\Lambda$. To study low regularity solutions of (E), the choice of $\mathcal{H}_\lambda$, $\lambda\in\Lambda$, has to be made judiciously. Typically, the so-called hierarchies, often associated with integrable PDEs/ΨDEs are not well-suited families. Our strategy is to choose such a family with the help of a Lax pair formulation of (E), $\partial_t L=[B,L]$, where $L\equiv L_u$ and $B\equiv B_u$ are typically differential or pseudo-differential operators acting on Hilbert spaces of functions, with symbols depending on u, and where $[B,L]$ denotes the commutator of B and L. At least formally, the spectrum of the operator L is conserved by the flow of (E). The goal is to find a Lax pair (L,B) for (E), with the property that the operator L is well defined for u of low regularity, and then choose functions $\mathcal{H}_\lambda$, encoding the spectrum of L, such as the (appropriately regularized) determinant of $L-\lambda$ or a perturbation determinant. We refer to such a function as a generating function. The key properties of $\mathcal{H}_\lambda$ to be established are the following ones: (i) the flows of the Hamiltonian vector fields $X_{\mathcal{H}_\lambda}$ are well defined for u of low regularity and can be integrated globally in time; (ii) for u sufficiently regular, $\mathcal{H}$ can be expressed in terms of the generating function; (iii) the generating function can be used to construct Birkhoff coordinates so that the Hamiltonian vector field $X_{\mathcal{H}}$, when expressed in these coordinates, extends to spaces of u of low regularity.

In the case of the Benjamin–Ono equation, this method is implemented as follows. In a first important step we prove that the operator L_u (cf. (2.9)) of the Lax pair for the Benjamin–Ono equation, found by Nakamura [31], has the property that it is well defined for u in the Sobolev spaces $H^{-s}_{r,0}$, $0<s<\frac{1}{2}$. See the paragraph *Ideas of the proof of Theorem* 2.3 in §2 for more details. By (3.11) in §3, our choice of the generating function is $\mathcal{H}_\lambda(u)=\langle (L_u+\lambda)^{-1}1|1\rangle$ and the Hamiltonian $\mathcal{H}$ of the Benjamin–Ono equation, when expressed in Birkhoff coordinates, is given by (2.6). The novel conservation laws of the Benjamin–Ono equation of Theorem 1.3, $I_{-s}: H^{-s}_r \to \mathbb{R}_{\geqslant 0}$, together with the results on the Lax operator L_u for u in $H^{-s}_{r,0}$ are the key ingredients to construct Birkhoff coordinates on $H^{-s}_{r,0}$ for any $0<s<\frac{1}{2}$. When expressed in these coordinates, equation (1.1) can be solved by quadrature.

Related work. Results on global well-posedness of the type stated in Theorem 1.3 have been obtained for other integrable PDEs such as the KdV, the KdV2 (also referred to as fifth-order KdV), the mKdV, and the cubic (defocusing) NLS equations. A detailed analysis of the frequencies of these equations allowed to prove in addition to the well-posedness results qualitative properties of solutions of these equations, among them properties corresponding to the ones stated in Theorem 1.7—see e.g. [19]–[22]. Very recently, sharp global well-posedness results for the cubic NLS, the mKdV, the KdV, and the KdV2 equations on the real line were obtained in [17], [23], and [9], respectively. They are based on novel integrals constructed in [24] (cf. also [25]). By the same method, Killip and Visan provide in [23] alternative proofs of the global well-posedness results for the KdV equation on the torus obtained in [22]. However, to the best of our knowledge, their method does not allow to deduce qualitative properties of solutions of the KdV equation on $\mathbb{T}$ such as almost periodicity nor to obtain coordinates which can be used to study perturbations of the KdV equation by KAM-type methods.

Subsequent work. One of the main novel features of the Benjamin–Ono equation, when compared from the point of view of integrable PDEs with the KdV equation or the cubic NLS equation, is that the Lax operator L_u (cf. (2.9)), appearing in the Lax pair formulation of (1.1) is *non-local*. One of the consequences of L_u being non-local is that the study of the regularity of the Birkhoff map and of its restrictions to the scale of Sobolev spaces $H^s_{r,0}$, $s\geqslant 0$, is quite involved. Further results on the Birkhoff map of the Benjamin–Ono equation in this direction will be reported on in subsequent work.

Organization. In §2, we state our results on the extension of the Birkhoff map Φ (cf. Theorem 2.3) and discuss first applications. All these results are proved in §3 and §4. In §5, we study the solution map $\mathcal{S}_B$ corresponding to the system of equations, obtained when expressing (1.1) in Birkhoff coordinates. These results are then used to study the solution map $\mathcal{S}$ of (1.1). In the same section, we also introduce the solution map $\mathcal{S}_c$ (cf. (5.8)), defined in terms of the solution map of the equation (1.1) in the affine space $H^s_{r,c}$, $c\in\mathbb{R}$, and study the solution map $\mathcal{S}_{c,B}$ obtained by expressing $\mathcal{S}_c$ in Birkhoff coordinates.

With all these preparations done, we prove Theorems 1.3, 1.7, and 1.9, in §6. The proof of Theorem 1.5 is presented in §7. Finally, in Appendix A we study the restriction of the Birkhoff map to the Sobolev spaces $H^s_{r,0}$ with $s>0$, and discuss applications to the Benjamin–Ono equation, while in Appendix B we discuss results on ill-posedness of the Benjamin–Ono equation in H^{-s}_r, with $s>\frac{1}{2}$.

Notation. By and large, we will use the notation established in [16]. In particular, the H^s-norm of an element v in the Sobolev space $H^s\equiv H^s(\mathbb{T},\mathbb{C})$, $s\in\mathbb{R}$, will be denoted by $\|v\|_s$. It is defined by

$$\|v\|_s = \left(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s}|\hat{v}(n)|^2\right)^{1/2}, \tag{1.5}$$

where $\langle n\rangle=\max\{1,|n|\}$. For $\|v\|_0$, we usually write $\|v\|$. By $\langle\cdot\,|\,\cdot\rangle$, we will also denote the extension of the L^2-inner product, introduced in (1.3), to $H^{-s}\times H^s$, $s\in\mathbb{R}$, by duality. By H_+ we denote the Hardy space consisting of elements $f\in L^2(\mathbb{T},\mathbb{C})\equiv H^0$ with the property that $\hat{f}(n)=0$ for any $n<0$. More generally, for any $s\in\mathbb{R}$, H^s_+ denotes the subspace of H^s consisting of elements $f\in H^s$ with the property that $\hat{f}(n)=0$ for any $n<0$.

2. The Birkhoff map Φ

In this section we present our results on Birkhoff coordinates which will be a key ingredient of the proofs of Theorems 1.3, 1.5, 1.7, and 1.9. We begin by reviewing the results on Birkhoff coordinates proved in [16]. Recall that, on appropriate Sobolev spaces, (1.1) can be written in Hamiltonian form

$$\partial_t u = \partial_x(\nabla\mathcal{H}(u)),$$

with

$$\mathcal{H}(u) := \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}(|\partial_x|^{1/2}u)^2 - \frac{1}{3}u^3\right)dx,$$

where $|\partial_x|^{1/2}$ is the square root of the Fourier multiplier operator $|\partial_x|$ given by

$$|\partial_x| f(x) = \sum_{n\in\mathbb{Z}} |n|\hat{f}(n)e^{inx}.$$

Note that the L^2-gradient $\nabla\mathcal{H}$ of $\mathcal{H}$ can be computed to be $|\partial_x|u-u^2$, and that $\partial_x\nabla\mathcal{H}$ is the Hamiltonian vector field corresponding to the Gardner bracket, defined, for any two functionals $F,G\colon H^0_r\to\mathbb{R}$ with sufficiently regular L^2-gradients, by

$$\{F,G\} := \frac{1}{2\pi}\int_0^{2\pi}(\partial_x\nabla F)\nabla G\,dx.$$

In [16], it is shown that (1.1) admits global Birkhoff coordinates, and hence is an integrable ΨDE in the strongest possible sense. To state this result in more detail, we first introduce some notation. For any subset $J\subset\mathbb{N}_0:=\mathbb{Z}_{\geqslant 0}$ and any $s\in\mathbb{R}$, $h^s(J)\equiv h^s(J,\mathbb{C})$ denotes the weighted ℓ^2-sequence space

$$h^s(J)=\{(z_n)_{n\in J}\subset\mathbb{C}:\|(z_n)_{n\in J}\|_s<\infty\},$$

where

$$\|(z_n)_{n\in J}\|_s:=\Big(\sum_{n\in J}\langle n\rangle^{2s}|z_n|^2\Big)^{1/2}$$

and $\langle n\rangle:=\max\{1,|n|\}$. By $h^s(J,\mathbb{R})$, we denote the real subspace of $h^s(J,\mathbb{C})$ consisting of real sequences $(z_n)_{n\in J}$. In the case where $J=\mathbb{N}:=\{n\in\mathbb{Z}:n\geqslant 1\}$, we write h^s_+ instead of $h^s(\mathbb{N})$. If $s=0$, we also write ℓ^2 instead of h^0, and ℓ^2_+ instead of h^0_+. In the sequel, we view h^s_+ as the $\mathbb{R}$-Hilbert space $h^s(\mathbb{N},\mathbb{R})\oplus h^s(\mathbb{N},\mathbb{R})$, by identifying a sequence $(z_n)_{n\in\mathbb{N}}\in h^s_+$ with the pair of sequences $((\operatorname{Re} z_n)_{n\in\mathbb{N}},(\operatorname{Im} z_n)_{n\in\mathbb{N}})$ in $h^s(\mathbb{N},\mathbb{R})\oplus h^s(\mathbb{N},\mathbb{R})$. We recall that $L^2_r=H^0_r$ and $L^2_{r,0}=H^0_{r,0}$. The following result was proved in [16].

THEOREM 2.1. ([16, Theorem 1.1]) *There exists a homeomorphism*

$$\Phi\colon L^2_{r,0}\longrightarrow h^{1/2}_+,\qquad u\longmapsto(\zeta_n(u))_{n\geqslant 1},$$

such that the following statements hold:

(B1) *For any* $n\geqslant 1$, $\zeta_n\colon L^2_{r,0}\to\mathbb{C}$ *is real analytic.*

(B2) *The Poisson brackets between the coordinate functions* ζ_n *are well defined and, for any* $n,k\geqslant 1$,

$$\{\zeta_n,\bar\zeta_k\}=-i\delta_{nk},\quad \{\zeta_n,\zeta_k\}=0. \tag{2.1}$$

It implies that the functionals $|\zeta_n|^2$, $n\geqslant 1$, *pairwise Poisson commute,*

$$\{|\zeta_n|^2,|\zeta_k|^2\}=0\quad \textit{for all } n,k\geqslant 1.$$

(B3) *On its domain of definition,* $\mathcal{H}\circ\Phi^{-1}$ *is a (real analytic) function, which only depends on the actions* $|\zeta_n|^2$, $n\geqslant 1$. *As a consequence, for any* $n\geqslant 1$, $|\zeta_n|^2$ *is an integral of* $\mathcal{H}\circ\Phi^{-1}$ *and* $\{\mathcal{H}\circ\Phi^{-1},|\zeta_n|^2\}=0$.

The coordinates ζ_n, $n\geqslant 1$, *are referred to as* complex Birkhoff coordinates *and the functionals* $|\zeta_n|^2$, $n\geqslant 1$, *as* action variables.

Remark 2.2. (i) When restricted to submanifolds of finite gap potentials (cf. [16, Definition 2.11]), the map Φ is a canonical, real analytic diffeomorphism onto corresponding Euclidean spaces — see [16, Theorem 7.1] for details.

(ii) For any bounded subset B of $L^2_{r,0}$, the image $\Phi(B)$ by Φ is bounded in $h^{1/2}_+$. This is a direct consequence of the trace formula, saying that, for any $u\in L^2_{r,0}$ (cf. [16, Proposition 3.1]),

$$\|u\|^2 = 2\sum_{n=1}^{\infty} n|\zeta_n|^2. \tag{2.2}$$

Theorem 2.1 together with Remark 2.2 (i) can be used to solve the initial value problem of (1.1) in $L^2_{r,0}$. Indeed, by approximating a given initial data in $L^2_{r,0}$ by finite gap potentials (cf. [16, Definition 2.11]), one concludes from [16, Theorem 7.1] and Theorem 2.1 that equation (1.1), when expressed in the Birkhoff coordinates $\zeta=(\zeta_n)_{n\geqslant 1}$, reads

$$\partial_t\zeta_n = \{\mathcal{H}\circ\Phi^{-1}, \zeta_n\} = i\omega_n\zeta_n \quad \text{for all } n\geqslant 1, \tag{2.3}$$

where ω_n, $n\geqslant 1$, are the BO frequencies:

$$\omega_n = \partial_{|\zeta_n|^2}\mathcal{H}\circ\Phi^{-1}. \tag{2.4}$$

Since the frequencies only depend on the actions $|\zeta_k|^2$, $k\geqslant 1$, they are conserved and hence (2.3) can be solved by quadrature,

$$\zeta_n(t) = \zeta_n(0)e^{i\omega_n(\zeta(0))t}, \quad t\in\mathbb{R}, \ \ n\geqslant 1. \tag{2.5}$$

By [16, Proposition 8.1]), $\mathcal{H}_B:=\mathcal{H}\circ\Phi^{-1}$ can be computed as

$$\mathcal{H}_B(\zeta) := \sum_{k=1}^{\infty} k^2|\zeta_k|^2 - \sum_{k=1}^{\infty}\bigg(\sum_{p=k}^{\infty}|\zeta_p|^2\bigg)^2, \tag{2.6}$$

implying that the frequencies, defined by (2.4), are given by

$$\omega_n(\zeta) = n^2 - 2\sum_{k=1}^{\infty}\min\{n,k\}|\zeta_k|^2 \quad \text{for all } n\geqslant 1. \tag{2.7}$$

Remarkably, for any $n\geqslant 1$, ω_n depends *linearly* on the actions $|\zeta_k|^2$, $k\geqslant 1$. Furthermore, while the Hamiltonian $\mathcal{H}_B$ is defined on h^1_+, the frequencies ω_n, $n\geqslant 1$, given by (2.7) for $\zeta\in h^1_+$, extend to bounded functionals on ℓ^2_+,

$$\begin{aligned}\omega_n\colon \ell^2_+ &\longrightarrow \mathbb{R},\\ \zeta=(\zeta_k)_{k\geqslant 1} &\longmapsto \omega_n(\zeta).\end{aligned} \tag{2.8}$$

We will prove that the restriction $\mathcal{S}_0$ of the solution map of (1.1) to $L^2_{r,0}$, when expressed in Birkhoff coordinates,

$$\begin{aligned}\mathcal{S}_B\colon h^{1/2}_+ &\longrightarrow C(\mathbb{R}, h^{1/2}_+),\\ \zeta(0) &\longmapsto [t\mapsto (\zeta_n(0)\, e^{i\omega_n(\zeta(0))t})_{n\geqslant 1}],\end{aligned}$$

is continuous — see Proposition 5.2 in §5. By Theorem 2.1, $\Phi: L^2_{r,0} \to h^{1/2}_+$ and its inverse $\Phi^{-1}: h^{1/2}_+ \to L^2_{r,0}$ are continuous. Since

$$\mathcal{S}_0 = \Phi^{-1} \mathcal{S}_B \Phi: L^2_{r,0} \longrightarrow C(\mathbb{R}, L^2_{r,0}),$$
$$u(0) \longmapsto \Phi^{-1} \mathcal{S}_B(\cdot, \Phi(u(0))),$$

it then follows that $\mathcal{S}_0: L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0})$ is continuous as well. We remark that, for any $u(0) \in L^2_{r,0}$, the solution $t \mapsto \mathcal{S}(t, u(0))$ can be approximated in $L^2_{r,0}$ by classical solutions of equation (1.1) (cf. Remark 2.2 (i)), and thus coincides with the solution, obtained by Molinet in [27] (cf. also [30]).

Starting point of the proof of Theorem 1.3 is formula (5.11) in §5. We will show that it extends to the Sobolev spaces $H^{-s}_{r,0}$ for any $0<s<\frac{1}{2}$. A key ingredient to prove Theorem 1.3 is therefore the following result on the extension of the Birkhoff map Φ to $H^{-s}_{r,0}$ for any $0<s<\frac{1}{2}$.

THEOREM 2.3. (Extension of Φ) *For any $0<s<\frac{1}{2}$, the map Φ of Theorem 2.1 admits an extension, also denoted by Φ,*

$$\Phi: H^{-s}_{r,0} \longrightarrow h^{1/2-s}_+,$$
$$u \longmapsto \Phi(u) := (\zeta_n(u))_{n\geqslant 1},$$

so that the following holds:

(i) *Φ is a homeomorphism*;

(ii) *Φ and its inverse map bounded subsets to bounded subsets*;

(iii) *for any $u \in H^{-s}_{r,0}$, one has*

$$\|u\|_{-s} \leqslant F_s(\|\Phi(u)\|_{1/2-s}),$$

where

$$F_s: \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0},$$
$$R \longmapsto \sup\{\|\Phi^{-1}(\zeta)\|_{1/2-s} : \|\zeta\|_{1/2-s} \leqslant R\}.$$

Remark 2.4. (i) The Birkhoff map does not continuously extend to $H^{-1/2}_{r,0}$ — see Corollary 7.4 at the end of §7.

(ii) Results developed in the course of the proof of Theorem 2.3 allow us to study the restriction of the Birkhoff map to H^s_r for any $s>0$. See Proposition A.1 in Appendix A for details.

(iii) Items (i) and (iii), combined with the Rellich compactness theorem, imply that, for $0\leqslant s<\frac{1}{2}$, the map $\Phi: H^{-s}_{r,0} \to h^{1/2-s}_+$ and its inverse $\Phi^{-1}: h^{1/2-s}_+ \to H^{-s}_{r,0}$ are weakly sequentially continuous on $H^{-s}_{r,0}$.

Remark 2.5. The above a-priori bound for $\|u\|_{-s}$ can be extended to the space H_r^{-s} as follows:

$$\|v\|_{-s} \leqslant F_s(\|\Phi(v-[v])\|_{1/2-s})+|[v]| \quad \text{for all } v\in H_r^{-s},$$

where $[v]:=\langle v|1\rangle$. For any $0<s<\frac{1}{2}$, the integral I_{-s} in Theorem 1.3 (iv) is defined as

$$I_{-s}(v):=F_s(\|\Phi(v-[v])\|_{1/2-s})+|[v]|.$$

Ideas of the proof of Theorem 2.3. At the heart of the proof of [16, Theorem 1] is the Lax operator L_u, appearing in the Lax pair formulation ([31], cf. also [8], [12], [15])

$$\partial_t L_u = [B_u, L_u]$$

of (1.1) — see [16, Appendix A] for a review. For any given $u\in L_r^2$, the operator L_u is the first-order operator acting on the Hardy space H_+,

$$L_u := -i\partial_x - T_u, \quad T_u(\cdot):=\Pi(u\cdot), \tag{2.9}$$

where Π is the orthogonal projector of L^2 onto H_+ and T_u is the Toeplitz operator with symbol u. We recall that $H_+\equiv H_+^0$ denotes the Hardy space and, more generally, that, for any $s\in\mathbb{R}$,

$$H_+^s = \{f\in H^s : \hat{f}(n)=0 \text{ for all } n<0\}.$$

The operator L_u is self-adjoint with domain H_+^1, bounded from below, and has a compact resolvent. Its spectrum consists of real eigenvalues which are bounded from below. When listed in increasing order, they form a sequence satisfying

$$\lambda_0\leqslant\lambda_1\leqslant\ldots, \quad \lim_{n\to\infty}\lambda_n=\infty.$$

For our purposes, the most important properties of the spectrum of L_u are that the eigenvalues are conserved along the flow of (1.1), and that they are all simple. More precisely, one has

$$\gamma_n := \lambda_n - \lambda_{n-1} - 1 \geqslant 0 \quad \text{for all } n\geqslant 1. \tag{2.10}$$

The non-negative number γ_n is referred to as the nth gap of the spectrum $\operatorname{spec}(L_u)$ of L_u — see [16, Appendix C] for an explanation of this terminology. For any $n\geqslant 1$, the complex Birkhoff coordinate ζ_n of Theorem 2.1 is related to γ_n by $|\zeta_n|^2=\gamma_n$, whereas its phase is defined in terms of an appropriately normalized eigenfunction f_n of L_u, corresponding to the eigenvalue λ_n.

