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Surface groups in uniform lattices of some semi-simple groups

by

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1. Introduction

As a corollary of our main theorem, we obtain the following easily stated result.

THEOREM A. *Let* G *be a center-free, complex semisimple Lie group and* Γ *a uniform lattice in* G. *Then,* Γ *contains a surface group.*

However, our main result is a quantitative version of this result.

By a surface group, we mean the fundamental group of a closed connected oriented surface of genus at least 2. We shall see, later on, that the restriction that G is complex can be relaxed: the theorem holds for a wider class of groups, for instance $\mathsf{PU}(p,q)$ with $q{>}p{>}0$. This theorem is a generalization of the celebrated Kahn–Marković theorem [14], [3] which deals with the case of $\mathsf{PSL}(2,\mathbb{C})$, and its proof follows a similar scheme: building pairs of pants, gluing them and showing that the group is injective, however the details vary greatly, notably in the injectivity part. Let us note that Hamenstädt [13] had followed a similar proof to show the existence of surfac++e subgroups of all uniform lattices in rank-1 groups, except $\mathsf{SO}(2n,1)$, while Kahn and Marković deals with reducible lattices for the case $\mathsf{G}{=}\mathsf{SL}(2,\mathbb{R})\times\mathsf{SL}(2,\mathbb{R})$ in their Ehrenpreis paper [15].

Also, in the non-uniform Kleinian case, a recent result of Kahn and Wright [16] extends [14].

Finally, let us recall that Kahn–Marković paper was preceded in the context of hyperbolic 3-manifolds by (non-quantitative) results of Lackenby [24] for lattices with torsion, and Cooper, Long and Reid [9] in the non-uniform case, both papers using very different techniques.

For higher rank, notably for $\mathsf{SL}(3,\mathbb{R})$, let us also quote Long–Reid [25] and Long–Reid–Thistlethwaite [26].

Kahn–Marković theorem has a quantitative version: the surface group obtained is K-quasi-symmetric where K can be chosen arbitrarily close to 1. Our theorem also has a quantitative version that needs some preparation and definitions to be stated properly: in particular, we need to define in this higher-rank context what is the analog of a K-quasi-symmetric map for K close to 1.

1.1. Sullivan maps

Given a semi-simple group G, we make the choice of an $\mathfrak{sl}_2$-*triple* in the Lie algebera $\mathfrak{g}$ of G, that is, an embedding of the Lie algebra of $\mathsf{SL}_2(\mathbb{R})$ with its standard generators (a,x,y) into the Lie algebra of G. For the sake of simplification, in this introduction, we suppose that this triple has a compact centralizer. Such an $\mathfrak{sl}_2$-triple defines a flag manifold $\mathbf{F}$: a compact G-transitive space on which the hyperbolic element a acts with a unique attractive fixed point (see §2 for details).

Most of the results and techniques of the proof involves the study of the following geometric objects in $\mathbf{F}$:

(i) *circles* in $\mathbf{F}$ which are maps from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}$ equivariant under a representation of $\mathsf{SL}_2(\mathbb{R})$ conjugate to the one defined by the $\mathfrak{sl}_2$-triple chosen above;

(ii) *perfect triangles* which are triple of distinct points on a circle; such a tripod τ defines — in a G-equivariant way — a metric d_τ on $\mathbf{F}$; later we will introduce related objects called *tripods*.

We can now define what is the generalization of a K-quasi-symmetric map, for K close to 1. Let ζ be a non-negative number. A ζ-*Sullivan map* is a map ξ from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}$, such that, for every triple of pairwise distinct points T in $\mathbf{P}^1(\mathbb{R})$, there is a circle $\eta_T\colon \mathbf{P}^1(\mathbb{R})\to\mathbf{F}$ such that

$$d_{\eta_T(T)}(\eta_T(x),\xi(x))\leqslant\zeta \quad \text{for all } x\in\mathbf{P}^1(\mathbb{R}).$$

We remark that circles are 0-Sullivan maps. Also, we insist that this notion is relative to the choice of some $\mathfrak{sl}_2$-triple, or more precisely of a conjugacy class of $\mathfrak{sl}_2$-triple. This notion is discussed in more details in §8.

Obviously, for this definition to make sense, ζ has to be small. We do not require any regularity nor continuity of the map ξ. Our first result actually guarantees some regularity. S

THEOREM B. (Hölder property) *There exist some positive numbers ζ and α such that any ζ-Sullivan map is α-Hölder.*

If we furthermore assume that the map ξ is equivariant under some representation ρ of a cocompact Fuchsian group Γ acting on $\mathbf{P}^1(\mathbb{R})$, we have the following result.

THEOREM C. (Sullivan implies Anosov) *There exists a positive number ζ such that if Γ is a cocompact Fuchsian group, ρ a representation of Γ in G such that there exists a ρ equivariant ζ-Sullivan map ξ from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}$, then ρ is $\mathbf{F}$-Anosov and ξ is its limit curve.*

When $\mathsf{G}{=}\mathsf{PSL}(2,\mathbb{C})$, $\mathbf{F}{=}\mathbf{P}^1(\mathbb{C}){=}\partial_\infty\mathbf{H}^3$, circles are boundaries at infinity of hyperbolic planes, and the theorems above translate into classical properties of quasi-symmetric maps. We refer to [22] and [12] for reference on Anosov representations and give a short introduction in §8.4.1. In particular recall that Anosov representations are faithful.

1.2. A quantitative surface subgroup theorem

We can now state what is our quantitative version of the existence of surface subgroup in higher-rank lattices.

THEOREM D. *Let G be a center-free, semisimple Lie group without compact factor and Γ a uniform lattice in G. Let us choose an $\mathfrak{sl}_2$-triple in G with a compact centralizer and satisfying the* flip assumption (*see below*) *with associated flag manifold $\mathbf{F}$.*

Let ζ be a positive number. Then, there exists a cocompact Fuchsian group Γ_0 and a $\mathbf{F}$-Anosov representation ρ of Γ_0 in G with values in Γ and whose limit map is ζ-Sullivan.

We recall that a Fuchsian group is a discrete subgroup of $\mathsf{PSL}(2,\mathbb{R})$.

The flip assumption is satisfied for all complex groups, all rank-1 groups except $\mathsf{SO}(1,2n)$, but not for real split groups. The precise statement is the following. Let (a,x,y) be an $\mathfrak{sl}_2$-triple and ζ_0 be the smallest real positive number such that

$$\exp(2i\zeta_0\cdot a)=1.$$

We say the (a,x,y) satisfies the *flip assumption* if the automorphism of G, $\mathbf{J}_0{:=}\exp(i\zeta_0\cdot a)$ belongs to the connected component of a compact factor of the centralizer of a. Ursula Hamenstädt also used the flip assumption for uniform lattices in [13].

We do hope the flip assumption is unnecessary. However, removing it is beyond the scope of the present article: it would involve in particular incorporating generalized arguments from [15] which deal with the (non-flip) case of $\mathsf{G}=\mathsf{SL}(2,\mathbb{R})\times\mathsf{SL}(2,\mathbb{R})$.

Finally, let us notice that Kahn and Wright [16] have proved a quantitative version of the surface subgroup theorem for non-uniform lattice in the case of $\mathsf{PSL}(2,\mathbb{C})$, leaving thus open the possibility to extend our theorem also for non-uniform lattices.

1.3. A tool: coarse geometry in flag manifolds

A classical tool for Gromov hyperbolic spaces is the Morse lemma: quasi-geodesics are at uniform distance to geodesics. Higher-rank symmetric spaces are not Gromov hyperbolic, but they do carry a version of the Morse lemma: see Kapovich–Leeb–Porti [17] and Bochi–Potrie–Sambarino [4]

Our approach in this paper is however to avoid as much as possible dealing with the (too rich) geometry of the symmetric space. We will only use the geometry of the flag manifolds that we defined above: circles, tripods and metrics assigned to tripods. In this new point of view, the analogs of geodesics will be coplanar path of triangles: roughly speaking, a coplanar path of triangles corresponds to a sequence of non-overlapping ideal triangles in some hyperbolic space such that two consecutive triangles are adjacent — see Figure 4.1a. We now have to describe a coarse version of that. First we need to define *quasi-tripods* which are deformation of tripods: roughly speaking these are tripods with deformed vertices (see Definition 4.1 for precisions). Then, we want to define almost coplanar sequence of quasi-tripods (see Definition 4.6). Finally, one of our main results (Theorem 7.2) guarantees some circumstances under which these “quasi-paths” converge “at infinity”, that is, shrink to a point in $\mathbf{F}$.

The Morse lemma by itself is not enough to conclude in the hyperbolic case, and we need a refined version. Our Theorem 7.2 is used at several points in the paper: to prove the main theorem and to prove the results about Sullivan maps. Although, this theorem requires too many definitions to be stated in the introduction, it is one of the main and new contributions of this paper.

While this paper was in its last stage, we learned that Ursula Hamenstädt has announced existence results for lattices in higher-rank group, without the quantitative part of our results, but with other very interesting features.

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1.4. A description of the content of this article

What follows is meant to be a reading guide of this article, while introducing informally the essential ideas. In order to improve readability, an index is produced at the end of this paper.

(i) §2 sets up the Lie theory background: it describes in more details $\mathfrak{sl}_2$-triples, the flip assumption, and the associated parabolic subgroups and flag manifolds.

(ii) §3 introduces the main tools of our paper: tripods. In the simplest case (for instance, principal $\mathfrak{sl}_2$-triples in complex simple groups), tripods are just preferred triples of points in the associated flag manifold. In the general case, tripods come with some extra decoration. They may be thought of as generalizations of ideal triangles in hyperbolic planar geometry and they reflect our choice of a preferred $\mathfrak{sl}_2$-triple. The space of tripods admits several actions that are introduced here and notably a *shearing flow*. Moreover, each tripod defines a metric on the flag manifold itself and we explore the relationships between the shearing flow and these metric assignments.

(iii) For the hyperbolic plane, (nice) sequences of non-overlapping ideal triangles, where two successive ones have a common edge, converges at infinity. This corresponds in our picture to *coplanar paths of tripods*. §4 deals with "coarse deformations" of these paths. First, we introduce quasi-tripods, which are deformation of tripods: in the simplest case, these are triples of points in the flag manifold which are not far from a tripod, with respect to the metric induced by the tripod. Then, we introduce *paths of quasi-tripods* that we see as deformation of coplanar paths of tripods. Our goal is now to show that these deformed paths converge under some nice hypotheses, result that we will achieve in §7.

(iv) For coplanar paths of tripods (which are sequences of ideal triangles), one see the convergence to infinity as a result of nesting of intervals in the boundary at infinity. This however is the consequence of the order structure on $\partial_\infty \mathbf{H}^2$ and very specific to planar geometry. In our case, we need to introduce "coarse deformations" of these intervals, that we call *slivers*, and introduce quantitative versions of the nesting property of intervals called *squeezing* and *controlling*. In §5 and §6, we define all of these objects and prove the confinement lemma. This lemma tells us that certain deformations of coplanar paths still satisfy our coarse nesting properties. These two sections are preliminary to the following one.

(v) In §7, we prove one of the main results of the papers, the limit point theorem that gives a condition under which a deformed sequence of quasi-tripods converges to a point in the flag manifold as well as some quantitative estimates on the rate of convergence. This theorem will be used several times in the sequel. Special instances of this theorem may be thought of as higher-rank versions of the Morse lemma. Our motto is to use the coarse geometry of path of quasi-tripods in the flag manifolds, rather than quasi-geodesics in the symmetric space.

(vi) In §8 we introduce *Sullivan curves*, which are analogs of quasi-circles. We show extensions of two classical results for Kleinian groups and quasi-circles: Sullivan curves are Hölder and if a Sullivan curve is equivariant under the representation of a surface group, this surface group is Anosov — the analog of quasi-Fuchsian. In the case of deformation of equivariant curves, we prove an improvement theorem that needs a Sullivan curve to be only defined on a smaller set.

(vii) So far, the previous sections were about the geometry of the flag manifold and did not make use of a lattice or discrete subgroups of G. We now move to the proof of existence of surface groups, that we shall build by gluing pairs of pants together. The following two sections deal with pairs of pants: §9 introduces the concept of an *almost closing pair of pants* that generalizes the idea of building a pair of pants out of two ideal triangles. We describe the structure of these pairs of pants in a structure theorem using a partially hyperbolic closing lemma. In Kahn–Marković original paper a central role is played by "connected pairs of tripods" which are (roughly speaking) three homotopy classes of paths joining two points. In §10, we introduce here the analog in our case, and call them *triconnected pair of tripods*, then describe *weight functions*. A triconnected pair of tripods on which the weight function is positive, gives rise to a nearby almost closing pair of pants. We also study an orientation inverting symmetry.

(viii) We study in the following two sections the boundary data that is needed to describe the gluing of pairs of pants. After having introduced *biconnected pair of tripods*, related to "bipods" in [14], which amounts to forget one of the paths in the triple of paths associated to the triconnected pair of tripods defining the pair of pants. In §11, we introduce spaces and measures for both triconnected and biconnected pairs of tripods, and show that the forgetting map almost preserve the measure using the mixing property of our mixing flow. Then, in §12, we move more closely to study the boundary data: we introduce the *feet spaces and projections* which is the higher-rank analog to the normal bundle to geodesics in hyperbolic 3-manifolds, and we prove Theorem 12.4 that describes under which circumstances a measure is not perturbed too much by a *Kahn–Marković* twist.

(ix) In §13, we wrap up the preceding two sections in proving the even distribution theorem, which essentially says that there are the same number of pairs of pants coming from "opposite sides" in the feet space. This makes use of the flip assumption which is discussed there with more details (with examples and counterexamples).

(x) As in Kahn–Marković original paper, we use the measured marriage theorem in §14 to produce *straight surface groups* which are pairs of pants glued nicely along their boundaries. It now remains to prove that these straight surface groups injects and are Sullivan.

(xi) Before starting that proof, we need to describe in §15 a little further the *R-perfect lamination* in the hyperbolic plane, and more importantly the *accessible points* in the boundary at infinity, which are roughly speaking those points which are limits of nice path of ideal triangles with respect to the lamination. This section is purely hyperbolic planar geometry.

(xii) We finally make a connection with the first part of the paper which leads to the limit point theorem. In §16, we consider the nice paths of tripods converging to accessible points described in the preceding section, and show that a straight surface (or more generally an *equivariant straight surface*) gives rise to a deformation of these paths of tripods into paths of quasi-tripods, these latter paths being well behaved enough to have limit points according to the limit point theorem. Then, using the improvement theorem of §8, we show that this gives rive to a Sullivan limit map for our surface.

(xiii) The last section is a wrap-up of the preceding results and finally, in an appendix, we present results and constructions dealing with the Lévy–Prokhorov distance between measures.

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2. Preliminaries: $\mathfrak{sl}_2$-triples

In this preliminary section, we recall some facts about the hyperbolic plane, $\mathfrak{sl}_2$-triples in a Lie algebra, and discuss the flip assumption that we need to state our result. We also give the construction of the parabolic group and flag manifold whose geometry is going to play a fundamental role in this paper.

2.1. $\mathfrak{sl}_2$-triples and the flip assumption

Let G be a semisimple center-free Lie group without compact factors, and let $\mathfrak{g}$ be its Lie algebra. In general, for a group H, we denote by $\mathsf{Z}_\mathsf{H}(\mathsf{M})$ the centralizer in H of the subset (or the element) M of H. We say that a hyperbolic element a of $\mathfrak{g}$ is *regular* if its centralizer in $\mathfrak{g}$ does not contain a subalgebra isomorphic to $\mathfrak{sl}_2$. Equivalently, a is regular if the semi-simple part of the centralizer in G of regular element is compact.

Definition 2.1. An $\mathfrak{sl}_2$*-triple* in $\mathfrak{g}$ [21] is $\mathfrak{s}{:=}(a,x,y)$ in $\mathfrak{g}^3$ such that

$$[a,x]=2x, \quad [a,y]=-2y \quad \text{and} \quad [x,y]=a.$$

An $\mathfrak{sl}_2$-triple (a,x,y) is *regular* if a is a regular element. The centralizer of a regular $\mathfrak{sl}_2$-triple is compact.

An $\mathfrak{sl}_2$-triple (a,x,y) is *even* if all the eigenvalues of a by the adjoint representation are even.

An $\mathfrak{sl}_2$-triple (a,x,y) generates a Lie algebra $\mathfrak{a}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ such that

$$a=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad x=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad \text{and} \quad y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{2.1}$$

For an even triple, the group whose Lie algebra is $\mathfrak{a}$ is isomorphic to $\mathsf{PSL}_2(\mathbb{R})$. For a triple which is not even, that group is isomorphic to $\mathsf{SL}_2(\mathbb{R})$. Note that $\mathfrak{s}$ might have a compact centralizer, while a is not regular.

We say that an element $\mathbf{J}_0$ of $\mathrm{Aut}(\mathsf{G})$ is a *reflection* for the $\mathfrak{sl}_2$-triple (a,x,y) if the following conditions hold:

- $\mathbf{J}_0$ is an involution and belongs to $\mathsf{Z}_{\mathsf{G}}(\mathsf{Z}_{\mathsf{G}}(a))$;
- $\mathbf{J}_0(a,x,y){=}(a,-x,-y)$, and in particular $\mathbf{J}_0$ normalizes the group generated by $\mathfrak{sl}_2$ isomorphic to $\mathsf{PSL}_2(\mathbb{R})$, and acts by conjugation by the matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

An example of a reflection in the case of complex group is $\mathbf{J}_0{:=}\exp\left(\frac{1}{2}i\zeta a\right)\in\mathsf{G}$, where ζ is the smallest non-zero real number such that $\exp(i\zeta a){=}1$. It follows that a reflection always exists (by passing to the complexified group), but is not necessarily an element of G.

Definition 2.2. (Flip assumption) We say that the $\mathfrak{sl}_2$-triple $\mathfrak{s}{=}(a,x,y)$ in G satisfies the *flip assumption* if $\mathfrak{s}$ is even and there exists a reflection $\mathbf{J}_0$ which is an inner automorphism, and which belongs to the connected component of the identity of $\mathsf{Z}_{\mathsf{G}}(\mathsf{Z}_{\mathsf{G}}(a))$ of an element a in $\mathfrak{g}$.

In the regular case, we have a weaker assumption.

Definition 2.3. (Regular flip assumption) If the even $\mathfrak{sl}_2$-triple $\mathfrak{s}$ is regular, we say that $\mathfrak{s}$ satisfies the *regular flip assumption* if $\mathfrak{s}$ is even and there exists a reflection $\mathbf{J}_0$ which belongs to the connected component of the identity of $\mathsf{Z}_{\mathsf{G}}(a)$.

The flip assumption for the $\mathfrak{sl}_2$-triple (a^0, x^0, y^0) in $\mathfrak{g}$ will only be assumed in order to prove the even distribution theorem (Theorem 13.2).

In §13.2.2, we shall give examples of groups and $\mathfrak{sl}_2$-triples satisfying the flip assumption.

2.2. Parabolic subgroups and the flag manifold

We recall standard facts about parabolic subgroups in real semi-simple Lie groups, for references see [5, Chapter VIII, §3, paragraphs 4 and 5].

2.2.1. Parabolic subgroups, flag manifolds, transverse flags

Let $\mathfrak{s}=(a, x, y)$ be an $\mathfrak{sl}_2$-triple. Let $\mathfrak{g}^\lambda$ be the eigenspace associated to the eigenvalue λ for the adjoint action of a, and let

$$\mathfrak{p} = \bigoplus_{\lambda \geqslant 0} \mathfrak{g}^\lambda.$$

Let P be the normalizer of $\mathfrak{p}$. By construction, P is a *parabolic subgroup* and its Lie algebra is $\mathfrak{p}$.

The associated *flag manifold* is the set $\mathbf{F}$ of all Lie subalgebras of $\mathfrak{g}$ conjugate to $\mathfrak{p}$. By construction, the choice of an element of $\mathbf{F}$ identifies $\mathbf{F}$ with G/P. The group G acts transitively on $\mathbf{F}$ and the stabilizer of a point — or *flag* — x (denoted by P_x) is a parabolic subgroup.

Given a, let now

$$\mathfrak{q} = \bigoplus_{\lambda \leqslant 0} \mathfrak{g}^\lambda.$$

By definition, the normalizer Q of $\mathfrak{q}$ is the *opposite parabolic* to P with respect to a. Since in $\mathsf{SL}_2(\mathbb{R})$, a is conjugate to $-a$, it follows that in our special case opposite parabolic subgroups are conjugate.

Two points x and y of $\mathbf{F}$ are *transverse* if their stabilizers are opposite parabolic subgroups. Then, the stabilizer L of the transverse pair of points is the intersection of two opposite parabolic subgroups, in particular its Lie algebra is $\mathfrak{g}_{\lambda_0}$, for the eigenvalue $\lambda_0=0$. Moreover, L is the Levi part of P.

PROPOSITION 2.4. *The group* L *is the centralizer of* a.

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the Lie algebras of the two opposite parabolic subgroups preserved by a. When $\mathsf{G}=\mathsf{SL}(m, \mathbb{R})$, the result follows from the explicit description of L as block diagonal group. More generally, it is enough to consider the case when G is center-free and simple, since both L and $\mathsf{Z}_{\mathsf{G}}(a)$ contain the center of G. Let $p{=}\dim(\mathfrak{p})$, let us consider $E{=}\Lambda^p(\mathfrak{g})$ and λ the induced representation of G on E. Let $(e_1, ..., e_p)$ be a basis of $\mathfrak{p}$ and $(f_1, ..., f_p)$ a basis of $\mathfrak{q}$ such that $(f_1, ..., f_m)$ is a basis of $\mathfrak{q}\cap\mathfrak{p}$. Let us considering the line

$$u := \langle e_1 \wedge ... \wedge e_p \rangle$$

in E, as a well as the hyperplane

$$V = \{v \in E : v \wedge f_{m+1} \wedge ... \wedge f_p = 0\}.$$

Then, observe that the element $\exp(a)$ acts as a non-trivial homothety on u and V. Let L_p be the stabilizer of u and V in $\mathsf{SL}(E)$. Then,

$$\lambda(\mathsf{L}) = \mathsf{L}_p \cap \lambda(\mathsf{G}) = \mathsf{Z}_{\mathsf{SL}(E)}(\lambda(a)) \cap \lambda(\mathsf{G}) = \lambda(\mathsf{Z}_{\mathsf{G}}(a)).$$

Since λ is faithful, $\mathsf{L}{=}\mathsf{Z}_{\mathsf{G}}(a)$. □

2.2.2. Loxodromic elements

We say that an element in G is P-*loxodromic*, if it has one attractive fixed point and one repulsive fixed point in $\mathbf{F}$, and these two points are transverse. We will denote by λ^- the repulsive fixed point of the loxodromic element λ, and by λ^+ its attractive fixed point in $\mathbf{F}$. By construction, for any non-zero real number s, $\exp(sa)$ is a loxodromic element.

2.2.3. Positive a-chambers

Let $\mathsf{C}{=}\mathsf{Z}_{\mathsf{G}}(\mathsf{L})$ be the centralizer of L in G. Since the 1-parameter subgroup generated by a belongs to $\mathsf{L}{=}\mathsf{Z}_{\mathsf{G}}(a)$, it follows that $\mathsf{C}{\subset}\mathsf{L}$ and C is an Abelian group. Let A be the maximal (connected) split torus in C. We now decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \bigoplus_{\lambda \in R} \mathfrak{p}^{\lambda},$$

where $R{\subset}\mathsf{A}^*$, and A acts on $\mathfrak{p}^{\lambda}$ by the weight λ. Let

$$R^+ = \{\lambda \in R : \lambda(a) > 0\}.$$

We define the *positive a-chamber* to be

$$W = \{b \in \mathsf{A} : \lambda(b) > 0 \text{ if } \lambda \in R^+\} \subset \mathsf{A}.$$

Observe that W is an open cone that contains a.

3. Tripods and perfect triangles

We define here *tripods* which are going to be one of the main tools of the proof. The first definition is not very geometric but we will give more flesh to it.

Namely, we will associate to a tripod a *perfect triangle* that is a certain type of triple of points in $\mathbf{F}$. We will define various actions and dynamics on the space of tripods. We will also associate to every tripod two important objects in $\mathbf{F}$: a *circle* (a certain class of embedding of $\mathbf{P^1}(\mathbb{R})$ in $\mathbf{F}$), as well as a metric on $\mathbf{F}$.

3.1. Tripods

Let G be a semi-simple Lie group with trivial center and Lie algebra $\mathfrak{g}$. Let us fix a group G_0 isomorphic to G.

Definition 3.1. (Tripod) A *tripod* is an isomorphism from G_0 to G.

So far, the terminology "tripod" is baffling. We will explain in the next section how tripods are related to triples of points in a flag manifolds.

We denote by $\mathcal{G}$ the *space of tripods.* To be more concrete, when one chooses

$$\mathsf{G}_0 := \mathsf{SL}_n(\mathbb{R})$$

in the case of $\mathsf{G}=\mathsf{SL}(V)$, the space of tripods is exactly the set of basis of $\mathbb{R}^n$. The space of tripods $\mathcal{G}$ is a left principal $\mathrm{Aut}(\mathsf{G})$-torsor as well a right principal $\mathrm{Aut}(\mathsf{G}_0)$-torsor where the actions are defined respectively by post-composition and pre-composition. These two actions commute.