A key step for the proof of Theorem 2.3 is to show that, for any $u\in H_r^{-s}$ with $0<s<\frac{1}{2}$, the Lax operator L_u can be defined as a self-adjoint operator with domain contained in

H_+^{1-s}, and that its spectrum has properties similar to the ones described above in the case where $u \in L_r^2$. In particular, the inequalities (2.10) continue to hold. Since the proof of Theorem 2.3 requires several steps, it is split up into two sections, namely §3 and §4.

A straightforward application of Theorem 2.3 is the following result on isospectral potentials. To state it, we need to introduce some additional notation. For any $\zeta \in h_+^{1/2-s}$, define

$$\mathrm{Tor}(\zeta) := \{z \in h_+^{1/2-s} : |z_n| = |\zeta_n| \text{ for all } n \geqslant 1\}. \tag{2.11}$$

Note that $\mathrm{Tor}(\zeta)$ is an infinite product of (possibly degenerate) circles and a compact subset of $h_+^{1/2-s}$. Furthermore, for any $u \in H_{r,0}^{-s}$, let

$$\mathrm{Iso}(u) := \{v \in H_{r,0}^{-s} : \mathrm{spec}(L_v) = \mathrm{spec}(L_u)\},$$

where, as above, $\mathrm{spec}(L_u)$ denotes the spectrum of the Lax operator $L_u := -i\partial_x - T_u$. The spectrum of L_u continues to be characterized in terms of its gaps γ_n, $n \geqslant 1$ (cf. (2.10)), and the extended Birkhoff coordinates continue to satisfy $|\zeta_n|^2 = \gamma_n$, $n \geqslant 1$. An immediate consequence of Theorem 2.3 then is that [16, Corollary 8.3] extends as follows.

COROLLARY 2.6. *For any $u \in H_{r,0}^{-s}$ with $0 < s < \frac{1}{2}$,*

$$\Phi(\mathrm{Iso}(u)) = \mathrm{Tor}(\Phi(u)).$$

Hence, by the continuity of Φ^{-1}, $\mathrm{Iso}(u)$ is a compact, connected subset of $H_{r,0}^{-s}$.

3. Extension of Φ. Part 1

In this section we prove the first part of Theorem 2.3, which we state as a separate result.

PROPOSITION 3.1. (Extension of Φ. Part 1) *For any $0 < s < \frac{1}{2}$, the following holds:*

(i) *For any $n \geqslant 1$, the formula in [16, (4.1)] of the Birkhoff coordinate $\zeta_n : L_{r,0}^2 \to \mathbb{C}$ extends to $H_{r,0}^{-s}$ and, for any $u \in H_{r,0}^{-s}$, $(\zeta_n(u))_{n \geqslant 1}$ is in $h_+^{1/2-s}$. The extension of the map Φ of Theorem 2.1, also denoted by Φ,*

$$\Phi : H_{r,0}^{-s} \longrightarrow h_+^{1/2-s}, \quad u \longmapsto \Phi(u) := (\zeta_n(u))_{n \geqslant 1},$$

maps bounded subsets of $H_{r,0}^{-s}$ to bounded subsets of $h_+^{1/2-s}$.

(ii) *Φ is sequentially weakly continuous and one-to-one.*

First, we need to establish some auxiliary results related to the Lax operator L_u.

Lemma 3.2. *Let* $u\in H^{-s}_{r,0}$, *with* $0\leqslant s<\frac{1}{2}$. *Then, for any* $f,g\in H^{1/2}_+$, *the following estimates hold*:

(i) *There exists a constant* $C_{1,s}>0$, *only depending on* s, *such that*

$$\|fg\|_s\leqslant C_{1,s}^2\|f\|_\sigma\|g\|_\sigma,\quad \sigma:=\tfrac{1}{2}\big(\tfrac{1}{2}+s\big). \tag{3.1}$$

(ii) *The expression* $\langle u|f\bar{f}\rangle$ *is well defined and satisfies the estimate*

$$\big|\langle u\,|\,f\bar{f}\rangle\big|\leqslant \tfrac{1}{2}\|f\|_{1/2}^2+\eta_s(\|u\|_{-s})\|f\|^2, \tag{3.2}$$

with

$$\eta_s(\|u\|_{-s}):=\|u\|_{-s}(2(1+\|u\|_{-s}))^\alpha C_{2,s}^2, \tag{3.3}$$

where $\alpha:=(1+2s)/(1-2s)$ *and* $C_{2,s}>0$ *is a constant, only depending on* s.

Proof. (i) Estimate (3.1) is obtained from standard estimates of paramultiplication (cf. e.g. [2, Exercise II.A.5] and [6, Theorems 2.82 and 2.85]).

(ii) By item (i), $\langle u|f\bar{f}\rangle$ is well defined by duality and satisfies

$$|\langle u\,|\,f\bar{f}\rangle|\leqslant\|u\|_{-s}\|f\bar{f}\|_s\leqslant\|u\|_{-s}C_{1,s}^2\|f\|_\sigma^2.$$

In order to estimate $\|f\|_\sigma^2$, note that, by interpolation, one has

$$\|f\|_\sigma\leqslant\|f\|_{1/2}^{1/2+s}\|f\|^{1/2-s},$$

and hence

$$C_{1,s}\|f\|_\sigma\leqslant\|f\|_{1/2}^{1/2+s}(C_{2,s}\|f\|)^{1/2-s} \tag{3.4}$$

for some constant $C_{2,s}>0$. Young's inequality then yields, for any $\varepsilon>0$,

$$(C_{1,s}\|f\|_\sigma)^2\leqslant\varepsilon\|f\|_{1/2}^2+\varepsilon^{-\alpha}(C_{2,s}\|f\|)^2, \tag{3.5}$$

where $\alpha=(1+2s)/(1-2s)$. Estimate (3.2) then follows from (3.5) by choosing

$$\varepsilon=\frac{1}{2(1+\|u\|_{-s})}.$$

□

Note that estimate (3.2) implies that the sesquilinear form $\langle T_uf\,|\,g\rangle$ on $H^{1/2}_+$, obtained from the Toeplitz operator $T_uf:=\Pi(uf)$ with symbol $u\in L^2_{r,0}$, can be defined, for any $u\in H^{-s}_{r,0}$ with $0\leqslant s<\frac{1}{2}$, by setting $\langle T_uf|g\rangle:=\langle u|g\bar{f}\rangle$, and that it is bounded. For any $u\in H^{-s}_{r,0}$, the sesquilinear form Q_u^+ on $H^{1/2}_+$ is then defined as follows:

$$Q_u^+(f,g):=\langle -i\partial_xf\,|\,g\rangle-\langle T_uf\,|\,g\rangle+(1+\eta_s(\|u\|_{-s}))\langle f\,|\,g\rangle, \tag{3.6}$$

where $\eta_s(\|u\|_{-s})$ is given by (3.3). The following lemma says that the quadratic form $Q_u^+(f,f)$ is equivalent to $\|f\|_{1/2}^2$. More precisely, the following holds.

LEMMA 3.3. *For any $u\in H^{-s}_{r,0}$, with $0\leqslant s<\frac{1}{2}$, Q^+_u is a positive, sesquilinear form satisfying, for any $f\in H^{1/2}_+$,*

$$\tfrac{1}{2}\|f\|^2_{1/2}\leqslant Q^+_u(f,f)\leqslant (3+2\eta_s(\|u\|_{-s}))\|f\|^2_{1/2}.$$

Proof. (i) Using that u is real valued, one verifies that Q^+_u is sesquilinear. The claimed estimates are obtained from (3.2) as follows: since $\langle n\rangle\leqslant 1+|n|$, one has

$$\|f\|^2_{1/2}\leqslant\langle -i\partial_x f\,|\,f\rangle+\|f\|^2,$$

and hence, by (3.2),

$$\big|\langle T_u f\,|\,f\rangle\big|\leqslant\tfrac{1}{2}\langle -i\partial_x f\,|\,f\rangle+\big(\tfrac{1}{2}+\eta_s(\|u\|_{-s})\big)\|f\|^2.$$

By the definition (3.6), the claimed estimates then follow. In particular, the lower bound for $Q^+_u(f,f)$ shows that Q^+_u is positive. □

Denote by $\langle f|g\rangle_{1/2}\equiv\langle f|g\rangle_{H^{1/2}_+}$ the inner product, corresponding to the norm $\|f\|_{1/2}$. It is given by

$$\langle f\,|\,g\rangle_{1/2}=\sum_{n\geqslant 0}\langle n\rangle\hat{f}(n)\overline{\hat{g}(n)}\quad\text{for all } f,g\in H^{1/2}_+.$$

Furthermore, denote by $D\colon H^t_+\to H^{t-1}_+$ and $\langle D\rangle\colon H^t_+\to H^{t-1}_+$, $t\in\mathbb{R}$, the Fourier multipliers, defined, for $f\in H^t_+$ with Fourier series $f=\sum_{n=0}^\infty\hat{f}(n)e^{inx}$, by

$$Df:=-i\partial_x f=\sum_{n=0}^\infty n\hat{f}(n)e^{inx},$$

$$\langle D\rangle f:=\sum_{n=0}^\infty\langle n\rangle\hat{f}(n)e^{inx}.$$

LEMMA 3.4. *For any $u\in H^{-s}_{r,0}$, with $0\leqslant s<\frac{1}{2}$, there exists a bounded linear isomorphism $A_u\colon H^{1/2}_+\to H^{1/2}_+$ such that*

$$\langle A_u f\,|\,g\rangle_{1/2}=Q^+_u(f,g)\quad\textit{for all } f,g\in H^{1/2}_+.$$

The operator A_u has the following properties:

(i) *A_u and its inverse A_u^{-1} are symmetric, i.e., for any $f,g\in H^{1/2}_+$,*

$$\langle A_u f\,|\,g\rangle_{1/2}=\langle f\,|\,A_u g\rangle_{1/2}\quad\textit{and}\quad\langle A_u^{-1}f\,|\,g\rangle_{1/2}=\langle f\,|\,A_u^{-1}g\rangle_{1/2}.$$

(ii) *The linear isomorphism B_u, given by the composition*

$$B_u:=\langle D\rangle A_u\colon H^{1/2}_+\longrightarrow H^{-1/2}_+$$

satisfies

$$Q^+_u(f,g)=\langle B_u f\,|\,g\rangle\quad\textit{for all } f,g\in H^{1/2}_+.$$

The operator norm of B_u and the one of its inverse can be bounded uniformly on bounded subsets of elements u in $H^{-s}_{r,0}$.

Proof. By Lemma 3.3, the sesquilinear form Q_u^+ is an inner product on $H_+^{1/2}$, equivalent to the inner product $\langle \cdot | \cdot \rangle_{1/2}$. Hence, by the theorem of Fréchet–Riesz, for any $g \in H_+^{1/2}$, there exists a unique element in $H_+^{1/2}$, which we denote by $A_u g$, such that

$$\langle A_u g \,|\, f \rangle_{1/2} = Q_u^+(g, f) \quad \text{for all } f \in H_+^{1/2}.$$

Then, $A_u: H_+^{1/2} \to H_+^{1/2}$ is a linear, injective operator which, by Lemma 3.3, is bounded, i.e., for any $f, g \in H_+^{1/2}$,

$$\big|\langle A_u g \,|\, f \rangle_{1/2}\big| = |Q_u^+(g, f)| \leqslant Q_u^+(g, g)^{1/2} Q_u^+(f, f)^{1/2} \leqslant (3 + 2\eta_s(\|u\|_{-s})) \|g\|_{1/2} \|f\|_{1/2},$$

implying that

$$\|A_u g\|_{1/2} \leqslant (3 + 2\eta_s(\|u\|_{-s})) \|g\|_{1/2}.$$

Similarly, by the theorem of Fréchet–Riesz, for any $h \in H_+^{1/2}$, there exists a unique element in $H_+^{1/2}$, which we denote by $E_u h$, such that

$$\langle h \,|\, f \rangle_{1/2} = Q_u^+(E_u h, f) \quad \text{for all } f \in H_+^{1/2}.$$

Then, $E_u: H_+^{1/2} \to H_+^{1/2}$ is a linear, injective operator which, by Lemma 3.3, is bounded, i.e.,

$$\tfrac{1}{2} \|E_u h\|_{1/2}^2 \leqslant Q_u^+(E_u h, E_u h) = \langle h \,|\, E_u h \rangle_{1/2} \leqslant \|h\|_{1/2} \|E_u h\|_{1/2},$$

implying that $\|E_u h\|_{1/2} \leqslant 2\|h\|_{1/2}$. Note that $A_u(E_u h) = h$, and hence E_u is the inverse of A_u. Therefore, $A_u: H_+^{1/2} \to H_+^{1/2}$ is a bounded linear isomorphism. Next, we show item (i). For any $f, g \in H_+^{1/2}$,

$$\langle g \,|\, A_u f \rangle_{1/2} = \overline{\langle A_u f \,|\, g \rangle}_{1/2} = \overline{Q_u^+(f, g)} = Q_u^+(g, f) = \langle A_u g \,|\, f \rangle_{1/2}.$$

The symmetry of A_u^{-1} is proved in the same way. Towards item (ii), note that, for any $f, g \in H_+^{1/2}$, $\langle f | g \rangle_{1/2} = \langle \langle D \rangle f | g \rangle$, and therefore

$$\langle A_u g \,|\, f \rangle_{1/2} = \langle \langle D \rangle A_u g \,|\, f \rangle,$$

implying that the operator $B_u = \langle D \rangle A_u: H_+^{1/2} \to H_+^{-1/2}$ is a bounded linear isomorphism and that

$$\langle B_u g \,|\, f \rangle = Q_u^+(g, f) \quad \text{for all } g, f \in H_+^{1/2}.$$

The last statement of (ii) follows from Lemma 3.3. □

We denote by L_u^+ the restriction of B_u to $\operatorname{dom}(L_u^+)$, where

$$\operatorname{dom}(L_u^+) := \{g \in H_+^{1/2} : B_u g \in H_+\}.$$

We view L_u^+ as an unbounded linear operator on H_+ and write $L_u^+: \operatorname{dom}(L_u^+) \to H_+$.

LEMMA 3.5. *For any* $u\in H^{-s}_{r,0}$, *with* $0\leqslant s<\frac{1}{2}$, *the following holds*:

(i) $\mathrm{dom}(L_u^+)$ *is a dense subspace of* $H^{1/2}_+$, *and hence of* H_+.

(ii) $L_u^+\colon \mathrm{dom}(L_u^+)\to H_+$ *is bijective and the right inverse of* L_u^+, $(L_u^+)^{-1}\colon H_+\to H_+$, *is compact. Hence*, L_u^+ *has discrete spectrum.*

(iii) $(L_u^+)^{-1}$ *is symmetric and* L_u^+ *is self-adjoint and positive.*

Proof. (i) Since H_+ is a dense subspace of $H^{-1/2}_+$ and $B_u^{-1}\colon H^{-1/2}_+\to H^{1/2}_+$ is a linear isomorphism, $\mathrm{dom}(L_u^+)=B_u^{-1}(H_+)$ is a dense subspace of $H^{1/2}_+$, and hence also of H_+.

(ii) Since L_u^+ is the restriction of the linear isomorphism B_u, it is one-to-one. By the definition of L_u^+, it is onto. The right inverse of L_u^+, denoted by $(L_u^+)^{-1}$, is given by the composition $\iota\circ B_u^{-1}|_{H_+}$, where $\iota\colon H^{1/2}_+\to H_+$ is the standard embedding which, by Sobolev's embedding theorem, is compact. It then follows that $(L_u^+)^{-1}\colon H_+\to H_+$ is compact as well.

(iii) For any $f,g\in H_+$,

$$\langle (L_u^+)^{-1}f\,|\,g\rangle = \langle A_u^{-1}\langle D\rangle^{-1}f\,|\,g\rangle = \langle A_u^{-1}\langle D\rangle^{-1}f\,|\,\langle D\rangle^{-1}g\rangle_{1/2}.$$

By Lemma 3.4, A_u^{-1} is symmetric with respect to the $H^{1/2}_+$-inner product. Hence,

$$\langle (L_u^+)^{-1}f\,|\,g\rangle = \langle \langle D\rangle^{-1}f\,|\,A_u^{-1}\langle D\rangle^{-1}g\rangle_{1/2} = \langle f\,|\,(L_u^+)^{-1}g\rangle,$$

showing that $(L_u^+)^{-1}$ is symmetric. Since, in addition, $(L_u^+)^{-1}$ is bounded, it is also self-adjoint. By Lemma 3.3, it then follows that

$$\langle L_u^+f\,|\,f\rangle = \langle \langle D\rangle A_u f\,|\,f\rangle = \langle A_u f\,|\,f\rangle_{1/2} = Q_u^+(f,f)\geqslant \tfrac{1}{2}\|f\|^2_{1/2},$$

implying that L_u^+ is a positive operator. □

We now define, for any $u\in H^{-s}_{r,0}$ with $0\leqslant s<\frac{1}{2}$, the operator L_u as a linear operator with domain $\mathrm{dom}(L_u):=\mathrm{dom}(L_u^+)$ by setting

$$L_u := L_u^+ - (1+\eta_s(\|u\|_{-s}))\colon \mathrm{dom}(L_u)\longrightarrow H_+.$$

Lemma 3.5 yields the following.

COROLLARY 3.6. *For any* $u\in H^{-s}_{r,0}$, *with* $0\leqslant s<\frac{1}{2}$, *the operator* $L_u\colon \mathrm{dom}(L_u)\to H_+$ *is densely defined, self-adjoint, bounded from below, and has discrete spectrum. It thus admits an* L^2*-normalized basis of eigenfunctions, contained in* $\mathrm{dom}(L_u)$, *and hence in* $H^{1/2}_+$.

Remark 3.7. Let $u\in H^{-s}_{r,0}$, with $0\leqslant s<\frac{1}{2}$ be given. Since $\mathrm{dom}(L_u^+)$ is dense in $H^{1/2}_+$ and L_u^+ is the restriction of $B_u\colon H^{1/2}_+\to H^{-1/2}_+$ to $\mathrm{dom}(L_u^+)$, the symmetry

$$\langle L_u^+f\,|\,g\rangle = \langle f\,|\,L_u^+g\rangle \quad \text{for all } f,g\in \mathrm{dom}(\mathrm{L}_\mathrm{u}^+)$$

can be extended by a straightforward density argument as follows:

$$\langle B_u f\,|\,g\rangle = \langle f\,|\,B_u g\rangle \quad \text{for all } f,g\in H^{1/2}_+.$$

Note that, for any $f, g \in H_+^{1/2}$,

$$\langle B_u f \mid g \rangle = \langle \langle D \rangle A_u f \mid g \rangle = \langle A_u f \mid g \rangle_{1/2},$$

and hence, by (3.6),

$$\langle B_u f \mid g \rangle = Q_u^+(f, g) = \langle Df - T_u f + (1 + \eta_s(\|u\|_{-s})) f \mid g \rangle,$$

yielding the following identity in $H_+^{-1/2}$:

$$B_u f = Df - T_u f + (1 + \eta_s(\|u\|_{-s})) f \quad \text{for all } f \in H_+^{1/2}. \tag{3.7}$$

Given $u \in H_{r,0}^{-s}$, with $0 \leqslant s < \frac{1}{2}$, let us consider the restriction of B_u to H_+^{1-s}.

LEMMA 3.8. *Let $u \in H_{r,0}^{-s}$, with $0 \leqslant s < \frac{1}{2}$. Then, $B_u(H_+^{1-s}) = H_+^{-s}$ and the restriction*

$$B_{u;1-s} := B_u|_{H_+^{1-s}} : H_+^{1-s} \longrightarrow H_+^{-s}$$

is a linear isomorphism. The operator norm of $B_u|_{H_+^{1-s}}$ and the one of its inverse are bounded uniformly on bounded subsets of elements $u \in H_{r,0}^{-s}$.

Proof. Since $1 - s > \frac{1}{2}$, H^{1-s} acts by multiplication on itself and on L^2, and hence, by interpolation, on H^r for $0 \leqslant r \leqslant 1 - s$. By duality, it also acts on H^{-r}, in particular with $r = s$. This implies that $B_u|_{H_+^{1-s}} : H_+^{1-s} \to H_+^{-s}$ is bounded. Being the restriction of an injective operator, it is injective as well. Let us prove that $B_u|_{H_+^{1-s}}$ has H_+^{-s} as its image. To this end, consider an arbitrary element $h \in H_+^{-s}$. We need to show that the solution $f \in H_+^{1/2}$ of $B_u f = h$ is actually in H_+^{1-s}. Write

$$Df = h + (1 + \eta_s(\|u\|_{-s})) f + T_u f. \tag{3.8}$$

Note that $h + (1 + \eta_s(\|u\|_{-s})) f$ is in H_+^{-s}, and it remains to study $T_u f$. By Lemma 3.2 (i) one infers that, for any $g \in H_+^{\sigma}$, with $\sigma = \frac{1}{2}\left(\frac{1}{2} + s\right)$,

$$\left|\langle T_u f \mid g \rangle\right| = \left|\langle u \mid g \bar{f} \rangle\right| \leqslant \|u\|_{-s} \|g \bar{f}\|_s \leqslant C_{1,s}^2 \|u\|_{-s} \|g\|_{\sigma} \|f\|_{\sigma},$$

implying that $T_u f \in H_+^{-\sigma}$, and hence, by (3.8), $f \in H_+^{1-\sigma}$. Since $1 - \sigma > \frac{1}{2}$, we argue as at the beginning of the proof to infer that $T_u f \in H_+^{-s}$. Thus, applying (3.8) once more, we conclude that $f \in H_+^{1-s}$. This shows that $B_u|_{H_+^{1-s}} : H_+^{1-s} \to H_+^{-s}$ is onto. Going through the arguments of the proof, one verifies that the operator norm of $B_u|_{H_+^{1-s}}$ and the one of its inverse are bounded uniformly on bounded subsets of elements $u \in H_{r,0}^{-s}$. This completes the proof of the lemma. □

Lemma 3.8 has the following important consequence.

COROLLARY 3.9. *For any* $u\in H^{-s}_{r,0}$, *with* $0\leqslant s<\frac{1}{2}$, $\mathrm{dom}(L_u^+)\subset H_+^{1-s}$. *In particular, any eigenfunction of* L_u^+ *(and hence of* L_u*) is in* H_+^{1-s}.

Proof. Since $H_+\subset H_+^{-s}$, one has $B_u^{-1}(H_+)\subset B_u^{-1}(H_+^{-s})$, and hence, by Lemma 3.8, $\mathrm{dom}(L_u^+)=B_u^{-1}(H_+)\subset H_+^{1-s}$. □

With the results obtained so far, it is straightforward to verify that many of the results in [16] extend to the case $u\in H^{-s}_{r,0}$. More precisely, let $u\in H^{-s}_{r,0}$, with $0\leqslant s<\frac{1}{2}$. We already know that the spectrum of L_u is discrete, bounded from below, and real. When listed in increasing order and with their multiplicities, the eigenvalues of L_u satisfy

$$\lambda_0\leqslant\lambda_1\leqslant\lambda_2\leqslant\ldots.$$

Arguing as in the proof of [16, Proposition 2.2], one verifies that $\lambda_n\geqslant\lambda_{n-1}+1$, $n\geqslant 1$, and following [16, (2.10)] we define

$$\gamma_n(u):=\lambda_n-\lambda_{n-1}-1\geqslant 0.$$

It then follows that, for any $n\geqslant 1$,

$$\lambda_n=n+\lambda_0+\sum_{k=1}^{n}\gamma_k\geqslant n+\lambda_0.$$

As [16, Lemmas 2.5 and 2.6] continue to hold for $u\in H^{-s}_{r,0}$, we can introduce eigenfunctions $f_n(x,u)$ of L_u, corresponding to the eigenvalues λ_n, which are normalized as in [16, Definition 2.8]. The identities [16, (2.13)] continue to hold,

$$\lambda_n\langle 1\,|\,f_n\rangle=-\langle u\,|\,f_n\rangle, \tag{3.9}$$

as does [16, Lemma 2.5], stating, among other results, that, for any $n\geqslant 1$,

$$\gamma_n=0 \quad \text{if and only if} \quad \langle 1\,|\,f_n\rangle=0. \tag{3.10}$$

Furthermore, the definition [16, (3.1)] of the generating function $\mathcal{H}_\lambda(u)$ extends to the case where $u\in H^{-s}_{r,0}$, with $0<s<\frac{1}{2}$,

$$\begin{aligned}\mathcal{H}_\lambda\colon H^{-s}_{r,0}&\longrightarrow\mathbb{C},\\ u&\longmapsto\langle(L_u+\lambda)^{-1}1\,|\,1\rangle.\end{aligned} \tag{3.11}$$

One also extends the identity [16, (3.2)], the product representation of $\mathcal{H}_\lambda(u)$, stated in [16, Proposition 3.1 (i)],

$$\mathcal{H}_\lambda(u)=\frac{1}{\lambda_0+\lambda}\prod_{n=1}^{\infty}\left(1-\frac{\gamma_n}{\lambda_n+\lambda}\right), \tag{3.12}$$

and the one for $|\langle 1|f_n\rangle|^2=\gamma_n\kappa_n$, $n\geqslant 1$, given in [16, Corollary 3.4],

$$\kappa_n=\frac{1}{\lambda_n-\lambda_0}\prod_{p\neq n}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_n}\right). \tag{3.13}$$

The formula (3.12) then yields the identity (cf. [16, Proposition 3.1 (ii)] and its proof)

$$-\lambda_0(u)=\sum_{n=1}^{\infty}\gamma_n(u). \tag{3.14}$$

Since $\gamma_n(u)\geqslant 0$ for any $n\geqslant 1$, one infers that, for any $u\in H^{-s}_{r,0}$ with $0\leqslant s<\frac{1}{2}$, the sequence $(\gamma_n(u))_{n\geqslant 1}$ is in $\ell^1_+\equiv\ell^1(\mathbb{N},\mathbb{R})$ and

$$\lambda_n(u)=n-\sum_{k=n+1}^{\infty}\gamma_k(u)\leqslant n. \tag{3.15}$$

By (3.6), Lemma 3.3, and (3.7), we infer that $-\lambda_0\leqslant\frac{1}{2}+\eta_s(\|u\|_{-s})$, yielding, when combined with (3.14) and (3.15), the estimate

$$n-\tfrac{1}{2}-\eta_s(\|u\|_{-s})\leqslant\lambda_n(u)\leqslant n\quad\text{for all } n\geqslant 0. \tag{3.16}$$

In a next step, we consider the linear isomorphism

$$B_{u;1-s}=B_u|_{H^{1-s}_+}\colon H^{1-s}_+\to H^{-s}_+$$

on the scale of Sobolev spaces. By duality, $B_{u;1-s}$ extends as a bounded linear isomorphism, $B_{u,s}\colon H^s_+\to H^{-1+s}_+$, and so, by complex interpolation, for any $s\leqslant t\leqslant 1-s$ the restriction of $B_{u,s}$ to H^t_+ gives also rise to a bounded linear isomorphism, $B_{u;t}\colon H^t_+\to H^{-1+t}_+$. All these operators satisfy the same bound as $B_{u;1-s}$ (cf. Lemma 3.8). To state our next result, it is convenient to introduce the notation $\mathbb{N}_0:=\mathbb{Z}_{\geqslant 0}$. Recall that $h^t(\mathbb{N}_0)=h^t(\mathbb{N}_0,\mathbb{C})$, $t\in\mathbb{R}$, and that we write $\ell^2(\mathbb{N}_0)$ instead of $h^0(\mathbb{N}_0)$.

LEMMA 3.10. *Let $u\in H^{-s}_{r,0}$, with $0\leqslant s<\frac{1}{2}$, and let $(f_n)_{n\geqslant 0}$ be the basis of L^2_+, consisting of eigenfunctions of L_u with f_n, $n\geqslant 0$, corresponding to the eigenvalue λ_n and normalized as in* [16, Definition 2.8]. *Then, for any $-1+s\leqslant t\leqslant 1-s$,*

$$K_{u;t}\colon H^t_+\longrightarrow h^t(\mathbb{N}_0),$$
$$f\longmapsto(\langle f\,|\,f_n\rangle)_{n\geqslant 0}$$

is a linear isomorphism. In particular, for $f=\Pi u\in H_+^{-s}$, *one obtains that*

$$(\langle \Pi u \mid f_n\rangle)_{n\geqslant 0}\in h^{-s}(\mathbb{N}_0).$$

The operator norm of $K_{u;t}$ *and the one of its inverse can be uniformly bounded for* $-1+s\leqslant t\leqslant 1-s$ *and for* u *in a bounded subset of* $H_{r,0}^{-s}$.

Proof. We claim that the sequence $(\tilde f_n)_{n\geqslant 0}$, defined by

$$\tilde f_n=\frac{f_n}{(\lambda_n+1+\eta_s(\|u\|_{-s}))^{1/2}},$$

is an orthonormal basis of the Hilbert space $H_+^{1/2}$, endowed with the inner product Q_u^+. Indeed, for any $n\geqslant 0$ and any $g\in H_+^{1/2}$, one has

$$Q_u^+(\tilde f_n, g)=\langle L_u^+\tilde f_n \mid g\rangle=(\lambda_n+1+\eta_s(\|u\|_{-s}))^{1/2}\langle f_n \mid g\rangle.$$

As a consequence, for any $n,m\geqslant 0$, $Q_u^+(\tilde f_n,\tilde f_m)=\delta_{nm}$, and the orthogonal complement of the subspace of $H_+^{1/2}$, spanned by $(\tilde f_n)_{n\geqslant 0}$, is the trivial vector space $\{0\}$, showing that $(\tilde f_n)_{n\geqslant 0}$ is an orthonormal basis of $H_+^{1/2}$. In view of (3.16), we then conclude that

$$\begin{aligned}K_{u;1/2}\colon H_+^{1/2}&\longrightarrow h^{1/2}(\mathbb{N}_0),\\ f&\longmapsto (\langle f \mid f_n\rangle)_{n\geqslant 0},\end{aligned}$$

is a linear isomorphism. Its inverse $K_{u;1/2}^{-1}\colon h^{1/2}(\mathbb{N}_0)\to H_+^{1/2}$ is given by

$$(z_n)_{n\geqslant 0}\longmapsto f:=\sum_{n=0}^{\infty} z_n f_n.$$

By interpolation, we infer that, for any $0\leqslant t\leqslant \frac12$, $K_{u;t}\colon H_+^t\to h^t(\mathbb{N}_0)$ is a linear isomorphism. Taking the transpose of $K_{u;t}^{-1}$, it then follows that, for any $0\leqslant t\leqslant\frac12$,

$$\begin{aligned}K_{u;-t}\colon H_+^{-t}&\longrightarrow h^{-t}(\mathbb{N}_0),\\ f&\longmapsto (\langle f \mid f_n\rangle)_{n\geqslant 0},\end{aligned}$$

is also a linear isomorphism. It remains to discuss the remaining range of t, stated in the lemma. By Lemma 3.8, the restriction of B_u^{-1} to H_+^{-s} gives rise to a linear isomorphism $B_{u;1-s}^{-1}\colon H_+^{-s}\to H_+^{1-s}$. For any $f\in H_+^{-s}$, one then has

$$B_{u;1-s}^{-1}f=\sum_{n=0}^{\infty}\frac{\langle f \mid f_n\rangle}{\lambda_n+1+\eta_s(\|u\|_{-s})}f_n.$$

Since, by our considerations above, $(\langle f|f_n\rangle)_{n\geqslant 0}\in h^{-s}(\mathbb{N}_0)$, one concludes that the sequence

$$\left(\frac{\langle f\,|\,f_n\rangle}{\lambda_n+1+\eta_s(\|u\|_{-s})}\right)_{n\geqslant 0}$$

is in $h^{1-s}(\mathbb{N}_0)$. Conversely, assume that $(z_n)_{n\geqslant 0}\in h^{1-s}(\mathbb{N}_0)$. Then,

$$((\lambda_n+1+\eta_s(\|u\|_{-s}))z_n)_{n\geqslant 0}$$

is in $h^{-s}(\mathbb{N}_0)$. Hence, by the considerations above on $K_{u;-s}$, there exists $g\in H^{-s}_+$ such that

$$\langle g\,|\,f_n\rangle=(\lambda_n+1+\eta_s(\|u\|_{-s}))z_n \quad \text{for all } n\geqslant 0.$$

Hence,

$$g=\sum_{n=0}^{\infty} z_n(\lambda_n+1+\eta_s(\|u\|_{-s}))f_n=\sum_{n=0}^{\infty} z_nB_uf_n,$$

implying that $f:=B_u^{-1}g$ is in H^{1-s}_+ and satisfies

$$f=\sum_{n=0}^{\infty} z_nf_n.$$

Altogether, we have thus proved that

$$\begin{aligned} K_{u;1-s}\colon H^{1-s}_+ &\longrightarrow h^{1-s}(\mathbb{N}_0),\\ f&\longmapsto(\langle f\,|\,f_n\rangle)_{n\geqslant 0},\end{aligned}$$

is a linear isomorphism. Interpolating between $K_{u;-s}$ and $K_{u;1-s}$ and between the adjoints of their inverses, shows that, for any $-1+s\leqslant t\leqslant 1-s$,

$$\begin{aligned} K_{u;t}\colon H^{t} &\longrightarrow h^{t}(\mathbb{N}_0),\\ f&\longmapsto(\langle f\,|\,f_n\rangle)_{n\geqslant 0}\end{aligned}$$

is a linear isomorphism. Going through the arguments of the proof, one verifies that the operator norm of $K_{u;t}$ and the one of its inverse can be uniformly bounded for $-1+s\leqslant t\leqslant 1-s$ and for bounded subsets of elements $u\in H^{-s}_{r,0}$. □

With these preparations done, we can now prove Proposition 3.1 (i).

Proof of Proposition 3.1 (i). Let $u\in H^{-s}_{r,0}$, with $0\leqslant s<\frac{1}{2}$. By (3.13), one has, for any $n\geqslant 1$, $\left|\langle 1|f_n\rangle\right|^2=\gamma_n\kappa_n$ and

$$\kappa_n=\frac{1}{\lambda_n-\lambda_0}\prod_{p\neq n}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_n}\right).$$

Note that the infinite product is absolutely convergent, since the sequence $(\gamma_n(u))_{n\geqslant 1}$ is in ℓ^1_+ (cf. (3.14)). Furthermore, since

$$1-\frac{\gamma_p}{\lambda_p-\lambda_n}=\frac{\lambda_{p-1}+1-\lambda_n}{\lambda_p-\lambda_n}>0 \quad \text{for all } p\neq n,$$

it follows that $\kappa_n>0$ for any $n\geqslant 1$. Hence, the formula [16, (4.1)] of the Birkhoff coordinates $\zeta_n(u)$, $n\geqslant 1$, defined for $u\in L^2_{r,0}$,

$$\zeta_n(u)=\frac{1}{\sqrt{\kappa_n(u)}}\langle 1\,|\,f_n(\cdot,u)\rangle, \tag{3.17}$$

extends to $H^{-s}_{r,0}$. By (3.9), one has (cf. also [16, (2.13)])

$$\lambda_n\langle 1\,|\,f_n\rangle=-\langle u\,|\,f_n\rangle=-\langle \Pi u\,|\,f_n\rangle.$$

Since, by Lemma 3.10, $(\langle \Pi u|f_n\rangle)_{n\geqslant 0}\in h^{-s}(\mathbb{N}_0)$ and, by (3.16),

$$n-\tfrac{1}{2}-\eta_s(\|u\|_{-s})\leqslant\lambda_n(u)\leqslant n \quad \text{for all } n\geqslant 0,$$

one concludes that

$$(\langle 1\,|\,f_n\rangle)_{n\geqslant 1}\in h^{1-s}_+ \quad \text{and} \quad \kappa_n^{-1/2}=\sqrt{n}+o(1),$$

and hence $(\zeta_n(u))_{n\geqslant 1}\in h^{1/2-s}_+$. In summary, we have proved that, for any $0<s<\frac{1}{2}$, the Birkhoff map $\Phi\colon L^2_{r,0}\to h^{1/2}_+$ of Theorem 2.1 extends to a map

$$\begin{aligned} H^{-s}_{r,0} &\longrightarrow h^{1/2-s}_+,\\ u &\longmapsto (\zeta_n(u))_{n\geqslant 1},\end{aligned}$$

which we again denote by Φ. Going through the arguments of the proof, one verifies that Φ maps bounded subsets of $H^{-s}_{r,0}$ into bounded subsets of $h^{1/2-s}_+$. □

To show Proposition 3.1 (ii), we first need to prove some additional auxilary results. By (3.11), the generating function is defined as

$$\begin{aligned} \mathcal{H}_\lambda\colon H^{-s}_{r,0} &\longrightarrow \mathbb{C},\\ u &\longmapsto \langle (L_u+\lambda)^{-1}1\,|\,1\rangle.\end{aligned}$$

For any given $u\in H^{-s}_{r,0}$, $\mathcal{H}_\lambda(u)$ is a meromorphic function in $\lambda\in\mathbb{C}$ with possible poles at the eigenvalues of L_u, and satisfies (cf. (3.12))

$$\mathcal{H}_\lambda(u)=\frac{1}{\lambda_0+\lambda}\prod_{n=1}^{\infty}\Big(1-\frac{\gamma_n}{\lambda_n+\lambda}\Big). \tag{3.18}$$

Lemma 3.11. *For any* $0\leqslant s<\frac{1}{2}$, *the following statements hold*:

(i) *For any* $\lambda\in\mathbb{C}\setminus\mathbb{R}$, $\mathcal{H}_\lambda: H^{-s}_{r,0}\to\mathbb{C}$ *is sequentially weakly continuous.*

(ii) $(\sqrt{\gamma_n})_{n\geqslant 1}: H^{-s}_{r,0}\to h^{1/2-s}_+$ *is sequentially weakly continuous. In particular, for any* $n\geqslant 0$, $\lambda_n: H^{-s}_{r,0}\to\mathbb{R}$ *is sequentially weakly continuous.*

Proof. (i) Let $(u^{(k)})_{k\geqslant 1}$ be a sequence in $H^{-s}_{r,0}$ with $u^{(k)}\rightharpoonup u$ weakly in $H^{-s}_{r,0}$ as $k\to\infty$. By the definition of $\zeta_n(u)$ (cf. (3.13)–(3.17)), one has $|\zeta_n(u)|^2=\gamma_n(u)$. Since, by Proposition 3.1 (i), Φ maps bounded subsets of $H^{-s}_{r,0}$ to bounded subsets of $h^{1/2-s}_+$, there exists $M>0$ such that, for any $k\geqslant 1$, one has $\|u\|, \|u^{(k)}\|\leqslant M$,

$$\sum_{n=1}^{\infty} n^{1-2s}\gamma_n(u) \quad \text{and} \quad \sum_{n=1}^{\infty} n^{1-2s}\gamma_n(u^{(k)})\leqslant M.$$

By passing to a subsequence, if needed, we may assume that

$$(\gamma_n(u^{(k)})^{1/2})_{n\geqslant 1}\rightharpoonup(\rho_n^{1/2})_{n\geqslant 1} \tag{3.19}$$

weakly in $h^{1/2-s}(\mathbb{N},\mathbb{R})$, where $\rho_n\geqslant 0$ for any $n\geqslant 1$. It then follows that

$$(\gamma_n(u^{(k)}))_{n\geqslant 1}\to(\rho_n)_{n\geqslant 1}$$

strongly in $\ell^1(\mathbb{N},\mathbb{R})$. Define

$$\nu_n := n-\sum_{p=n+1}^{\infty}\rho_p \quad \text{for all } n\geqslant 0.$$

Then, for any $n\geqslant 1$, $\nu_n=\nu_{n-1}+1+\rho_n$, and $\lambda_n(u^{(k)})\to\nu_n$ uniformly in $n\geqslant 0$. Since

$$L_{u^{(k)}}\geqslant\lambda_0(u^{(k)}),$$

we infer that there exists $c>|-\nu_0+1|$ such that, for any $k\geqslant 1$ and $\lambda\geqslant c$,

$$B_{u^{(k)};1-s}-(1+\eta_s(\|u^{(k)}\|_{-s}))+\lambda: H^{1-s}_+\longrightarrow H^{-s}_+$$

is a linear isomorphism whose inverse is bounded uniformly in k. Since $L_{u^{(k)}}$ is the restriction of $B_{u^{(k)};1-s}-(1+\eta_s(\|u\|_{-s}))$ to $\operatorname{dom}(L_{u^{(k)}})$, one then has that

$$w_\lambda^{(k)} := (L_{u^{(k)}}+\lambda)^{-1}[1] \quad \text{for all } k\geqslant 1,$$

is a well-defined, bounded sequence in H^{1-s}_+. Let us choose an arbitrary countable subset $\Lambda\subset[c,\infty)$ with one cluster point. By a diagonal procedure, we extract a subsequence of

$(w_\lambda^{(k)})_{k\geqslant 1}$, again denoted by $(w_\lambda^{(k)})_{k\geqslant 1}$, such that, for every $\lambda\in\Lambda$, the sequence $(w_\lambda^{(k)})$ converges weakly in H_+^{1-s} to some element $v_\lambda\in H_+^{1-s}$. By Rellich's theorem,

$$(L_{u^{(k)}}+\lambda)w_\lambda^{(k)} \rightharpoonup (L_u+\lambda)v_\lambda$$

weakly in H_+^{-s} as $k\to\infty$. Since, by definition, $(L_{u^{(k)}}+\lambda)w_\lambda^{(k)}=1$ for any $k\geqslant 1$, it follows that, for any $\lambda\in\Lambda$, $(L_u+\lambda)v_\lambda=1$, and thus, by the definition of the generating function,

$$\mathcal{H}_\lambda(u^{(k)}) = \langle w_\lambda^{(k)} \mid 1\rangle \to \langle v_\lambda \mid 1\rangle = \mathcal{H}_\lambda(u) \quad \text{for all } \lambda\in\Lambda.$$

Since $\mathcal{H}_\lambda(u^{(k)})$ and $\mathcal{H}_\lambda(u)$ are meromorphic functions whose poles are on the real axis, it follows that the convergence holds for any $\lambda\in\mathbb{C}\setminus\mathbb{R}$. This proves item (i).