3.1.1. Connected components

Let us consider the map from G to $\mathrm{Aut}(\mathsf{G})$ which associates to g in G, the inner automorphism (by conjugation) $\mathrm{Inn}(g)$. Let us fix a tripod τ_0 in $\mathcal{G}$, that is an isomorphism $\tau_0 \colon \mathsf{G}_0 \to \mathsf{G}$. Then, the map defined from G to $\mathcal{G}$ by $g \mapsto \mathrm{Inn}(g) \cdot \tau_0$, realizes an isomorphism from G to the connected component of $\mathcal{G}$ containing τ_0. Obviously, $\mathrm{Aut}(\mathsf{G})$ acts transitively on $\mathcal{G}$. We thus obtain the following result.

Proposition 3.2. *Every connected component of $\mathcal{G}$ is identified* (*as a G-torsor*) *with G. Moreover*, *the number of connected components of $\mathcal{G}$ is equal to the cardinality of* $\mathrm{Out}(\mathsf{G})$.

3.1.2. Correct $\mathfrak{sl}_2$-triples and circles

Throughout this paper, we fix an $\mathfrak{sl}_2$-triple $\mathfrak{s}_0=(a_0,x_0,y_0)$ in $\mathfrak{g}_0$. Let $\boldsymbol{i}_0$ be a Cartan involution that extends the standard Cartan involution of $\mathsf{SL}_2(\mathbb{R})$, that is such that

$$\boldsymbol{i}_0(a_0,x_0,y_0)=(-a_0,y_0,x_0). \tag{3.1}$$

Let then

- S_0 be the connected subgroup of G_0 whose Lie algebra is generated by $\mathfrak{s}_0$; the group S_0 is isomorphic either to $\mathsf{SL}_2(\mathbb{R})$ or $\mathsf{PSL}_2(\mathbb{R})$;
- $\mathsf{Z}_{\mathsf{G}_0}$ be the centralizer of (a_0,x_0,y_0) in G_0;
- L_0 be the centralizer of a_0;
- P_0^+ be the parabolic subgroup associated to a_0 in G_0 and P_0^- be the opposite parabolic subgroup;
- $\mathsf{N}_0^\pm$ be the respective unipotent radicals of $\mathsf{P}_0^\pm$.

Definition 3.3. (Correct $\mathfrak{sl}_2$-triple) A *correct* $\mathfrak{sl}_2$*-triple* — with respect to the choice of $\mathfrak{s}_0$ — is the image of $\mathfrak{s}_0$ by a tripod τ. The space of correct $\mathfrak{sl}_2$-triples forms an orbit under the action of $\mathrm{Aut}(\mathsf{G})$ on the space of conjugacy classes of $\mathfrak{sl}_2$-triples.

A correct $\mathfrak{sl}_2$-triple $\mathfrak{s}$ is thus identified with an embedding $\xi^{\mathfrak{s}}$ of $\mathfrak{s}_0$ in G in a given orbit of $\mathrm{Aut}(\mathsf{G})$.

Definition 3.4. (Circles) The *circle map* associated to the correct $\mathfrak{sl}_2$-triple $\mathfrak{s}$ is the unique $\xi^{\mathfrak{s}}$-equivariant map $\phi^{\mathfrak{s}}$ from $\mathbf{P^1}(\mathbb{R})$ to $\mathbf{F}$. The image of a circle map is a *circle.*

Since we can associate a correct $\mathfrak{sl}_2$-triple to a tripod, we can associate a circle map to a tripod.

We define a right $\mathsf{SL}_2(\mathbb{R})$-action on $\mathcal{G}$ by restricting the G_0 action to S_0.

Definition 3.5. (Coplanar tripods) Two tripods are *coplanar* if they belong to the same $\mathsf{SL}_2(\mathbb{R})$-orbit.

3.2. Tripods and perfect triangles of flags

This subsection will justify our terminology. We introduce perfect triangles which generalize ideal triangles in the hyperbolic plane and relate them to tripods.

Definition 3.6. (Perfect triangle) Let $\mathfrak{s}=(a,x,y)$ be a correct $\mathfrak{sl}_2$-triple. The associated *perfect triangle* is the triple of flags $t_{\mathfrak{s}}:=(t^-,t^+,t^0)$ which are the attractive fixed points of the 1-parameter subgroups generated respectively by $-a$, a and $a+2y$. We denote by $\mathcal{T}$ the space of perfect triangles.

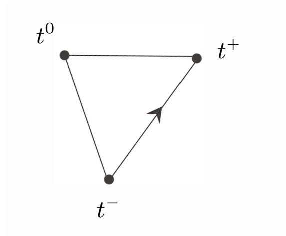


Figure 3.1. A perfect triangle.

We represent in Figure 3.1 graphically a perfect triangle (t^-, t^+, t^0) as a triangle whose vertices are (t^-, t^+, t^0) drawing an arrow from t^- to t^+.

If $\mathsf{G}{=}\mathsf{SL}_2(\mathbb{R})$, then the perfect triangle associated to the standard $\mathfrak{sl}_2$-triple (a_0, x_0, y_0) described in equation (2.1) is $(0, \infty, 1)$, the perfect triangle associated to $(a_0, -x_0, -y_0)$ is $(0, \infty, -1)$. As a consequence, we can give the following definition.

Definition 3.7. (Vertices of a tripod) Let ϕ_τ be the circle map associated to a tripod. The *set of vertices* associated to τ is the perfect triangle $\partial\tau{:=}\phi_\tau(0, \infty, 1)$.

Observe that any triple of distinct points in a circle is a perfect triangle and that, if two tripods are coplanar, their vertices lie in the same circle.

3.2.1. Space of perfect triangles

The group G acts on the space of tripods, the space of $\mathfrak{sl}_2$-triples and the space of perfect triangles.

Proposition 3.8. (Stabilizer of a perfect triangle) *Let $t{=}(u, v, w)$ be a perfect triangle associated to a correct $\mathfrak{sl}_2$-triple $\mathfrak{s}$. Then, the stabilizer of t in G is the centralizer $\mathsf{Z}_\mathsf{G}(\mathfrak{s})$ of $\mathfrak{s}$.*

Proof. Let u, v and w be as above. Denote by $\mathsf{L}_{x,y}$ the stabilizer of a pair of transverse points (x, y) in $\mathbf{F}$. Let also $\mathsf{A}_{x,y}{=}\mathsf{L}_{x,y}{\cap}\mathsf{S}$, where S is the group generated by $\mathfrak{s}$. Observe that $\mathsf{A}_{x,y}$ is a 1-parameter subgroup. By Proposition 2.4, $\mathsf{L}_{x,y}$ is the centralizer in G of $\mathsf{A}_{x,y}$. Now given three distinct points in the projective line, the group generated by the three diagonal subgroups $\mathsf{A}_{u,v}$, $\mathsf{A}_{v,w}$ and $\mathsf{A}_{u,w}$ is $\mathsf{SL}_2(\mathbb{R})$. Thus, the stabilizer of the perfect triangle t is the centralizer of $\mathfrak{s}$, that is $\mathsf{Z}_\mathsf{G}(\mathfrak{s})$. ☐

Corollary 3.9. (i) *The map $\mathfrak{s}{\mapsto}t_\mathfrak{s}$ defines a G-equivariant homeomorphism from the space of correct triples to the space of perfect triangles.*

(ii) *We have $\mathcal{T}{=}\mathcal{G}/\mathsf{Z}_{\mathsf{G}_0}$ and the map $\partial{:}\mathcal{G}{\to}\mathcal{T}$ is a (right) $\mathsf{Z}_{\mathsf{G}_0}$-principal bundle.*

A perfect triangle t, then defines a correct $\mathfrak{sl}_2$-triple and thus an homomorphism denoted ξ^t from $\mathsf{SL}_2(\mathbb{R})$ to G.

It will be convenient in the sequel to describe a tripod τ as a quadruple (H,t^-,t^+,t^0), where $t{=}(t^-,t^+,t^0){=}{:}\partial\tau$ is a perfect triangle and H is the set of all tripods coplanar to τ. We write

$$\partial\tau=(t^-,t^+,t^0),\quad \partial^-\tau=t^-,\quad \partial^+\tau=t^+,\quad \partial^0\tau=t^0.$$

3.3. Structures and actions

We have already described commuting left $\mathrm{Aut}(\mathsf{G})$ and right $\mathrm{Aut}(\mathsf{G}_0)$ actions on $\mathcal{G}$. Since G and G_0 embed as inner automorphisms in $\mathrm{Aut}(\mathsf{G})$ and in $\mathrm{Aut}(\mathsf{G}_0)$, respectively, we obtain by restriction actions of G and G_0 on $\mathcal{G}$.

Since $\mathsf{Z}_{\mathsf{G}_0}$ is the centralizer of $\mathfrak{s}_0$, we also obtain a right action of $\mathsf{SL}_2(\mathbb{R})$ on $\mathcal{T}$, as well as a left G-action, commuting together.

We summarize the properties of the actions (and specify some notation) in the following list.

(1) Actions of G and G_0 coming from inner automorphisms.

   (a) The transitive left G-action on $\mathcal{T}$ is given — in the interpretation of triangles by triple of points in $\mathbf{F}$ — by

$$g(f_1,f_2,f_3):=(g(f_1),g(f_2),g(f_3)).$$

   Interpreting perfect triangles as morphisms τ from $\mathsf{SL}_2(\mathbb{R})$ to G, sending the canonical $\mathfrak{sl}_2$-triple of $\mathsf{SL}_2(\mathbb{R})$ to a correct $\mathfrak{sl}_2$-triple, we have

$$g\cdot\tau(x):=(\mathrm{Inn}(g)\circ\tau)(x)=g\cdot\tau(x)\cdot g^{-1}.$$

   (b) The (right)-action of an element g_0 of G_0 on $\mathcal{G}$ is denoted by R_{g_0}.
   We have the relation

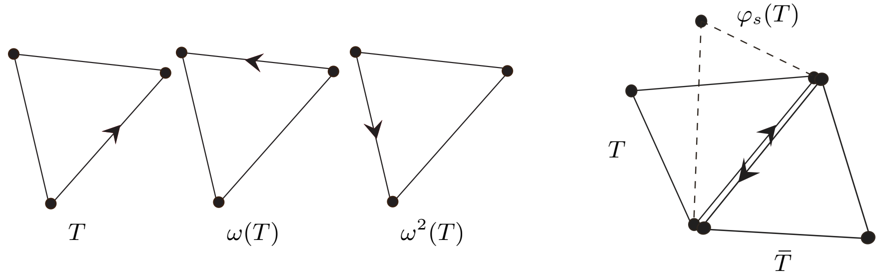
$$R_{g_0}\cdot\tau=\tau(g_0)\cdot\tau.$$

(2) The right $\mathsf{SL}_2(\mathbb{R})$-action on $\mathcal{G}$ and $\mathcal{T}$ gives rises to a flow, an involution and an order-3 symmetry as follows.

   (a) The *shearing flow* $\{\varphi_s\}_{s\in\mathbb{R}}$ is given by $\varphi_s{:=}R_{\exp(sa_0)}$ on $\mathcal{G}$ (see Figure 3.2b). If we denote by ξ the embedding of $\mathsf{SL}(2,\mathbb{R})$ given by the perfect triangle $t{=}(t^-,t^+,t^0)$, then

$$\varphi_s(H,t^-,t^+,t^0):=(H,t^-,t^+,\exp(sa)\cdot t^0)),$$

   where $a{=}\mathsf{T}\xi(a^0)$ and $\mathsf{T}f$ denote the tangent map to a map f. We say that $\varphi_R(\tau)$ is *R-sheared* from τ .



(a) Permutation of order 3.

(b) Flow and involution.

Figure 3.2. Some actions.

(b) The *reflection* $\sigma: t \mapsto \bar{t}$ is given on $\mathcal{G}$ by $\bar{\tau} = \tau \cdot \sigma$, where $\sigma \in \mathsf{SL}_2(\mathbb{R})$ is the involution defined by $\sigma(\infty, 0, 1) = (0, \infty, -1)$. For the point of view of tripods via perfect triangles

$$\overline{(H, t^+, t^-, t^0)} = (H, t^-, t^+, s^0),$$

where t^-, t^-, t^0 and s^0 form a harmonic division on a circle (see Figure 3.2b). With the same notation the involution on $\mathcal{T}$ is given by

$$\overline{(t^+, t^-, t^0)} = (t^-, t^+, s^0).$$

(c) The *rotation* ω of order 3 (see Figure 3.2a) is defined on $\mathcal{G}$ by $\omega(\tau) = \tau \cdot r_\omega$, where $r_\omega \in \mathsf{PSL}_2(\mathbb{R})$ is defined by

$$r_\omega(0, 1, \infty) = (1, \infty, 0).$$

For the point of view of tripods via perfect triangles,

$$\omega(H, t^-, t^+, t^0) = (H, t^+, t^0, t^-).$$

Similarly, the action of ω on T is given by $\omega(t^-, t^+, t^0) = (t^+, t^0, t^-)$.

(3) Two foliations $\mathcal{U}^-$ and $\mathcal{U}^+$ on $\mathcal{G}$ and $\mathcal{T}$ called respectively the *stable* and *unstable* foliations. The leaf of $\mathcal{U}^\pm$ is defined as the right orbit of, respectively, N_0^+ and N_0^- (normalized by Z_0), and alternatively by

$$\mathcal{U}_\tau^\pm := \mathsf{U}^\pm(\tau),$$

where $\mathsf{U}^\pm(\tau)$ is the unipotent radical of the stabilizer of $\partial^\pm\tau$ under the left action of G. We also define the *central stable* and *central unstable* foliations by the right actions of $\mathsf{P}_0^\pm$, respectively, or alternatively by

$$\mathcal{U}_\tau^{\pm,0} := \mathsf{U}^{\pm,0}(\tau),$$

where $\mathsf{U}^{\pm,0}(t)$ is the stabilizer of $\partial^\pm\tau$ under the left action of G. Observe that $\mathsf{U}^{\pm,0}(t)$ are both conjugate to P_0.

(4) A foliation, called the *central foliation*, $\mathcal{L}^0$ whose leaves are the right orbits of L_0 on $\mathcal{G}$, naturally invariant under the action of the flow $\{\varphi_s\}_{s\in\mathbb{R}}$. Alternatively,

$$\mathcal{L}^0_\tau = \mathsf{L}^0(\tau),$$

where $\mathsf{L}^0(\tau)$ is the stabilizer in G of $(\partial^+\tau, \partial^-\tau)$.

Then, we have the following result.

Proposition 3.10. (i) *The action of* G *commutes with the flow* $\{\varphi_s\}_{s\in\mathbb{R}}$, *the involution* σ *and the permutation* ω.

(ii) *For any real number* s *and tripod* τ,

$$\overline{\varphi_s(\tau)} = \varphi_{-s}(\bar{\tau}).$$

(iii) *The foliations* $\mathcal{U}^+$ *and* $\mathcal{U}^-$ *are invariant by the left action of* G.

(iv) *Moreover, the leaves of* $\mathcal{U}^+$ *and* $\mathcal{U}^-$ *on* $\mathcal{G}$ *are, respectively, uniformly contracted* (*with respect to any left* G*-invariant Riemannian metric*) *and dilated by the action of* $\{R_{\exp(tu)}\}_{t\in\mathbb{R}}$ *for* u *in the interior of the positive Weyl chamber and* $t>0$.

(v) *The flow* $\{\varphi_t\}_{t\in\mathbb{R}}$ *acts by isometries along the leaves of* $\mathcal{L}_0$.

(vi) *We have* $\tau\in\mathcal{U}^{0,-}_\eta$ *if and only if* $\partial^-\tau=\partial^-\eta$.

Proof. The first three assertions are immediate.

Let us choose a tripod τ, which then gives an identification of $\mathcal{G}$ with $\mathsf{G}=\mathsf{G}_0$. If d is a left-invariant metric associated to a norm $\|\cdot\|$ on $\mathfrak{g}$, the image of d under the right action of an element g is associated to the norm $\|\cdot\|_g$ such that $\|u\|_g=\|\mathrm{ad}(g)\cdot u\|$. The fourth and fifth assertions follow from that description.

For the last assertion, we have $\mathcal{U}^{0,-}_\tau=\mathcal{U}^{0,-}_\sigma$ if and only if the stabilizer of $\partial^-\tau$ and $\partial^-\sigma$ are the same. The result follows. □

Corollary 3.11. (Contracting along leaves) *For any left-invariant Riemannian metric* d *on* G, *there exists a constant* $\mathbf{M}$ *depending only on* G *such that if* ε *is small enough then, for all positive* R, *the following two properties hold*:

$$\begin{aligned} d(u,v)\leqslant\varepsilon,\ d(\varphi_R(u),\varphi_R(v))\leqslant\varepsilon \quad &\Longrightarrow\quad d(\varphi_t(u),\varphi_t(v))\leqslant\mathbf{M}\varepsilon \ \textit{ for all } t\in[0,R],\\ \partial^-u=\partial^-v, d(u,v)\leqslant\varepsilon \quad &\Longrightarrow\quad d(\varphi_t(u),\varphi_t(v))\leqslant\mathbf{M}\varepsilon \ \textit{ for all } t<0. \end{aligned}$$

3.3.1. A special map

We consider the map K (see Figure 3.3) defined from $\mathcal{T}$ or $\mathcal{G}$ to itself by

$$K(x) := \omega(\bar{x}).$$

Later on, we shall need the following property of this map K.

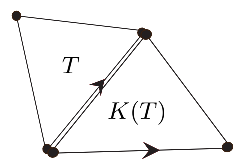


Figure 3.3. The map K.

PROPOSITION 3.12. *For any* (x,y,z) *in* $\mathcal{T}$, $K(x,y,z)=(x,t,y)$ *for some* t *in* $\mathbf{F}$. *The map* K *preserves each leaf of the foliation* $\mathcal{U}^{0,-}$.

Proof. This follows from Proposition 3.10 (vi). □

3.4. Tripods, measures and metrics

Let us equip once and for all $\mathcal{G}$ with a Riemannian metric d invariant under the left action of G, as well as the action of ω. We will denote by d_0 the metric on G_0 such that $d(\tau\cdot g, \tau\cdot h)=d_0(g,h)$ for all tripods τ, and observe that d_0 is left invariant. The associated Lebesgue measure is now both left invariant by $\mathrm{Aut}(\mathsf{G})$ and right invariant by $\mathrm{Aut}(\mathsf{G}_0)$.

We denote by $\mathrm{Sym}(\mathsf{G})$ the symmetric space of G seen as the space of Cartan involutions of $\mathfrak{g}$. Let us first recall some facts about the totally geodesic space $\mathrm{Sym}(\mathsf{G})$.

Let H be a reductive subgroup of G. There is a Cartan involution i in $\mathrm{Sym}(\mathsf{G})$ such that $i(\mathfrak{h})=\mathfrak{h}$. Then, the H-orbit of i is a totally geodesic subspace of $\mathrm{Sym}(\mathsf{G})$ isometric to $\mathrm{Sym}(\mathsf{H})$ — we then say *of type* H.

Any two totally geodesic spaces H_1 and H_2 of the same type are *parallel*: that is for all $x_i\in H_i$, $\inf(d(x_i,y):y\in H_{i+1})$ is constant and equal by definition to the distance $h(H_1,H_2)$.

The space of parallel totally geodesic subspaces to a given one is isometric to $\mathrm{Sym}(\mathsf{Z}_\mathsf{G})$ if Z_G is the centralizer of H, and in particular reduced to a point if Z_G is compact.

3.4.1. Totally geodesic hyperbolic planes

By assumption (3.1), if τ is a tripod, the Cartan involution

$$\boldsymbol{i}_\tau := \tau\circ\boldsymbol{i}_0\circ\tau^{-1},$$

sends the correct $\mathfrak{sl}_2$-triple (a,x,y) associated to the tripod τ to $(-a,y,x)$. It follows that the image of a right $\mathsf{SL}_2(\mathbb{R})$-orbit gives rise to a totally geodesic embedding of the hyperbolic plane denoted by η_τ, which we call *correct* and which is equivariant under the action of a correct $\mathsf{SL}_2(\mathbb{R})$.

Observe also that a totally geodesic embedding of $\mathbf{H}^2$ in $\mathrm{Sym}(\mathsf{G})$ is the same thing as a totally geodesic hyperbolic plane H in $\mathrm{Sym}(\mathsf{G})$, with three given points in the boundary at infinity in H.

Let us consider the space $\mathcal{H}$ of correct totally geodesic maps from $\mathbf{H}^2$ to the symmetric space $\mathrm{Sym}(\mathsf{G})$.

PROPOSITION 3.13. *The space $\mathcal{H}$ is equipped with a transitive action of* $\mathrm{Aut}(\mathsf{G})$ *and a right action of* $\mathsf{SL}_2(\mathbb{R})$*.*

We have also have $\mathsf{SL}_2(\mathbb{R})\times\mathsf{G}$ *equivariant maps*

$$\mathcal{G}\longrightarrow\mathcal{H}\longrightarrow\mathcal{T},$$

$$\tau\longmapsto\eta_\tau\longmapsto\partial\tau,$$

such that the composition is the map ∂ which associates to a tripod its vertices. Moreover, if the centralizer of the correct $\mathfrak{sl}_2$-triple is compact, then the map $\mathcal{H}\to\mathcal{T}$ is bijective and we shall identify $\mathcal{H}=\mathcal{T}$.

Proof. We described above that map $\tau\mapsto\eta_\tau$. By construction, this map is $\mathsf{SL}_2(\mathbb{R})\times\mathsf{G}$ equivariant.The map ∂ from $\mathcal{G}$ to $\mathcal{T}$ obviously factors through this map.

If the centralizer of a correct $\mathsf{SL}_2(\mathbb{R})$ in G is compact, then all correct parallel hyperbolic planes are identical. The result follows. □

From this point of view, a tripod τ defines the following:

(i) a totally geodesic hyperbolic plane $\mathbf{H}^2_\tau$ in $S(\mathsf{G})$, with three preferred points denoted by $\tau(0)$, $\tau(\infty)$ and $\tau(1)$ in $\partial_\infty\mathbf{H}^2_\tau$;

(ii) an $\mathsf{SL}_2(\mathbb{R})$-equivariant map ξ^τ from $\partial_\infty\mathbf{H}^2_\tau$ to $\mathbf{F}$, such that

$$\xi^\tau((\tau(0),\tau(\infty),\tau(1))=\partial\tau.$$

3.4.2. Metrics, cones, and projection on the symmetric space

Definition 3.14. (Projection and metrics) We define the *projection* from $\mathcal{G}$ to $\mathrm{Sym}(\mathsf{G})$ to be the map

$$s\colon\tau\longmapsto s(\tau):=\eta_\tau(i).$$

In other words, $s(\tau)$ is the orthogonal projection of $\tau(1)$ on the geodesic $]\tau(0),\tau(\infty)[$. See Figure 3.4.

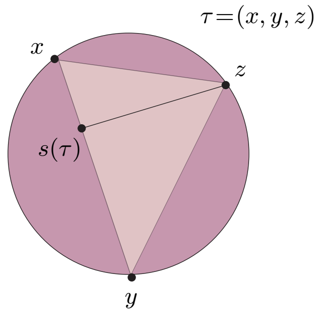


Figure 3.4. Projection.

The metric on $\mathfrak{g}$ associated to $s(\tau)$ is denoted by d_τ and so are the associated Riemannian metric on $\mathbf{F}$ — seeing $\mathbf{F}$ as a subset of the Grassmannian of $\mathfrak{g}$ — and the right-invariant metric on G defined by

$$d_\tau(g, h) = \sup\{d_\tau(g(x), h(x)) : x \in \mathbf{F}\}. \tag{3.2}$$

As a particular case, a triple τ of three pairwise distinct points in $\mathbf{P}^1(\mathbb{R})$ defines a metric d_τ on $\mathbf{P}^1(\mathbb{R})$ — such that $\mathbf{P}^1(\mathbb{R})$ is isometric to S^1 — that is called the *visual metric* of τ. The following properties of the assignment $\tau \mapsto d_\tau$, for d_τ a metric on $\mathbf{F}$ will be crucial:

(i) for every g in G, $d_{g\tau}(g(x), g(y)) = d_\tau(x, y)$;

(ii) the circle map associated to any tripod τ is an isometry from $\mathbf{P}^1(\mathbb{R})$ equipped with the visual metric of $(0, 1, \infty)$ to $\mathbf{F}$ equipped with d_τ.