(ii) We apply item (i) (and its proof) as follows. As mentioned above, $\lambda_n(u^{(k)})\to\rho_n$, uniformly in $n\geqslant 0$. By the proof of item (i), one has, for any $c\leqslant\lambda<\infty$,

$$\mathcal{H}_\lambda(u^{(k)}) \to \frac{1}{\nu_0+\lambda}\prod_{n=1}^{\infty}\left(1-\frac{\rho_n}{\nu_n+\lambda}\right),$$

and hence, for any $\lambda\in\Lambda$,

$$\mathcal{H}_\lambda(u) = \frac{1}{\nu_0+\lambda}\prod_{n=1}^{\infty}\left(1-\frac{\rho_n}{\nu_n+\lambda}\right).$$

Since $\mathcal{H}_\lambda(u)$ and the infinite product on the right-hand side of the latter identity are both meromorphic functions in λ, the functions are equal. In particular, they have the same zeroes and the same poles. Since the sequences $(\lambda_n(u))_{n\geqslant 0}$ and $(\nu_n)_{n\geqslant 0}$ are both listed in increasing order, it then follows from (3.18) that $\lambda_n(u)=\nu_n$ for any $n\geqslant 0$, implying that, for any $n\geqslant 1$,

$$\gamma_n(u) = \lambda_n(u)-\lambda_{n-1}(u)-1 = \nu_n-\nu_{n-1}-1 = \rho_n.$$

By (3.19), we then conclude that

$$(\gamma_n(u^{(k)})^{1/2})_{n\geqslant 1} \rightharpoonup (\gamma_n(u)^{1/2})_{n\geqslant 1}$$

weakly in $h^{1/2-s}(\mathbb{N},\mathbb{R})$. □

Corollary 3.12. *For any* $0 \leqslant s < \frac{1}{2}$ *and* $n \geqslant 1$, *the functional* $\kappa_n : H^{-s}_{r,0} \to \mathbb{R}$, *introduced in* (3.13), *is sequentially weakly continuous.*

Proof. Let $(u^{(k)})_{k \geqslant 1}$ be a sequence in $H^{-s}_{r,0}$ with $u^{(k)} \rightharpoonup u$ weakly in $H^{-s}_{r,0}$ as $k \to \infty$. By (3.15), one has, for any $p < n$,

$$\lambda_p(u^{(k)}) - \lambda_n(u^{(k)}) = p - n - \sum_{j=p+1}^{n} \gamma_j(u^{(k)}),$$

whereas, for $p > n$,

$$\lambda_p(u^{(k)}) - \lambda_n(u^{(k)}) = p - n + \sum_{j=n+1}^{p} \gamma_j(u^{(k)}).$$

By Lemma 3.11, one then concludes that

$$\lim_{k \to \infty} (\lambda_p(u^{(k)}) - \lambda_n(u^{(k)})) - (\lambda_p(u) - \lambda_n(u)) = 0$$

uniformly in $p, n \geqslant 0$. By the product formula (3.13) for κ_n, it then follows that, for any $n \geqslant 1$, one has

$$\lim_{k \to \infty} \kappa_n(u^{(k)}) = \kappa_n(u). \qquad \square$$

Furthermore, we need to prove the following lemma concerning the eigenfunctions $f_n(\cdot, u)$, $n \geqslant 0$, of L_u.

Lemma 3.13. *Given* $0 \leqslant s < \frac{1}{2}$, $M > 0$, *and* $n \geqslant 0$, *there exists a constant* $C_{s,M,n} \geqslant 1$ *such that, for any* $u \in H^{-s}_{r,0}$ *with* $\|u\|_{-s} \leqslant M$ *and any* $n \geqslant 0$,

$$\|f_n(\cdot, u))\|_{1-s} \leqslant C_{s,M,n}. \tag{3.20}$$

Proof. By the normalization of f_n, $\|f_n\| = 1$. Since f_n is an eigenfunction, corresponding to the eigenvalue λ_n, one has

$$-i\partial_x f_n = L_u f_n + T_u f_n = \lambda_n f_n + T_u f_n,$$

implying that

$$\|\partial_x f_n\|_{-s} \leqslant |\lambda_n| + \|T_u f_n\|_{-s}. \tag{3.21}$$

Note that, by the estimates (3.16),

$$|\lambda_n| \leqslant \max\{n, |\lambda_0|\} \leqslant n + |\lambda_0| \quad \text{for all } n \geqslant 0, \tag{3.22}$$

and

$$|\lambda_0| \leqslant 1 + \eta_s(\|u\|_{-s}),$$

where $\eta_s(\|u\|_{-s})$ is given by (3.3). Furthermore, since $\sigma=\frac{1}{2}\left(\frac{1}{2}+s\right)$ (cf. (3.1)), one has

$$1-s>1-\sigma>\tfrac{1}{2},$$

implying that $H_+^{1-\sigma}$ acts on H_+^t for every t in the open interval $(-(1-\sigma),1-\sigma)$. Hence,

$$\|T_u f_n\|_{-s}\leqslant C_s\|u\|_{-s}\|f_n\|_{1-\sigma}. \tag{3.23}$$

Using interpolation and Young's inequality (cf. (3.4), (3.5)), (3.23) yields an estimate which, together with (3.21) and (3.22), leads to the claimed estimate (3.20). □

With these preparations done, we can now prove Proposition 3.1 (ii).

Proof of Proposition 3.1 (ii). First, we prove that, for any $0\leqslant s<\frac{1}{2}$, $\Phi\colon H_{r,0}^{-s}\to h_+^{1/2-s}$ is sequentially weakly continuous: assume that $(u^{(k)})_{k\geqslant 1}$ is a sequence in $H_{r,0}^{-s}$ with $u^{(k)}\rightharpoonup u$ weakly in $H_{r,0}^{-s}$ as $k\to\infty$. Let $\zeta^{(k)}:=\Phi(u^{(k)})$ and $\zeta:=\Phi(u)$. Since $(u^{(k)})_{k\geqslant 1}$ is bounded in $H_{r,0}^{-s}$, and Φ maps bounded subsets of $H_{r,0}^{-s}$ to bounded subsets of $h_+^{1/2-s}$, the sequence $(\zeta^{(k)})_{k\geqslant 1}$ is bounded in $h_+^{1/2-s}$. To show that $\zeta^{(k)}\rightharpoonup\zeta$ weakly in $h_+^{1/2}$, it then suffices to prove that, for any $n\geqslant 1$, $\lim_{k\to\infty}\zeta_n^{(k)}=\zeta_n$. By the definition of the Birkhoff coordinates (3.17),

$$\zeta_n^{(k)}=\frac{\langle 1\,|\,f_n^{(k)}\rangle}{(\kappa_n^{(k)})^{1/2}},$$

where $\kappa_n^{(k)}:=\kappa_n(u^{(k)})$ and $f_n^{(k)}:=f_n(\cdot\,,u^{(k)})$. By Corollary 3.12,

$$\lim_{k\to\infty}\kappa_n^{(k)}=\kappa_n$$

and, by Lemma 3.13, saying that, for any $n\geqslant 0$, $\|f_n\|_{1-s}$ is uniformly bounded on bounded subsets of $H_{r,0}^{-s}$,

$$\lim_{k\to\infty}\langle 1\,|\,f_n^{(k)}\rangle=\langle 1\,|\,f_n\rangle,$$

where $\kappa_n:=\kappa_n(u)$ and $f_n:=f_n(\cdot\,,u)$. This implies that

$$\lim_{k\to\infty}\zeta_n^{(k)}=\zeta_n\quad\text{for any } n\geqslant 1.$$

It remains to show that, for any $0<s<\frac{1}{2}$, $\Phi\colon H_{r,0}^{-s}\to h_+^{1/2-s}$ is one-to-one. In the case where $u\in L_{r,0}^2$, it was verified in the proof of [16, Proposition 4.3] that the Fourier coefficients $\hat{u}(k)$, $k\geqslant 1$, of u can be explicitly expressed in terms of the components $\zeta_n(u)$ of the sequence $\zeta(u)=\Phi(u)$. These formulas continue to hold for $u\in H_{r,0}^{-s}$. This completes the proof of Proposition 3.1 (ii). □

4. Extension of Φ. Part 2

In this section we prove the second part of Theorem 2.3, which we again state as a separate proposition.

PROPOSITION 4.1. (Extension of Φ. Part 2) *For any* $0<s<\frac{1}{2}$, *the map*

$$\Phi: H^{-s}_{r,0} \longrightarrow h^{1/2-s}_+$$

has the following additional properties:

(i) *The inverse image of* Φ *of any bounded subset of* $h^{1/2-s}_+$ *is a bounded subset in* $H^{-s}_{r,0}$.

(ii) Φ *is onto, and the inverse map* $\Phi^{-1}: h^{1/2-s}_+ \to H^{-s}_{r,0}$ *is sequentially weakly continuous.*

(iii) *For any* $0<s<\frac{1}{2}$, *the Birkhoff map and its inverse,*

$$\Phi: H^{-s}_{r,0} \longrightarrow h^{1/2-s}_+ \quad \textit{and} \quad \Phi^{-1}: h^{1/2-s}_+ \longrightarrow H^{-s}_{r,0},$$

are continuous.

Remark 4.2. As mentioned in Remark 2.4, the map $\Phi: L^2_{r,0} \to h^{1/2}_+$ and its inverse $\Phi^{-1}: h^{1/2}_+ \to L^2_{r,0}$ are sequentially weakly continuous.

Proof of Proposition 4.1 (i). Let $0<s<\frac{1}{2}$ and $u \in H^{-s}_{r,0}$. Recall that, by Corollary 3.6, L_u is a self-adjoint operator with domain $\operatorname{dom}(L_u) \subset H_+$, has discrete spectrum and is bounded from below. Thus, $L_u - \lambda_0(u) + 1 \geqslant 1$, where $\lambda_0(u)$ denotes the smallest eigenvalue of L_u. By the considerations in §3 (cf. Lemma 3.8), L_u extends to a bounded operator $L_u: H^{1/2}_+ \to H^{-1/2}_+$ and satisfies

$$\langle L_u f \mid f\rangle = \langle Df \mid f\rangle - \langle u \mid f\bar{f}\rangle \quad \text{for all } f \in H^{1/2}_+.$$

By Lemma 3.2 (i), one has

$$|\langle u \mid f\bar{f}\rangle| \leqslant C^2_{1,s} \|u\|_{-s} \|f\|^2_{1/2} \quad \text{for all } f \in H^{1/2}_+,$$

and hence

$$\|f\|^2 \leqslant \langle (L_u - \lambda_0(u) + 1) f \mid f\rangle \leqslant \langle Df \mid f\rangle + C^2_{1,s} \|u\|_{-s} \|f\|^2_{1/2} + (-\lambda_0(u) + 1)\|f\|^2,$$

yielding the estimate

$$\|f\|^2 \leqslant \langle (L_u - \lambda_0(u) + 1) f \mid f\rangle \leqslant M_u \|f\|^2_{1/2},$$

where

$$M_u := C_{1,s}^2 \|u\|_{-s} + (2-\lambda_0(u)). \tag{4.1}$$

To shorten notation, we will, for the remainder of the proof, no longer indicate the dependence of spectral quantities such as λ_n or γ_n on u, whenever appropriate. The square root of the operator $L_u - \lambda_0 + 1$,

$$R_u := (L_u - \lambda_0 + 1)^{1/2} \colon H_+^{1/2} \longrightarrow H_+,$$

can then be defined in terms of the basis $f_n \equiv f_n(\cdot, u)$, $n \geqslant 0$, of eigenfunctions of L_u in a standard way as follows: by Lemma 3.10, any $f \in H_+^{1/2}$ has an expansion of the form

$$f = \sum_{n=0}^{\infty} \langle f \mid f_n \rangle f_n,$$

where $(\langle f | f_n \rangle)_{n \geqslant 0}$ is a sequence in $h^{1/2}(\mathbb{N}_0)$. $R_u f$ is then defined as

$$R_u f := \sum_{n=0}^{\infty} (\lambda_n - \lambda_0 + 1)^{1/2} \langle f \mid f_n \rangle f_n.$$

Since $(\lambda_n - \lambda_0 + 1)^{1/2} \sim \sqrt{n}$ (cf. (3.16)), one has

$$((\lambda_n - \lambda_0 + 1)^{1/2} \langle f \mid f_n \rangle)_{n \geqslant 0} \in \ell^2(\mathbb{N}_0),$$

implying that $R_u f \in H_+$ (cf. Lemma 3.10). Note that, for any $f \in H_+^{1/2}$,

$$\|f\|^2 \leqslant \langle R_u f \mid R_u f \rangle = \langle R_u^2 f \mid f \rangle \leqslant M_u \|f\|_{1/2}^2,$$

and that R_u is a positive, self-adjoint operator, when viewed as an operator with domain $H_+^{1/2}$, acting on H_+. By complex interpolation (cf. e.g. [37, §1.4]), one then concludes that, for any $0 \leqslant \theta \leqslant 1$, $R_u^\theta \colon H_+^{\theta/2} \to H_+$ and

$$\|R_u^\theta f\|^2 \leqslant M_u^\theta \|f\|_{\theta/2}^2 \quad \text{for all } f \in H_+^{\theta/2}.$$

Since, by duality, $R_u^\theta \colon H_+ \to H_+^{-\theta/2}$ and

$$\|R_u^\theta g\|_{-\theta/2}^2 \leqslant M_u^\theta \|g\|^2 \quad \text{for all } g \in H_+,$$

one infers, using that $R_u^\theta \colon H_+ \to H_+^{-\theta/2}$ is boundedly invertible, that, for any $f \in H_+^{-\theta/2}$,

$$\|f\|_{-\theta/2}^2 \leqslant M_u^\theta \|R_u^{-\theta} f\|^2,$$

where $R_u^{-\theta}:=(R_u^\theta)^{-1}$. Applying the latter inequality to $f=\Pi u$ and $\theta=2s$, and using that

$$\Pi u = \sum_{n=1}^{\infty} \langle \Pi u \,|\, f_n\rangle f_n \quad \text{and} \quad \langle \Pi u \,|\, f_n\rangle = -\lambda_n \langle 1 \,|\, f_n\rangle,$$

one sees that

$$\tfrac{1}{2}\|u\|_{-s}^2 = \|\Pi u\|_{-s}^2 \leqslant M_u^{2s}\Sigma, \tag{4.2}$$

where

$$\Sigma := \sum_{n=1}^{\infty} \lambda_n^2 (\lambda_n - \lambda_0 + 1)^{-2s} \; |\langle 1 \,|\, f_n\rangle|^2.$$

We would like to deduce from (4.2) an estimate of $\|u\|_{-s}$ in terms of the γ_n's. Let us first consider M_u^{2s}. By (4.1), one has

$$M_u^{2s} \leqslant 2^{2s} \max\{(C_{1,s}^2 \|u\|_{-s})^{2s}, (2-\lambda_0(u))^{2s}\},$$

yielding

$$M_u^{2s} \leqslant (\|u\|_{-s}^2)^s (2C_{1,s}^2)^{2s} + (2(2-\lambda_0(u)))^{2s}. \tag{4.3}$$

Applying Young's inequality with $1/p=s$ and $1/q=1-s$, one obtains

$$(\|u\|_{-s}^2)^s (2C_{1,s}^2)^{2s}\Sigma \leqslant \tfrac{1}{4}\|u\|_{-s}^2 + ((4C_{1,s}^2)^{2s}\Sigma)^{1/(1-s)}, \tag{4.4}$$

which, when combined with (4.2) and (4.3), leads to

$$\tfrac{1}{4}\|u\|_{-s}^2 \leqslant ((4C_{1,s}^2)^{2s}\Sigma)^{1/(1-s)} + (2(2-\lambda_0(u)))^{2s}\Sigma.$$

The latter estimate is of the form

$$\|u\|_{-s}^2 \leqslant C_{3,s}\Sigma^{1/(1-s)} + C_{4,s}(2-\lambda_0(u))^{2s}\Sigma, \tag{4.5}$$

where $C_{3,s}, C_{4,s}>0$ are constants, only depending on s. Next, let us turn to

$$\Sigma = \sum_{n=1}^{\infty} \lambda_n^2 (\lambda_n - \lambda_0 + 1)^{-2s} \; |\langle 1 \,|\, f_n\rangle|^2.$$

Since, for any $n \geqslant 1$,

$$\lambda_n = n - \sum_{k=n+1}^{\infty} \gamma_k, \quad \big|\langle 1 \,|\, f_n\rangle\big|^2 = \gamma_n \kappa_n,$$

and, by (3.13),

$$\kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right),$$

the series Σ can be expressed in terms of the γ_n's. To obtain a bound for Σ, it remains to estimate the κ_n's. Note that, for any $n\geqslant 1$,

$$\prod_{p\neq n}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_n}\right)\leqslant\prod_{p<n}\left(1+\frac{\gamma_p}{\lambda_n-\lambda_p}\right)\leqslant e^{\sum_{p=1}^{n}\gamma_p}\leqslant e^{-\lambda_0}.$$

Since

$$(\lambda_n-\lambda_0)^{-1}=\left(n+\sum_{k=1}^{n}\gamma_k\right)^{-1}\leqslant n^{-1},$$

it then follows that

$$0<\kappa_n\leqslant\frac{e^{-\lambda_0}}{n}\quad\text{for all } n\geqslant 1.$$

Combining the estimates above, we get

$$\Sigma\leqslant e^{-\lambda_0}\sum_{n=1}^{\infty}\lambda_n^2 n^{-2s-1}\gamma_n.$$

By splitting the sum Σ into two parts, $\Sigma=\sum_{n<-\lambda_0(u)}+\sum_{n\geqslant-\lambda_0(u)}$, and taking into account that $0\leqslant\lambda_n\leqslant n$ for any $n\geqslant-\lambda_0$ and $|\lambda_n|\leqslant-\lambda_0$ for any $1\leqslant n<-\lambda_0$, one has

$$\Sigma\leqslant(1-\lambda_0)^2 e^{-\lambda_0}\sum_{n=1}^{\infty}n^{1-2s}\gamma_n.$$

Together with the estimate (4.5), this shows that the inverse image by Φ of any bounded subset of sequences in $h^{1/2-s}$ is bounded in $H^{-s}_{r,0}$. □

Proof of Proposition 4.1 (ii). First, we prove that, for any $0<s<\frac{1}{2}$, $\Phi\colon H^{-s}_{r,0}\to h^{1/2-s}_+$ is onto. Given $z=(z_n)_{n\geqslant 1}$ in $h^{1/2-s}_+$, consider the sequence $\zeta^{(k)}=(\zeta^{(k)}_n)_{n\geqslant 1}$ defined, for any $k\geqslant 1$, by

$$\zeta^{(k)}_n=\begin{cases} z_n, & \text{for all } 1\leqslant n\leqslant k,\\ 0, & \text{for all } n>k.\end{cases}$$

Clearly, $\zeta^{(k)}\to z$ strongly in $h^{1/2-s}$. Since $\zeta^{(k)}\in h^{1/2}_+$ for any $k\geqslant 1$, Theorem 2.1 implies that there exists a unique element $u^{(k)}\in L^2_{r,0}$ with $\Phi(u^{(k)})=\zeta^{(k)}$. By Proposition 4.1 (i),

$$\sup_{k\geqslant 1}\|u^{(k)}\|_{-s}<\infty.$$

Choose a weakly convergent subsequence $(u^{(k_j)})_{j\geqslant 1}$ of $(u^{(k)})_{k\geqslant 1}$, and denote its weak limit in $H^{-s}_{r,0}$ by u. Since, by Proposition 3.1, $\Phi\colon H^{-s}_{r,0}\to h^{1/2-s}_+$ is sequentially weakly continuous, $\Phi(u^{(k_j)})\rightharpoonup\Phi(u)$ weakly in $h^{1/2-s}_+$. On the other hand, $\Phi(u^{(k_j)})=\zeta^{(k_j)}\to z$ strongly in $h^{1/2-s}$, implying that $\Phi(u)=z$. This shows that Φ is onto.

It remains to prove that, for any $0 \leqslant s < \frac{1}{2}$, Φ^{-1} is sequentially weakly continuous. Assume that $(\zeta^{(k)})_{k\geqslant 1}$ is a sequence in $h^{1/2-s}$, weakly converging to $\zeta \in h^{1/2-s}$. Let $u^{(k)} := \Phi^{-1}(\zeta^{(k)})$. By Proposition 4.1 (i) (in the case $0 < s < \frac{1}{2}$) and Remark 2.2 (ii) (in the case $s=0$), $(u^{(k)})_{k\geqslant 1}$ is a bounded sequence in $H^{-s}_{r,0}$, and thus admits a weakly convergent subsequence $(u^{k_j})_{j\geqslant 1}$. Denote its limit in $H^{-s}_{r,0}$ by u. Since, by Proposition 3.1, Φ is sequentially weakly continuous, $\Phi(u^{k_j}) \rightharpoonup \Phi(u)$ weakly in $h^{1/2-s}$. On the other hand, by assumption, $\Phi(u^{k_j}) = \zeta^{(k_j)} \rightharpoonup \zeta$, and hence $u = \Phi^{-1}(\zeta)$ and u is independent of the chosen subsequence $(u^{k_j})_{j\geqslant 1}$. This shows that $\Phi^{-1}(\zeta^{(k)}) \rightharpoonup \Phi^{-1}(\zeta)$ weakly in $H^{-s}_{r,0}$. □

Proof of Proposition 4.1 (iii). By Proposition 3.1, $\Phi\colon H^{-s}_{r,0} \to h^{1/2-s}_{+}$ is sequentially weakly continuous for any $0 \leqslant s < \frac{1}{2}$. To show that this map is continuous, it then suffices to prove that the image $\Phi(A)$ of any relatively compact subset A of $H^{-s}_{r,0}$ is relatively compact in $h^{1/2-s}_{+}$. For any given $\varepsilon > 0$, choose $N \equiv N_\varepsilon \geqslant 1$ and $R \equiv R_\varepsilon > 0$ as in Lemma 4.3, stated below. Decompose $u \in A$ as $u = u_N + u_\perp$, where

$$u_N := \sum_{0<|n|\leqslant N_\varepsilon} \hat{u}(n) e^{inx} \quad \text{and} \quad u_\perp := \sum_{|n|>N_\varepsilon} \hat{u}(n) e^{inx}.$$

By Lemma 4.3, $\|u_N\| < R_\varepsilon$ and $\|u_\perp\|_{-s} < \varepsilon$. By Lemma 3.10, applied with $\theta = -s$, one has

$$K_{u;-s}(\Pi u) = K_{u;-s}(\Pi u_N) + K_{u;-s}(\Pi u_\perp) \in h^{-s}(\mathbb{N}_0),$$

where $K_{u;-s}(\Pi u_N) = K_{u;0}(\Pi u_N)$, since $\Pi u_N \in H_+$. Lemma 3.10 then implies that there exists $C_A > 0$, independent of $u \in A$, such that

$$\|K_{u;0}(\Pi u_N)\| \leqslant C_A R_\varepsilon \quad \text{and} \quad \|K_{u;-s}(\Pi u_\perp)\|_{-s} \leqslant C_A \varepsilon.$$

As $\varepsilon > 0$ can be chosen arbitrarily small, it then follows by Lemma 4.3 that $K_{u;-s}(\Pi(A))$ is relatively compact in $h^{-s}(\mathbb{N}_0)$. Since, by definition,

$$(K_{u;-s}(\Pi u))_n = \langle \Pi u \,|\, f_n(\cdot, u) \rangle \quad \text{for all } n \geqslant 0,$$

and since, by (3.9),

$$\zeta_n(u) \simeq -\frac{1}{\sqrt{n}} \langle \Pi u \,|\, f_n(\cdot, u) \rangle \quad \text{as} \quad n \to \infty$$

uniformly with respect to $u \in A$, it follows that $\Phi(A)$ is relatively compact in $h^{1/2-s}_{+}$. Now, let us turn to Φ^{-1}. By Proposition 4.1 (ii), $\Phi^{-1}\colon h^{1/2-s}_{+} \to H^{-s}_{r,0}$ is sequentially weakly continuous. To show that this map is continuous, it then suffices to prove that the image $\Phi^{-1}(B)$ of any relatively compact subset B of $h^{1/2-s}_{+}$ is relatively compact in $H^{-s}_{r,0}$. By the same arguments as above, one sees that $\Phi^{-1}\colon h^{1/2-s}_{+} \to H^{-s}_{r,0}$ is also continuous. □

It remains to state Lemma 4.3, used in the proof of Proposition 4.1 (iii). It concerns the well-known characterization of relatively compact subsets of $H^{-s}_{r,0}$ in terms of the Fourier expansion

$$u(x)=\sum_{n\neq 0}\hat{u}(n)e^{inx}$$

of an element u in $H^{-s}_{r,0}$.

LEMMA 4.3. *Let $0<s<\frac{1}{2}$ and $A\subset H^{-s}_{+}$. Then, A is relatively compact in H^{-s}_{+} if and only if, for any $\varepsilon>0$, there exist $N_\varepsilon\geqslant 1$ and $R_\varepsilon>0$ such that, for any $f\in A$, the sequence $\xi_n:=\hat{f}(n)$, $n\geqslant 0$, satisfies*

$$\bigg(\sum_{n>N_\varepsilon}|n|^{-2s}|\xi_n|^2\bigg)^{1/2}<\varepsilon \quad \textit{and} \quad \bigg(\sum_{0<n\leqslant N_\varepsilon}|\xi_n|^2\bigg)^{1/2}<R_\varepsilon.$$

The two latter conditions on $(\xi_n)_{n\geqslant 0}$ characterize relatively compact subsets of $h^{-s}(\mathbb{N}_0)$.

Proof of Theorem 2.3. The claimed statements follow from Propositions 3.1 and 4.1. In particular, it follows from Proposition 4.1 (i) that

$$F_s(R)=\sup_{\|\xi\|_{1/2-s}\leqslant R}\|\Phi^{-1}(\xi)\|_{-s}<\infty \quad \text{for all } R\geqslant 0,$$

and hence F_s takes values in $\mathbb{R}_{\geqslant 0}$ as stated in Theorem 2.3 (iii). □

5. Solution maps $\mathcal{S}_0$, $\mathcal{S}_B$, $\mathcal{S}_c$, and $\mathcal{S}_{c,B}$

In this section we provide results related to the solution map of (1.1), which will be used to prove Theorem 1.3 in the subsequent section.

Solution map $\mathcal{S}_B$ and its extension. First, we study the map $\mathcal{S}_B$, defined in §2 on $h^{1/2}_{+}$. Recall that, by (2.7) and (2.8), the nth frequency of (1.1) is a real-valued map defined on ℓ^2_+ by

$$\omega_n(\zeta):=n^2-2\sum_{k=1}^{n}k|\zeta_k|^2-2n\sum_{k=n+1}^{\infty}|\zeta_k|^2.$$

For any $0<s\leqslant\frac{1}{2}$, the map $\mathcal{S}_B$ naturally extends to $h^{1/2-s}_{+}$, mapping the initial data $\zeta(0)\in h^{1/2-s}_{+}$ to the curve

$$\begin{aligned}\mathcal{S}_B(\cdot,\zeta(0))\colon \mathbb{R}&\longrightarrow h^{1/2-s}_{+},\\ t&\longmapsto \mathcal{S}_B(t,\zeta(0)):=(\zeta_n(0)e^{it\omega_n(\zeta)})_{n\geqslant 1}.\end{aligned} \tag{5.1}$$

We first record the following properties of the frequencies.

LEMMA 5.1. (i) *For any* $n\geqslant 1$, $\omega_n\colon \ell^2_+\to\mathbb{R}$ *is continuous and*

$$|\omega_n(\zeta)-n^2|\leqslant 2n\|\zeta\|_0^2 \quad \textit{for all } \zeta\in\ell^2_+,$$
$$|\omega_n(\zeta)-n^2|\leqslant 2\|\zeta\|_{1/2}^2 \quad \textit{for all } \zeta\in h^{1/2}_+.$$

(ii) *For any* $0\leqslant s<\frac{1}{2}$, $\omega_n\colon h^{1/2-s}_+\to\mathbb{R}$ *is sequentially weakly continuous.*

Proof. Item (i) follows in a straightforward way from the formula (2.7) of ω_n. Since, for any $0\leqslant s<\frac{1}{2}$, $h^{1/2-s}_+$ compactly embeds into ℓ^2_+, item (ii) follows from (i). □

From Lemma 5.1 one infers the following properties of $\mathcal{S}_B$. We leave the easy proof to the reader.

PROPOSITION 5.2. *For any* $0\leqslant s\leqslant\frac{1}{2}$, *the following holds*:
(i) *For any initial data* $\zeta(0)\in h^{1/2-s}_+$,

$$\begin{aligned}\mathbb{R}&\longrightarrow h^{1/2-s}_+,\\ t&\longmapsto \mathcal{S}_B(t,\zeta(0)),\end{aligned}$$

is continuous.

(ii) *For any* $T>0$,

$$\begin{aligned}\mathcal{S}_B\colon h^{1/2-s}_+&\longrightarrow C([-T,T],h^{1/2-s}_+),\\ \zeta(0)&\longmapsto[t\mapsto\mathcal{S}_B(t,\zeta(0))],\end{aligned}$$

is continuous and, for any $t\in\mathbb{R}$,

$$\begin{aligned}\mathcal{S}_B^t\colon h^{1/2-s}_+&\longrightarrow h^{1/2-s}_+,\\ \zeta(0)&\longmapsto\mathcal{S}_B(t,\zeta(0)),\end{aligned}$$

is a homeomorphism.

Solution map $\mathcal{S}_0$ *and its extension.* Recall that in §2 we introduced the solution map $\mathcal{S}_0$ of (1.1) on the subspace space $L^2_{r,0}$ of L^2_r, consisting of elements in L^2_r with average zero, in terms of the Birkhoff map Φ:

$$\mathcal{S}_0=\Phi^{-1}\mathcal{S}_B\Phi\colon L^2_{r,0}\longrightarrow C(\mathbb{R},L^2_{r,0}). \tag{5.2}$$

Theorem 2.3 will now be applied to prove the following result about the extension of $\mathcal{S}_0$ to the Sobolev space $H^{-s}_{r,0}$, with $0<s<\frac{1}{2}$, consisting of elements in H^{-s}_r with average zero. It will be used in §6 to prove Theorem 1.3.

PROPOSITION 5.3. *For any* $0\leqslant s<\frac{1}{2}$, *the following holds*:
(i) *The Benjamin–Ono equation is globally* C^0*-well-posed on* $H^{-s}_{r,0}$.
(ii) *There exists an increasing function* $F_s\colon\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ *such that*

$$\|u\|_{-s}\leqslant F_s(\|\Phi(u)\|_{1/2-s})\quad \textit{for all } u\in H^{-s}_{r,0}.$$

In particular, for any initial data $u(0)\in H^{-s}_{r,0}$,

$$\sup_{t\in\mathbb{R}}\|\mathcal{S}^t_0(u(0))\|_{-s}\leqslant F_s(\|\Phi(u(0))\|_{1/2-s}). \tag{5.3}$$

Remark 5.4. (i) By the trace formula (2.2), for any $u(0)\in L^2_{r,0}$, estimate (5.3) can be improved as follows:

$$\|u(t)\|=\sqrt{2}\|\Phi(u(0))\|_{1/2}=\|u(0)\|\quad\text{for all } t\in\mathbb{R}.$$

(ii) We refer to the comments of Theorem 1.3 in §1 for a discussion of the recent results of Talbut [35], related to (5.3).

Proof. Statement (i) follows from the corresponding statements for S_B in Proposition 5.2, and the continuity properties of Φ and Φ^{-1} stated in Theorem 2.3.

(ii) By Theorem 2.3, there exists an increasing function $F_s\colon\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ such that, for any $u\in H^{-s}_{r,0}$,

$$\|u\|_{-s}\leqslant F_s(\|\Phi(u)\|_{1/2-s}).$$

Since the norm of $h^{1/2-s}$ is left-invariant by the flow $\mathcal{S}^t_B$, it follows that, for any initial data $u(0)\in H^{-s}_{r,0}$, one has

$$\sup_{t\in\mathbb{R}}\|\mathcal{S}^t(u(0))\|_{-s}\leqslant F_s(\|\Phi(u(0))\|_{1/2-s}).\qquad\square$$

Solution map $\mathcal{S}_c$. Next, we introduce the solution map $\mathcal{S}_c$, where c is a real parameter. Let $v(t,x)$ be a solution of (1.1) with initial data $v(0)\in H^s_r$ and $s>\frac{3}{2}$, satisfying the properties (S1) and (S2) stated in §1. By the uniqueness property in (S1), it then follows that

$$v(t,x)=u(t,x-2ct)+c, \tag{5.4}$$

where $c=\langle v(0)|1\rangle$ and $u\in C(\mathbb{R},H^s_{r,0})\cap C^1(\mathbb{R},H^{s-2}_{r,0})$ is the solution of the initial value problem

$$\partial_t u=H\partial_x^2u-\partial_x(u^2),\quad u(0)=v(0)-c, \tag{5.5}$$

satisfying (S1) and (S2). It then follows that $w(t,x):=u(t,x-2ct)$ satisfies $w(0)=u(0)$ and

$$\partial_t w=H\partial_x^2 w-\partial_x(w^2)+2c\partial_x w. \tag{5.6}$$

By (5.4), the solution map of (5.6), denoted by $\mathcal{S}_c$, is related to the solution map $\mathcal{S}$ of (1.1) (cf. property (S2) stated in §1) by

$$\mathcal{S}(t, v(0)) = \mathcal{S}_{[v(0)]}(t, v(0)-[v(0)])+[v(0)], \tag{5.7}$$

where $[v(0)]:=\langle v(0)|1\rangle$. In particular, for any $s>\frac{3}{2}$,

$$\begin{aligned} \mathcal{S}_c\colon H^s_{r,0} &\longrightarrow C(\mathbb{R}, H^s_{r,0}),\\ w(0) &\longmapsto \mathcal{S}_c(\,\cdot\,, w(0)), \end{aligned} \tag{5.8}$$

is well defined and continuous. Molinet's results in [27] (cf. also [30]) imply that the solution map $\mathcal{S}_c$ continuously extends to any Sobolev space $H^s_{r,0}$ with $0\leqslant s\leqslant\frac{3}{2}$. More precisely, for any such s, $\mathcal{S}_c\colon H^s_{r,0}\to C(\mathbb{R}, H^s_{r,0})$ is continuous and, for any $v_0\in H^s_{r,0}$, $\mathcal{S}_c(t, w_0)$ satisfies equation (1.1) in H^{s-2}_r.

Solution map $\mathcal{S}_{c,B}$ and its extension. Arguing as in §2, we use Theorem 2.1 to express the solution map $\mathcal{S}_{c,B}$ corresponding to the equation (5.6) in Birkhoff coordinates. Note that (5.6) is Hamiltonian,

$$\partial_t w = \partial_x \nabla \mathcal{H}_c,$$

where the Hamiltonian $\mathcal{H}_c\colon H^s_{r,0}\to\mathbb{R}$ is given by

$$\mathcal{H}_c(w) := \mathcal{H}(w)+2c\mathcal{H}^{(0)}(w).$$

By (1.2), one has

$$\mathcal{H}^{(0)}(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2} w^2\, dx.$$

Since, by the (non-linear) Parseval formula,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2} w^2\, dx = \sum_{n=1}^{\infty} n|\zeta_n|^2$$

(cf. [16, Proposition 3.1]), it follows that $\mathcal{H}_{c,B}(\zeta):=\mathcal{H}_c(\Phi^{-1}(\zeta))$ satisfies

$$\mathcal{H}_{c,B}(\zeta) = \mathcal{H}_B(\zeta)+2c\sum_{n=1}^{\infty} n|\zeta_n|^2,$$

implying that the corresponding frequencies $\omega_{c,n}$, $n\geqslant 1$, are given by

$$\omega_{c,n}(\zeta) = \partial_{|\zeta_n|^2}\mathcal{H}_{c,B}(\zeta) = \omega_n(\zeta)+2cn. \tag{5.9}$$

For any $c\in\mathbb{R}$, denote by $\mathcal{S}_{c,B}$ the solution map of (5.6), when expressed in Birkhoff coordinates,

$$\begin{aligned} \mathcal{S}_{c,B}\colon h^{1/2}_+ &\longrightarrow C(\mathbb{R}, h^{1/2}_+),\\ \zeta(0) &\longmapsto [t\mapsto (\zeta_n(0)e^{it\omega_{c,n}(\zeta(0))})_{n\geqslant 1}]. \end{aligned} \tag{5.10}$$

Note that $\omega_{0,n}=\omega_n$, and hence $\mathcal{S}_{0,B}=\mathcal{S}_B$. Using the same arguments as in the proof of Proposition 5.2, one obtains the following.

Corollary 5.5. *The statements of Proposition 5.2 continue to hold for* $\mathcal{S}_{c,B}$ *with* $c\in\mathbb{R}$ *arbitrary.*

Extension of the solution map $\mathcal{S}_c$. Above, we introduced the solution map $\mathcal{S}_c$ on the subspace space $L^2_{r,0}$. One infers from (5.7) that

$$\mathcal{S}_c = \Phi^{-1}\mathcal{S}_{c,B}\Phi\colon L^2_{r,0} \longrightarrow C(\mathbb{R}, L^2_{r,0}). \tag{5.11}$$

Using the same arguments as in the proof of Proposition 5.3, one infers from Corollary 5.5 the following results, concerning the extension of $\mathcal{S}_c$ to the Sobolev space $H^{-s}_{r,0}$ with $0<s<\frac{1}{2}$.