3.4.3. Elementary properties

PROPOSITION 3.15. (i) *For all tripod τ, $d_\tau = d_{\overline{\tau}}$.*

(ii) *If the stabilizer of $\mathfrak{s}$ is compact, d_τ only depends on $\partial\tau$.*

Proof. The first item comes from the fact that d_τ only depends on $s(\tau)$. For the second item, in that case the map $\eta_\tau \mapsto \partial_\tau$ is an isomorphism by Proposition 3.13. ☐

PROPOSITION 3.16. (Metric equivalences) *For every positive numbers A and ε, there exists a positive number B such that, if $\tau, \tau' \in \mathcal{T}$ are tripods and g is in G, then*

$$d_\tau(g, \mathrm{Id}) \leqslant \varepsilon \text{ and } d(\tau, \tau') \leqslant A \quad \Longrightarrow \quad d_\tau(g, \mathrm{Id}) \leqslant B \cdot d_{\tau'}(g, \mathrm{Id}).$$

Similarly, for all u, v in $\mathbf{F}$ and $g \in \mathsf{G}$,

$$d(\tau, \tau') \leqslant A \quad \Longrightarrow \quad d_\tau(u, v) \leqslant B \cdot d_{\tau'}(u, v), \tag{3.3}$$

$$d(\tau, g\tau) \leqslant \varepsilon \quad \Longrightarrow \quad d_\tau(g, \mathrm{Id}) \leqslant B \cdot d(\tau, g\tau), \tag{3.4}$$

Proof. Let $U(\varepsilon)$ be a compact neighborhood of Id. The G-equivariance of the map $d\colon\tau\mapsto d_\tau$ implies the continuity of d seen as a map from $\mathcal{G}$ to $C^1(U(\varepsilon)\times U(\varepsilon))$ equipped with uniform convergence. The first result follows. The second assertion follows by a similar argument. For the inequality (3.4), let us fix a tripod τ_0. The metrics

$$(g,h)\longmapsto d_{\tau_0}(g,h) \quad\text{and}\quad (g,h)\longmapsto d(h^{-1}\cdot\tau_0, g^{-1}\cdot\tau_0),$$

are both right-invariant Riemannian metrics on G. In particular, they are locally biLipschitz, and thus there exists some B such that

$$d(\tau_0, g\tau_0)\leqslant\varepsilon \quad\Longrightarrow\quad d_{\tau_0}(g,\mathrm{Id})\leqslant B\cdot d(g_0^{-1}\cdot\tau_0,\tau_0)=B\cdot d(\tau_0, g\cdot\tau_0).$$

We now propagate this inequality to any tripod using the equivariance: writing $\tau=h\cdot\tau_0$, and assuming $d(\tau, g\cdot\tau)\leqslant\varepsilon$, we get that

$$d(\tau_0, h^{-1}gh\cdot\tau_0)=d(h\cdot\tau_0, gh\cdot\tau)=d(\tau, g\cdot\tau)\leqslant\varepsilon.$$

Thus, according to the previous implication,

$$d_{\tau_0}(h^{-1}gh,\mathrm{Id})\leqslant B\cdot d(\tau_0, h^{-1}gh\cdot\tau_0)=B\cdot d(\tau, g\cdot\tau).$$

The result follows from the equalities

$$d_{\tau_0}(h^{-1}gh,\mathrm{Id})=d_{h\cdot\tau_0}(gh,h)=d_\tau(g,\mathrm{Id}).$$ □

As a corollary, we have the following.

Corollary 3.17. (ω is uniformly Lipschitz) *There exists a constant C such that, for all τ,*

$$\frac{1}{C}d_\tau\leqslant d_{\omega(\tau)}\leqslant C\cdot d_\tau.$$

3.4.4. Aligning tripods

We explain a slightly more sophisticated way to control tripod distances.

Let τ_0 and τ_1 be two coplanar tripods associated to a totally geodesic hyperbolic plane $\mathbf{H}^2$ and a circle C identified with $\partial_\infty\mathbf{H}^2$ such that $z_1, z_0\in C$. We say that

$$(z_0,\tau_0,\tau_1,z_1)$$

are *aligned* if there exists a geodesic γ in $\mathbf{H}^2$, passing through $s(\tau_0)$ and $s(\tau_1)$, starting at z_0 and ending in z_1. In the generic case where $s(\tau_0)$ is different from $s(\tau_1)$, z_1 and z_0 are uniquely determined.

We first have the following property which is standard for $\mathsf{G}=\mathsf{SL}(2,\mathbb{R})$.

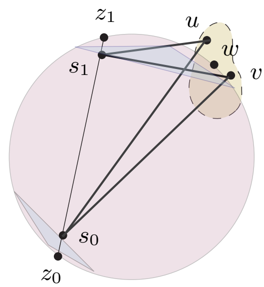


Figure 3.5. Aligning tripods.

Proposition 3.18. (Aligning tripods) *There exist positive constants $\boldsymbol{K}$, c and α_0 depending only on G such that, if (z_0,τ_0,τ_1,z_1) are aligned and associated to a circle $C\subset\mathbf{F}$ the following holds: if w in C satisfies $d_{\tau_1}(w,z_1)\leqslant\frac{3}{4}\pi$, then we have*

$$d_{\tau_1}(w,u)\leqslant\alpha_0,\ d_{\tau_1}(w,v)\leqslant\alpha_0 \quad\Longrightarrow\quad d_{\tau_0}(u,v)\leqslant\tfrac{1}{4}\boldsymbol{K}e^{-cd(\tau_0,\tau_1)}\cdot d_{\tau_1}(u,v). \tag{3.5}$$

Proof. There exists a correct $\mathfrak{sl}_2$-triple $s=(a,x,y)$ preserving the totally geodesic plane $\mathbf{H}^2_{\tau_0}$ such that the 1-parameter group $\{\lambda_t\}_{t\in\mathbb{R}}$ generated by a fixes C and has z_1 as an attractive fixed point and z_0 as a repulsive fixed point in $\mathbf{F}$. Let t_1 the positive number defined by $\lambda_{t_1}(s(\tau_0))=s(\tau_1)$.

Recall that, by construction, d_τ only depends on $s(\tau)$. Let $B\subset C$ be the closed ball of center z_1 and radius $\frac{3}{4}\pi$ with respect to d_{τ_1}. Observe that B lies in the basin of attraction of H, and so does a closed neighborhood U of B. In particular, we have that the 1-parameter group H converges C^1-uniformly to a constant on U. Thus, there exist $K_0, d>0$ such that

$$d_{\tau_1}(\lambda_t(u),\lambda_t(v))\leqslant K_0e^{-dt}\cdot d_{\tau_1}(u,v)\quad\text{for all } u,v\in U \text{ and all } t\geqslant 0.. \tag{3.6}$$

Recall that, for all u,v in $\mathbf{F}$, since $s(\lambda_{-t_1}(\tau_1))=s(\tau_0)$,

$$d_{\tau_1}(\lambda_{t_1}(u),\lambda_{t_1}(u))=d_{\lambda_{-t_1}(\tau_1)}(u,v)=d_{\tau_0}(u,v). \tag{3.7}$$

Finally, there exists $\alpha>0$ depending only on G such that, for any w in B, the ball B_w of radius α with respect to d_{τ_1} lies in U. Thus, combining (3.6) and (3.7), we get

$$d_{\tau_0}(u,v)\leqslant K_0\cdot e^{-dt}d_{\tau_1}(u,v).$$

This concludes the proof of statement (3.5), since there exists constants B and C such that

$$d(\tau_0,\tau_1)\leqslant Bt_1+C. \qquad\square$$

3.5. The contraction and diffusion constants

The constant $\boldsymbol{K}$ defined in Proposition 3.18 is called the *diffusion constant* and $\boldsymbol{\kappa}:=\boldsymbol{K}^{-1}$ is called the *contraction constant*.

4. Quasi-tripods and finite paths of quasi-tripods

We now want to describe a coarse geometry in the flag manifold; our main devices will be the following: paths of quasi-tripods and coplanar paths of tripods. Since not all triple of points lie in a circle in $\mathbf{F}$, we need to introduce a deformation of the notion of tripods. This is achieved through the definition of *quasi-tripod* (Definition 4.1).

A *coplanar path of tripods* is just a sequence of non-overlapping ideal triangles in some hyperbolic plane such that any ideal triangle have a common edge with the next one. Then, a path of quasi-tripods is a deformation of such a coplanar path which can also be described as a *model*, deformed by a sequence of specific elements of G.

Our goal is the following. The common edges of a coplanar path of tripods, considered as intervals in the boundary at infinity of the hyperbolic plane, defines a sequence of nested intervals. We want to show that, in certain circumstances, the corresponding chords of the deformed path of quasi-tripods are still nested in the deformed sense that we will introduce in the following sections.

One of our main result is then the confinement lemma (Lemma 6.1) which guarantees nesting.

4.1. Quasi-tripods

Quasi-tripods will make sense of the notion of a "deformed ideal triangle". Related notions are defined: swished quasi-tripods and the *foot map*.

Definition 4.1. (Quasi-tripod) An ε-*quasi-tripod* is a quadruple

$$\theta=(\dot{\theta},\theta^-,\theta^+,\theta^0)\in\mathcal{G}\times\mathbf{F}^3$$

such that

$$d_{\dot{\theta}}(\partial^+\dot{\theta},\theta^+))\leqslant\varepsilon,\quad d_{\dot{\theta}}(\partial^-\dot{\theta},\theta^-))\leqslant\varepsilon\quad\text{and}\quad d_{\dot{\theta}}(\partial^0\dot{\theta},\theta^0)\leqslant\varepsilon.$$

The set $\partial\theta:=\{\theta^+,\theta^-,\theta^0\}$ is the set of *vertices* of θ, and $\dot{\theta}$ is the *interior* of θ . An ε-quasi-tripod τ is *reduced* if $\partial^\pm\dot{\tau}=\tau^\pm$.

Obviously, a tripod defines an ε-quasi-tripod for all ε. Moreover, some of the actions defined on tripods in §3.3 extend to ε-quasi-tripods. Most notably, we have an action of a cyclic permutation ω of order 3 on the set of quasi-tripods, given by

$$\omega(\dot{\theta}, \theta^-, \theta^+, \theta^0) = (\omega(\dot{\theta}), \theta^+, \theta^0, \theta^-).$$

By Corollary 3.17, we have the following result.

PROPOSITION 4.2. *There is a constant* $\mathbf{M}$ *depending only on* G *such that, if* θ *is an* ε*-quasi-tripod then,* $\omega(\theta)$ *is an* $\mathbf{M}\varepsilon$*-quasi-tripod.*

4.1.1. A foot map

For any positive β, let us consider the following G-stable set:

$$W_\beta := \{(\tau, a^+, a^-) : \tau \in \mathcal{G}, a^\pm \in \mathbf{F}, d_\tau(a^\pm, \partial^\pm \tau) \leqslant \beta\} \subset \mathcal{G} \times \mathbf{F}^2.$$

We equip W_β with the Riemannian metric which is d_τ for every fiber $\{\tau\} \times F^2$ and d on the first factor.

LEMMA 4.3. *There exist positive numbers* β *and* $\mathbf{M}_1$, *and a smooth* G*-equivariant map* $\Psi: W_\beta \to \mathcal{G}$ *such that the following statements hold*:

(i) $\partial^\pm \Psi(\tau, a^+, a^-) = a^\pm$;

(ii) $d(\tau, \Psi(\tau, a^+, a^-)) \leqslant M \sup(d_\tau(a^\pm, \partial^\pm \tau))$;

(iii) Ψ *is* $\mathbf{M}_1$*-Lipschitz.*

Moreover, the choice of Ψ *only depends on the choice of a* G*-invariant metric on* $\mathcal{G}$. *We choose* Ψ *in such a way that given a finite group of isometries* F *of* $\mathcal{G}$, *if* f *in* F *is such that for all tripods* τ *we have* $\partial^\pm f(\tau) = \partial^\mp \tau$, *then*

$$\Psi(f(\tau), x, y) = f(\Psi(\tau, y, x)). \tag{4.1}$$

Proof. For a transverse pair $a = (a^+, a^-)$ in $\mathbf{F}$, let $\mathcal{G}_a$ be the set of tripods τ in $\mathcal{G}$ such that $\partial^\pm \tau = a^\pm$. Let G_a the stabilizer of the pair (a^+, a^-). Let us choose a left-invariant metric on $\mathcal{G}$ invariant by the finite group F. Let us fix (in a G-equivariant way) a small enough tubular neighborhood N_a of $\mathcal{G}_a$ in $\mathcal{G}$ for all transverse pairs $a = (a^+, a^-)$, as well as a G_a-equivariant projection Π_a from N_a to $\mathcal{G}_a$.

Observe that, fixing τ_0, for all α there exists ε such that, if $b = (b^+, b^-)$ is such that $d_{\tau^0}(\partial^\pm \tau^0, b^\pm)$ is less than ε, then there exists τ_1 in $\mathcal{G}_b$ such that

$$d(\tau_1, \tau_0) \leqslant \alpha.$$

Thus, using the G-equivariance of the metric, there exists β such that

$$d_\tau(\partial^\pm\tau, a^\pm) \leqslant \beta \quad \text{implies} \quad \tau \in N_a.$$

We now define

$$\Psi(\tau, a^+, a^-) := \Pi_a(\tau).$$

By G-equivariance, Ψ is uniformly Lipschitz. □

Definition 4.4. (Foot map and feet) A map Ψ satisfying the conclusion of the lemma is called a *foot map*. For ε small enough, we define the *feet* $\psi_1(\theta)$, $\psi_2(\theta)$ and $\psi_3(\theta)$ of the ε-quasi-tripod $\theta=(\dot{\theta}, \theta^-, \theta^+, \theta^0)$ as the three tripods which are respectively defined by

$$\psi_1(\theta) := \Psi(\dot{\theta}, \theta^-, \theta^+), \quad \psi_2(\theta) := \psi_1(\omega(\theta)) \quad \text{and} \quad \psi_3(\theta) := \psi_1(\omega^2(\theta)),$$

where Ψ is the foot map defined in the preceding section.

By the last item of Lemma 4.3, for an ε-quasi-tripod θ, we have

$$d(\psi_i(\theta), \omega^{i-1}(\dot{\theta})) \leqslant \mathbf{M}_1\varepsilon, \tag{4.2}$$

Observe also that, for ε small enough, there exists a constant $\mathbf{M}_2$ depending only on G such that, for ε small enough, if θ is an ε-quasi-tripod, then

$$d(\omega(\psi_1(\theta)), \psi_2(\theta)) \leqslant \mathbf{M}_2\varepsilon \quad \text{and} \quad d(\omega(\psi_2(\theta)), \psi_3(\theta)) \leqslant \mathbf{M}_2\varepsilon. \tag{4.3}$$

Using the triangle inequality, this is a consequence of the previous inequality and the assumption that ω is an isometry for d.

4.1.2. Foot map and flow

The following property explains how well the foot map behaves with respect to the flow action.

Proposition 4.5. (Foot and flow) *There exist positive constants β_1 and $\mathbf{M}_3$ with the following property. Let $\varepsilon \leqslant \beta_1$, $x_0 \in \mathcal{G}$ and $x_1 := \varphi_R(x_0)$ for some R. Let $a=(a^+, a^-)$ be a transverse pair of flags $\mathbf{F}$ such that $d_{x_i}(a^\pm, \partial^\pm x_i) \leqslant \varepsilon$. Then,*

$$d(y_1, \varphi_R(y_0)) \leqslant \mathbf{M}_3\varepsilon,$$

where $y_i = \Psi(x_i, a^+, a^-)$.

Proof. In the proof, $\mathbf{N}_i$ will denote a constant depending only on G.

It is enough to prove the weaker result that there exist z_0 and z_1, with $z_1{=}\varphi_R(z_0)$, in $\mathcal{G}_a$ such that $d(z_i,x_i)\leqslant\mathbf{N}_1\varepsilon$. Indeed, it first follows that $d(z_i,y_i)\leqslant\mathbf{N}_2\varepsilon$, by the triangle inequality. Secondly, G_a is a central leaf of the foliation, and the flow acts by isometries on it (see property (v) of Proposition 3.10). As a consequence, $d(y_1,\varphi_R(y_0))\leqslant\mathbf{N}_3\varepsilon$, and the result follows.

Observe first that $d(x_i,y_i)\leqslant\mathbf{N}_4\varepsilon$, by definition of a foot map. Assume $R{>}0$. Let $x^\pm{=}\partial^\pm x_0{=}\partial^\pm x_1$. Let us first assume that $x^+{=}a^+$. Thus, by the contraction property (see Proposition 3.10),

$$d(\varphi_R(x_0),\varphi_R(y_0))\leqslant\mathbf{N}_5\varepsilon.$$

It follows by the triangle inequality that

$$d(\varphi_R(y_0),y_1)\leqslant\mathbf{N}_6\varepsilon.$$

Thus, this works with $z_0{=}y_0$ and $z_1{=}\varphi_R(z_0)$.

The same results hold symmetrically whenever $x^-{=}a^-$ by taking

$$z_1=y_1\quad\text{and}\quad z_0=\varphi_{-R}(z_1).$$

The general case follows by considering intermediate projections. First (as a consequence of our initial argument) we find w_0 and $w_1{=}\varphi_R(w_0)$ in $\mathcal{G}_{a^+,x^-}$ with

$$d(w_i,x_i)\leqslant\mathbf{N}_7\varepsilon.$$

Applying now the symmetric argument with the pair (w_0,w_1) and projection on $\mathcal{G}_{a^+,x^-}$, we get z_0 and $z_1{:=}\varphi_R(z_0)$ such that $d(w_i,z_i)\leqslant\mathbf{N}_8\varepsilon$.

A simple combination of triangle inequalities yields the result. □

4.1.3. Swishing quasi-tripods

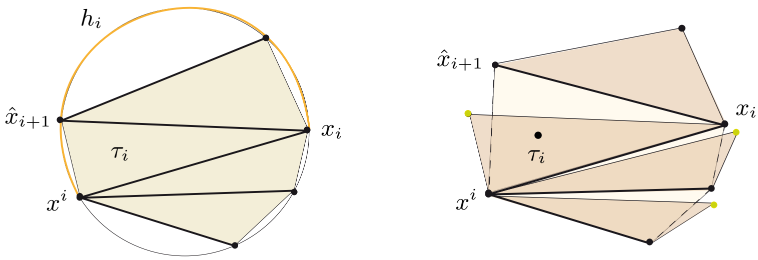
Definition 4.6. (Swishing quasi-tripods) The ε-quasi-tripod θ' is (R,α)-*swished* from the ε-quasi-tripod θ if the following conditions hold:

(i) $\partial^\pm\theta{=}\partial^\mp\theta'$;

(ii) the tripods $\psi_1(\theta')$ and $\overline{\varphi_R(\psi_1(\theta))}$ are α-close.

Being swished is a reciprocal condition.

Proposition 4.7. *If θ' is (R,α)-swished from θ, then θ is (R,α)-swished from θ'.*



(a) Coplanar path of tripods. (b) Path of reduced quasi-tripods.

Figure 4.1. A deformation of a path of quasi-tripods.

Proof. We have

$$d(\sigma(\varphi_R(\theta)),\theta') = d(\varphi_R(\theta),\sigma(\theta')).$$

Since $\partial^{\pm}\theta = \partial^{\mp}\theta'$, we have $\sigma(\theta') = (\theta)g$ for some $g \in L_0$. Since, by Proposition 3.10 (v), φ_R acts by isometries on the orbits of L_0, we get

$$d(\sigma(\varphi_R(\theta)),\theta') = d(\varphi_R(\theta),\sigma(\theta')) = d(\theta,\varphi_{-R}(\sigma(\theta'))).$$

But, by Proposition 3.10 again, $\varphi_{-R}\circ\sigma = \sigma\circ\varphi_R$. The result follows. $\square$

4.2. Paths of quasi-tripods and coplanar paths of tripods

4.2.1. Swished paths of quasi-tripods and their model

Let $\underline{R}(N) = (R_0, ..., R_N)$ be a finite sequence of positive numbers.

Definition 4.8. (Coplanar path of tripods) An $\underline{R}(N)$*-swished coplanar path of tripods* is a sequence of tripods $\underline{\tau}(N) = (\tau_0, \ldots \tau_N)$ such that τ_{i+1} is R_i-swished from $\omega^{n_i}\tau_i$, where $n_i \in \{1,2\}$. The sequence $(n_1, ..., n_N)$ is the *combinatorics* of the path. See Figure 4.1a.

We remark that a coplanar path of tripods consists of pairwise coplanar tripods and is totally determined up to the action of G by $\underline{R}(N)$ and the combinatorics. These coplanar paths of tripods will represent the model situation and we need to deform them.

Definition 4.9. (Path of quasi-tripods) An $(\underline{R}(N),\varepsilon)$*-swished path of quasi-tripods* is a sequence of ε-quasi-tripods $\underline{\theta}(N) = (\theta_0, \ldots \theta_N)$ such that θ_{i+1} is (R_i,ε)-swished from $\omega^{n_i}\theta_i$, where $n_i \in \{1,2\}$. The sequence $(n_1, ..., n_N)$ is the *combinatorics* of the path.

A *model* of an $(\underline{R}(N),\varepsilon)$-swished path of quasi-tripods is an $\underline{R}(N)$-swished coplanar path of tripods with the same combinatorics. See Figure 4.1b.

Let us introduce some notation and terminology: $\partial\theta_i$, $\partial\theta_{i+1}$ and $\partial\theta_{i-1}$ have exactly one point in common denoted x_i and called the *pivot* of θ_i.

Remarks. Observe that, given a path of quasi-tripods, the following hold.

(i) There exists some constant M such that any $(\underline{R}(N),\varepsilon)$-swished path of quasi-tripods give rise to an $(\underline{R}(N), M\varepsilon)$-swished path of quasi-tripods with the same vertices, but which are all reduced. In the sequel, we shall mostly consider such reduced paths of quasi-tripods.

(ii) From the previous items, in the case of a reduced path, the sequence of triangles $(\theta_0, ..., \theta_N)$ is actually determined by the sequence of (not necessarily coplanar) tripods $(\dot{\theta}_0, ..., \dot{\theta}_N)$.

One immediately has the following.

Proposition 4.10. *Any $(\underline{R}(N),\varepsilon)$-swished path of quasi-tripods admits a model which is unique up to the action of* G.

4.2.2. Coplanar paths of tripods and sequence of chords

To a reduced path of quasi-tripods $\underline{\theta}(N)$ we associate a *path of chords*

$$\underline{h}(N) = (h_0, ..., h_N),$$

such that $h_i := h_{\dot{\theta}_i}$ has x_i and x^i as extremities. Observe, that the subsequence of triangles $(\theta_0, ..., \theta_{N-1})$ is actually determined by the sequence of chords $(h_0, ..., h_N)$.

In the sequel, by an abuse of language, we shall call the sequence of chords $\underline{h}(N)$ a path of quasi-tripods as well.

Observe that, for a coplanar path of tripods, the associated path of chords is such that (h_i, h_{i+1}) is nested.

4.2.3. Deformation of coplanar paths of tripods

Let $\underline{\tau} = (\tau_0, ..., \tau_N)$ be a coplanar path of tripods.

Definition 4.11. (Deformation of a path) A *deformation of* $\underline{\tau}$ is a sequence

$$v = (g_0, ..., g_{N_1})$$

with $g_i \in \mathsf{P}_{x_i}$, the stabilizer of x_i in G, where x_i is the pivot of τ_i. The deformation is an *ε-deformation* if furthermore $d_{\tau_i}(g_i, \mathrm{Id}) \leqslant \varepsilon$.

Given a deformation $v=(g_0, ..., g_{N-1})$, the *deformed path of quasi-tripods* is the path of quasi-tripods $\underline{\theta}^v=(\theta_0^v, ..., \theta_N^v)$ where

$$\begin{aligned} \theta_i^v &= (b_i\tau_i, b_i\tau_i^-, b_i\tau_i^+, b_{i+1}\tau_i^0), && \text{for } i<N, \\ \theta_N^v &= (b_N\tau_N, b_N\tau_N^-, b_N\tau_N^+, b_N\tau_N^0), && \text{for } i=N, \end{aligned}$$

where $b_0=\mathrm{Id}$ and $b_i=g_0\circ...\circ g_{i-1}$.

From the point of view of sequence of chords, the sequence of chords associated to the deformed coplanar path of tripods as above is

$$\underline{h}^v := (h_0^v, ..., h_N^v) := (b_0\cdot h_0, ..., b_N\cdot h_N),$$

where $(h_0, ..., h_N)$ is the sequence of chords associated to $\underline{\tau}$.

4.3. Deformation of coplanar paths of tripods and swished path of quasi-tripods

We want to relate our various notions and we have the following two propositions.

PROPOSITION 4.12. *There exists a constant* $\mathbf{M}$ *depending only on* G *such that, given an* $(\underline{R}(N), \varepsilon)$*-swished path of reduced quasi-tripods* $\underline{\theta}$ *with model* $\underline{\tau}$*, there exists a unique* $\mathbf{M}\varepsilon$*-deformation* v *such that* $\underline{\theta}=g\cdot\underline{\tau}^v$ *for some* g *in* G.

Proof. Given any path of quasi-tripods $\underline{\theta}$. Let x_i be the pivot of θ_i. We know that θ_{i+1} is (R_i, ε)-swished from $\omega^{n_i}\theta_i$.

Let then τ_i be such that $\psi_1(\theta_{i+1})$ is R_i-swished from τ_i, and symmetrically let τ_{i+1} be the tripod R_i-swished from $\omega^{n_i}\psi_1(\theta_i)$.