Corollary 5.6. *The statements of Proposition 5.3 continue to hold for* $\mathcal{S}_{c,B}$ *with* $c\in\mathbb{R}$ *arbitrary.*

6. Proofs Theorems 1.3, 1.7, and 1.9

Proof of Theorem 1.3. Theorem 1.3 is a straightforward consequence of Proposition 5.3 and Corollary 5.6. □

Proof of Theorem 1.7. We argue similarly as in the proof of [16, Theorem 1.3]. Since the case $c\neq 0$ is proved by the same arguments, we only consider the case $c=0$. Let $u_0\in H^{-s}_{r,0}$ with $0\leqslant s<\frac{1}{2}$, and let $u(t):=\mathcal{S}_0(t,u_0)$. By formula (5.1), $\zeta(t):=\mathcal{S}_B(t,\Phi(u_0))$ evolves on the torus $\operatorname{Tor}(\Phi(u_0))$, defined by (2.11).

(i) Since $\operatorname{Tor}(\Phi(u_0))$ is compact in $h^{1/2-s}_+$ and $\Phi^{-1}\colon h^{1/2-s}_+ \to H^{-s}_{r,0}$ is continuous, $\{u(t): t\in\mathbb{R}\}$ is relatively compact in $H^{-s}_{r,0}$.

(ii) In order to prove that $t\mapsto u(t)$ is almost periodic, we appeal to Bochner's characterization of such functions (cf. e.g. [26]): a bounded continuous function $f\colon\mathbb{R}\to X$ with values in a Banach space X is almost periodic if and only if the set $\{f_\tau:\tau\in\mathbb{R}\}$ of functions defined by $f_\tau(t):=f(t+\tau)$ is relatively compact in the space $\mathcal{C}_b(\mathbb{R},X)$ of bounded continuous functions on $\mathbb{R}$ with values in X. Since $\Phi\colon H^{-s}_{r,0}\to h^{1/2-s}_+$ is a homeomorphism, it suffices to prove that for every sequence $(\tau_k)_{k\geqslant 1}$ of real numbers, the sequence $f_{\tau_k}(t):=\Phi(u(t+\tau_k))$, $k\geqslant 1$, in $C_b(\mathbb{R}, h^{1/2-s}_+)$ admits a subsequence which converges uniformly in $C_b(\mathbb{R}, h^{1/2-s}_+)$. Notice that

$$f_{\tau_k}(t) = (\zeta_n(u(0))e^{i\omega_n(t+\tau_k)})_{n\geqslant 1}.$$

By Cantor's diagonal process, and since the circle is compact, there exists a subsequence of $(\tau_k)_{k\geqslant 1}$, again denoted by $(\tau_k)_{k\geqslant 1}$, such that, for any $n\geqslant 1$, $\lim_{k\to\infty} e^{i\omega_n\tau_k}$ exists, implying that the sequence of functions f_{τ_k} converges uniformly in $C_b(\mathbb{R}, h^{1/2-s}_+)$. □

Proof of Theorem 1.9. Since the general case can be proved by the same arguments, we consider only the case $c=0$. By [16, Proposition B.1], the traveling wave solutions of the BO equation on $\mathbb{T}$ coincide with the 1-gap solutions. Without further reference, we use notations and results from [16, Appendix B], where 1-gap potentials have been analyzed. Let u_0 be an arbitrary 1-gap potential. Then, u_0 is C^∞-smooth and there exists $N\geqslant 1$ such that $\gamma_N(u_0)>0$ and $\gamma_n(u_0)=0$ for any $n\neq N$. Furthermore, the orbit of the corresponding 1-gap solution is given by $\{u_0(\cdot+\tau):\tau\in\mathbb{R}\}$. Let $0\leqslant s<\frac{1}{2}$. It is to prove that, for any $\varepsilon>0$, there exists $\delta>0$ such that, for any $v(0)\in H^{-s}_{r,0}$ with $\|v(0)-u_0\|_{-s}<\delta$, one has

$$\sup_{t\in\mathbb{R}}\inf_{\tau\in\mathbb{R}}\|v(t)-u_0(\cdot+\tau)\|_{-s}<\varepsilon. \tag{6.1}$$

To prove the latter statement, we argue by contradiction. Assume that there exists $\varepsilon>0$, a sequence $(v^{(k)}(0))_{k\geqslant 1}$ in $H^{-s}_{r,0}$, and a sequence $(t_k)_{k\geqslant 1}$ such that

$$\lim_{k\to\infty}\|v^{(k)}(0)-u_0\|_{-s}=0$$

and

$$\inf_{\tau\in\mathbb{R}}\|v^{(k)}(t_k)-u_0(\cdot+\tau)\|_{-s}\geqslant\varepsilon \quad \text{for all } k\geqslant 1.$$

Since

$$A:=\{v^{(k)}(0):k\geqslant 1\}\cup\{u_0\}$$

is compact in $H^{-s}_{r,0}$ and Φ is continuous, $\Phi(A)$ is compact in $h^{1/2-s}_+$ and

$$\lim_{k\to\infty}\|\Phi(v^{(k)}(0))-\Phi(u_0)\|_{1/2-s}=0.$$

It means that

$$\lim_{k\to\infty}\sum_{n=1}^{\infty}n^{1-2s}|\zeta_n(v^{(k)}(0))-\zeta_n(u_0)|^2=0.$$

Note that, for any $k\geqslant 1$,

$$\zeta_n(v^{(k)}(t_k))=\zeta_n(v^{(k)}(0))\,e^{it_k\omega_n(v^{(k)}(0))} \quad \text{for all } n\geqslant 1,$$

and $\zeta_n(u(t_k))=\zeta_n(u_0)=0$ for any $n\neq N$. Hence,

$$\lim_{k\to\infty}\sum_{n\neq N}n^{1-2s}|\zeta_n(v^{(k)}(t_k))|^2=0 \tag{6.2}$$

and, since $|\zeta_N(v^{(k)}(t_k))|=|\zeta_N(v^{(k)}(0))|$, one has

$$\lim_{k\to\infty}\big||\zeta_N(v^{(k)}(t_k))|-|\zeta_N(u_0)|\big|=0, \tag{6.3}$$

implying that

$$\sup_{k\geqslant 1}|\zeta_N(v^{(k)}(t_k))|<\infty.$$

It thus follows that the subset $\{\Phi(v^{(k)}(t_k)):k\geqslant 1\}$ is relatively compact in $h^{1/2-s}$, and hence $\{v^{(k)}(t_k):k\geqslant 1\}$ relatively compact in $H^{-s}_{r,0}$. Choose a subsequence $(v^{(k_j)}(t_{k_j}))_{j\geqslant 1}$ which converges in $H^{-s}_{r,0}$ and denote its limit by $w\in H^{-s}_{r,0}$. By (6.2) and (6.3), one infers that there exists $\theta\in\mathbb{R}$ such that

$$\zeta_n(w)=0 \text{ for all } n\neq N \quad\text{and}\quad \zeta_N(w)=\zeta_N(u_0)e^{i\theta}.$$

As a consequence, $w(x)=u_0(x+\theta/N)$, contradicting the assumption that

$$\inf_{\tau\in\mathbb{R}}\|v^{(k)}(t_k)-u_0(\cdot+\tau)\|_{-s}\geqslant\varepsilon \quad\text{for any } k\geqslant 1. \qquad\square$$

7. Proof of Theorem 1.5

In this section we prove Theorem 1.5. First, we need to to make some preparations. We consider potentials of the form

$$u(x)=v(e^{ix})+\overline{v(e^{ix})},$$

where v is a Hardy function, defined in the unit disc by

$$v(z)=\frac{\varepsilon qz}{1-qz},\quad |z|<1,$$

and $0<\varepsilon<q<1$. Note that

$$\|u\|^2_{-1/2}=2\varepsilon^2\sum_{n=1}^{\infty}n^{-1}q^{2n}=-2\varepsilon^2\log(1-q^2). \tag{7.1}$$

We want to investigate properties of the Birkhoff coordinates of u. To this end, we consider the Lax operator $L_u=D-T_u$, for u of the above form. Since, for any $f\in H^1_+$ and $z\in\mathbb{C}$ in the unit disc,

$$T_uf(z)=\Pi((v+\bar v)f)(z)=v(z)f(z)+\varepsilon q\frac{f(z)-f(q)}{z-q},$$

the eigenvalue equation $L_uf-\lambda f=0$, $\lambda\in\mathbb{R}$, when formulated as an equation in the unit disc $|z|<1$, reads

$$zf'(z)-\left(\frac{\varepsilon qz}{1-qz}+\frac{\varepsilon q}{z-q}-\mu\right)f(z)=f(q)\frac{\varepsilon q}{q-z}, \tag{7.2}$$

where we have set $\mu := -\lambda$. Note that, if $-\mu$ is an eigenvalue, then the eigenfunction $f(z)$ is holomorphic in $|z|<1$. Evaluating (7.2) in such a case at $z=0$, one infers that

$$f(q) = \frac{\varepsilon+\mu}{\varepsilon} f(0). \tag{7.3}$$

Define, for $|z|<q^{-1}$, $z\notin[0,q^{-1})$,

$$\psi(z) := \frac{z^{\varepsilon+\mu}(1-qz)^{\varepsilon}}{(q-z)^{\varepsilon}},$$

with the branches of the fractional powers chosen as

$$\arg(z)\in(0,2\pi), \quad \arg(q-z)\in(-\pi,\pi), \quad \text{and} \quad \arg(1-qz)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

Then, (7.2) reads

$$\frac{d}{dz}(\psi(z)f(z)) = f(q)\frac{\varepsilon q\psi(z)}{z(q-z)}.$$

As a consequence, one has $f(q)\neq 0$, since otherwise $\psi(z)f(z)$ would be constant and hence $\psi(z)^{-1}$ holomorphic on the whole unit disc, which is impossible as

$$\frac{\psi(t+i0)}{\psi(t-i0)} = \frac{(t+i0)^{\varepsilon+\mu}(q-t+i0)^{\varepsilon}}{(q-t-i0)^{\varepsilon}(t-i0)^{\varepsilon+\mu}} = \begin{cases} e^{-2\pi i(\varepsilon+\mu)}, & \text{if } t\in(0,q), \\ e^{-2\pi i\mu}, & \text{if } t\in(q,q^{-1}), \end{cases} \tag{7.4}$$

and $e^{2\pi i\varepsilon}\neq 1$ for any $0<\varepsilon<1$. For what follows, it is convenient to normalize the eige-function f by $f(q)=1$. Then, the eigenvalue equation reads

$$\frac{d}{dz}(\psi(z)f(z)) = g(z) \quad \text{for all } |z|<1, \tag{7.5}$$

where

$$g(z) := \frac{\varepsilon q\psi(z)}{z(q-z)} \quad \text{for all } |z|<1.$$

Our goal is to prove that, for an appropriate choice of the parameters ε and q, $\mu=-\lambda_0(u)$ becomes arbitrarily large. So, from now on, we assume that $\mu>0$. Note that, by (7.5), one has, for any $|z|<q^{-1}$ with $z\notin[0,q^{-1})$,

$$f(z) = \frac{1}{\psi(z)}\int_0^z g(\zeta)\,d\zeta.$$

Hence, $-\mu$ will be an eigenvalue of L_u, if the right-hand side of the latter expression extends to a holomorphic function in the disc $\{|z|<q^{-1}\}$. It means that f can be continuously extended to $z=0$ and $z=q$, and that, for any $t\in(0,q)\cup(q,q^{-1})$,

$$f(t+i0) = f(t-i0). \tag{7.6}$$

It is straightforward to check that, in the case $\mu>0$, f extends continuously to $z=0$. Furthermore, the identity (7.6) is verified for any $t\in(0,q)$ since, for any $0<s<t$,

$$\frac{g(s+i0)}{g(s-i0)}=e^{-2\pi i(\varepsilon+\mu)}=\frac{\psi(t+i0)}{\psi(t-i0)}.$$

It is then straightforward to verify that f extends continuously to $z=q$ and that $f(q)=1$. Next, let us examine the condition $f(t+i0)=f(t-i0)$ for $t\in(q,1)$. Notice that

$$g(z)=\frac{d}{dz}\left(\frac{z^\varepsilon}{(q-z)^\varepsilon}\right)z^\mu(1-qz)^\varepsilon,$$

so that, with

$$h(z):=\frac{z^\varepsilon}{(q-z)^\varepsilon}\frac{d}{dz}(z^\mu(1-qz)^\varepsilon),$$

one has, integrating by parts,

$$\int_0^z g(\zeta)\,d\zeta=\frac{z^\varepsilon}{(q-z)^\varepsilon}z^\mu(1-qz)^\varepsilon-\int_0^z h(\zeta)\,d\zeta=\psi(z)-\int_0^z h(\zeta)\,d\zeta.$$

As a consequence, for any $t\in(q,q^{-1})$, the condition $f(t+i0)=f(t-i0)$ reads

$$\int_0^t h(\zeta+i0)\,d\zeta=\frac{\psi(t+i0)}{\psi(t-i0)}\int_0^t h(\zeta-i0)\,d\zeta.$$

Since, by (7.4), for any $q<\zeta<q^{-1}$,

$$\psi(\zeta+i0)=e^{-2\pi i\mu}\psi(\zeta-i0)\quad\text{and}\quad h(\zeta+i0)=e^{-2\pi i\mu}h(\zeta-i0),$$

we conclude that the condition

$$\int_0^q h(\zeta+i0)\,d\zeta=e^{-2\pi i\mu}\int_0^q h(\zeta-i0)\,d\zeta$$

is necessary and sufficient for $-\mu<0$ to be an eigenvalue of L_u. After simplification and using again that $e^{2i\pi\varepsilon}\neq1$, this condition reads

$$F(\mu,\varepsilon,q):=\int_0^q\frac{t^\varepsilon}{(q-t)^\varepsilon}\frac{d}{dt}(t^\mu(1-qt)^\varepsilon)\,dt=0 \tag{7.7}$$

or

$$F(\mu,\varepsilon,q)=\int_0^q\frac{t^{\varepsilon+\mu}(1-qt)^\varepsilon}{(q-t)^\varepsilon}\left(\frac{\mu}{t}-\frac{\varepsilon q}{1-qt}\right)dt=0. \tag{7.8}$$

Notice that, if

$$\mu\geqslant\frac{\varepsilon q^2}{1-q^2},$$

then the latter integrand is strictly positive for any $0<t<q$. In particular, one has

$$F\left(\frac{\varepsilon q^2}{1-q^2},\varepsilon,q\right)>0. \tag{7.9}$$

On the other hand, let us fix $\mu>0$ and study the limit of $F(\mu,\varepsilon,q)$ as $(\varepsilon,q)\to(0,1)$. Clearly, one has

$$\lim_{(\varepsilon,q)\to(0,1)}\int_0^q \frac{t^{\varepsilon+\mu}(1-qt)^\varepsilon}{(q-t)^\varepsilon}\frac{\mu}{t}\,dt=\int_0^1 \mu t^{\mu-1}\,dt=1. \tag{7.10}$$

To compute the limit of the remaining part of $F(\mu,\varepsilon,q)$ is more involved. For any given fixed positive parameter $1-q^2<\theta<1$, split the integral

$$I(\mu,\varepsilon,q):=\int_0^q \frac{t^{\varepsilon+\mu}(1-qt)^\varepsilon}{(q-t)^\varepsilon}\frac{\varepsilon q}{1-qt}\,dt$$

into three parts:

$$I(\mu,\varepsilon,q)=I_1(\mu,\varepsilon,q;\theta)+I_2(\mu,\varepsilon,q;\theta)+I_3(\mu,\varepsilon,q),$$

where $I_1\equiv I_1(\mu,\varepsilon,q;\theta)$, $I_2\equiv I_2(\mu,\varepsilon,q;\theta)$, and $I_3\equiv I_3(\mu,\varepsilon,q)$ are defined as

$$I_1:=\int_0^{q(1-\theta)},\quad I_2:=\int_{q(1-\theta)}^{q^3},\quad \text{and}\quad I_3:=\int_{q^3}^{q}.$$

It is easy to check that

$$0\leqslant I_1\leqslant C_1(\theta)\varepsilon,$$

and that, with the change of variable $t:=q-q(1-q^2)y$ in I_3, one has

$$0\leqslant I_3=\varepsilon\, q^{\mu+2}\int_0^1 (1-(1-q^2))^{\varepsilon+\mu}(1+q^2y)^\varepsilon y^{-\varepsilon}\frac{dy}{1+q^2y}\leqslant C_3\varepsilon.$$

Using the same change of variable in I_2, we obtain

$$I_2=\varepsilon\, q^{\mu+2}\int_1^{\theta/(1-q^2)}(1-(1-q^2)y)^{\varepsilon+\mu}\frac{dy}{y^\varepsilon(1+q^2y)^{1-\varepsilon}}.$$

Note that, for $1\leqslant y\leqslant\theta/(1-q^2)$, one has $1-\theta\leqslant 1-(1-q^2)y\leqslant 1$, and hence

$$(1-\theta)^{\mu+1}\leqslant(1-(1-q^2)y)^{\mu+\varepsilon}\leqslant 1.$$

Since $\frac{1}{2}(1+y)\leqslant y\leqslant 1+y$ and

$$q^2(1+y)\leqslant 1+q^2y\leqslant 1+y,$$

we then infer that

$$I_2 \geqslant \varepsilon\, q^{\mu+2}(1-\theta)^{\mu+1} \int_1^{\theta/(1-q^2)} \frac{dy}{1+y}$$

and

$$I_2 \leqslant \varepsilon\, q^{2\varepsilon+\mu}\, 2^{\varepsilon} \int_1^{\theta/(1-q^2)} \frac{dy}{1+y}.$$

Using that, as $q \to 1$,

$$\frac{1}{-\log(1-q^2)} \int_1^{\theta/(1-q^2)} \frac{dy}{1+y} = 1 - \frac{\log((1-q^2+\theta)/2)}{\log(1-q^2)} \to 1,$$

we then obtain

$$(1-\theta)^{\mu+1} \leqslant \liminf_{(\varepsilon,q)\to(0,1)} \frac{I_2(\mu,\varepsilon,q;\theta)}{-\varepsilon\log(1-q^2)} \leqslant \limsup_{(\varepsilon,q)\to(0,1)} \frac{I_2(\mu,\varepsilon,q;\theta)}{-\varepsilon\log(1-q^2)} \leqslant 1.$$

Summarizing, we have proved that, for any $0<\theta<1$,

$$(1-\theta)^{\mu+1} \leqslant \liminf_{(\varepsilon,q)\to(0,1)} \frac{I(\mu,\varepsilon,q)}{-\varepsilon\log(1-q^2)} \leqslant \limsup_{(\varepsilon,q)\to(0,1)} \frac{I(\mu,\varepsilon,q)}{-\varepsilon\log(1-q^2)} \leqslant 1.$$

Letting $\theta \to 0$, we conclude that

$$I(\mu,\varepsilon,q) = -\varepsilon\log(1-q^2)(1+o(1)) \to +\infty \tag{7.11}$$

for (ε,q) satisfying

$$\varepsilon\log(1-q^2) \to -\infty. \tag{7.12}$$

Therefore, if (7.12) holds, then, by (7.8), (7.10), and (7.11),

$$F(\mu,\varepsilon,q) \to -\infty \quad \text{for all } \mu>0.$$

For any $k \geqslant 1$, let

$$q_k^2 := 1 - e^{-\varepsilon_k^{-3/2}}$$

with $0<\varepsilon_k<q_k$ so small that $F(k,\varepsilon_k,q_k)<0$ and

$$\frac{\varepsilon_k q_k^2}{1-q_k^2} = \varepsilon_k e^{\varepsilon_k^{-3/2}}(1-e^{-\varepsilon_k^{-3/2}}) > k. \tag{7.13}$$

Set

$$u^{(k)}(x) := 2\,\mathrm{Re}\left(\frac{\varepsilon_k q_k e^{ix}}{1-q_k e^{ix}}\right). \tag{7.14}$$

Lemma 7.1. *For any* $k\geqslant 1$,

$$u^{(k)}\in\bigcap_{n\geqslant 1}H^n_{r,0},\quad \lambda_0(u^{(k)})<-k,\quad \textit{and}\quad \lim_{k\to\infty}\|u^{(k)}\|_{-1/2}=0.$$

Proof. Expanding $u^{(k)}$, $k\geqslant 1$, in its Fourier series, it is straightforward to check that

$$u^{(k)}\in\bigcap_{n\geqslant 1}H^n_{r,0}.$$

By (7.1), we have

$$\|u^{(k)}\|^2_{H^{-1/2}}=\sqrt{\varepsilon_k}\to 0$$

and, by (7.9) and (7.13), $\lambda_0(u^{(k)})<-k$. □

In a next step, we prove that, for u of the form

$$u=2\operatorname{Re}\left(\frac{\varepsilon q e^{ix}}{1-qe^{ix}}\right),$$

L_u has only one negative eigenvalue. More precisely, the following holds.

Lemma 7.2. *For any* $0<\varepsilon<q<1$, $F(\cdot,\varepsilon,q)$ *has precisely one zero in* $\mathbb{R}_{>0}$. *It means that* $\lambda_0(u)$ *is the only negative eigenvalue of* L_u, *and thus* $\lambda_1(u)\geqslant 0$. *Furthermore,*

$$\lambda_1(u)=1-\sum_{k\geqslant 2}\gamma_k(u)\leqslant 1$$

and

$$0\leqslant\sum_{k=2}^{\infty}\gamma_k(u)\leqslant 1,\quad \gamma_k(u)>0\ \textit{for all}\ k\geqslant 2. \tag{7.15}$$

Proof. The proof relies on the formula for $F(\mu,\varepsilon,q)$, obtained from (7.7) by integrating by parts. Choosing $q^\mu(1-q^2)^\varepsilon-t^\mu(1-qt)^\varepsilon$ as antiderivative of

$$\frac{d}{dt}(t^\mu(1-qt)^\varepsilon),$$

one gets

$$F(\mu,\varepsilon,q)=\varepsilon q\int_0^q\frac{t^\varepsilon}{(q-t)^\varepsilon}\,\frac{q^\mu(1-q^2)^\varepsilon-t^\mu(1-qt)^e}{t(q-t)}\,dt. \tag{7.16}$$

Consequently, $\partial_\mu F(\mu,\varepsilon,q)$ is given by

$$\varepsilon q\int_0^q\frac{t^\varepsilon}{(q-t)^\varepsilon}\,\frac{q^\mu(1-q^2)^\varepsilon\log q-t^\mu(1-qt)^\varepsilon\log t}{t(q-t)}\,dt.$$

Assume that $F(\mu,\varepsilon,q)=0$ for some $\mu>0$. Subtracting $F(\mu,\varepsilon,q)\log q$ from the above expression for $\partial_\mu F(\mu,\varepsilon,q)$, we infer from (7.16) that

$$\frac{\partial F}{\partial\mu}(\mu,\varepsilon,q)=\varepsilon q\int_0^q \frac{t^\varepsilon}{(q-t)^\varepsilon}\,\frac{t^\mu(1-qt)^\varepsilon}{t(q-t)}\log\Big(\frac{q}{t}\Big)\,dt>0.$$

This implies that $F(\,\cdot\,,\varepsilon,q)=0$ cannot have more than one zero in $\mathbb{R}_{>0}$. It means that $\lambda_0(u)$ is the only negative eigenvalue of L_u, and hence $\lambda_1(u)\geqslant 0$. Since, by (3.15),

$$\lambda_n(u)=n-\sum_{k\geqslant n+1}\gamma_k(u)$$

for any $n\geqslant 0$, it follows that

$$0\leqslant\sum_{k=2}^\infty\gamma_k(u)\leqslant 1.$$

For any $n\geqslant 0$, denote by $\tilde f_n(\,\cdot\,,u)$ the eigenfunction of L_u corresponding to $\lambda_n(u)$, normalized by $\tilde f_n(q,u)=1$. By (7.3), it then follows that

$$\langle \check f_n(\,\cdot\,,u)\,|\,1\rangle=\tilde f_n(0,u)\neq 0\quad\text{for all } n\geqslant 1,$$

and hence, by (3.10), that $\gamma_n(u)>0$. □

In a next step, given an arbitrary potential $u\in\bigcap_{n\geqslant 1}H^n_{r,0}$, we want to express the Fourier coefficient $\hat u(1)=\langle u|e^{ix}\rangle$ in terms of the Birkhoff coordinates $\zeta_n(u)$, $n\geqslant 1$, of u. Denote by $(f_p)_{p\geqslant 0}$ the orthonormal basis of eigenfunctions of L_u with our standard normalization

$$\langle f_0\,|\,1\rangle>0\quad\text{and}\quad\langle f_{n+1}\,|\,Sf_n\rangle>0\ \text{for all } n\geqslant 0.$$

Then, we get

$$\hat u(1)=\sum_{p=0}^\infty\langle u\,|\,f_p\rangle\langle f_p\,|\,e^{ix}\rangle=\sum_{p=0}^\infty-\lambda_p\langle 1\,|\,f_p\rangle\sum_{n=0}^\infty\langle S^*f_p\,|\,f_n\rangle\langle f_n\,|\,1\rangle. \tag{7.17}$$

The following lemma provides a formula for $\hat u(1)$ in terms of the Birkhoff coordinates $\zeta_n(u)$, $n\geqslant 1$. To keep the exposition as simple as possible, we restrict to the case where $\gamma_n(u)>0$ for any $n\geqslant 1$.

LEMMA 7.3. *For any $u\in\bigcap_{k\geqslant 1}H^k_{r,0}$, with $\gamma_n(u)>0$ for any $n\geqslant 1$,*

$$\hat u(1)=-\sum_{n=0}^\infty\sqrt{\frac{\mu_{n+1}\kappa_n}{\kappa_{n+1}}}\,\zeta_{n+1}\bar\zeta_n\bigg|_u. \tag{7.18}$$

Proof. Recall that, in the case $\gamma_{n+1}(u)\neq 0$, the matrix coefficients

$$M_{np} := \langle S^* f_p \mid f_n\rangle$$

are given by [16, formulas (4.7) and (4.9)]. Using that

$$\big|\langle f_{n+1} \mid 1\rangle\big|^2 = \kappa_{n+1}\gamma_{n+1} \quad \text{and} \quad \zeta_{n+1} = \frac{\langle 1 \mid f_{n+1}\rangle}{\sqrt{\kappa_{n+1}}},$$

one then gets

$$M_{np} = \sqrt{\frac{\mu_{n+1}}{\kappa_{n+1}}}\frac{\langle f_p \mid 1\rangle}{\lambda_p - \lambda_n - 1}\zeta_{n+1}.$$

Substituting this formula into the expression (7.17) for $\hat{u}(1)$ yields

$$\hat{u}(1) = -\sum_{n=0}^{\infty}\sqrt{\frac{\mu_{n+1}\kappa_n}{\kappa_{n+1}}}\zeta_{n+1}\bar{\zeta}_n\sum_{p=0}^{\infty}\frac{\lambda_p\big|\langle 1 \mid f_p\rangle\big|^2}{\lambda_p - \lambda_n - 1},$$

where, for convenience, $\zeta_0 := 1$. Since(1)

$$\sum_{p=0}^{\infty}\frac{\lambda_p\big|\langle 1 \mid f_p\rangle\big|^2}{\lambda_p - \lambda_n - 1} = \sum_{p=0}^{\infty}\big|\langle 1 \mid f_p\rangle\big|^2 + (\lambda_n + 1)\mathcal{H}_{-\lambda_n - 1},$$

and $\mathcal{H}_{-\lambda_n-1} = 0$ due to $\gamma_{n+1} > 0$, formula (7.18) follows. □

Let us now consider the sequence $\mathcal{S}_0(t, u^{(k)})$, $k \geqslant 1$, with $u^{(k)}$ given by (7.14). Since $\gamma_1(u^{(k)}) > k$ (Lemma 7.1), $\gamma_n(u^{(k)}) > 0$, $n \geqslant 2$ (Lemma 7.2), and since γ_n, $n \geqslant 1$, are conserved quantities of (1.1), it follows that formula (7.18) is valid for $\mathcal{S}_0(t, u^{(k)})$ for any $t \in \mathbb{R}$ and $k \geqslant 1$. Hence,

$$\xi_k(t) := \langle \mathcal{S}_0(t, u^{(k)}) \mid e^{ix}\rangle \tag{7.19}$$

is given by

$$-\sum_{n=0}^{\infty}\sqrt{\frac{\mu_{n+1}\kappa_n}{\kappa_{n+1}}}\,\bigg|_{u^{(k)}}\zeta_{n+1}(\mathcal{S}_0(t, u^{(k)}))\overline{\zeta_n(\mathcal{S}_0(t, u^{(k)}))}.$$

For any potential $u \in L^2_{r,0}$ and any $n \geqslant 1$, one has, by (2.5) and (2.7),

$$\zeta_n(\mathcal{S}_0(t, u)) = \zeta_n(u)e^{it\omega_n(u)}$$

and

$$\omega_n(u) := n^2 - 2\sum_{k=1}^{\infty}\min\{k, n\}\gamma_k(u),$$

(1) We are grateful to Louise Gassot for drawing our attention to this cancellation.

implying that $\omega_1(u)=1+2\lambda_0(u)$ and, for any $n\geqslant 1$,

$$\omega_{n+1}(u)-\omega_n(u)=2n+1-2\sum_{k=n+1}^{\infty}\gamma_k(u)=1+2\lambda_n(u),$$

so that, for any $n\geqslant 0$,

$$\zeta_{n+1}(\mathcal{S}_0(t,u))\overline{\zeta_n(\mathcal{S}_0(t,u))}=\zeta_{n+1}(u)\overline{\zeta_n(u)}e^{it(1+2\lambda_n(u))}.$$

Therefore,

$$\xi_k(t)=-\sum_{n=0}^{\infty}\sqrt{\frac{\mu_{n+1}\kappa_n}{\kappa_{n+1}}}\zeta_{n+1}\bar{\zeta}_n e^{it(1+2\lambda_n)}\bigg|_{u^{(k)}}. \tag{7.20}$$

Proof of Theorem 1.5. First, we consider the case $c=0$. Our starting point is formula (7.20). By [16, Corollary 3.4],

$$\kappa_0=\prod_{p\geqslant 1}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_0}\right)$$

and, for any $n\geqslant 1$,

$$\kappa_n=\frac{1}{\lambda_n-\lambda_0}\prod_{1\leqslant p\neq n}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_n}\right).$$

Furthermore (cf. [16, formula (4.9)]), for any $n\geqslant 0$,

$$\frac{\mu_{n+1}}{\kappa_{n+1}}=\frac{\lambda_n+1-\lambda_0}{\displaystyle\prod_{1\leqslant p\neq n+1}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_n-1}\right)}.$$

These formulas yield

$$\frac{\mu_1\kappa_0}{\kappa_1}=\prod_{p\geqslant 1}\frac{1-\dfrac{\gamma_p}{\lambda_p-\lambda_0}}{1-\dfrac{\gamma_{p+1}}{\lambda_{p+1}-\lambda_0-1}}=\prod_{p\geqslant 1}\left(1-\left(\frac{\gamma_p}{\lambda_p-\lambda_0}\right)^2\right),$$

$$\frac{\mu_2\kappa_1}{\kappa_2}=\left(1+\frac{1}{\lambda_1-\lambda_0}\right)(1+\gamma_1)^{-1}\prod_{p\geqslant 2}\frac{1-\dfrac{\gamma_p}{\lambda_p-\lambda_1}}{1-\dfrac{\gamma_{p+1}}{\lambda_{p+1}-\lambda_1-1}},$$

and, for any $n\geqslant 2$, $\mu_{n+1}\kappa_n/\kappa_{n+1}$ equals

$$\left(1+\frac{1}{\lambda_n-\lambda_0}\right)\frac{1+\dfrac{\gamma_1}{\lambda_n-\lambda_1}}{1+\dfrac{\gamma_1}{\lambda_n+1-\lambda_1}}\frac{\displaystyle\prod_{2\leqslant p\neq n}\left(1-\frac{\gamma_p}{\lambda_p-\lambda_n}\right)}{\displaystyle\prod_{1\leqslant p\neq n}\left(1-\frac{\gamma_{p+1}}{\lambda_{p+1}-\lambda_n-1}\right)}.$$

Since, by the definition of γ_1,

$$\lambda_1(u^{(k)}) = \lambda_0(u^{(k)}) + 1 + \gamma_1(u^{(k)})$$

and since, by Lemma 7.2, $0 \leqslant \lambda_1(u^{(k)}) \leqslant 1$, it follows from Lemma 7.1 that

$$\gamma_1(u^{(k)}) = \lambda_1(u^{(k)}) - (\lambda_0(u^{(k)}) + 1) \sim -\lambda_0(u^{(k)}) \to +\infty$$

and

$$1 - \left(\frac{\gamma_1}{\lambda_1 - \lambda_0}\right)^2\bigg|_{u^{(k)}} = \left(1 - \frac{\gamma_1}{\lambda_1 - \lambda_0}\right)\left(1 + \frac{\gamma_1}{\lambda_1 - \lambda_0}\right)\bigg|_{u^{(k)}} \sim \frac{2}{\gamma_1(u^{(k)})}.$$

Furthermore by (7.15),

$$\sup_{k \geqslant 1} \sum_{n \geqslant 2} \gamma_n(u^{(k)}) \leqslant 1.$$

One then concludes that

$$\frac{\mu_1 \kappa_0}{\kappa_1}\bigg|_{u^{(k)}} \sim \frac{2}{\gamma_1(u^{(k)})}, \qquad \frac{\mu_2 \kappa_1}{\kappa_2}\bigg|_{u^{(k)}} = O\left(\frac{1}{\gamma_1(u^{(k)})}\right),$$

and

$$\frac{\mu_{n+1} \kappa_n}{\kappa_{n+1}}\bigg|_{u^{(k)}} = O(1) \quad \text{for all } n \geqslant 2.$$

From the above estimates, we infer that the sequences $(\xi_k(t))_{k \geqslant 1}$, given by (7.20), are uniformly bounded for $t \in \mathbb{R}$. Assume that the function $t \mapsto (\xi_k(t))_{k \geqslant 1}$ converges pointwise to zero on an interval I of positive length. Then, by the dominated convergence theorem,

$$\lim_{k \to \infty} \int_I \xi_k(t) e^{-it(1+2\lambda_0(u^{(k)}))}\, dt = 0.$$

On the other hand, since

$$\xi_k(t) e^{-it(1+2\lambda_0)}|_{u^{(k)}} = -\sqrt{\frac{\mu_1 \kappa_0}{\kappa_1}}\zeta_1\bigg|_{u^{(k)}} - \sum_{n \geqslant 1}^{\infty} \sqrt{\frac{\mu_{n+1} \kappa_n}{\kappa_{n+1}}}\zeta_{n+1}\bar{\zeta}_n e^{it2(\lambda_n - \lambda_0)}\bigg|_{u^{(k)}}$$

and

$$\lambda_n(u^{(k)}) - \lambda_0(u^{(k)}) \geqslant |\lambda_0(u^{(k)})| \quad \text{for all } n \geqslant 1,$$

the above estimates yield

$$\int_I \xi_k(t) e^{-it(1+2\lambda_0)}\bigg|_{u^{(k)}} dt = -\frac{\sqrt{2}\zeta_1}{\sqrt{\gamma_1}}\bigg|_{u^{(k)}} |I| + O\left(\frac{1}{|\lambda_0(u^{(k)})|}\right),$$

which does not converge to zero as $k \to \infty$, since $|\zeta_1(u^{(k)})| = \sqrt{\gamma_1(u^{(k)})}$ and $|I| > 0$. Hence, the assumption that $(\xi_k(\cdot))_{k \geqslant 1}$ converges pointwise to zero on an interval I of positive length leads to a contradiction, and thus cannot be true.

Finally, let us treat the case where $c\in\mathbb{R}$ is arbitrary. Consider the sequence

$$(u^{(k)}+c)_{k\geqslant 1},$$

with $(u^{(k)})_{k\geqslant 1}$ given by (7.14). By Lemma 7.1, $\lim_{k\to\infty}(u^{(k)}+c)=c$ in $H^{-1/2}_{r,c}$. Furthermore, by (5.7) and (5.8), one has $\mathcal{S}(t,u^{(k)}+c)=\mathcal{S}_c(t,u^{(k)})+c$ and, by (5.9) and (5.10),

$$\zeta_n(\mathcal{S}_c(t,u^{(k)}))=\zeta_n e^{it(2cn+\omega_n)}|_{u^{(k)}}.$$

Hence, with $\xi_k(t)$ given by (7.19), one has, for any $k\geqslant 1$,

$$\xi_{c,k}(t):=\langle\mathcal{S}(t,u^{(k)}+c)|\ e^{ix}\rangle=e^{i2ct}\xi_k(t).$$

Since

$$\int_I \xi_{c,k}(t)e^{-i2ct}e^{-it(1+2\lambda_0)}\Big|_{u^{(k)}}dt=\int_I \xi_k(t)e^{-it(1+2\lambda_0)}\Big|_{u^{(k)}}dt,$$

Theorem 1.5 follows from the proof of the case $c=0$ treated above. □

An immediate consequence of Theorem 1.5 is the following.

COROLLARY 7.4. *The Birkhoff map Φ does not continuously extend to a map*

$$H^{-1/2}_{r,0}\longrightarrow h^0_+.$$

Appendix A. Restriction of Φ to $H^s_{r,0}$, $s>0$

The purpose of this appendix is to study the restriction of the Birkhoff map Φ to $H^s_{r,0}$ with $s>0$, and to discuss applications to the flow map of the Benjamin–Ono equation.

PROPOSITION A.1. *For any $s\geqslant 0$, the restriction of the Birkhoff map Φ to $H^s_{r,0}$ is a homeomorphism from $H^s_{r,0}$ onto $h^{s+1/2}_+$. Furthermore, $\Phi: H^s_{r,0}\to h^{s+1/2}_+$ and its inverse $\Phi^{-1}: h^{s+1/2}_+\to H^s_{r,0}$ map bounded subsets to bounded subsets.*

Proof. The case $s=0$ is proved in [16]. We first treat the case $s\in]0,1]$. Assume that A is a bounded subset of $L^2_{r,0}$. Given $u\in A$, let $K_{u;s}$, $0\leqslant s\leqslant 1$, be the linear isomorphism of Lemma 3.10,

$$f\in H^s_+\longrightarrow(\langle f\,|\,f_n\rangle)_{n\geqslant 0}\in h^s(\mathbb{N}_0),$$

where $f_n\equiv f_n(\cdot,u)$. By Lemma 3.10, the operator $K_{u;s}$ is uniformly bounded with respect to $u\in A$. Since $L_u 1=-\Pi u$, one has

$$K_{u;s}(\Pi u)=(\langle\Pi u\,|\,f_n\rangle)_{n\geqslant 0}=(-\lambda_n(u)\langle 1\,|\,f_n\rangle)_{n\geqslant 0}=0(-\lambda_n(u)\sqrt{\kappa_n(u)}\zeta_n(u))_{n\geqslant 0},\quad\text{(A.1)}$$

where $\zeta_0(u){:=}1$ and (cf. [16, Corollary 3.4])

$$\kappa_0(u)=\prod_{p\geqslant 1}\left(1-\frac{\gamma_p(u)}{\lambda_p(u)-\lambda_0(u)}\right). \tag{A.2}$$

By the proof of Proposition 3.1 (i), $\lambda_n\equiv\lambda_n(u)$ and $\kappa_n\equiv\kappa_n(u)$ satisfy

$$\lambda_n=n+o\left(\frac{1}{n}\right) \quad\text{and}\quad \sqrt{\kappa_n}=\frac{1}{\sqrt{n}}(1+o(1)), \tag{A.3}$$

uniformly with respect to $u\in A$. Therefore, for any $s\in[0,1]$, an element $u\in L^2_{r,0}$ belongs to $H^s_{r,0}$ if and only if $K_{u;0}(\Pi u)\in h^s(\mathbb{N}_0)$ or, equivalently, $(\sqrt{n}\zeta_n(u))_{n\geqslant 1}\in h^s_+$, implying that

$$\Phi(u)\in h^{s+1/2}_+.$$

Furthermore, if A is a bounded subset of $H^s_{r,0}$, then $\Phi(A)$ is a bounded subset of $h^{s+1/2}_+$. Conversely, if B is a bounded subset of $h^{s+1/2}_+$, a fortiori it is bounded in $h^{1/2}_+$, and hence $\Phi^{-1}(B)$ is bounded in $L^2_{r,0}$. As a consequence, the norm of $K^{-1}_{u;s}: h^s(\mathbb{N}_0)\to H^s_+$ is uniformly bounded with respect to $u\in\Phi^{-1}(B)$, and hence $\Phi^{-1}(B)$ is bounded in $H^s_{r,0}$.