Since G acts transitively on the space of tripods and commutes with the right $\mathsf{SL}_2(\mathbb{R})$-action, there exists a unique g_i in P_{x_i} such that

$$g_i(\omega^{n_i}\psi_1(\theta_i)) = \tau_i \quad \text{and} \quad g_i(\tau_{i+1}) = \psi_1(\theta_{i+1}).$$

We have thus recovered $\underline{\theta}$ as a $(g_0, ..., g_{N-1})$-deformation of its model. It remains to show that this is an $\mathbf{M}\varepsilon$-deformation, for some $\mathbf{M}$.

Since θ_{i+1} is (R_i, ε)-swished from $\omega^{n_i}\theta_i$, we have $d(\tau_i, \psi_1(\omega^{n_i}\theta_i))\leqslant\varepsilon$. Moreover, since θ_i is a quasi-tripod, by inequality (4.3), we have

$$d(\omega^{n_i}\psi_1(\theta_i), \psi_1(\omega^{n_i}\theta_i)) \leqslant \mathbf{M}_2\varepsilon.$$

Thus, from the triangular inequality

$$d(g_i(\omega^{n_i}\psi_1(\theta_i)), \omega^{n_i}\psi_1(\theta_i)) \leqslant d(\tau_i, \psi_1(\omega^{n_i}\theta_i)) + d(\omega^{n_i}\psi_1(\theta_i), \psi_1(\omega^{n_i}\theta_i)),$$

we get

$$d(g_i(\omega^{n_i}\psi_1(\theta_i), \omega^{n_i}\psi_1(\theta_i)) \leqslant (\mathbf{M}_2+1)\varepsilon.$$

Then, inequality (3.4) and Corollary 3.17 yield

$$d_{\psi_1(\theta_i)}(g_i, \mathrm{Id}) \leqslant C^2 d_{\omega^{n_i}\psi_1(\theta_i)}(g_i, \mathrm{Id}) \leqslant B_0\varepsilon$$

for some constant B_0 depending only on G. Using Proposition 3.16, this yields that there exists $\mathbf{M}$ depending only on G such that

$$d_{\dot{\theta}_i}(g_i, \mathrm{Id}) \leqslant \mathbf{M}\varepsilon.$$

Hence the result. □

5. Cones, nested tripods and chords

We will describe geometric notions that generalize the inclusion of intervals in $\mathbf{P}^1(\mathbb{R})$ (which corresponds to the case of $\mathsf{SL}_2(\mathbb{R})$): we will introduce *chords* which generalize intervals, as well as the notions of squeezing and nesting which replace — in a quantitative way — the notion of inclusion for intervals. We will study how nesting and squeezing is invariant under perturbations.

Our motto in this paper is that we can phrase all the geometry that we need using the notions of tripods and their associated dynamics, circles and the assignment of a metric to a tripod. These will be the basic geometric objects that we will manipulate throughout all the paper.

5.1. Cones and nested tripods

Definition 5.1. (Cone and nested tripods) Given a tripod τ and a positive number α, the α-*cone* of τ is the subset of $\mathbf{F}$ defined by

$$C_\alpha(\tau) := \{u \in \mathbf{F} : d_\tau(\partial^0\tau, u) \leqslant \alpha\}.$$

Let α and κ be positive numbers. A pair of tripods (τ_0, τ_1) is (α,κ)-*nested* if

$$C_\alpha(\tau_1) \subset C_{\kappa\cdot\alpha}(\tau_0), \tag{5.1}$$

$$d_{\tau_0}(u,v) \leqslant \kappa \cdot d_{\tau_1}(u,v) \quad \text{for all } u, v \in C_\alpha(\tau_1). \tag{5.2}$$

We write this symbolically as

$$C_\alpha(\tau_1) \prec \kappa \cdot C_{\kappa\cdot\alpha}(\tau_0).$$

The following immediate transitivity property justifies our symbolic notation.

LEMMA 5.2. (Composing cones) *Assume* (τ_0,τ_1) *is* $(\alpha\cdot\kappa_2,\kappa_1)$*-nested and* (τ_1,τ_2) *is* (α,κ_2)*-nested, then* (τ_0,τ_2) *is* $(\alpha,\kappa_1\cdot\kappa_2)$*-nested. in other words,*

$$C_\alpha(\tau_2)\prec\kappa_2 C_{\kappa_2\alpha}(\tau_1)\ \textit{and}\ C_{\kappa_2\alpha}(\tau_1)\prec\kappa_1 C_{\kappa_1\kappa_2\alpha}(\tau_0)\implies C_\alpha(\tau_2)\prec\kappa_1\kappa_2 C_{\kappa_1\kappa_2\alpha}(\tau_0).$$

5.1.1. Convergent sequence of cones

We say that a sequence of tripods $\{\tau_i\}_{i\in\{1,\ldots,N\}}$ — where N is finite of infinite — defines a (α,κ)-*contracting sequence of cones* if, for all i, the pair (τ_i,τ_{i+1}) is (α,κ)-nested and $\kappa<\frac{1}{2}$.

As a corollary of Lemma 5.2, one gets the following result.

COROLLARY 5.3. (Convergence corollary) *There exists a positive constant* α_3 *such that, if* $\{\tau_i\}_{i\in\mathbb{N}}$ *defines an infinite* (α,κ)*-contracting sequence of cones, with* $\kappa<\frac{1}{2}$ *and* $\alpha\leqslant\alpha_3$*, then there exists a point* $x\in\mathbf{F}$ *called the* limit of the contracting sequence of cones *such that*

$$\bigcap_{i=1}^{\infty}C_\alpha(\tau_i)=\{x\}.$$

Moreover, for all n*, all* q *and all* u,v *in* $C_\alpha(\tau_{n+q})$*, we have*

$$d_{\tau_n}(u,v)\leqslant\frac{1}{2^q}d_{\tau_{n+q}}(u,v)\leqslant\frac{1}{2^{q-1}}\alpha. \tag{5.3}$$

We then write

$$x=\lim_{i\to\infty}\tau_i.$$

Proof. This follows at once from the fact that

$$C_\alpha(\tau_{n+p})\prec\frac{1}{2^p}C_{\alpha/2^p}(\tau_n).$$ □

5.1.2. Deforming nested cones

The next proposition will be very helpful in the sequel by proving that the notion of being nested is stable under sufficiently small deformations.

LEMMA 5.4. (Deforming nested pair of tripods) *There exists a constant* β_0 *depending only on* G *such that, for all* $\beta\leqslant\beta_0$*, if*

- *the pair of tripod* (τ_0,τ_1) *is* $\left(\beta,\frac{1}{2}\kappa\right)$*-nested,*
- *the element* g *in* G *is such that* $d_{\tau_0}(\mathrm{Id},g)\leqslant\frac{1}{2}\kappa\cdot\beta$*,*

then, the pair $(\tau_0,g(\tau_1))$ *is* (β,κ)*-nested*

Proof. Let $z=\partial^0\tau_0$. It is equivalent to prove that $(g^{-1}(\tau_0),\tau_1)$ is (β,κ)-nested. Let $u\in C_\beta(\tau_1)\subset C_{\kappa\cdot\beta/2}(\tau_0)$. In particular, $d_{\tau_0}(u,z)\leqslant\frac{1}{2}\kappa\cdot\beta$. It follows that

$$d_{g^{-1}(\tau_0)}(u,g^{-1}(z))=d_{\tau_0}(g(u),z)\leqslant d_{\tau_0}(g(u),u)+d_{\tau_0}(u,z)\leqslant\tfrac{1}{2}\kappa\cdot\beta+\tfrac{1}{2}k\cdot\beta=k\cdot\beta.$$

Thus,

$$C_\beta(\tau_1)\subset C_{k\cdot\beta/2}(\tau_0)\subset C_{k\cdot\beta}(g^{-1}(\tau_0)).$$

Moreover, for β small enough, by Proposition 3.16, we have $d_{\tau_0}\leqslant 2d_{g^{-1}(\tau_0)}$. Thus, for all $(u,v)\in C_\beta(\tau_1)$,

$$d_{g^{-1}\tau_0}(u,v)\leqslant 2d_{\tau_0}(u,v)\leqslant\kappa d_{\tau_1}(u,v).$$

It follows that $(g^{-1}(\tau_0),\tau_1)$ is (β,k)-nested. □

5.1.3. Sliding out

LEMMA 5.5. *There exist constants k and δ_0 depending only on the group G such that, if τ_0 is a tripod R-swished from τ_1, then*

$$d_{\tau_0}(u,v)\leqslant k\cdot d_{\tau_1}(u,v)\quad \textit{for all } u,v\in C_{\delta_0}(\tau_1).$$

Proof. This is an immediate consequence of Proposition 3.18, with (for $R>0$)

$$z_0=\partial^-\tau_1,\quad z_1=\partial^+\tau_1\quad\text{and}\quad w=\partial^0\tau_1.$$

The case $R<0$ being symmetric. □

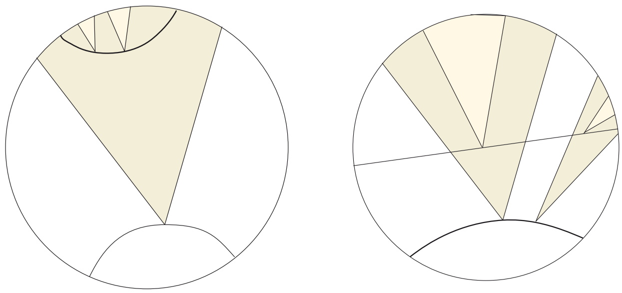
5.2. Chords and slivers

A *chord* is an orbit of the shearing flow. We denote by h_τ the chord associated to a tripod τ, and denote $\check{h}_\tau:=h_{\sigma(\tau)}$. Observe that all pairs of tripods in $\check{h}_\tau\times h_\tau$ are coplanar. We also say that h_τ goes from $\partial^-\tau$ to $\partial^+\tau$ which are its *endpoints*.

The α-*sliver* of a chord H is the subset of $\mathbf{F}$ defined by

$$S_\alpha(H):=\bigcup_{\tau\in H}C_\alpha(\tau)\subset\mathbf{F}.$$

In particular, $S_0(H)=\{\partial^0\tau:\tau\in H\}$. Observe that two points a and b in the closure of $S_0(H)$ define a unique chord H_{ab} which is coplanar to H and is such that $S_0(H_{a,b})$ is a subinterval of $S_0(H)$ with endpoints a and b.



(a) Squeezed chords.

(b) Controlled chords.

Figure 5.1. Controlled and squeezed chords.

5.2.1. Nested, squeezed and controlled pairs of chords

We shall need the following definitions.

(i) The pair (H_0, H_1) of chords is *nested* if $H_0 \neq H_1$, H_0 and H_1 are coplanar and $S_0(H_1) \subset S_0(H_0)$. Given a nested pair (H_0, H_1)—with no endpoints in common—the *projection* of H_1 on H_0 is the tripod $\tau_0 \in H_0$ such that $s(\tau_0)$ is the closest point in the geodesic joining the endpoints of H_0, to the geodesic joining the endpoints of H_1. Observe finally that, if (H_0, H_1) is nested, then every pair of tripods in $H_0 \cup H_1$ is coplanar.

(ii) The pair (H_0, H_1) of chords is (α, k)-*squeezed* if there exists $\tau_0 \in H_0$ such that

$$(\tau_0, \tau_1) \text{ is } (\alpha, k)\text{-nested} \quad \text{for all } \tau_1 \in H_1.$$

The tripod τ_0 is called a *commanding tripod* of the pair.

(iii) The pair (H_0, H_1) of chords is (α, k)-*controlled* if, for all $\tau_1 \in H_1$, there exists $\tau_0 \in H_0$ such that

$$(\tau_0, \tau_1) \text{ is } (\alpha, k)\text{-nested}.$$

(iv) The *shift* of two chords H_0 and H_1 is

$$\delta(H_0, H_1)) := \inf\{d(\tau_0, \tau_1) : \tau_0 \in H_0 \text{ and } \tau_1 \in H_1\}.$$

5.2.2. Squeezing nested pair of chords

In all the sequel $\boldsymbol{K}$ is the diffusion constant defined in Proposition 3.18 and $\boldsymbol{\kappa} = \boldsymbol{K}^{-1}$ is the contraction constant.

The following proposition provides our first example of nested pairs of chords in the coplanar situation.

PROPOSITION 5.6. (Nested pair of chords) *There exists β_1 depending only on G, and a decreasing function*

$$\ell\colon]0,\beta_1] \longrightarrow \mathbb{R},$$

such that, for any positive number β, with $\beta \leqslant \beta_1$, any nested pair (H_0, H_1) with

$$\delta(H_0, H_1) \geqslant \ell(\beta)$$

is $(\boldsymbol{K}\beta, \boldsymbol{\kappa}^9)$-squeezed. The projection τ_0 of H_1 on H_0 is a commanding tripod of (H_0, H_1).

Observe in particular that $S_0(H_1) \subset S_{\boldsymbol{K}\beta}(H_1) \subset C_{\boldsymbol{\kappa}^8\beta}(\tau_0)$. The choice of $\boldsymbol{\kappa}^9$ is rather arbitrary in this proposition, but will make our life easier later on.

Proof. Let $\tau_1 \in H_1$. Let then $\check{\tau}_0 \in H_0$, with $\partial^0\check{\tau}_0 = \partial^0\tau_1$. As in §3.4.4, let s_0, s_1, z_0 and z_1 be constructed from $\check{\tau}_0$ and τ_1. One notices that $d_{\tau_1}(z, z_1) \leqslant \frac{1}{2}\pi$. Then, given ε, for $\delta(H_0, H_1)$ large enough the second part of Proposition 3.18 yields that $(\check{\tau}_0, \tau_1)$ is (α, ε)-nested.

Observe now that, for any β, there exists δ_1 such that $\delta(H_0, H_1) > \delta_1$ yields

$$d(\tau_0, \check{\tau}_0) \leqslant \beta,$$

where τ_0 is the projection of H_1 on H_0. Thus, using Proposition 3.16 for β small enough, we have that the pair of tripods $(\check{\tau}_0, \tau_1)$ is $(\alpha, 2\cdot\varepsilon)$-nested. In other words, since τ_0 is independent from the choice of τ_1, we have proved that the pair of chords (H_0, H_1) is $(\alpha, 2\cdot\varepsilon)$-squeezed for $\delta(H_0, H_1)$ large enough. ☐

5.2.3. Controlling nested pair of chords

Our second result about coplanar pair of chords is the following lemma.

LEMMA 5.7. (Controlling diffusion) *There exists a positive number β_2, with $\beta_2 \leqslant \beta_3$, depending only on G, such that, given a positive $\beta \leqslant \beta_2$ and a nested pair (H_0, H_1), then (H_0, H_1) is $(\xi\beta, \boldsymbol{K})$-controlled for all $\xi \leqslant 1$.*

Assume furthermore that $\ell_0 \geqslant \delta(H_0, H_1)$, where $\ell_0 \geqslant \ell(\beta)$. Then, given $\tau_1 \in H_1$, there exists H_2 such that

- *(H_1, H_2) is nested;*
- *$0 < \delta(H_0, H_2) \leqslant \ell_0$;*
- *(τ_0, τ_1) is $(\boldsymbol{\kappa}^2\beta, \boldsymbol{K})$-nested, where τ_0 is the projection of H_2 on H_0.*

Let us first prove the following proposition.

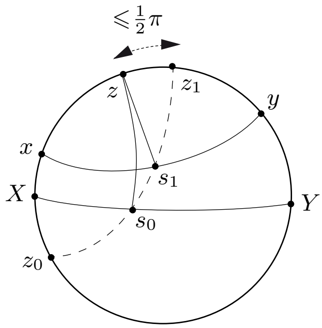


Figure 5.2. Aligned points and angle.

Proposition 5.8. *There exists α_4 with the following property. Let (H_0, H_1) be two nested chords and let $\tau_0 \in H_0$ and $\tau_1 \in H_1$ be such that, for some $\alpha \leqslant \alpha_4$,*

$$C_\alpha(\tau_0) \cap C_\alpha(\tau_1) \neq \varnothing.$$

Then, (τ_0, τ_1) are $(\alpha, \boldsymbol{K})$-nested.

Proof. Observe first that, if $(z_0, \tau_0, \tau_1, z_1)$ are aligned, then $d_{\tau_1}(\partial^0\tau_1, z_1) \leqslant \frac{1}{2}\pi$. See Figure 5.2.

Let $u, v \in C_\alpha(\tau_1)$ and $w \in C_\alpha(\tau_0) \cap C_\alpha(\tau_1)$. Then, by Proposition 3.18.

$$d_{\tau_0}(u, v) \leqslant \tfrac{1}{4}\boldsymbol{K} d_{\tau_1}(u, v), \tag{5.4}$$

$$d_{\tau_0}(u, \partial^0\tau_0) \leqslant d_{\tau_0}(u, w) + d_{\tau_0}(w, \partial^0\tau_0) \leqslant \tfrac{1}{4}\boldsymbol{K} d_{\tau_1}(u, w) + \alpha \leqslant \boldsymbol{K}\alpha. \tag{5.5}$$

Thus, from the second equation,

$$C_\alpha(\tau_1) \leqslant C_{\boldsymbol{K}\alpha}(\tau_1).$$

This concludes the proof of the proposition. □

Let us now move to the proof of Lemma 5.7.

Proof of Lemma 5.7. Let $\tau_1 \in H_1$. Let $\mathbf{H}^2$ be the hyperbolic associated plane to the coplanar pair (H_0, H_1). Then, there exists τ_0 in H_0 such that $\partial^0\tau_0 = \partial^0\tau_1$. In particular, $\partial^0\tau^0 \in C_{\xi\beta}(\tau_1) \cap C_{\xi\beta}(\tau_0)$ is non-empty, since it contains $\partial^0\tau_0$. Using Proposition 5.8, (τ_0, τ_1) is $(\xi\beta, \boldsymbol{K})$-nested, and this concludes the proof of the first assertion which is that (H_0, H_1) are nested.

Assume now that $\delta(H_0, H_1) \leqslant \ell_0$. Let H_3 be such that

$$S_0(H_3) = C_{\boldsymbol{\kappa}^2\beta}(\tau_1) \cap \partial_\infty \mathbf{H}^2.$$

We have two cases.

(1) If $\delta(H_0,H_3)\leqslant\ell_0$, we can take $H_2=H_3$, and let τ_0 be the projection of H_2 on H_0. Thus,

$$\partial^0\tau^0\in C_{\boldsymbol{\kappa}^2\beta}(\tau_1)\cap C_{\boldsymbol{\kappa}^2\beta}(\tau_0)\neq\varnothing,$$

and we conclude by Proposition 5.8: (τ_0,τ_1) is $(\boldsymbol{\kappa}^2\beta,\boldsymbol{K})$-nested.

(2) If $\delta(H_0,H_3)\geqslant\ell_0\geqslant\ell(\beta)$, a continuity argument shows the existence of H_2 such that the pairs (H_1,H_2) and (H_2,H_3) are nested, and $\delta(H_0,H_2)=\ell_0$. Let τ_0 be the projection of H_2 on H_0. Then, we have,

$$(C_{\boldsymbol{\kappa}^2\beta}(\tau_1)\cap\mathbf{H}^2)=S_0(H_3)\subset S_0(H_2)\subset(C_{\boldsymbol{\kappa}^8\beta}(\tau_0)\cap\mathbf{H}^2)\subset(C_{\boldsymbol{\kappa}^2\beta}(\tau_0)\cap\mathbf{H}^2),$$

where the first inclusion follows from the definition of H_3, the second by the fact of (H_2,H_3) is nested, and the last one by Proposition 5.6, since $\delta(H_0,H_2)\geqslant\ell(\beta)$. In particular,

$$C_{\boldsymbol{\kappa}^2\beta}(\tau_1)\cap C_{\boldsymbol{\kappa}^2\beta}(\tau_0)\neq\varnothing.$$

Again, we conclude by Proposition 5.8: (τ_0,τ_1) is $(\boldsymbol{\kappa}^2\beta,\boldsymbol{K})$-nested. □

6. The confinement lemma

The main results of this section are the confinement lemma and the weak confinement lemma that guarantee that a deformed path of quasi-tripods is squeezed or controlled, provided that the deformation is small enough.

We say that a coplanar path of tripods associated to a path of chords $(h_i)_{0\leqslant i\leqslant N}$ is a *weak (ℓ,N)-coplanar path of tripods* if

$$\delta(h_0,h_i)\leqslant\ell\quad\text{for } i<N.$$

A coplanar path of tripods associated to a sequence of chords $(h_i)_{0\leqslant i\leqslant N}$ is a *strong (ℓ,N)-coplanar path of tripods* if furthermore

$$\delta(h_0,h_N)\geqslant\ell.$$

The main result of this section is the following,

LEMMA 6.1. (Confinement) *There exists β_3 depending only on G such that, for every α, with $\alpha\leqslant\beta_3$, there exists $\ell_0(\alpha)$ such that, for all $\ell_0\geqslant\ell_0(\alpha)$, there is η_0 such that, for all N,*

- *for all weak (ℓ_0,N)-coplanar paths of tripods $\underline{\tau}=(\tau_0,\dots,\tau_N)$, associated to a path of chords $\underline{h}(N)=(h_0,\dots,h_N)$,*
- *for all ε/N-deformation $v=(g_0,\dots,g_{N-1})$ with $\varepsilon\leqslant\eta_0$,*

the following statements hold:

(i) *the pair* (h_0^v, h_N^v) *is* $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{K}^2)$*-controlled*;

(ii) *if furthermore* $\underline{h}$ *is a strong coplanar path of tripods, then* (h_0^v, h_N^v) *is* $(\alpha, \boldsymbol{\kappa}^7)$*-squeezed*; *moreover*, (h_0^v, h_N^v) *and* (h_0, h_N) *both have the same commanding tripod*;

(iii) *if, finally*, $\underline{h}$ *is a strong coplanar path with* $\delta(h_0, h_N)=\ell_0$, *then* $\check{\tau}_0$, *the projection of* h_N *on* h_0, *is a commanding tripod of* (h_0^v, h_N^v).

In the sequel, we shall refer the first case as the *weak confinement lemma* and the second case as the *strong confinement lemma.*

6.0.1. Controlling deformations from a tripod

We first prove a proposition that allows us to control the size of deformation from a tripod depending only on the last and first chords.

Proposition 6.2. (Barrier) *For any positive* ℓ_0, *there exist positive constants* k *and* η_1 *such that*,

- *for all integers* N,
- *for all weak* (ℓ_0, N)*-coplanar paths of tripods* $\underline{\tau}=(\tau_0, ..., \tau_N)$, *associated to a path of chords* $\underline{h}(N)=(h_0, ..., h_N)$,
- *for all chord* H *such that* (h_N, H) *is nested with* $0<\delta(h_0, H)\leqslant\ell_0$,
- *for all* (ε/N)*-deformation* $v=(g_0, ..., g_{N-1})$ *with* $\varepsilon\leqslant\eta_1$,

we have

$$d_{\check{\tau}_0}(\mathrm{Id}, b_N)\leqslant k\varepsilon, \tag{6.1}$$

where $b_N=g_0 ... g_{N-1}$ *and* $\check{\tau}_0$ *is the projection of* H *on* h_0.

In this proposition, the position of h_N plays no role.

6.0.2. The confinement control

We shall use in the sequel the following proposition.

Proposition 6.3. (Confinement control) *There exists a positive* ε_0 *such that*, *for every positive* ℓ_0, *there exists a constant* k *with the following property.*

- *Let* (H, h) *be a pair of nested chords*, *associated to the circle* $C\subset\mathbf{F}$ *such that* $0<\delta(h, H)\leqslant\ell_0$, *and let* τ_0 *be the projection of* h *on* H.
- *Let* (X, Y) *and* (x, y) *be the extremities of* H *and* h, *respectively.*
- *Let* $u, v, w\in C\subset\mathbf{F}$ *be pairwise distinct such that* (X, u, v, x, y, w, Y) *is cyclically oriented — possibly with repetition — in* C, *and* τ *be the tripod coplanar to* H *such that* $\partial\tau=(u, w, v)$.

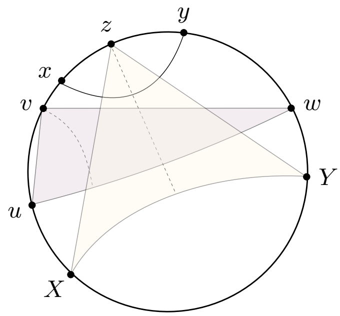


Figure 6.1. Confinement control.

- *Let* $g \in \mathsf{P}_w$ *with* $d_\tau(g, \mathrm{Id}) \leqslant \varepsilon_0$.

Then,

$$d_{\tau_0}(g, \mathrm{Id}) \leqslant k \cdot d_\tau(g, \mathrm{Id}).$$

Figure 6.1 illustrates the configuration of this proposition.

Proof. It is no restriction to assume that $\delta(h, H) = \ell_0$. Let τ and τ_0 be as in the statement and τ_1 be the tripod coplanar to H, such that $\partial\tau_1 = (u, w, z)$. Observe that, there is a positive t such that $\varphi_t(\tau) = \tau_1$. Let $a = \mathsf{T}\tau(a_0) \in \mathfrak{g}$. Then, we have

$$d_{\varphi_t(\tau)}(g, \mathrm{Id}) = d_{\exp(ta)(\tau)}(g, \mathrm{Id}) = d_\tau(\exp(-ta)g\exp(ta), \mathrm{Id}).$$

Let $\mathfrak{p}_+$ be the Lie algebra of $\mathsf{P}_+ := \mathsf{P}_{\partial^+\tau} = P_w$, that we consider also equipped with the Euclidean norm $\|\cdot\|_\tau$. By construction, $\mathsf{P}_+ = \tau(\mathsf{P}_0^+)$, and thus

$$\sup_{t>0} \|\mathrm{ad}(\exp(-ta))\| \, |_{\mathfrak{p}_+} < \infty.$$

For ε small enough and independent of $\partial^+\tau$, exp is k_1-biLipschitz from the ball of radius ε in $\mathfrak{p}_+$ onto its image in P_+ for some constant k_1 independent of $\partial^+\tau$. Thus, for ε_0-small enough, there exists a constant k_1 such that

$$d_{\tau_1}(g, \mathrm{Id}) \leqslant k_1 d_\tau(g, \mathrm{Id}). \tag{6.2}$$

Now, the set K of tripods σ coplanar to τ_0, with $\partial\sigma = (u, w, z)$, where z is fixed, and u and w are as above, is compact. In particular, there exists k_2 depending only on ℓ_0 such that, for any tripod σ in K,

$$d(\tau_1, \tau_0) \leqslant k_3.$$

Thus, by Proposition 3.16, there exists k_4 such that

$$d_{\tau_1}(g, \mathrm{Id}) \leqslant k_4 \cdot d_\sigma(g, \mathrm{Id}).$$

The proposition now follows by combining with inequality (6.2). $\square$

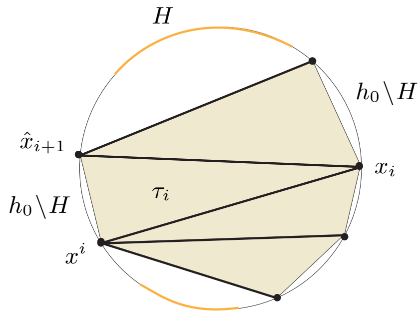


Figure 6.2

Proof of Proposition 6.2. Let (x_i, x^i) be the extremities of h_i where x_i is the pivot. Let $\hat{x}_{i+1}$ the vertex of τ_i different from x_i and x^i.

Let $\check{\tau}_0$ be the projection of H on h_0. Observe that x_i lies in one of the connected component of $h_0\setminus H$, while x^i lies in the other (see Figure 6.2).

Thus, according to Proposition 6.3 for ε small enough there exists k, depending only on ℓ such that

$$d_{\tau_0}(g_i, \mathrm{Id}) \leqslant k \cdot d_{\tau_i}(g_i, \mathrm{Id}) \leqslant k\frac{\varepsilon}{N}.$$

Thus, using the right-invariance of d_{τ_0},

$$d_{\tau_0}(\mathrm{Id}, b_N) \leqslant \sum_{i=1}^{N} d_{\tau_0}\left(\prod_{j=i}^{N} g_j, \prod_{j=i+1}^{N} g_j\right) = \sum_{i=1}^{N} d_{\tau_0}(g_i, \mathrm{Id}) \leqslant k\varepsilon.$$

This proves inequality (6.1) and concludes the proof of Proposition 6.2. $\square$

Proof of Lemma 6.1. Let β_1 be as in Proposition 5.6. Let then α be such that $\alpha \leqslant \beta_1$. According to Proposition 5.6, there exists $\ell = \ell_0(\alpha)$ such that if (H_0, H_1) is a nested pair of chords with $\delta(H_0, H_1) \geqslant \ell$, then for any $\sigma_1 \in H_1$, the pair (τ_0, σ_1) is $(\boldsymbol{K}\alpha, \boldsymbol{\kappa}^9)$-nested, where τ_0 is the projection of H_1 on H_0. Let now fix $\ell_0 \geqslant \ell_0(\alpha)$.

First step (Strong coplanar). Consider first the case where $\delta(h_0, h_N) \geqslant \ell_0$. By continuity, we may find a chord $\check{h}_N$ such that the pairs $(h_{N-1}, \check{h}_N)$ and $(\check{h}_N, h_N)$ are nested, and such that $\delta(\check{h}_N, h_0) = \ell_0$.

Let $\check{\tau}_0$ be the projection of $\check{h}_N$ on h_0. Then, by Proposition 5.6, for any σ_1 in $\check{h}_N$, we have that $(\check{\tau}_0, \sigma_1)$ is $(\boldsymbol{K}\alpha, \boldsymbol{\kappa}^9)$-nested.

By Lemma 5.7, for any σ_N in h_N, there exist σ_1 in $\check{h}_N$ such that (σ_1, σ_N) is $(\alpha, \boldsymbol{K})$-nested, and thus $(\check{\tau}_0, \sigma_N)$ is $(\alpha, \boldsymbol{\kappa}^8)$-nested.

By the barrier proposition (Proposition 6.2) applied to $\underline{h}(N)$ and $H = \check{h}_N$, we get that

$$d_{\check{\tau}_0}(\mathrm{Id}, b_N) \leqslant k\varepsilon,$$

for k depending only on G and where ℓ and b_N are defined in the barrier proposition.

We now furthermore assume that $\alpha \leqslant \beta_0$, where β_0 comes from Proposition 5.4. For ε is small enough, Proposition 5.4 shows that, for any σ_1 in h_N, we have that $(\check{\tau}_0, b_N(\sigma_1))$ is $(\alpha, 2\boldsymbol{\kappa}^8)$-nested. Thus, $(h_0, b_N(h_N))$ is $(\alpha, 2\boldsymbol{\kappa}^8)$-squeezed, and hence $(\alpha, \boldsymbol{\kappa}^7)$-squeezed, since $2\boldsymbol{\kappa} \leqslant 1$, with $\check{\tau}_0$ as a commanding tripod.

This applies of course if the deformation is trivial and we see that (h_0, h_N), and (h_0^v, h_N^v) both have $\check{\tau}_0$ as a commanding tripod.

This concludes this first step and the proofs of the second and third statements in Lemma 6.1.

Second step. Let us consider the remaining case when $\delta(h_0, h_N) \leqslant \ell$. Let us apply Proposition 5.7 to $(H_0, H_1) = (h_0, h_N)$ and τ_1 in h_N. Thus, there exists H_2 such that (h_N, H_2) is nested, $0 < \delta(H_0, H_2) \leqslant \ell$, and (τ_0, τ_1) is $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{K})$-nested, where τ_0 is the projection of H_2 on h_0.

Applying the barrier proposition to $h = H_2$ and $H = H_0$ yields that

$$d_{\tau_0}(\mathrm{Id}, b_N) \leqslant k \cdot \varepsilon.$$

Thus, for ε small enough, then Proposition 5.4 yields that $(\tau_0, b_N(\tau_1)$ is $(\boldsymbol{\kappa}^2\alpha, 2\boldsymbol{K})$-nested, and hence $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{K}^2)$-nested.

This shows that $(h_0, b_N(h_N))$ is $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{K}^2)$-controlled. This concludes the proof of Lemma 6.1. □

7. Infinite paths of quasi-tripods and their limit points

The goal of this section is to make sense of the limit point of an infinite sequence of quasi-tripods and to give a condition under which such a limit point exists. The ad-hoc definitions are motivated by the last section of this paper, as well as by the discussion of Sullivan maps.

One may think of the main theorem of this section, Theorem 7.2, as a refined version of a Morse lemma in higher rank: instead of working with quasi-geodesic paths in the symmetric space, we work with sequence of quasi-tripods in the flag manifold; instead of making the quasi-geodesic converge to a point at infinity, we make the sequence of quasi-tripods shrink to a point in the flag manifold. This is guaranteed by some local conditions that will allow us to use our nesting and squeezing concepts defined in the preceding section.

Theorem 7.2 is the goal of our efforts in this first part.

7.1. Definitions: Q-sequences and their deformations

Definition 7.1. (i) A *Q-coplanar-sequence of tripods* is an infinite sequence of tripods $\underline{T}=\{T_m\}_{m\in\mathbb{N}}$ such that the associated sequence of coplanar chords $\underline{c}=\{c_i\}_{i\in\mathbb{N}}$ satisfies the following: for all integers m and p we have

$$|m-p| \leqslant Q\delta(c_m, c_p)+Q,$$

where $\delta(\cdot,\cdot)$ is the shift defined in §5.2.1.

(ii) A sequence of quasi-tripod $\underline{\tau}=\{\tau_m\}_{m\in\mathbb{N}}$ is a *(Q,ε)-sequence of quasi-tripods* if there exists a Q-coplanar-sequence of tripods $T=\{T_m\}_{m\in\mathbb{N}}$ such that, for every n, $\{\tau_m\}_{m\in[0,n]}$ is an ε-deformation of $\{T_m\}_{m\in[0,n]}$.

(iii) The associated sequence of chords to a (Q,ε)-sequence of quasi-tripods is called a *(Q,ε)-sequence of chords.*

7.2. Main result: existence of a limit point

Our main theorem asserts the existence of limit points for some deformed (Q,ε)- sequence and their quantitative properties.

THEOREM 7.2. (Limit point) *There exist some positive constants* A *and* q *depending only on* G, *with* $\mathrm{q}<1$, *such that for every positive number* β *and* ℓ_0 *with* $\beta\leqslant\mathrm{A}$, *there exist a positive constant* $\varepsilon>0$ *such that, for any* $R>1$:

For any $(\ell_0 R, \varepsilon/R)$-deformed sequence of quasi-tripods $\underline{\theta}=\{\theta_m\}_{m\in\mathbb{N}}$, with associated sequence of chords $\underline{\Gamma}=\{\Gamma_m\}_{m\in\mathbb{N}}$, there exists some $\delta>0$ such that

$$\bigcap_{m}^{\infty} S_\delta(\Gamma_m) := \{\xi(\underline{\theta})\},$$

with

$$\xi(\underline{\theta}) = \lim_{m\to\infty}(\partial^j\theta_m) \quad \textit{for } j\in\{+,-,0\}.$$

Moreover, we have the following quantitative estimates:

(i) *for any τ in Γ_0, and $m>(\ell_0+1)^2R$,*

$$d_\tau(\xi(\underline{\theta}), \partial^j\theta_m)) \leqslant \mathrm{q}^m\beta \quad \textit{for } j\in\{+,-,0\}. \tag{7.1}$$

(ii) *Let τ in Γ_0. Assume that $\{\theta_m\}_{m\in\mathbb{N}}$ is the deformation of a sequence of coplanar tripods $\underline{\tau}=\{\tau_m\}_{m\in\mathbb{N}}$ with $\tau_0=\dot{\theta}_0$. Then,*

$$d_\tau(\xi(\underline{\theta}), \xi(\underline{\tau})) \leqslant \beta\,. \tag{7.2}$$

(iii) *Finally, let* $\{\theta'_m\}_{m\in\mathbb{N}}$ *be another* $(\ell_0 R, \varepsilon/R)$*-deformed sequence of quasi-tripods. Assume that* $\{\theta'_m\}_{m\in\mathbb{N}}$ *and* $\{\theta_m\}_{m\in\mathbb{N}}$ *coincide up to the* n*-th chord with* $n>(\ell_0+1)^2 R$*. Then, for all* $\tau\in\Gamma_0$,

$$d_\tau(\xi(\underline{\theta'}), \xi(\underline{\theta})) \leqslant \mathrm{q}^n \beta. \tag{7.3}$$

The limit point theorem will be the consequence of the following more technical one.

THEOREM 7.3. (Squeezing chords) *There exists some constant* A, *depending only on* G, *such that, for every positive number* δ *with* $\delta\leqslant \mathrm{A}$, *there exist positive constants* R_0, ℓ_0 *and* ε *with the following property*:

If $\underline{\Gamma}$ *is an* $(\ell_0 R, \varepsilon/R)$*-deformed sequence of chords of the coplanar sequence of chords* $\underline{c}$ *with* $R\geqslant R_0$, *and if* $j>i$ *are such that* $\delta(c_i, c_j)\geqslant \ell_0$, *then* (Γ_i, Γ_j) *is* $(\delta, \boldsymbol{\kappa})$*-squeezed.*

7.3. Proof of the squeezing chords theorem

As a preliminary, we make the choice of constants, then we cut a sequence of chords into small more manageable pieces. Finally, we use the confinement lemma (Lemma 6.1) to obtain the proof.

7.3.1. Fixing constants and choosing a threshold

Let α_3 be as in Corollary 5.3, and let β_3 be as in the confinement lemma. We now choose α such that

$$\alpha \leqslant \inf(\beta_3, \alpha_3).$$

Then, let $\ell_0=\ell_0(\alpha)$ be the *threshold*, and let η_0 be obtained by the confinement lemma. Let finally

$$\varepsilon \leqslant \frac{\eta_0}{\ell_0\cdot(\ell_0+1)}.$$

7.3.2. Cutting into pieces

Let $\underline{c}$ be a sequence of coplanar chords admitting an $\ell_0 R$-coplanar path of tripods.

LEMMA 7.4. *We can cut* $\underline{\tau}$ *into successive pieces* $\underline{\tau}^n := \{\tau_p\}_{p_n\leqslant p< p_{n+1}}$ *for* $n\in\{0, M\}$ *such that the following statements hold*:

(i) *for* $n\in\{0, M-1\}$, $\underline{\tau}^n$ *is a strong* (ℓ_0, N)*-coplanar path of tripods*;

(ii) $\underline{\tau}^M$ *is a weak* (ℓ_0, N)*-coplanar path of tripods.*

In both cases,

$$N \leqslant L := \lfloor (\ell_0+1)(\ell_0 R) \rfloor + 1,$$

where $\lfloor x \rfloor$ *denotes the integer value of the real number* x.

Proof. Let $\underline{c}$ be the corresponding sequence of chords. Recall that the function $q \mapsto \delta(c_p, c_q)$ is increasing for $q>p$. Thus, we can further cut into (maximal) pieces such that

$$\delta(c_{p_n}, c_{p_{n+1}-1})) \leqslant \ell_0 \quad \text{and} \quad \delta(c_{p_n}, c_{p_{n+1}})) \geqslant \ell_0.$$

This gives the lemma: the bound on N comes from the fact that $\underline{\tau}$ is a $\ell_0 R$-sequence. In particular, since $\delta(c_{p_n}, c_{p_{n+1}-1}) \leqslant \ell_0$, then

$$|p_{n+1}-p_n|-1 \leqslant (\ell_0 R)(\ell_0+1). \qquad \square$$

7.3.3. Completing the proof

Let $\underline{\theta}$ be an $(\ell_0 R, \varepsilon/R)$-sequence of quasi-tripods, with $R>R_0$. Let $\underline{\Gamma}$ be the associated sequence of chords. Assume $\underline{\theta}$ is the deformation of an $\ell_0 R$- coplanar sequence of tripods $\underline{\tau}$, cut in smaller sub-pieces as in Lemma 7.4.

Proposition 7.5. (i) *For* $n<M$, $(\Gamma_{p_n}, \Gamma_{p_{n+1}})$ *is* $(\alpha, \boldsymbol{\kappa}^7)$*-squeezed.*
(ii) *Moreover,* $(\Gamma_{p_M}, \Gamma_{p_{M+1}})$ *is* $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{K}^2)$*-controlled.*

Proof. If $n<M$, $\underline{\tau}^n$ is a strong (ℓ_0, L)-path. Then, according to the confinement lemma (Lemma 6.1) and the choice of our constants $(\Gamma_{p_n}, \Gamma_{p_{n+1}})$ is $(\alpha, \boldsymbol{\kappa}^7)$-squeezed.

Since $\underline{\tau}^M$ is a weak (ℓ_0, L)-path, it follows by our choice of constants and the confinement lemma that $(\Gamma_{p_M}, \Gamma_{p_{M+1}})$ is $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{K}^2)$-controlled. $\square$

We now prove the squeezing chord theorem (Theorem 7.3) with $\delta = \boldsymbol{\kappa}^2\alpha$

Proposition 7.6. *Assuming,* $\delta(c_i, c_j) > \ell_0$ *and* $j>i$, *the pair* (Γ_i, Γ_j) *is* $(\boldsymbol{\kappa}^2\alpha, \boldsymbol{\kappa})$*-squeezed.*

Proof. We will use freely the observation that $(\alpha, \boldsymbol{\kappa}^n)$-nesting implies $(\boldsymbol{\kappa}^p\alpha, \boldsymbol{\kappa}^q)$-nesting for $p, q \geqslant 0$ with $p+q \leqslant n$.

Recall that, due to the composition proposition (Proposition 5.2), if the pairs of chords (H_0, H_1) and (H_1, H_2) are both $(\alpha, \boldsymbol{\kappa}^7)$-squeezed (in particular (H_1, H_2) is $(\alpha, \boldsymbol{\kappa}^5)$-squeezed), then (H_0, H_2) is $(\alpha, \boldsymbol{\kappa}^7)$-squeezed.

We cut $\underline{\tau}$ as above in pieces and control every sub-piece using Proposition 7.5.

Thus, by induction, $(\Gamma_{p_0}, \Gamma_{p_M})$ is $(\alpha, \boldsymbol{\kappa}^7)$-squeezed and thus $(\boldsymbol{K}^2(\boldsymbol{\kappa}^2\alpha), \boldsymbol{\kappa}^7)$-squeezed since $\boldsymbol{\kappa}\boldsymbol{K}=1$.

Finally, since $(\Gamma_{p_M},\Gamma_{p_{M+1}})$ is $(\boldsymbol{\kappa}^2\alpha,\boldsymbol{K}^2)$-controlled, the composition proposition (Proposition 5.2) yields that $(\Gamma_{p_0},\Gamma_{p_{M+1}})$ is $(\boldsymbol{\kappa}^2\alpha,\boldsymbol{\kappa}^7\boldsymbol{K}^2)$-squeezed, and thus $(\boldsymbol{\kappa}^2\alpha,\boldsymbol{\kappa})$-squeezed. This finishes the proof. □

7.4. Proof of the existence of limit points, Theorem 7.2

Let $\{\theta_m\}_{m\in\mathbb{N}}$ and $\{\tau_m\}_{m\in\mathbb{N}}$ be sequences of quasi-tripods and tripods as in Theorem 7.2.

Let $\underline{\Gamma}$ be the sequence of chords associated to $\{\theta_m\}_{m\in\mathbb{N}}$, and similarly $\underline{c}$ associated to $\{\tau_m\}_{m\in\mathbb{N}}$ as in Theorem 7.2, then, according to Theorem 7.3, if $j>i$ are such that $\delta(c_i,c_j)\geqslant\ell_0$, then (Γ_i,Γ_j) is $(\delta,\boldsymbol{\kappa})$-squeezed. Since $\underline{c}$ is Q_0-controlled (with $Q_0=\ell_0 R$), we have

$$\delta(c_i,c_j)\geqslant\frac{|i-j|}{\ell_0 R}-1.$$

Thus,

$$j-i\geqslant L \quad\Longrightarrow\quad \delta(c_i,c_j)\geqslant\ell_0,$$

We can summarize this discussion in the following statement:

$$j-i\geqslant L \quad\Longrightarrow\quad S_\delta(\Gamma_j)\subset S_{\boldsymbol{\kappa}\delta}(\Gamma_i), \tag{7.4}$$

7.4.1. Convergence for lacunary subsequences

We first prove an intermediate result.

COROLLARY 7.7. *There exists a constant* M *depending only on* G, *with* q<1, *such that, for* δ *small enough, if* $\underline{l}=\{l_m\}_{m\in\mathbb{N}}$ *is a sequence such that* $l_{m+1}\geqslant l_m+L$ *and* $l_0=0$, *then*

$$S_\delta(\Gamma_{l_{m+1}})\subset S_{\boldsymbol{\kappa}\delta}(\Gamma_{l_m}),$$

and furthermore there exists a unique point $\xi(\underline{l})\in\mathbf{F}$ *such that*

$$\bigcap_{m=1}^{\infty} S_\delta(\Gamma_{l_m})=\{\xi(\underline{l})\}\subset C_\delta(\check{\tau}_0),$$

where $\check{\tau}_0$ *is a commanding tripod for* (Γ_0,Γ_{l_1}).

Finally, if $\tau\in\Gamma_0$ *then, for all* u *in* $S_\delta(\Gamma_{l_m})$ *with* $m\geqslant1$, *we have*

$$d_\tau(u,\xi(\underline{l}))\leqslant 2^{-m}\mathbf{M}\delta. \tag{7.5}$$

Proof. From the squeezed condition for chords, we obtain that there exists $\check{\tau}_m$ in Γ_{l_m} such that

$$S_\delta(\Gamma_{l_{m+1}}) \subset C_{\boldsymbol{\kappa}\delta}(\check{\tau}_m) \subset S_{\boldsymbol{\kappa}\delta}(\Gamma_{l_m}).$$

This proves the first assertion. As a consequence, $C_\delta(\check{\tau}_{m+1}) \subset C_{\boldsymbol{\kappa}\delta}(\check{\tau}_m)$. Combining with the convergence corollary (Corollary 5.3), we get the second assertion, with

$$\{\xi(\underline{l})\} := \bigcap_{m=1}^{\infty} C_\delta(\check{\tau}_m) = \bigcap_{m=1}^{\infty} S_\delta(\check{\tau}_m).$$

Using the second assertion of the convergence corollary, we obtain that, if $u, v \in S_\delta(\Gamma_{l_m}) \subset C_\delta(\check{\tau}_{m-1})$, then

$$d_{\tau_0}(u,v) \leqslant 2^{2-m}\delta,$$

and in particular $u, \xi(\underline{l}) \in C_\delta(\check{\tau}_0)$ and

$$d_{\tau_0}(u, \xi(\underline{l})) \leqslant 2^{2-m}\delta. \tag{7.6}$$

We now extend the previous inequality when we replace τ_0 by any $\tau \in C_n$. We use Lemma 5.5 which produces constants δ_0 and k depending only on G such that, if δ is smaller than δ_0, then, since $u, \xi(\underline{l}) \in C_\delta(\check{\tau}_0)$,

$$d_\tau(u, \xi(\underline{l})) \leqslant k.d_{\tau_0}(u, \xi(\underline{l})). \tag{7.7}$$

This concludes the proof of the corollary, since we now get from inequalities (7.7) and (7.6) that

$$d_\tau(u, \xi(\underline{l})) \leqslant k.2^{2-m}\delta.$$

□

7.4.2. Completion of the proof

Let $\{l_m\}_{m\in\mathbb{N}}$ and $\{l'_m\}_{m\in\mathbb{N}}$ be two subsequences. It follows from inclusion (7.4), that

$$\bigcap_{m=1}^{\infty} S_\delta(\Gamma_{l_m}) = \bigcap_{m=1}^{\infty} S_\delta(\Gamma_{l'_m}).$$

As an immediate consequence, we get that

$$\bigcap_{m=1}^{\infty} S_\delta(\Gamma_m) = \{\xi(\underline{\lambda})\},$$

where $\underline{\lambda} = \{\lambda_m\}_{m\in\mathbb{N}}$, with $\lambda_m = m.L$. Thus, we may write $\xi(\underline{\lambda}) =: \xi(\underline{\Gamma})$. The existence of $\xi(\underline{c})$ follows form the fact that $\underline{c}$ is also a $(\ell_0 R, \varepsilon/R)$-deformed sequence of cuffs.

By construction — and see the second item in the confinement lemma (Lemma 6.1) — the commanding tripod $\check{\tau}_0$ of (Γ_0, Γ_L) is the commanding tripod of (c_0, c_L).

It follows that both $\xi(\underline{l})$ and $\xi(\underline{\lambda})$ belong to $C_\delta(\check{\tau}_0)$. Thus, using the triangle inequality and Lemma 5.5, for all $\tau \in \Gamma_0$,

$$d_\tau(\xi(\underline{l}), \xi(\underline{\lambda})) \leqslant k\delta, \tag{7.8}$$

where k only depends on G. By inequality (7.5), if $\{l_m\}_{m\in\mathbb{N}}$ is a lacunary subsequence, for any $\tau \in \gamma_0$, for $u \in S_\delta(\Gamma_{l_m})$ with $m \geqslant 1$,

$$d_\tau(\xi(\underline{\lambda}), u) \leqslant 2^{-m}\mathbf{M}\delta. \tag{7.9}$$

In particular, taking $l_m = mL$, one gets

$$d_\tau(\xi(\underline{\lambda}), \theta^j_{m.L}) \leqslant 2^{-m}\mathbf{M}\delta. \tag{7.10}$$

Let now $n = (m+1)L + p$, with $p \in [0, L]$. The inclusion (7.4), gives the first inclusion below, whereas the second is a consequence of the fact that, for $\boldsymbol{\kappa} < 1$,

$$S_\delta(\Gamma_n) \subset S_{\boldsymbol{\kappa}\delta}(\Gamma_{m.L}) \subset S_\delta(\Gamma_{m.L}). \tag{7.11}$$

Thus, combining the previous assertion with assertion (7.9), for all $u \in S_\delta(\Gamma_n)$, with $n > L$, we have

$$d_\tau(\xi(\underline{\lambda}), u) \leqslant 2^{-m}\mathbf{M}\delta \leqslant (2^{-1/L})^n 4\mathbf{M}\delta.$$

Taking $\mathrm{q} = 2^{-1/L}$, $\beta = 4\mathbf{M}\delta$ and $u = \theta^j_n$ yields the inequality

$$d_\tau(\xi(\underline{\lambda}), \theta^j_n) \leqslant \mathrm{q}^n \beta.$$

This completes the proof of inequality (7.1) for $n > L$.