Next, we prove that, for any $0<s\leqslant 1$, $\Phi: H^s_{r,0}\to h^{s+1/2}_+$ and its inverse are continuous. Since $\Phi: L^2_{r,0}\to h^{1/2}_+$ is a homeomorphism, we infer from Rellich's theorem and the boundedness properties of Φ and its inverse, derived above, that, for any $s\in(0,1]$, $\Phi: H^s_{r,0}\to h^{s+1/2}_+$ and $\Phi^{-1}: h^{s+1/2}_+\to H^s_{r,0}$ are sequentially weakly continuous. As in the proof of Proposition 4.1 (iii), to prove that $\Phi: H^s_{r,0}\to h^{s+1/2}_+$ is continuous, it then suffices to show that Φ maps any relatively compact subset A of $H^s_{r,0}$ to a relatively compact subset of $h^{s+1/2}_+$. For $s=1$, this is straightforward. Indeed, given any bounded subset A of $H^1_{r,0}$, A is relatively compact in $H^1_{r,0}$ if and only if $\{D\Pi u: u\in A\}$ is relatively compact in L^2_+. As $\{T_u\Pi u: u\in A\}$ is bounded in H^1_+, this amounts to say that $\{L_u(\Pi u): u\in A\}$ is relatively compact subset in L^2_+. Since

$$\|L_u(\Pi u)\|^2=\sum_{n\geqslant 0}\lambda_n(u)^2|\langle 1\,|\,f_n(\cdot,u)\rangle|^2,$$

we infer from (A.3) that

$$\|L_u(\Pi u)\|^2=\|\Phi(u)\|^2_{3/2}+R(u),$$

where R is a weakly continuous functional on $H^1_{r,0}$. As a consequence, $\{L_u(\Pi u): u\in A\}$ is relatively compact in L^2_+ if and only if $\Phi(A)$ is relatively compact in $h^{3/2}_+$. This shows that $\Phi: H^1_{r,0}\to h^{3/2}_+$ is continuous. In a similar way, one proves that $\Phi^{-1}: h^{3/2}_+\to H^1_{r,0}$ is continuous. This completes the proof for $s=1$.

To treat the case $s\in(0,1)$, we will use the following standard variant of Lemma 4.3.

LEMMA A.2. *Let $s\in(0,1)$. A bounded subset $A_+\subset H^s_+$ is relatively compact in H^s_+ if and only if, for any $\varepsilon>0$, there exist $N_\varepsilon\geqslant 1$ and $R_\varepsilon>0$ such that, for any $f\in A_+$, the sequence $\xi_n:=\hat f(n)$, $n\geqslant 0$, satisfies*

$$\Big(\sum_{n>N_\varepsilon} n^{2s}|\xi_n)|^2\Big)^{1/2}<\varepsilon$$

and

$$\Big(\sum_{0\leqslant n\leqslant N_\varepsilon} n^2|\xi_n|^2\Big)^{1/2}<R_\varepsilon.$$

The latter conditions on $(\xi_n)_{n\geqslant 0}$ characterize relatively compact subsets of $h^s(\mathbb{N}_0)$.

We then argue as in the proof of Proposition 4.1 (iii), using the operators $K_{u;s}$ and $K_{u;s}^{-1}$, to complete the proof of Proposition A.1 in the case $s\in]0,1]$.

In order to deal with the case $s>1$, we need the following lemma.

LEMMA A.3. *Let k be a non-negative integer, and assume that A is a bounded subset of $H^k_{r,0}$. Then, for any $u\in A$ and $s\in[k,k+1]$, an element $f\in H^k_+$ is in H^s_+ if and only if, for any $0\leqslant j\leqslant k$, $L_u^j f\in H_+^{s-k}$, with bounds which are uniform with respect to $u\in A$. Furthermore, A is compact in $H^s_{r,0}$ if and only if, for any $0\leqslant j\leqslant k$, $\{L_u^j\Pi u:u\in A\}$ is compact in H_+^{s-k}.*

Proof. The statement is trivial for $k=0$. Let us first prove it for $k=1$. Assume that A is a bounded subset of $H^1_{r,0}$ and write $s=t+1$ with $t\in[0,1]$. Then, for any $f\in H^1_+$, one has $f\in H^s_+$ if and only if f and Df belong to H^t_+, with uniform bounds and with correspondence of compact subsets. For any u in $H^1_{r,0}$, the operator T_u maps bounded subsets of H^t_+ to bounded subsets of H^t_+. Hence, for any $f\in H^1_+$, we have $f,Df\in H^t_+$ if and only if $f,L_uf\in H^t_+$ with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. This completes the proof for $k=1$. For $k\geqslant 2$, we argue by induction. Assume that $k\geqslant 2$ is such that the statement is true for $k-1$. Let A be a bounded subset of $H^k_{r,0}$, and let $s\in[k,k+1]$. Then, for any $f\in H^k_+$, one has $f\in H^s_+$ if and only if $f,Df\in H^{s-1}_+$, with uniform bounds and correspondence of compact subsets. Since $s-1\geqslant k-1\geqslant 1$, H^{s-1}_+ is an algebra, and hence T_u maps bounded subsets of H^{s-1}_+ to bounded subsets of H^{s-1}_+. Consequently, $f,Df\in H^{s-1}_+$ if and only if $f,L_uf\in H^{s-1}_+$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. By the induction hypothesis, this is equivalent to $L_u^jf\in H^{s-k}_+$ for any $0\leqslant j\leqslant k$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. □

Let us now come back to the proof of Proposition A.1. We start with the case where $s=k$ is a positive integer, and assume that A is a bounded subset of $H^k_{r,0}$. Applying

Lemma A.3, one easily shows by induction on k that, for any $u\in A$, one has $L_u^j\Pi u\in L_+^2$ for any $0\leqslant j\leqslant k$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. In other words, for any $u\in A$, Πu belongs to the domain of L_u^k, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. On the other hand, if u belongs to a bounded subset of $L^2_{r,0}$, $K_{u,0}$ maps the domain of L_u^k into the space of sequences $(\xi_n)_{n\geqslant 0}$ such that $(\lambda_n(u)^\ell\xi_n)_{n\geqslant 0}\in h^0(\mathbb{N}_0)$ for any $0\leqslant\ell\leqslant k$, and hence into $h^k(\mathbb{N}_0)$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. Hence, for any $u\in A$, one has $\Phi(u)\in h_+^{k+1/2}$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets.

Finally, let $s\in(k,k+1)$ and assume that A is a bounded subset of $H^k_{r,0}$. It then follows by Lemma A.3 that, for any $u\in A$, one has $L_u^j\Pi u\in H_+^{s-k}$ for $j=0,...,k$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. Applying $K_{u;s-k}$ and Lemma 3.10, this means that $K_{u;s-k}(L_u^j\Pi u)\in h^{s-k}(\mathbb{N}_0)$ for any $0\leqslant j\leqslant k$, with bounds which are uniform with respect to $u\in A$ and with correspondence of compact subsets. Since

$$K_{u;0}(L_u^j f)=(\lambda_n(u)^j\langle f\mid f_n\rangle)_{n\geqslant 0},$$

this implies $K_{u;0}(\Pi u)\in h^s(\mathbb{N}_0)$, and hence $\Phi(u)\in h_+^{s+1/2}$ with bounds which are uniform with respect to $u\in A$. The proof of the converse is similar, taking into account that, by the case $s=k$ treated above, for any bounded subset B of $h_+^{k+1/2}$, the set $\Phi^{-1}(B)$ is bounded in $H^k_{r,0}$. The correspondence of compact subsets is established similarly. □

Arguing as in the proofs of Theorems 1.3, 1.7 and 1.9, one deduces from Proposition A.1 the following.

Corollary A.4. *Let $s>0$ and $c\in\mathbb{R}$. Then, for any $t\in\mathbb{R}$, the flow map $S^t=\mathcal{S}(t,\cdot)$ of the Benjamin–Ono equation leaves the affine space $H^s_{r,c}$, introduced in (1.4), invariant. Furthermore, there exists an integral $I_s\colon H^s_r\to\mathbb{R}_{\geqslant 0}$ of (1.1) satisfying*

$$\|v\|_s\leqslant I_s(v)\quad\text{for all } v\in H^s_r.$$

In particular, one has

$$\sup_{t\in\mathbb{R}}\|\mathcal{S}(t,v_0)\|_s\leqslant I_s(v_0)\quad\text{for all } v_0\in H^s_r.$$

In addition, for any $v_0\in H^s_{r,c}$, the solution $t\mapsto\mathcal{S}(t,v_0)$ is almost periodic in $H^s_{r,c}$, and the traveling wave solutions are orbitally stable in $H^s_{r,c}$.

Remark A.5. Corollary A.4 significantly improves [18, Theorem 1.3], which provides polynomial (in t) bounds of solutions of (1.1) in H^s_r for $\frac{1}{2}<s\leqslant 1$.

Appendix B. Ill-posedness in H^s_r for $s<-\frac{1}{2}$

The goal of this appendix is to construct, for any $c\in\mathbb{R}$, a sequence $(v^{(k)}(t,x))_{k\geqslant 1}$ of smooth solutions of (1.1) such that

(i) $(v^{(k)}(0,\cdot))_{k\geqslant 1}$ has a limit in $H^{-s}_{r,c}$ for any $s>\frac{1}{2}$;

(ii) for any $t\neq 0$, $(v^{(k)}(t,\cdot))_{k\geqslant 1}$ diverges in the sense of distributions, even after renormalizing the flow by a translation in the spatial variable.

Since the same arguments work for any $c\in\mathbb{R}$, we consider only the case $c=0$. In a first step, let us review the results from [4], using the setup developed in this paper: the authors of [4] construct a sequence of 1-gap potentials $v^{(k)}(t,\cdot)$, $k\geqslant 1$, of the Benjamin–Ono equation such that $v^{(k)}(0,\cdot)$ converges in $H^{-s}_{r,0}$ for any $s>\frac{1}{2}$, whereas, for any $t\neq 0$, $(v^{(k)}(t,\cdot))_{k\geqslant 1}$ diverges in the sense of distributions. Without further reference, we use notation and results from [16, Appendix B], where 1-gap potentials have been analyzed. Consider the following family of 1-gap potentials of average zero,

$$u_{0,q}(x)=2\mathrm{Re}\left(\frac{qe^{ix}}{1-qe^{ix}}\right),\quad 0<q<1.$$

The gaps $\gamma_n(u_{0,q})$, $n\geqslant 1$, of $u_{0,q}$ can be computed as

$$\gamma_{1,q}:=\gamma_1(u_{0,q})=\frac{q^2}{1-q^2},\quad \gamma_n(u_{0,q})=0 \text{ for all } n\geqslant 2.$$

The frequency $\omega_{1,q}:=\omega_1(u_{0,q})$ is given by (cf. (2.7))

$$\omega_{1,q}=1-2\gamma_{1,q}=\frac{1-3q^2}{1-q^2}.$$

The 1-gap solution, also referred to as traveling wave solution, of the BO equation with initial data $u_{0,q}$ is then given by

$$u_q(t,x)=u_{0,q}(x+\omega_{1,q}t)\quad \text{for all } t\in\mathbb{R}.$$

Note that, for any $s>\frac{1}{2}$,

$$\lim_{q\to 1}u_{0,q}=2\mathrm{Re}\left(\sum_{k=1}^{\infty}e^{ikx}\right)=\delta_0-1$$

strongly in $H^{-s}_{r,0}$, where δ_0 denotes the periodic Dirac δ-distribution, centered at zero. Since $\omega_{1,q}\to-\infty$ as $q\to 1$, it follows that, for any $t\neq 0$, $u_q(t,\cdot)$ diverges in the sense of distributions as $q\to 1$.

Note that, by the trace formula (cf. (2.2)), $\|u_{0,q}\|^2=2\gamma_{1,q}$, and hence

$$\omega_{1,q}+\|u_{0,q}\|^2=1,$$

implying that a renormalization of the spatial variable by $\eta_q t$ with $\eta_q=\|u_{0,q}\|^2=1-\omega_{1,q}$ removes the divergence, i.e., for any $s>\frac{1}{2}$, $u_{0,q}(\cdot+t)$ converges in $H^{-s}_{r,0}$ for any t as $q\to 1$. In order to construct examples where no such renormalization is possible, we consider 2-gap solutions. To keep the exposition as simple as possible, let u be a 2-gap potential with $\gamma_1>0$ and $\gamma_2>0$, and hence $\gamma_n=0$ for any $n\geqslant 3$. From [16, §7], we know that

$$\Pi u(z)=-z\frac{Q'(z)}{Q(z)},$$

where

$$Q(z)=\det(I-zM)\quad\text{for all } z\in\mathbb{C},\ |z|<1, \tag{B.1}$$

and $M:=(M_{np})_{0\leqslant n,p\leqslant 1}$ is a 2×2 matrix with

$$M_{np}=\sqrt{\frac{\mu_{n+1}\kappa_p}{\kappa_{n+1}}}\,\frac{\zeta_{n+1}\bar{\zeta}_p}{\lambda_p-\lambda_n-1}$$

or

$$M_{np}=\sqrt{\frac{(\lambda_n+1-\lambda_0)\kappa_p}{\displaystyle\prod_{1\leqslant q\neq n+1}\left(1-\frac{\gamma_q}{\lambda_q-\lambda_n-1}\right)}}\,\frac{\zeta_{n+1}\bar{\zeta}_p}{\lambda_p-\lambda_n-1}.$$

Here, $\zeta_0=1$ and $\zeta_n=\sqrt{\gamma_n}e^{i\varphi_n}$, $n=1,2$, whereas $\zeta_n=0$ for any $n\geqslant 3$. Furthermore, κ_n, $n\geqslant 0$, are given by (A.2) and (3.13). Along the flow of the Benjamin–Ono equation, we have $\partial_t\varphi_n=\omega_n$, $1\leqslant n\leqslant 2$, with (cf. (2.7))

$$\omega_1=1-2\gamma_1-2\gamma_2\quad\text{and}\quad\omega_2=4-2\gamma_1-4\gamma_2.$$

Let us express the entries of M in terms of γ_1, γ_2, φ_1, and φ_2. By (A.2) and (3.13),

$$\kappa_0=\left(1-\frac{\gamma_1}{\lambda_1-\lambda_0}\right)\left(1-\frac{\gamma_2}{\lambda_2-\lambda_0}\right)=\frac{2+\gamma_1}{(1+\gamma_1)(2+\gamma_1+\gamma_2)}$$

and

$$\kappa_1=\frac{1}{\lambda_1-\lambda_0}\left(1-\frac{\gamma_2}{\lambda_2-\lambda_1}\right)=\frac{1}{(1+\gamma_1)(1+\gamma_2)}.$$

Then, the above formula for M_{np} yields

$$M_{00}=-\sqrt{\frac{\gamma_1(2+\gamma_1)(1+\gamma_1+\gamma_2)}{2+\gamma_1+\gamma_2}}\,\frac{e^{i\varphi_1}}{1+\gamma_1},$$
$$M_{01}=\sqrt{\frac{1+\gamma_1+\gamma_2}{1+\gamma_2}}\,\frac{1}{1+\gamma_1},$$
$$M_{10}=-\sqrt{\frac{\gamma_2}{2+\gamma_1+\gamma_2}}\,\frac{e^{i\varphi_2}}{1+\gamma_1},$$
$$M_{11}=-\sqrt{\frac{\gamma_1\gamma_2(2+\gamma_1)}{1+\gamma_2}}\,\frac{e^{i(\varphi_2-\varphi_1)}}{1+\gamma_1}.$$

Consequently, $Q(z)=1+\alpha z+\beta z^2$ with $\alpha=-\mathrm{Tr}(M)$ and $\beta=\det(M)$, or

$$\alpha=\frac{\sqrt{\gamma_1(2+\gamma_1)}}{1+\gamma_1}\left(\sqrt{\frac{1+\gamma_1+\gamma_2}{2+\gamma_1+\gamma_2}}e^{i\varphi_1}+\sqrt{\frac{\gamma_2}{1+\gamma_2}}e^{i(\varphi_2-\varphi_1)}\right),$$
$$\beta=\sqrt{\frac{(1+\gamma_1+\gamma_2)\gamma_2}{(1+\gamma_2)(2+\gamma_1+\gamma_2)}}\;e^{i\varphi_2}.$$

Taking the limit for $\gamma_1\to\infty$ and $\gamma_2\to\infty$, one sees that

$$\alpha=e^{i\varphi_1}+e^{i(\varphi_2-\varphi_1)}+o(1)\quad\text{and}\quad\beta=e^{i\varphi_2}+o(1),$$

which means that $Q(z)=(1+q_1z)(1+q_2z)$, where

$$q_1=e^{i\varphi_1}+o(1)\quad\text{and}\quad q_2=e^{i(\varphi_2-\varphi_1)}+o(1).$$

Since $|q_1|<1$ and $|q_2|<1$, the family u is bounded in $H^{-s}_{r,0}$ for any $s>\frac{1}{2}$.

We define our sequence $(v_0^{(k)})_{k\geqslant1}$ of initial data to be a sequence of 2-gap potentials where, for any $k\geqslant1$,

$$\varphi_1^{(k)}(0):=0,\qquad\varphi_2^{(k)}(0):=0,$$
$$\gamma_1^{(k)}:=\gamma^{(k)},\qquad\gamma_2^{(k)}:=\gamma^{(k)},$$

where $(\gamma^{(k)})_{k\geqslant1}$ is an increasing sequence of positive numbers with $\lim_{k\to\infty}\gamma^{(k)}=\infty$. At $t=0$, we infer that, for any z in the unit disc,

$$\lim_{k\to\infty}\Pi v_0^{(k)}(z)=\frac{2z}{1+z},$$

and hence $v_0^{(k)}\to2(1-\delta_\pi)$ in $H^{-s}_{r,0}$ for any $s>\frac{1}{2}$. For $t\neq0$, we have

$$\varphi_1^{(k)}(t)=t(1-4\gamma^{(k)})\quad\text{and}\quad\varphi_2^{(k)}(t)-\varphi_1^{(k)}(t)=t(3-2\gamma^{(k)}),$$

and therefore

$$\Pi v^{(k)}(z,t)=\frac{ze^{i\varphi_1^{(k)}(t)}}{1+ze^{i\varphi_1^{(k)}(t)}}+\frac{ze^{i(\varphi_2^{(k)}-\varphi_1^{(k)})(t)}}{1+ze^{i(\varphi_2^{(k)}-\varphi_1^{(k)})(t)}}+o(1).$$

We thus conclude that $(\Pi v^{(k)}(z,t))_{k\geqslant 1}$ does not have a limit if $t\neq 0$ for any z with $0<|z|<1$, even after renormalizing z by a phase factor $e^{i\eta_k(t)}$, with $\eta_k(t)$ being a function of our choice. This proves that, for $t\neq 0$, $(v^{(k)}(t,\cdot))_{k\geqslant 1}$ has no limit in $\mathcal{D}'(\mathbb{T})$, even after renormalizing it by a t-dependent translation. In particular, the renormalization of the spatial variable by $\|v_0^{(k)}\|^2t=(4-\omega_2^{(k)})t$ does not make $(v^{(k)}(t,\cdot))_{k\geqslant 1}$ convergent, even in the sense of distributions.

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