The second item comes from inequality (7.8), after possibly changing β.

The third one comes form the first and the triangle inequality, again after changing β.

8. Sullivan limit curves

The purpose of this section is to define and describe some properties of an analog of the Kleinian property: being a K-quasi-circle with K close to 1.

This is achieved in Definition 8.1. We then show, under the hypothesis of a compact centralizer for $\mathfrak{sl}_2$, three main theorems of independent interest: Sullivan maps are Hölder (Theorem 8.2), a representation with a Sullivan limit map is Anosov (Theorem 8.3), and finally one can weaken the notion of being Sullivan under some circumstances (Theorem 8.14).

In this section, as usual, G will be a semisimple group, $\mathfrak{s}$ an $\mathfrak{sl}_2$-triple, $\mathbf{F}$ the associate flag manifold. We will furthermore assume in this section that

$$\text{The centralizer of } \mathfrak{s} \text{ is compact.} \tag{8.1}$$

We will comment on the case of non-compact centralizer later.

Let us start with a comment on our earlier definition of circle maps (Definition 3.4). Let $T=(x^-,x^+,x_0)$ be a triple of pairwise distinct points in $\mathbf{P}^1(\mathbb{R})$—also known as a tripod for $\mathsf{SL}_2(\mathbb{R})$—and τ a tripod in $\mathcal{G}$. Such a pair (T,τ) defines uniquely

- an associated circle map η from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}$ such that $\eta(T)=\partial\tau$;
- an associated *extended circle map*, which is a map ν from the space of triples of pairwise distinct points (x,y,z) in $\mathbf{P}^1(\mathbb{R})$ to $\mathcal{G}$ whose image consists of coplanar tripods and such that

$$\partial\nu(x,y,z)=(\eta(x),\eta(y),\eta(z)) \quad \text{and} \quad \nu(x^-,x^+,x_0)=\tau.$$

8.1. Sullivan curves: definition and main results

Definition 8.1. (Sullivan curve) Let $\zeta>0$. We say that a map ξ from $\mathbf{P^1}(\mathbb{R})$ to $\mathbf{F}$ is a ζ-*Sullivan curve* with respect to $\mathfrak{s}$ if the following property holds.

Let $T=(x^-,x^+,x^0)$ be any triple of pairwise distinct point in $\mathbf{P}^1(\mathbb{R})$. Then, there exists a tripod τ—called a *compatible tripod*—a circle map η, with $\eta(T)=\partial\tau$, such that, for all $y\in\mathbf{P^1}(\mathbb{R})$,

$$d_\tau(\xi(y),\eta(y))\leqslant\zeta. \tag{8.2}$$

Obviously if ζ is large, for instance greater than $\operatorname{diam}(\mathbf{F})$, the definition is pointless: every map is a ζ-Sullivan. We will however show that the definition makes sense for ζ small enough.

We also leave to the reader to check that in the case of $\mathsf{G}=\mathsf{PSL}_2(\mathbb{C})$—in which case $\mathbf{F}$ is $\mathbf{P}^1(\mathbb{C})$—the following holds: for $K>1$ and any compact interval C containing -1, there exists a positive ε such that, if ξ is ε-Sullivan, then, for all (x,y,z,t) in $\mathbf{P}^1(\mathbb{R})$ such that $[x,y,z,t]$ belongs to C,

$$\frac{1}{K}\leqslant\left|\frac{[\xi(x),\xi(y),\xi(z),\xi(t)]}{[x,y,z,t]}\right|\leqslant K.$$

This readily implies that ξ is a quasicircle. Thus, in that case, an ε-Sullivan map is quasi-symmetric for ε-small enough. The following results of independent interest justify our interest of ζ-Sullivan maps.

THEOREM 8.2. (Hölder property) *There exist some positive numbers ζ and α such that any ζ-Sullivan map is α-Hölder.*

We prove a more quantitative version of this theorem with an explicit modulus of continuity in §8.3. This modulus of continuity will be needed in other proofs, for example in §8.5 and §16.6.

The existence of ζ-Sullivan limit maps implies some strong dynamical properties. We refer to [22] and [12] for background and references on Anosov representations.

THEOREM 8.3. (Sullivan implies Anosov) *There exists some positive ζ_1 with the following property. Let S be a closed hyperbolic surface and ρ be a representation of $\pi_1(S)$ in G. Assume there exists a ρ-equivariant ζ_1-Sullivan map*

$$\xi\colon \partial_\infty \pi_1(S) = \mathbf{P}^1(\mathbb{R}) \longrightarrow \mathbf{F}.$$

Then, ρ is P-Anosov and ξ is its limit curve.

Recall that P is the stabilizer of a point in $\mathbf{F}$. We recall that a P-Anosov representation is in particular faithful and a quasi-isometric embedding and that all its elements are loxodromic [22], [12] . Recall also that, in that context, the parabolic is isomorphic to its opposite. We prove this theorem in §8.4.

The following result is worth stating, although we will not use it in the proof.

PROPOSITION 8.4. *Let ρ_n be a family of P-Anosov representations of a cocompact Fuchsian group, whose limit maps are ζ_n-Sullivan, with ζ_n converging to zero. Then, after conjugation, ρ_n converges to a representation whose limit curve is a circle.*

Proof. Let ξ_n be the limit curve of ρ_n. By definition, there exists a tripod τ_n and an associated circle map η_n such that

$$\eta_n(0,1,\infty) = \partial\tau_n \quad \text{and} \quad d_{\tau_n}(\eta_n, \xi_n) \leqslant \zeta_n.$$

After conjugating ρ_n by an element g_n, we may assume that τ_n and η_n are constant and equal to τ and η, respectively. Thus, we have

$$\eta(0,1,\infty) = \partial\tau \quad \text{and} \quad d_{\tau}(\eta, \xi_n) \leqslant \zeta_n.$$

In particular, it follows that ξ_n converges uniformly to η. Let ρ_0 be the cocompact Fuchsian representation associated to η. Let now γ^i be generators of Γ. The same argument as above shows that $\xi_n \circ \gamma_i = \rho_n(\gamma^i) \circ \xi_n$ converges to $\eta \circ \gamma_i = \rho_0(\gamma_i)$. It follows that, using the fact that the centralizer of a circle is compact, we may extract a subsequence such that $\rho_n(\gamma_i)$ converges for all i to $\rho(\gamma^i)$, where ρ is a representation in $\mathsf{H} \times \mathsf{K}$ of the form (ρ_0, ρ_1), where H the group isomorphic to $\mathsf{SL}_2(\mathbb{R})$ associated to η, and K its centralizer. $\square$

In §8.1.1, we single out the consequence of the "compact stabilizer hypothesis" that we shall use.

8.1.1. The compact stabilizer hypothesis

Our standing hypothesis will have the following consequence.

LEMMA 8.5. (i) *There exists a positive constant* $\boldsymbol{\zeta}$ *such that, for every positive real number* M*, there exists a positive real number* N *such that, if* ξ *is a* $\boldsymbol{\zeta}$*-Sullivan map, if* T_1 *and* T_2 *are two triples of distinct points in* $\mathbf{P}^1(\mathbb{R})$ *with* $d(T_1,T_2)\leqslant M$*, and if* τ_1 *and* τ_2 *are the respective compatible tripods with respect to* ξ*, then*

$$d(\tau_1,\tau_2)\leqslant N.$$

(ii) *For any positive* ε *and* M*, then for* ζ *small enough, for any* ζ*-Sullivan map* ξ*, if* T_1 *and* T_2 *are two triples of distinct points in* $\mathbf{P}^1(\mathbb{R})$ *with* $d(T_1,T_2)\leqslant M$*, and if* τ_1 *and* ν_1 *are, respectively, the compatible tripod and the extended circle map of* T_1 *with respect to* ξ*, then we may choose a compatible tripod* τ_2 *for* T_2 *such that*

$$d(\tau_2,\nu_1(T_2))\leqslant \varepsilon.$$

Actually, this lemma will be the unique consequence of our standard hypothesis which will be used in the proof. This lemma is itself a corollary of the following proposition.

PROPOSITION 8.6. (i) *There exist positive constants* $\boldsymbol{A}$ *and* $\boldsymbol{\zeta}_0$ *such that, if* τ_1 *and* τ_2 *are two tripods and* X *is a triple of points in* $\mathbf{F}$*, then we have the implication*

$$d_{\tau_1}(X,\partial\tau_1)\leqslant\boldsymbol{\zeta}_0 \ \text{ and } \ d_{\tau_2}(X,\partial\tau_2)\leqslant\boldsymbol{\zeta}_0 \quad\Longrightarrow\quad d(\tau_1,\tau_2)\leqslant\boldsymbol{A}.$$

(ii) *Moreover, given* $\alpha>0$*, there exists* $\varepsilon>0$ *such that*

$$\begin{aligned}&d_{\tau_1}(X,\partial\tau_1)\leqslant\varepsilon \ \text{ and } \ d_{\tau_2}(X,\partial\tau_2)\leqslant\varepsilon\\ &\qquad\Longrightarrow\quad \text{there exists } \tau_3 \text{ with } \partial\tau_3=\partial\tau_2 \text{ and } d(\tau_1,\tau_3)\leqslant\alpha.\end{aligned}$$

We first prove Lemma 8.5 from Proposition 8.6.

Proof. Let $\boldsymbol{\zeta}_0$ and $\mathbf{A}$ be as in Proposition 8.6. Let first ν_0 be an extended circle map with associated map η_0. By continuity of η_0, there exists M such that

$$d(T_1,T_2)\leqslant M \quad\Longrightarrow\quad d_{\nu_0(T_1)}(\eta_0(T_1),\eta_0(T_2))\leqslant\tfrac{1}{2}\boldsymbol{\zeta}_0.$$

The equivariance under the action of G then shows that the previous inequality holds for all $\nu = g\nu_0$.

Let $\boldsymbol{\zeta} = \frac{1}{2}\boldsymbol{\zeta}_0$ and ξ be a $\boldsymbol{\zeta}$-Sullivan map.

Proof of the first assertion. Let T_1 and T_2 be two tripods with $d(T_1, T_2) < M$. Let us denote by η_1 and η_2 the corresponding compatible circle maps, by τ_1 and τ_2 the corresponding compatible tripods, and by ν_1 and ν_2 the corresponding extended circle maps such that $\tau_i = \nu_i(T_i)$. Let $X = \xi(T_2)$.

Then, the $\boldsymbol{\zeta}$-Sullivan property implies that $d_{\tau_1}(X, \eta_1(T_2)) \leqslant \boldsymbol{\zeta}$. Then,

$$d_{\tau_1}(X, \partial\tau_1) \leqslant d_{\tau_1}(X, \eta_1(T_2)) + d_{\nu_1(T_1)}(\eta_1(T_2), \eta_1(T_1)) \leqslant 2\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathbf{0}}.$$

From the $\boldsymbol{\zeta}$-Sullivan condition, we get

$$d_{\tau_2}(X, \partial\tau_2) = d_{\tau_2}(\xi(T_2), \nu_2(T_2)) \leqslant \boldsymbol{\zeta} \leqslant \boldsymbol{\zeta}_{\mathbf{0}}.$$

Thus, Proposition 8.6 implies $d(\tau_2, \tau_1) \leqslant \boldsymbol{A}$. This proves the first assertion with $N = \boldsymbol{A}$.

Proof of the second assertion. Let ξ be a ζ-Sullivan map, and let T_i, η_i and τ_i be as above. Let again $X = \xi(T_2) \in \mathbf{F}^3$ and $\tau_0 = \eta_1(T_2)$. Using the definition of a ζ-Sullivan map, we have

$$d_{\tau_1}(X, \partial\tau_0) \leqslant \zeta \quad \text{and} \quad d_{\tau_2}(X, \partial\tau_2) \leqslant \zeta.$$

Moreover, $d(\tau_0, \tau_1) = d(T_1, T_2) \leqslant M$. Thus, by Proposition 3.16, d_{τ_0} and d_{τ_1} are uniformly equivalent. It follows that, for any positive β, for ζ small enough we have

$$d_{\tau_0}(X, \partial\tau_0) \leqslant \beta \quad \text{and} \quad d_{\tau_2}(X, \partial\tau_2) \leqslant \beta.$$

The second part of Lemma 8.6 guarantees us that, for any positive α, for ζ small enough, we may choose τ_3 with the same vertices as τ_2 such that

$$d(\tau_3, \eta_1(T_2)) = d(\tau_3, \tau_0) \leqslant \alpha.$$

Observe finally that τ_3 is a compatible tripod, recalling that in the case of the compact stabilizer hypothesis d_τ and the circle maps associated to τ, only depends on $\partial\tau$, by Proposition 3.15. Thus, choosing τ_3 concludes the proof of the lemma. $\square$

Proof of Proposition 8.6. Let us first prove that G acts properly on some open subset of $\mathbf{F}^3$ containing the set V of vertices of tripods.

We shall use the geometry of the associated symmetric space $\mathsf{S}(\mathsf{G})$. Let x be an element of $\mathbf{F}$, let A_x be the family of hyperbolic elements conjugate to a fixing x; observe that A_x is a $\operatorname{Stab}(x)$-orbit under conjugacy.

The family of hyperbolic elements in A_x corresponds in the symmetric space to an asymptotic class of geodesics at $+\infty$. Thus, A_x defines a Busemann function h_x well defined up to a constant. Each gradient line of h_x is one of the above described geodesic. The function h_x is convex on every geodesic γ, or in other words $\mathrm{D}^2_w h_x(u,u)\geqslant 0$ for all tangent vectors u. Moreover, $\mathrm{D}^2_w h_x(u,u)=0$ if and only if the 1-parameter subgroup associated to the geodesic γ in the direction of u commutes with the 1-parameter group associated to the gradient line of H_x through the point w. If now (x,y,z) are three points on a circle C, then the function

$$C := h_x+h_y+h_z$$

is geodesically convex. If $\mathbf{H}^2_C$ is the hyperbolic geodesic plane associated to the circle C, then x, y and z correspond to three points at infinity in $\mathbf{H}^2_C$, and all gradient lines of h_x, h_y and h_z along $\mathbf{H}^2_C$ are tangent to $\mathbf{H}^2_C$. There is a unique point M in $\mathbf{H}^2_C$ which is a critical point of H restricted to $\mathbf{H}^2$. Every vector u normal to $\mathbf{H}^2$ at M is then also normal to the gradient lines of h_x, h_y and h_z which are tangent to $\mathbf{H}^2$, and, as a consequence, $\mathrm{D}_M H(u)=0$. Thus, M is a critical point of H. By the above discussion, $\mathrm{D}^2 H(v,v)=0$, if and only if the 1-parameter subgroup generated by u commutes with the $\mathsf{SL}_2(\mathbb{R})$ associated to $\mathbf{H}^2_C$. Since, by hypothesis, this $\mathsf{SL}_2(\mathbb{R})$ has a compact centralizer, M is a non-degenerate critical point.

The map $G\colon (x,y,z)\mapsto M$ is G equivariant and extends continuously to some G-invariant neighborhood U of V in $\mathbf{F}^3$ with values in $\mathsf{S}(\mathsf{G})$: to have a non-degenerate minimum is an open condition on convex functions of class C^2. It follows that the action of G on U is proper, since the action of G on the symmetric space $\mathsf{S}(\mathsf{G})$ is proper.

We now prove the first assertion of the proposition. let us work by contradiction, and assume that, for all n, there exist tripods τ_1^n and τ_2^n, and triples of points X_n such that

$$d_{\tau_1^n}(X_n,\partial\tau_1^n)<\frac{1}{n},\quad d_{\tau_2^n}(X_n,\partial\tau_2^n)<\frac{1}{n}\quad\text{and}\quad n<d(\tau_1^n,\tau_2^n).$$

We may as well assume that τ_n^1 is constant and equal to τ, and consider $g_n\in G$ such that $g_n(\tau_1^n)=\tau_n^2$. Thus, we have

$$d_\tau(X_n,\partial\tau)\to 0,\quad d_\tau(g_n(X_n),\partial\tau)\to 0\quad\text{and}\quad d(\tau,g_n(\tau))\to\infty.$$

However, this last assertion contradicts the properness of the action of G on a neighborhood of $\partial\tau\in\mathbf{F}^3$.

For the second assertion, working by contradiction again and taking limits as in the proof of the first part, we obtain two tripods τ_1 and τ_2 such that $d_{\tau_1}(\partial\tau_1,\partial\tau_2)=0$ and, for all τ_3 with $\partial\tau_3=\partial\tau_2$, $d(\tau_1,\tau_3)>0$. This is obviously a contradiction. $\square$

8.2. Paths of quasi-tripods and Sullivan maps

Let in this subsection ξ be a ζ-Sullivan map from a dense set W of $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}$. To make life simpler, assuming the axiom of choice, we may extend ξ — a priori non-continuously — to a ζ-Sullivan map defined on all of $\mathbf{P}^1(\mathbb{R})$. We choose, for every element z of $\mathbf{P}^1(\mathbb{R})\setminus W$, a sequence $(w_n)_{n\in\mathbb{N}}$ in W converging to z such that $\xi(w_n)$ converges, and for $\xi(z)$ the limit of $(\xi(w_n))_{n\in\mathbb{N}}$.

Our technical goal is, given a point z_0 in $\mathbf{H}^2$ and two (possibly equal) close points x_1 and x_2 with respect to z_0 in $\mathbf{P}^1(\mathbb{R})$, we construct paths of quasi-tripods "converging" to $\xi(x_i)$. This is achieved in Proposition 8.9 and in its consequence Lemma 8.10. This preliminary construction will be used for the main results of this section: Theorems 8.2 and 8.3

8.2.1. Two paths of tripods for the hyperbolic plane

We start with the model situation in $\mathbf{H}^2$ and prove the following lemma which only uses hyperbolic geometry and concerns tripods for $\mathsf{SL}_2(\mathbb{R})$, which in that case are triple of pairwise distinct points in $\mathbf{P}^1(\mathbb{R})$.

LEMMA 8.7. *There exist universal positive constants $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ with the following property.*

Let z_0 be a point in $\mathbf{H}^2$, x_1 and x_2 be two points in $\mathbf{P}^1(\mathbb{R})$, such that $d_{z_0}(x_1,x_2)$ is small enough (possibly zero), then there exists two 2-sequences of tripods $\underline{T}^1$ and $\underline{T}^2$, where z_0 belongs to the geodesic arc corresponding to the initial chord of both $\underline{T}^1$ and $\underline{T}^2$, with the following properties (see Figure 8.1):

(i) *we have that $\lim \underline{T}_i = x_i$;*

(ii) *the sequences $\underline{T}_1$ and $\underline{T}_2$ coincide for the first n tripods, for n greater than*

$$-\boldsymbol{\kappa}_1 \log d_{z_0}(x_1,x_2);$$

(iii) *two successive tripods T^i_m and T^i_{m+1} are at most $\boldsymbol{\kappa}_2$-swished;*

(iv) *defining the $\mathsf{SL}_2(\mathbb{R})$-tripods $x^i_m := (\partial^- T^i_m, \partial^+ T^i_m, x^i)$, one has $d(x^i_m, T^i_m) \leqslant \boldsymbol{\kappa}_2$.*

In item (iii) of this lemma, we make several abuses of language: firstly, since we are on $\mathsf{SL}_2(\mathbb{R})$, we write $\boldsymbol{K}$-swished instead of $(\boldsymbol{K},0)$-swished, as in Definition 4.6. Secondly, we say that T and T' are $\boldsymbol{K}$-swished whenever actually $\omega^p\tau$ and $\omega^q T'$ are $\boldsymbol{K}$-swished for some integers p and q.

In the proof of the Anosov property for equivariant Sullivan curve, we will use the "degenerate construction", when $x_1 = x_2 =: x_0$, in which case $\underline{T}^1 = \underline{T}^2 =: \underline{\tau}$, whereas we shall use the full case for the proof of the Hölder property.

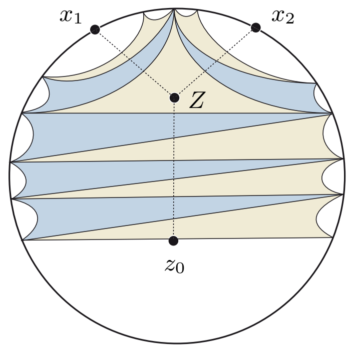


Figure 8.1. Two paths of tripods.

Proof. The process is clear from Figure 8.1. Let us make it formal. Let x_1 and x_2 be two points in $\mathbf{P}^1(\mathbb{R})$ and assume that $d_{z_0}(x_1, x_2)$ is small enough. If $x_1 \neq x_2$, we can now find three geodesic arcs γ_0, γ_1 and γ_2 joining in a point Z in $\mathbf{H}^2$ with angles $\frac{2}{3}\pi$ such that their other extremities are respectively z_0, x_1 and x_2. The arc γ_0 is oriented from z_0 to Z, whilst the others are from Z to x_i, respectively. The tripod τ^0 orthogonal to all three geodesic arcs γ_0, γ_1 and γ_2 will be referred in this proof as the *forking tripod*, and the point of intersection of γ_i with τ^0 is denoted by y_i.

Observe now that there exists a universal positive constant $\boldsymbol{\kappa}_1$ such that

$$\operatorname{length}(\gamma_0) = d_{\mathbf{H}^2}(z_0, Z) \geqslant -2\boldsymbol{\kappa}_1 \log(d_{z_0}(x_1, x_2)), \tag{8.3}$$

where $d_{\mathbf{H}^2}$ is the hyperbolic distance. We now construct a (discrete) lamination Γ with the following properties:

(i) Γ contains the three sides of the forking tripod, and z_0 is in the support of Γ;

(ii) all geodesics in Γ intersect orthogonally, either γ_0, γ_1 or γ_2; let X be the set of these intersection points;

(iii) the distance between any two successive points in X (for the natural ordering of γ_0, γ_1 and γ_2) is greater than 1 and less than 2.

We orient each geodesic in Γ such that its intersection with γ_0, γ_1 or γ_2 is positive. We may now construct two sequences of geodesics Γ^1 and Γ^2 such that Γ^i contains all the geodesics in Γ that are encountered successively when going from z_0 to x_i.

For two successive geodesics γ_i and γ_{i+1} — in either Γ^1 or Γ^2 — we consider the associated finite paths of tripods given by the following construction:

(i) in the case γ_i and γ_{i+1} are both sides of the forking tripod, the path consists of just one tripod: the forking tripod;

(ii) in the other case, we consider the path of tripods with two elements τ_i and $\hat{\tau}_i$, where

$$\tau_i = (\gamma_i(-\infty), \gamma_i(+\infty), \gamma_{i+1}(+\infty)),$$
$$\tau_{i+1} = (\gamma_i(-\infty), \gamma_{i+1}(+\infty), \gamma_{i+1}(-\infty)).$$

Combining these finite paths of tripods in infinite sequences, one obtains two sequences of tripods $\underline{T}^i = \{T^i_m\}_{m\in\mathbb{N}}$, with $i\in\{1,2\}$ which coincides up to the first

$$-\boldsymbol{\kappa}_1 \log(d_{z_0}(x_1, x_2))$$

tripods. Moreover, the swish between T^i_m and T^i_{m+1} is bounded by a universal constant $\boldsymbol{\kappa}_2$ — of whose actual value we do not care, since obviously the set of configurations (T^i_m, T^i_{m+1}) is compact (up to the action of $\mathsf{SL}_2(\mathbb{R})$).

An easy check shows that these sequences of tripods are 3-sequences. The last condition is immediate after possibly enlarging the value of $\boldsymbol{\kappa}_2$ obtained previously. □

8.2.2. Sullivan curves as deformations

Let z_0, x_1, x_2, $\underline{T}^1$ and $\underline{T}^2$ be as in Lemma 8.7. Let ξ be an ζ-Sullivan map. The main idea is that ξ will define a deformation of the sequences of tripods. Our first step is the following lemma.

Lemma 8.8. *For every positive ε, there exists ζ such that, for every $i\in\{1,2\}$ and $m\in\mathbb{N}$, there exists a compatible tripod τ^i_m for T^i_m with respect to ξ, with associated circle maps η^i_m and extended circle maps ν^i_m, such that, denoting by d^i_m the metric $d_{\tau^i_m}$, we have*

$$\partial^\pm \tau^i_m = \xi(\partial^\pm T^i_m), \tag{8.4}$$
$$d^i_m(\xi, \eta^i_m) \leqslant \varepsilon, \tag{8.5}$$
$$d(\tau^i_m, \nu^i_{m-1}(T^i_m)) \leqslant \varepsilon. \tag{8.6}$$

Moreover, for all m smaller than $-\boldsymbol{\kappa}_1 \log d_{z_0}(x_1, x_2)$, we have $\tau^1_m = \tau^2_m$.

Proof. Let us construct inductively the sequence $\underline{\tau}_i$. Let us first construct $\tau^1_0 = \tau^2_0$. We first choose a compatible tripod for T_0, with associated circle maps $\eta^1_0 = \eta^2_0$ and extended circle maps $\nu^1_0 = \nu^2_0$. Let $\tau^i_0 = \eta^i_0(T_0)$ be such that, denoting by d^i_0 the metric $d_{\tau^i_0}$, we have the inequality

$$d^i_0(\xi, \eta^i_0) \leqslant \zeta. \tag{8.7}$$

In particular,

$$d^i_0(\partial^\pm \tau^i_0, \xi(\partial^\pm T^i_0)) \leqslant \zeta.$$

we may thus slightly deform η_0^i (with respect to the metric d_0^i) such that assertion (8.4) holds. Then, for ζ small enough, the relation (8.5) holds for $m=0$, where $\varepsilon=2\zeta$.

Assume now that we have built the sequence up to τ_{m-1}^i. Let then

$$\tau_1=\tau_{m-1}^i, \quad T_1=T_{m-1}^i, \quad T_2=T_m^i,$$

and finally $\nu_1=\nu_{m-1}^i$. Recall that, by the construction of T_1 and T_2,

$$d(T_1,T_2)\leqslant \boldsymbol{\kappa}_2=:M \quad \text{and} \quad d(\nu_1(T_1),\tau_1)\leqslant\varepsilon.$$

We may now apply the second part of Lemma 8.5, which shows that, given ε and ζ small enough, we may choose a compatible tripod τ_2 for T_2 with respect to ξ such that

$$d(\tau_2,\nu_1(T_2))\leqslant\tfrac{1}{2}\varepsilon.$$

We now set $\tau_m^i=:\tau_2$, possibly deforming it a little such that assertion (8.4) holds. Then, by the definition of compatibility, assertion (8.5) holds, while assertion (8.6) holds by construction. The last part of the lemma follows from the inductive nature of our construction and some bookkeeping. □

8.2.3. Main result

Let ξ be a ζ-Sullivan curve. We use the notation of the two previous lemmas. Our main result is the following.

Proposition 8.9. *For all positive ε and for ζ small enough, the following statements hold.*

(i) *The quadruples $\theta_m^i:=(\tau_m^i,\xi(\partial T_m^i))$ are reduced ε-quasi-tripods.*

(ii) *If T_m^i and T_{m+1}^i are R_m^i-swished, then θ_m^i and θ_{m+1}^i are (R_m^i,ε)-swished.*

(iii) *The sequences $\underline{\theta}^1$ and $\underline{\theta}^2$ are ε-deformations of the sequence $\nu_0(\underline{T}^1)$ and $\nu_0(\underline{T}^2)$, respectively.*

(iv) *The n first elements of $\underline{\theta}^1$ and $\underline{\theta}^2$ coincide for $n=-\boldsymbol{\kappa}_1\log(d_{z_0}(x_1,x_2))$.*

(v) *For all m, $\xi(x_i)$ belongs to the ε-sliver $S_\varepsilon(\tau_m^i)$ (see definition in §5.2).*

Proof. Equation (8.7) guarantees that θ_m^i is a ζ-quasi-tripod and reduced by condition (8.4). Furthermore, since T_m^i is at most $\boldsymbol{\kappa}_2$-swished from T_{m-1}^i by Proposition 8.7, inequality (8.6) implies that θ_{m+1}^i is at most (R_m^i,ε)-swished from θ_m^i, and thus $\underline{\tau}^i$ is a model for $\underline{\theta}^i$. Statement (iii) then follows from Proposition 4.12. The coincidence up to $-\boldsymbol{\kappa}_1\log(d_{z_0}(x_1,x_2))$ follows from the last part of Lemma 8.8. Let us prove the last item in the proposition. By the ζ-Sullivan condition, we have

$$d_m^i(\xi(x^i),\eta_m^i(x^i))\leqslant\zeta.$$

Let x^i_m be the $\mathsf{SL}_2(\mathbb{R})$-tripod as in Proposition 8.7, let $\sigma^i_m = \nu^i_m(x^i_m)$ and $\underline{d}^i_m := d_{\sigma^i_m}$. By construction, σ^i_m and τ^m_i are coplanar. By statement (iv) of Lemma 8.7 , $d(x^i_m, T^i_m)$ is bounded by a constant $\boldsymbol{\kappa}_2$, and hence, by Proposition 3.16, d^i_m and $\underline{d}^i_m$ are uniformly equivalent by constants depending only on G and $\boldsymbol{\kappa}_2$. Thus, for ζ small enough, we have

$$\underline{d}^i_m(\xi(x^i), \partial^0\sigma^i_m) = \underline{d}^i_m(\xi(x^i), \eta^i_m(x^i)) \leqslant \varepsilon.$$

In other words, $\xi(x^i)$ belongs to the cone $C_\varepsilon(\sigma^i_m)$, and hence to the sliver $S_\varepsilon(\hat{\tau}^i_m)$ as required, since σ^i_m and τ^m_i are coplanar and $\partial^\pm\sigma^i_m = \partial^\pm\tau^i_m$. □

8.2.4. Limit points

Let then Γ^i_m be the chords generated by the tripods $\hat{\theta}^i_{2m}$, and let us consider the sequences of chords $\underline{\Gamma}_i := \{\Gamma^i_m\}_{m\in\mathbb{N}}$. The final part of our construction is the following lemma.

Lemma 8.10. *The sequence of chords $\underline{\Gamma}^i$ are $(1,\varepsilon)$-deformed sequences of cuffs for ζ small enough. Furthermore, these two sequences coincides up to*

$$N > -\boldsymbol{\kappa}_1 d_\tau(z_1, z_2).$$

Finally,

$$\bigcap_{m=0}^{\infty} \underline{\Gamma}^i = \{\xi(x_i)\}. \tag{8.8}$$

Proof. The first two items of Proposition 8.9, together with Proposition 4.12, imply that, for ζ small enough, the sequences $\underline{\theta}^i$ are ε-deformations of the model sequences $\nu_0(\underline{T}^i)$. This implies the first two assertions. Equation (8.8) follows by Theorem 7.2 (taking $\ell_0 = R = 1$ and $\beta = \mathrm{A}$) and the last item of Proposition 8.9. □

8.3. Sullivan curves and the Hölder property

We prove a more precise version of Theorem 8.2 that we state as follows.

Theorem 8.11. (Hölder modulus of continuity) *There exist positive constants* M, *ζ_0 and α such that, given a ζ_0-Sullivan map ξ from $\mathbf{P}^1(\mathbb{R})$ to $\mathbf{F}$, for every tripod T in $\mathbf{P}^1(\mathbb{R})$, with associated G-tripod τ, with respect to ξ, we have*

$$d_\tau(\xi(x), \xi(y)) \leqslant \mathrm{M} \cdot d_T(x,y)^\alpha.$$

Proof. Since d_τ has uniformly bounded diameter, It is enough to prove this inequality for T such that $d_T(x,y)$ is small enough. Let then $x_1 = x$ and $x_2 = y$ be in $\mathbf{P}^1(\mathbb{R})$, $z_0 = s(T)$,

ξ be a ζ-Sullivan map (for ζ small enough), $\underline{T}^i$ and $\underline{\tau}_i$ be the sequences of $\mathsf{SL}_2(\mathbb{R})$-tripods and G-tripods constructed in the preceding section, and $\underline{\Gamma}_i$ be the sequence of chords satisfying Lemma 8.10. Let

$$\tau_0 := \tau_0^1 = \tau_0^2, \quad \nu_0 := \nu_0^1 = \nu_0^2 \quad \text{and} \quad T_0 := T_0^1 = T_0^2.$$

Let $N > -\boldsymbol{\kappa}_1 \log(d_{T_0}(x_1, x_2))$ be such that $\underline{\tau}^1$ and $\underline{\tau}^2$ coincide up to the first N tripods. By Theorem 7.2 and using Lemma 8.10, we have

$$d_{\tau_0}(\xi(x_1), \xi(x_2)) \leqslant \mathrm{q}^N \cdot \mathrm{A} \leqslant \mathrm{B} \cdot d_{z_0}(x_1, x_2)^{\boldsymbol{\alpha}} = \mathrm{B} \cdot d_{T_0}(x_1, x_2)^{\boldsymbol{\alpha}}, \tag{8.9}$$

for some positive constants B and $\boldsymbol{\alpha}$ depending only on q, A and $\boldsymbol{\kappa}_1$.

Here, τ_0 is associated to T_0. But since $d(T_0, T)$ is uniformly bounded, by the first assertion in Lemma 8.5, $d(\tau_0, \tau)$ is uniformly bounded (for ζ small enough). Thus, by Proposition 3.16, d_τ and d_{τ_0} are uniformly equivalent. In particular,

$$d_\tau(\xi(x), \xi(y)) \leqslant \mathrm{F} \cdot d_{\tau_0}(\xi(x), \xi(y)) \leqslant \mathrm{M} \cdot d_T(x, y)^\alpha.$$

This concludes the proof. □

8.4. Sullivan curves and the Anosov property

In this section, let ξ be a ζ-Sullivan map equivariant under the action of a cocompact Fuchsian group Γ for a representation ρ of Γ in G.

8.4.1. A short introduction to Anosov representations

Intuitively, a Gromov hyperbolic discrete subgroup of G is P-Anosov if every element is P-loxodromic, with "contraction constant" comparable with the word length of the group.

To be more precise, let P^+ be a parabolic subgroup and P^- be its opposite associated to the decompositions

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{l} \oplus \mathfrak{n}^- \quad \text{and} \quad \mathfrak{p}^\pm = \mathfrak{n}^\pm \oplus \mathfrak{l}.$$

For a hyperbolic surface S, let $\mathsf{U}S$ be its unit tangent bundle equipped with its geodesic flow $\{h_t\}_{t\in\mathbb{R}}$. Let ρ be a representation of $\pi_1(S)$ into G. Let $\mathfrak{G}_\rho$ be the flat Lie algebra bundle over S with monodromy $\mathrm{Ad} \circ \rho$. The action of $\{h_t\}_{t\in\mathbb{R}}$ lifts by parallel transport the action of a flow $\{H_t\}_{t\in\mathbb{R}}$ on $\mathfrak{g}_\rho$. We say that the action is *Anosov* if we can find a continuous splitting into vector sub-bundles

$$\mathfrak{G}_\rho = \mathfrak{N}^+ \oplus \mathfrak{l} \oplus \mathfrak{N}^-,$$

, invariant under the action of $\{H_t\}_{t\in\mathbb{R}}$, such that the following conditions hold:

- at each point $x\in\mathsf{U}S$, the splitting is conjugate to the splitting $\mathfrak{g}=\mathfrak{p}^+\oplus\mathfrak{l}\oplus\mathfrak{p}^-$;
- the action of $\{H_t\}_{t\in\mathbb{R}}$ is contracting towards the future on $\mathfrak{N}^+$ and contracting towards the past on $\mathfrak{N}^-$.

Equivalently, let $\mathcal{F}^{\pm}_{\rho}$ be the associated flat bundles to ρ with fibers $\mathsf{G}/\mathsf{P}^{\pm}$. The action of h_t lifts to an action denoted H_t by parallel transport. Then, the representation ρ is Anosov, if we can find continuous ρ-equivariant maps $\xi^{\pm}$ from $\partial_\infty\pi_1(S)$ into $\mathsf{G}/\mathsf{P}^{\pm}$ such that the following conditions hold:

- for $x\neq y$, $\xi^+(x)$ is transverse to $\xi^-(y)$;
- the associated sections $\Xi^{\pm}$ of $\mathsf{F}^{\pm}$ over $\mathsf{U}S$ by ρ are attracting points towards the future and the past, respectively, for the action of $\{H_t\}_{t\in\mathbb{R}}$ on the space of sections endowed with the uniform topology.

8.4.2. A preliminary lemma

For a tripod τ, let $\tau^{\perp}$ be the coplanar tripod to τ such that $\tau^{\perp}$ is obtained after a $\frac{1}{2}\pi$ rotation of τ with respect to $s(\tau)$. In other words, $\partial\tau^{\perp}=(\partial^0\tau,x,\partial^+\tau)$, where x is the symmetric of $\partial^0\tau$ with respect to the geodesic whose endpoints are $\partial^-\tau$ and $\partial^+\tau$. Observe that $s(\tau)=s(\tau^{\perp})$, and thus $d_\tau=d_{\tau^{\perp}}$.

Our key lemma is the following.

Lemma 8.12. *There exists $\boldsymbol{\zeta}$ with such that, if ξ is a $\boldsymbol{\zeta}$-Sullivan map, then there exist positive constants R and c such that, if T is a tripod in $\mathsf{SL}_2(\mathbb{R})$, then for any τ and σ compatible tripods (with respect to ξ) to T and $\varphi_R(T)$ satisfying*

$$\partial^+\sigma=\partial^+\tau=\xi(\partial^+T),$$

we have

$$d_{\tau^{\perp}}(x,y)\leqslant\tfrac{1}{2}\cdot d_{\sigma^{\perp}}(x,y)\quad \textit{for all } x,y\in C_c(\sigma^{\perp}).$$

In this lemma, ξ does not have to be equivariant. Also note that, with the notation of the lemma, we have $\partial^0(\sigma^{\perp})=\xi(\partial^+T)$.

Proof. We will use the confinement lemma (Lemma 6.1). Using the notation of that lemma, let $b:=\beta_3$, let ℓ_0 be an integer greater than $\ell(\beta_3)$ and let η_0 be as in the conclusion of the lemma.

Let $z_0:=s(T)$ be the orthogonal projection of ∂^0T on the geodesic joining ∂^-T to ∂^+T. Let $x_1=x_2:=\partial^+T$. Let us now construct, for $\varepsilon\leqslant\eta_0/(2\ell_0)$ and $\boldsymbol{\zeta}$ small enough as in Theorem 8.2 and Proposition 8.9, the following objects.

- The sequence of $\mathsf{SL}_2(\mathbb{R})$-tripods $\underline{T}:=\underline{T}^1=\underline{T}^2$, with $T_0=T^{\perp}$, associated to the coplanar sequence of chords $\underline{h}$.

• The tripods $\tau_m := \tau_m^1 = \tau_m^2$, and the corresponding sequence of reduced ε-quasi-tripods $\underline{\theta} := \underline{\theta}^1 = \underline{\theta}^2$, which is an ε-deformation of $\nu_0(\underline{\tau})$—according to Proposition 8.9—and associated to the deformed sequence of chords $\underline{\Gamma}$.

• We also denote by ν_i the extended circle map associated to T_i that satisfies $\nu(T_i) = \tau_i$. Let us also denote by μ_i the $\mathsf{SL}_2(\mathbb{R})$-tripods which is the projection of $h_{2(i+1)}$ on h_{2i}, and

$$\lambda_i := \nu_{2i\ell_0}(\mu_{i\ell_0}).$$

It follows that $T_{2\ell_0 m}, ..., T_{2\ell_0(m+1)}$ is a strong $(\ell_0, 2\ell_0)$-coplanar path of tripods. Thus, according to the confinement lemma and our choice of constants, $(\Gamma_{2\ell_0 m}, \Gamma_{2\ell_0(m+1)})$ is $(b, \boldsymbol{\kappa}^7)$-squeezed, and its commanding tripod is the projection of $\nu_{2\ell_0 m}(h_{2\ell_0 m+1})$ on $\nu_{2\ell_0 m}(h_{2\ell_0 m})$, that is λ_m. In other words, since $\lambda_{m+1} \in S_0(\nu_{2\ell_0 m}(h_{2\ell_0 m}))$, we have, for all m,

$$C_b(\lambda_m) \prec \boldsymbol{\kappa}^7 C_{\boldsymbol{\kappa}^7 b}(\lambda_{m+1}).$$

Thus, by Corollary 5.3, using the fact that $\beta_3 \leqslant \alpha_3$, where α_3 is the constant in the proposition, we have

$$d_{\lambda_0}(u, v) \leqslant \frac{1}{2^n} d_{\lambda_n}(u, v) \quad \text{for all } u, v \in C_b(\lambda_n). \tag{8.10}$$

We now make the following claim.

Claim 1. *There exists a constant N only depending on G such that, for any tripod β compatible with $\varphi_{2n\ell_0}(T)$, one has*

$$d(\beta^\perp, \lambda_n) \leqslant N. \tag{8.11}$$

Elementary hyperbolic geometry first shows that there exist positive constants N_1 and $\mathbf{M}_2$ such that

$$\begin{aligned} d(\lambda_n, \tau_{2n\ell_0}) = d(\mu_{n\ell_0}, T_{2n\ell_0}) &\leqslant N_1, \\ d(\varphi_{2n\ell_0}(T), T_{2n\ell_0}) &\leqslant \mathbf{M}_2. \end{aligned}$$

By Lemma 8.5, there exists a constant N_2 such that

$$d(\beta, \tau_{2n\ell_0}) \leqslant N_2.$$

Since there exists a constant N_3 such that $d(\beta, \beta^\perp) \leqslant N_3$, the triangle inequality yields the claim.

Inequality (8.11) and Proposition 3.16 yield that there exists a constant C such that, if σ_n is compatible with $\varphi_{n\ell_0}(T)$, then

$$\frac{1}{C} d_{\sigma_n^\perp} \leqslant d_{\lambda_n} \leqslant C d_{\sigma_n^\perp}. \tag{8.12}$$

Then, taking n_0 such that $2^{n_0-1}>C^2$ and $R=n_0\ell_0$, from inequality (8.10) we get

$$d_{\tau^\perp}(x,y) \leqslant \tfrac{1}{2}\cdot d_{\sigma^\perp}(x,y) \quad \text{for all } x,y\in C_b(\lambda_{n_0}).$$

To conclude, it is therefore enough to prove the following.

CLAIM 2. *There exists a constant c depending only on G such that*

$$C_c(\sigma^\perp)\subset C_b(\lambda_{n_0}).$$

Recall that, by hypothesis, $\partial^+(\sigma)=\xi(x)$. By the last item in Proposition 8.9, for ζ small enough, we have

$$\partial^0(\sigma^\perp)=\xi(x)\in S_{b/2}(h_n),$$

for all n. By the squeezing property, it follows that $\xi(x)\in C_{b/2}(\lambda_m)$ for all m.

Since $d_{\lambda_{n_0}}$ and $d_{\sigma^\perp}$ are uniformly equivalent by inequality (8.12), taking $c=b(2C)^{-1}$ we obtain

$$C_c(\sigma^\perp)=\{u\in\mathbf{F} : d_{\sigma^\perp}(u,\xi(x))\leqslant c\}\subset\left\{u\in\mathbf{F} : d_{\lambda_{n_0}}(u,\xi(x))\leqslant\tfrac{1}{2}b\right\}=C_b(\lambda_{n_0}).$$

This concludes the proof of the Claim 2, and hence of the lemma. □

8.4.3. Completion of the proof of Theorem 8.3

The proof is now standard. Let ρ be a representation of a cocompact torsion-free Fuchsian group Γ. Let $\mathcal{S}$ be the space of $\mathsf{SL}_2(\mathbb{R})$ tripods, $U=\Gamma\backslash\mathcal{S}$ be the space of tripods in the quotient equipped with the flow φ. The space U with its flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is naturally conjugate to (a multiple of) the geodesic flow $\{h_t\}_{t\in\mathbb{R}}$ of the underlying hyperbolic surface. Let finally $\mathcal{F}_\rho$ be the ρ-associated flat bundle over U with fiber $\mathbf{F}$. This fiber bundle is equipped with a flow $\{\Phi_t\}_{t\in\mathbb{R}}$ lifting the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ by parallel transport along the orbit. This flow is a multiple of the $\{H_t\}_{t\in\mathbb{R}}$ in §8.4.1.

Let ξ be a ρ-equivariant ζ-Sullivan map for ζ small enough such that Lemma 8.12 holds. Observe that ξ gives now rise to two transverse Φ_t-invariant sections of $\mathbf{F}$:

$$\sigma^+(T):=\xi(\partial^+T) \quad\text{and}\quad \sigma^-(T):=\xi(\partial^-T).$$

These sections are transverse for ζ small enough, and more precisely for $\zeta < \frac{1}{2}k$, where $k = d_\tau(\partial^+\tau, \partial^-\tau)$ for any tripod τ.

We now choose a fiberwise metric d on $\mathcal{F}$ as follows. For every $T \in \mathcal{S}$, let $\tau(T)$ be a compatible tripod. We may choose the assignment $T \mapsto \tau(T)$ to be Γ-invariant. We define our fiberwise metric at T to be $d_T := d_{\tau(T)}$. This metric may not be continuous transversely to the fibers, but it is locally bounded: locally at a finite distance to a continuous metric, since the set of compatible tripods has a uniformly bounded diameter by Lemma 8.5.

Now, Lemma 8.12 exactly tells us that σ^+ is an attracting fixed section of $\{\phi_t\}_{t \in \mathbb{R}}$ towards the future, and by symmetry σ^- is an attracting fixed section of $\{\phi_t\}_{t \in \mathbb{R}}$ towards the past. By definition, ρ is $\mathbf{F}$-Anosov, and this concludes the proof of Theorem 8.3.

8.4.4. Anosov and Sullivan lemma

As another relation of the Anosov property and Sullivan curves, let us prove the following lemma of independent interest.

Lemma 8.13. *Let ρ_0 be an Anosov representation of a cocompact Fuchsian group Γ. Assume that the limit map ξ_0 is ζ-Sullivan. Then, for any positive ε, any nearby representation ρ is Anosov with a $(\zeta+\varepsilon)$-Sullivan limit map.*

Proof. By the stability property of Anosov representations [22], [12] any nearby representation ρ is Anosov. Let ξ_ρ be its limit map.

By Guichard–Wienhard [12] (see also [6]), ξ_ρ depends continuously on ρ in the uniform topology. More precisely, for any positive ε, for any tripod τ for G, there exists a neighborhood U of ρ_0 such that, for all ρ in U and all $x \in \partial_\infty \mathbf{H}^2$,

$$d_\tau(\xi_\rho(x), \xi_{\rho_0}(x)) \leqslant \varepsilon. \tag{8.13}$$

Instead of fixing τ, we may as well assume that τ belongs to a bounded set K of $\mathcal{G}$, using for instance Proposition 3.16.

Let us consider a compact fundamental domain D for the action of Γ on the space of tripods with respect to $\mathbf{H}^2$. For every tripod T in D, we have a compatible G-tripod τ_T with circle map ν_T with respect to ξ_0. Then, by Lemma 8.5, the set

$$D_\mathsf{G} := \{\tau_T : T \in D\},$$

is bounded. Thus, inequality (8.13) holds for all τ in D_G . It follows that, for all $T \in D$,

$$d_{\tau_T}(\xi_\rho(x), \eta_T(x)) \leqslant \zeta + \varepsilon. \tag{8.14}$$

Using the equivariance under Γ, the inequality (8.14) now holds for all tripods T for $\mathbf{H}^2$. In other words, ξ_ρ is $(\zeta+\varepsilon)$-Sullivan. $\square$

8.5. Improving Hölder derivatives

Our goal is to explain that under certain hypothesis we can can promote a Sullivan curve with respect to a smaller subset to a full Sullivan curve. We need a series of technical definitions before actually stating our theorem.

(i) For every tripod T for $\mathbf{H}^2$, let d_T be the visual distance on $\partial_\infty\mathbf{H}^2$ associated to T. We say that a subset W of $\partial_\infty\mathbf{H}^2$ is (a,T)-*dense* if

$$\text{for all } x\in\partial_\infty\mathbf{H}^2 \text{ there exists } y\in W \text{ such that } d_T(x,y)\leqslant a.$$

(ii) Let a and ζ be positive numbers and Z be a dense subset of $\partial_\infty\mathbf{H}^2$. We say that a map ξ from $\partial_\infty\mathbf{H}^2$ to $\mathbf{F}$ is (a,ζ)-*Sullivan* if, given any tripod T in $\mathbf{H}^2$, there exist a circle map ξ_T and an (a,T)-dense subset W_T of Z such that, writing $\tau{:=}\xi_T(T)$, we have $d_\tau(\xi_T(x),\xi(x))\leqslant\zeta$ for all x in W_T.

(iii) Let Γ a be cocompact Fuchsian group and let ρ be a representation of Γ in G. Let ξ be a ρ-equivariant map from $\partial_\infty\mathbf{H}^2$ to F. We say that ξ is *attractively coherent for* ρ if, given any y point in $\partial_\infty\mathbf{H}^2$, there exists a sequence $\{\gamma_m\}_{m\in\mathbb{N}}$ of elements of Γ such that the following conditions hold:

- the limit of $\{\gamma_m^+\}_{m\in\mathbb{N}}$ is y;
- $\xi(y)$ is the limit of $\{z_m\}_{m\in\mathbb{N}}$, where z_m is an attractive fixed point for $\rho(\gamma_m)$.

The improvement theorem is the following.

THEOREM 8.14. (Improvement theorem) *Let Γ be a cocompact Fuchsian group. Then, there exists a positive constant ζ_2 such that, for every ζ less than ζ_2, there exists a positive a_0 satisfying the following property. Suppose that we are given*

(i) *a continuous family $\{\rho_t\}_{t\in[0,1]}$ of representations of Γ in G,*

(ii) *a family $\{\xi_t\}_{t\in[0,1]}$ of (a_0,ζ)-Sullivan maps,*

and assume that every ξ_t is ρ_t-equivariant and attractively coherent, and that the map ξ_0 is 2ζ-Sullivan. Then, for all t, ξ_t is a 2ζ-Sullivan map.

8.5.1. Bootstrapping and the proof of Theorem 8.14

Let us first start with a preliminary lemma.

LEMMA 8.15. *Let ρ be an Anosov representation of a cocompact Fuchsian group. Let ξ be a map from $\partial_\infty\mathbf{H}^2$ to $\mathbf{F}$ which is attractively coherent for ρ. Then, ξ is the limit map of ρ.*

Proof. Let η be the limit map of ρ. Let $y\in\partial_\infty\mathbf{H}^2$ and let $\{\gamma_m\}_{m\in\mathbb{N}}$ be as in the definition of attractively continuous. Since γ_m^+ is the attractive fixed point of γ, it follows that $\eta(\gamma_m^+)=z_m$. The continuity of η shows that $\eta(y)=\xi(y)$. □

We may now proceed to the proof. Let $\{\xi_t\}_{t\in[0,1]}$, $\{\rho_t\}_{t\in[0,1]}$ and Γ be as in the hypothesis of the theorem that we want to prove. Let ζ_0, M and α be as in Theorem 8.11. Let ζ_1 be such that Theorem 8.3 holds. Let finally $\zeta_2=\frac{1}{4}\min(\zeta_1,\zeta_0)$ and $\zeta\leqslant\zeta_2$.

Let us consider the subset K of $[0,1]$ of those parameters t such that ξ_t is 2ζ-Sullivan.

Lemma 8.16. *The set K is closed.*

Proof. Let $\{t_m\}_{m\in\mathbb{N}}$ be a sequence of elements of K converging to s. For all n, $\{\xi_{t_m}\}_{m\in\mathbb{N}}$ forms an equicontinuous family by Theorem 8.11, as $2\zeta_2\leqslant\zeta_0$. We may extract a subsequence converging to a map $\hat{\xi}$ which is ρ_s equivariant and ζ_2-Sullivan. In particular, since $2\zeta_2\leqslant\zeta_1$, it follows that ρ_s is Anosov and $\hat{\xi}$ is the limit map of ρ_s. By hypothesis, ξ_s is attractively continuous, and thus $\xi_s=\hat{\xi}$ by Lemma 8.15. This proves that $s\in K$. $\square$

We prove that K is open in two steps.

Lemma 8.17. *Assume that ξ_t is 2ζ-Sullivan. Then, there exists a neighborhood U of t such that, for $s\in U$, ξ_s is ζ_0-Sullivan.*

Proof. Our assumptions guarantee that ρ_t is Anosov and, by the stability condition for Anosov representations [22], [12], the representation ρ_s is Anosov for s close to t. Lemma 8.15 implies that ξ_s is the limit curve of ρ_s. Lemma 8.13 then shows that, for s close enough to t, ξ_s is ζ_0-Sullivan since $2\zeta<2\zeta_2<\zeta_0$. $\square$

We now prove a bootstrap lemma.

Lemma 8.18. (Bootstrap) *Given ζ, there exists some constant A such that, for $a_0<A$, if ξ_s is ζ_0-Sullivan, then ξ_s is 2ζ-Sullivan.*

Proof. This is an easy consequence of the triangle inequality together with Theorem 8.11. Since ξ_s is (a_0,ζ)-Sullivan, for every tripod T for $\mathbf{H}^2$, there exist an a_0-dense subset W and a circle map η such that, for all $y\in W$, $d_\tau(\xi_s(y),\eta(y))\leqslant\zeta$, where $\tau=\eta(T)$. Let then $x\in\partial_\infty\mathbf{H}^2$ and $y\in W$ be such that $d_T(x,y)\leqslant a_0$. Then,

$$d_\tau(\xi_s(x),\eta(x))\leqslant d_\tau(\xi(x),\xi(y))+d_\tau(\eta(x),\eta(y))+d_\tau(\xi(y),\eta(y)\leqslant \mathrm{M}a_0^{\alpha_0}+a_0+\zeta.$$

The last quantity is less than 2ζ, for a_0 small enough. This concludes the proof. $\square$

Thus, K is open. Let $t\in K$. Then, by Lemma 8.17, for any nearby s in K, ξ_s is ζ_0-Sullivan, and hence 2ζ-Sullivan by the bootstrap lemma (Lemma 8.18). Since K is non-empty, closed and open, $K=[0,1]$ and this concludes the proof of the theorem.

9. Pair of pants from triangles

The purpose of this section is to define *almost closing pairs of pants*. These almost closing pairs of pants will play the role of almost Fuchsian pair of pants in [14]. §13 will reveal they are ubiquitous in $\Gamma\backslash\mathcal{G}$.

These almost closing pair of pants are the building blocks for the construction of surfaces whose fundamental group injects. Themselves are built out of two tripods using symmetries, a construction reminiscent of building hyperbolic pair of pants using ideal triangles.

Our main results here will be a result describing the structure of a pair of pants (Theorem 9.5), whose proof relies on the closing lemma (Lemma 9.4). We will also prove Proposition 9.9 that gives information onf the boundary loops.

In all this section, ε and R are positive constants.

9.1. Almost closing pair of pants

Let Γ be a subgroup of G. We will consider not only the case of a discrete Γ, but also the case $\Gamma{=}\mathsf{G}$.

Given a tripod τ_0, the *R-perfect pair of pants* associated to τ_0 is the quintuple $(\alpha,\beta,\gamma,\tau_0,\tau_1)$ such that τ_0 and τ_1 are tripods, α, β and γ are elements of G such that $\alpha\gamma\beta{=}\mathrm{Id}$, and moreover the pairs $(\tau_0,\omega^2\tau_1)$, $(\omega(\tau_0),\omega\beta(\tau_1))$ and $(\omega^2(\tau_0),\alpha^{-1}\tau_1)$ are all R-sheared.

We also consider alternatively an R-perfect pair of pants to be a quadruple of tripods (T,S_0,S_1,S_2) such that

$$S_0 = K\varphi_R T, \quad S_2 = \omega K\phi_R(\omega T) \quad \text{and} \quad S_1 = \omega^2 K\phi_R(\omega^2 T),$$

with the *boundary loops* α, β and γ such that $S_0{=}\alpha S_1$, $S_2{=}\beta S_0$ and $S_1{=}\gamma S_2$, and such that $(\alpha,\beta,\gamma,T,S_0)$ is a perfect pair of pants with respect to the previous definition.

More generally, we will navigate freely between quadruple of tripods (T,S_0,S_1,S_2) with *boundary loops* α, β and γ such that $S_0{=}\alpha S_1$, $S_2{=}\beta S_0$ and $S_1{=}\gamma S_2$, and quintuple $(\alpha,\beta,\gamma,T,S_0)$, where α, β and γ are elements of G such that $\alpha\gamma\beta{=}\mathrm{Id}$, using the construction described above in the particular case of R-perfect pair of pants.

We now wish to deform that perfect situation.

Let K be the map $x{\to}\omega(\bar{x})$ defined in §3.3.1.

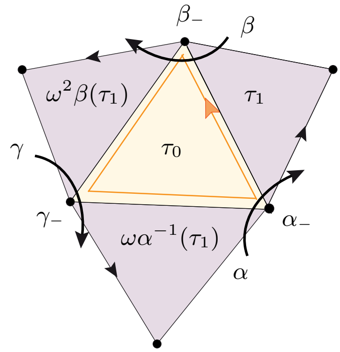


Figure 9.1. Pair of pants from triangles.

Definition 9.1. (Almost closing) (i) Let T and S be two tripods in $\mathcal{G}$, α be an element in G and μ be a positive constant. We say that (T,S) is (μ,R)-*almost closing* for α if there exist tripods u and v such that

$$d(u,T)\leqslant\mu, \qquad\qquad d(v,S)\leqslant\mu, \tag{9.1}$$

$$d(K\circ\varphi_R(u),S)\leqslant\mu, \quad d(K\circ\varphi_R(v),\alpha(T))\leqslant\mu. \tag{9.2}$$

(ii) Let $P{=}(\alpha,\beta,\gamma,\tau_0,\tau_1)$ such that $\alpha,\beta,\gamma\in\mathsf{G}$ with $\alpha\gamma\beta{=}1$, and τ_0,τ_1 are tripods, we say P is a (μ,R)-*almost closing pair of pants* if

(a) τ_0 and τ_1 are (μ,R)-almost closing for α,
(b) $\omega^2\tau_0$ and $\omega\alpha^{-1}(\tau_1)$ are (μ,R)-almost closing for γ,
(c) $\omega\tau_0$ and $\omega^2\beta(\tau_1)$ are (μ,R)-almost closing for β.

Let us first make immediate remarks.

Proposition 9.2. *If* (T,S) *is* (μ,R)*-almost closing for* α*, then* $(S,\alpha(T))$ (*and then* $(\alpha^{-1}(S),T)$) *is also almost closing for* α*.*

Proof. For the first item, observe that, if (u,v) is the pair of tripods working in the definition for (T,S), then $(v,\alpha(u))$ works for $(S,\alpha(T))$. For the second item, observe that the pair $(\alpha^{-1}\tau_1,\tau_0)$ is (μ,R)-almost closing for α: we use $(\alpha_*^{-1}\tau_1^*,\tau_0^*)$ for (u,v) in the definition. We apply the first item to get that (τ_0,τ_1) is (μ,R)-almost closing. The other results for other pairs follows from symmetric considerations. $\square$

We observe also the following symmetries.

Proposition 9.3. (Symmetries) *If* $(\alpha,\beta,\gamma,\tau_0,\tau_1)$ *is an* (μ,R)*-almost closing of pants, then both*

$$\omega(\alpha,\beta,\gamma,\tau_0,\tau_1):=(\beta,\gamma,\alpha,\omega(\tau_0),\omega^2(\beta\tau_1)) \quad \textit{and} \quad (\alpha,\beta^{-1}\alpha^{-1},\beta,\tau_1,\alpha(\tau_0)),$$

are also (μ,R)*-almost closing.*

Proof. Using the definition of almost closing, we have the following.

(i) $(\tau_1, \alpha(\tau_0))$ is (μ, R)-almost closing for α, from the first item in the previous proposition.

(ii) After taking the image by α, we have that $(\omega^2\alpha(\tau_0), \omega(\tau_1))$ is (μ, R)-almost closing for $\alpha\gamma\alpha^{-1}{=}\beta^{-1}\alpha^{-1}$, and thus, from the first item in the previous proposition, $(\omega(\tau_1), \omega^2\beta^{-1}(\tau_0))$ is also (μ, R)-almost closing for $\beta^{-1}\alpha^{-1}$,

(iii) $(\omega\tau_0, \omega^2\beta(\tau_1))$ is $(\varepsilon/R, R)$-almost closing for β, and thus, from the first item in the previous proposition, $(\omega^2\tau_1, \omega\tau_0)$ is $(\varepsilon/R, R)$-almost closing for β.

This proves the result. ☐

9.2. Closing lemma for tripods

The first step in the proof of the closing pant theorem is the following lemma.

LEMMA 9.4. (Closing lemma) *There exist constants $\mathbf{M}_2$, ε_2 and R_2 such that, assuming that (T, S) is (μ, R)-almost closing for α, for $R{>}R_2$ and $\mu{\leqslant}\varepsilon_2$, then the following statements hold*:

(i) *α is P-loxodromic*;

(ii) *$d_T(T^\pm, \alpha^\pm){\leqslant}\mathbf{M}_2(\mu{+}e^{-R})$*;

(iii) *moreover, if $\tau_\alpha{=}\psi(T, \alpha^-, \alpha^+)$ and $\sigma_\alpha{=}\Psi(S, \alpha^-, \alpha^+)$, then*

$$d(\varphi_{2R}(\tau_\alpha), \alpha(\tau_\alpha)) \leqslant \mathbf{M}_2(\mu{+}e^{-R}), \tag{9.3}$$

$$d(\varphi_R(\tau_\alpha), \sigma_\alpha) \leqslant \mathbf{M}_2(\mu{+}e^{-R}); \tag{9.4}$$

(iv) *$d(T, S){\leqslant}2R$.*

In the sequel, $\mathbf{M}_i$, R_i and ε_i will denote positive constants depending only on G.

As an immediate consequence and using Proposition 9.3, we get the following structure theorem for almost closing pair of pants.

THEOREM 9.5. (Structure of pair of pants) *There exist positive constants $\mathbf{M}_0$, ε_0 and R_0, depending only on G, with the following property. Let $\varepsilon{\leqslant}\varepsilon_0$ and $R{\geqslant}R_0$. Then, for any (ε, R)-almost closing pair of pants $(\alpha, \beta, \gamma, \tau_0, \tau_1)$, we have the following*:

(i) *α, β and γ are all P-loxodromic*;

(ii) *the two quadruples (τ_0, x, y, z), with $x{\in}\{\alpha^-, \gamma^+\}$, $y{\in}\{\alpha^+, \beta^-\}$ and $z{\in}\{\gamma^-, \beta^+\}$, and (τ_1, u, v, w), with $u{\in}\{\alpha^-, \beta^+\}$, $v{\in}\{\alpha^+, \beta^{-1}(\gamma^-)\}$ and $w{\in}\{\beta^-, \beta^{-1}(\gamma^+)\}$, are both $\mathbf{M}_0(\varepsilon{+}e^{-R})$-quasi-tripod*;

(iii) *moreover, if $\tau_\alpha{=}\Psi(\tau_0, \alpha^-, \alpha^+)$ and $\sigma_\alpha{=}\Psi(\tau_1, \alpha^-, \alpha^+)$, then*

$$d(\varphi_{2R}(\tau_\alpha), \alpha(\tau_\alpha)) \leqslant \mathbf{M}_0(\varepsilon{+}e^{-R}), \tag{9.5}$$

$$d(\varphi_R(\tau_\alpha), \sigma_\alpha) \leqslant \mathbf{M}_0(\varepsilon{+}e^{-R}). \tag{9.6}$$

Proposition 9.9 will give further information on the boundary loops.

9.3. Preliminaries

Our first lemma is essentially a result on hyperbolic plane geometry.

LEMMA 9.6. *There exist constants* R_3 *and* $\mathbf{M}_3$ *such that, for* $R \geqslant R_3$, *the following holds. Let* u *be any tripod. Then,* $v := \varphi_{-2R}((K\circ\varphi)^2(u))$ *satisfies*

$$d(v,u) \leqslant \mathbf{M}_3 e^{-R},$$
$$d(\varphi_R(v), K\circ\varphi_R(u)) \leqslant \mathbf{M}_3 e^{-R},$$
$$\partial^- v = \partial^- u.$$

Proof. There exists a constant M such that, for all w,

$$d(w, K(w)) \leqslant M. \tag{9.7}$$

Recall that w and $K(w)$ are coplanar. In the upper half-plane model where

$$\partial^- w = \partial^- K(w) = \infty,$$

$K(w)$ is obtained from w by an horizontal translation. Thus, for R large enough,

$$d(\varphi_{-R}(w), \varphi_{-R}(K(w))) \leqslant \mathbf{M}_3 e^{-R}.$$

Applying this inequality to $w = \varphi_R(K\circ\varphi_R(u))$ gives

$$d(K\circ\varphi_R(u), \varphi_R(v)) = d(K\circ\varphi_R(u), \varphi_{-R}(K\circ\varphi_R)^2(u)) \leqslant \mathbf{M}_3 e^{-R}, \tag{9.8}$$

and thus the second assertion. Proceeding further, for R large enough, the previous inequality and (9.7), together with the triangle inequality, give

$$d(\varphi_R(u), \varphi_{-R}(K\circ\varphi_R)^2(u)) \leqslant 2M. \tag{9.9}$$

Then, for R large enough,

$$d(u, \varphi_{-2R}(K\circ\varphi_R)^2(u)) \leqslant \mathbf{M}_3 e^{-R}. \tag{9.10}$$

This concludes the proof. □

The second lemma gives a way to prove an element is loxodromic.

LEMMA 9.7. *There exist constants* $\mathbf{M}_4$, R_4 *and* ε_4 *depending only on* G *such that, for any* $\varepsilon \leqslant \varepsilon_4$ *and* $R \geqslant R_4$, *given* $\alpha \in \mathsf{G}$ *and assuming that there exists a tripod* v *such that*

$$d(\varphi_{2R}(v), \alpha(v)) \leqslant \varepsilon,$$

then α *is loxodromic.*

Proof. Let ξ be the isomorphism from G_0 to G associated to v, it follows that, for some constant B depending only on G, by inequality (3.4),

$$d_0(\xi^{-1}(\alpha), \exp(2Ra_0)) \leqslant B\varepsilon.$$

Thus, α is P-loxodromic and $d_v(\alpha^{\pm}, \partial^{\pm} v) \leqslant \varepsilon$ for R large enough. □

9.4. Proof of Lemma 9.4

We now start the proof of the closing lemma (Lemma 9.4), referring to "(T, S) is (μ, R)-almost closing for α" as assumption $(*)$.

9.4.1. A better tripod

PROPOSITION 9.8. *There exist constants* $\mathbf{M}_2$, ε_2 *and* R_1 *such that, assuming* $(*)$, $\mu \leqslant \varepsilon_1$ *and* $R > R_1$, *there exist*

(i) *a tripod* u_0 *such that* u_0, $K \circ \varphi_R(u_0)$ *and* $(K \circ \varphi_R)^2(u_0)$ *are respectively* $\mathbf{M}_2\mu$-*close to* T, S *and* $\alpha(T)$;

(ii) *a tripod* u_1 *such that* u_1, $\varphi_R(u_1)$ *and* $\varphi_{2R}(u_1)$ *are* $\mathbf{M}_2(\mu + e^{-R})$-*close respectively to* T, S *and* $\alpha(T)$.

Proof. Let u and v be associated to S and T by assumption $(*)$. Recall that $K \circ \varphi_t$ is contracting on $\mathcal{U}^+$ for positive t (large enough)—see Proposition 3.10. Similarly, by Proposition 3.12, K preserves each leaf of $\mathcal{U}^{0,-}$, and thus $\varphi_{-t} \circ K^{-1}$ is uniformly κ-Lipschitz (for some κ) along $\mathcal{U}^{0,-}$ for all positive t.

By hypotheses (9.1), (9.2) and the triangle inequality,

$$d(K \circ \varphi_R(u), v) \leqslant 2\mu.$$

Thus, if μ is small enough, $\mathcal{U}^{0,-}_{K(\varphi_R(u))}$ intersects $\mathcal{U}^+_v$ in a unique point w which is 4μ-close to both v and $K \circ \varphi_R(u)$ —and hence 5μ close to S—as in Figure 9.2.

Recall that K preserves each leaf of $\mathcal{U}^{0,-}$ by Proposition 3.12. Thus,

$$\mathcal{U}^{0,-}_{K\varphi_R(u)} = K(\mathcal{U}^{0,-}_{\varphi_R(u)}).$$

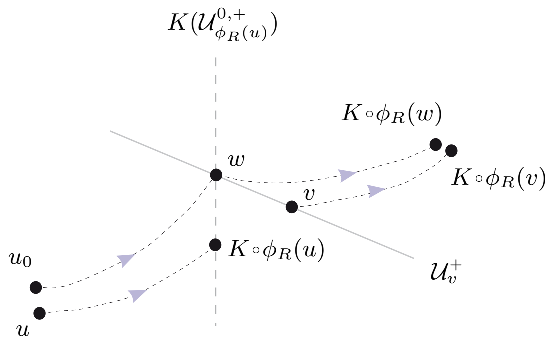


Figure 9.2. Closing quasi-orbits.

Let now u_0 be such that $K\circ\varphi_R(u_0)=w$. According to our initial remark that $\varphi_{-R}\circ K^{-1}$ is κ-Lipschitz, since

$$d(K\circ\varphi_R(u_0), K\circ\varphi_R(u))\leqslant 2\mu,$$

we get that

$$d(u_0,u)\leqslant\kappa(2\mu)\quad\text{and}\quad d(u_0,T)\leqslant(2\kappa+1)\mu,$$

where the second inequality used hypothesis (9.1).

Symmetrically, using now that $K\circ\varphi_R$ is contracting for R large enough along the leaves of $\mathcal{U}^+$, it follows that

$$d((K\circ\varphi_R)^2(u_0),(K\circ\varphi_R)(v))\leqslant\mu.$$

Combining with hypothesis (9.2), this yields

$$d((K\circ\varphi_R)^2(u_0),\alpha(T))\leqslant 2\mu.$$

Thus, with $M=2\kappa+1$, we obtain a tripod u_0 such that u_0, $K\circ\varphi_R(u_0)$ and $(K\circ\varphi_R)^2(u_0)$ are $\mathbf{M}_1\mu$-close to T, S and $\alpha(T)$, respectively.

Now, according to Lemma 9.6, it is enough to take $u_1=\varphi_{-2R}(K\circ\varphi_R)^2(u_0)$, then apply the triangle inequality. □

9.4.2. Proof of the closing Lemma 9.4

Combining Proposition 9.8 and Lemma 9.7, we obtain that, for ε small enough and R large enough, α is loxodromic, and moreover

$$d_{u_1}(\partial^- u_1,\alpha^-)\leqslant M_3(\mu+e^{-R}).$$

Since u_1 is $\mathbf{M}_2(\mu+e^{-R})$-close to T, applications of Proposition 3.16 yields

$$d_T(\partial^- T, \alpha^-) \leqslant M_4(\mu+e^{-R}). \tag{9.11}$$

Observe that $(\bar{T}, \bar{S})$ is $(\mu, -R)$-almost closed with respect to α. Thus, reversing the signs in the proof, one gets symmetrically that

$$d_{\bar{T}}(\partial^+ T, \alpha^+) \leqslant M_4(\mu+e^{-R}),$$

and thus

$$d_T(\partial^+ T, \alpha^+) \leqslant M_5(\mu+e^{-R}). \tag{9.12}$$

It remains to prove the last statement in the lemma. Since

$$d(T, u_1) \leqslant \mathbf{M}_2(\mu+e^{-R}) \quad \text{and} \quad d(\alpha(T), \varphi_{2R}(u_1) \leqslant \mathbf{M}_2(\mu+e^{-R}), \tag{9.13}$$

it follows that u_1, $\alpha^\pm$ satisfies the hypothesis of Proposition 4.5. Thus, setting

$$u_\alpha := \Psi(u_1, \alpha^-, \alpha^+),$$

one has

$$d(\Psi(\varphi_{2R}(u_1), \alpha^-, \alpha^+), \varphi_{2R}(u_\alpha))) \leqslant M_6(\mu+e^{-R}). \tag{9.14}$$

Using inequalities (9.13) a second time and Lemma 4.3, we obtain that

$$d(\Psi(\varphi_{2R}(u_1), \alpha^-, \alpha^+), \alpha(\tau_\alpha)) \leqslant M_6(\mu+e^{-R}), \tag{9.15}$$

$$d(\varphi_{2R}(\tau_\alpha), \varphi_{2R}(u_\alpha)) = d(\tau_\alpha, u_\alpha) \leqslant M_7(\mu+e^{-R}), \tag{9.16}$$

where the equality in the line (9.16) comes from the fact that the flow acts by isometry on the leaves of the central foliation (cf. property (v)). The triangle inequality yields, from inequalities (9.14) and (9.15),

$$d(\varphi_{2R}(u_\alpha), \alpha(\tau_\alpha)) \leqslant M_8(\mu+e^{-R}).$$

Combining finally with (9.16), we get

$$d(\varphi_{2R}(\tau_\alpha), \alpha(\tau_\alpha)) \leqslant M_9(\mu+e^{-R}). \tag{9.17}$$

This proves inequality (9.3). A similar argument shows inequality (9.4).

The last statement is an obvious consequence of the previous ones.

9.5. Boundary loops

We show that the boundary loops of an almost closing pair of pants are close to be perfect in a precise sense.

Let us say that a triple of tripods (S_1^*, T^*, S_0^*) is *R-perfect* if $T^* = K\phi_R(S_1)$ and $S_0^* = K\phi_R T^*$. Observe that there is a unique α_* such that $S_0^* = \alpha_*(S_1^*)$. As a shorthand, we say then that (S_1^*, T^*, S_0^*) is *R-perfect for* α_*.

We finally say α is *R-perfect* if it is conjugate to $\exp(Ra_0)$.

PROPOSITION 9.9. (Boundary loop) *There exists a constant C such that, given ε small enough, and then R large enough, the following holds.*

For all (ε, R)-almost closing pair of pants $W=(T, S_0, S_1, S_2)$ with boundary loop α, β and γ, there exists an R- perfect triple (S_1^*, T^*, S_0^*) for α_* such that

$$d(S_0^*, S_0) \leqslant C\frac{\varepsilon}{R}, \quad d(S_1^*, S_1) \leqslant C\frac{\varepsilon}{R}, \quad d(T^*, T) \leqslant C\frac{\varepsilon}{R},$$

and, if k is such that $\alpha=\alpha_*\cdot k$, then k is in L_{α_*} and

$$\sup(d_{S_0^*}(k, \mathrm{Id}), d_{T^*}(k, \mathrm{Id}), d_{S_1^*}(k, \mathrm{Id})) \leqslant C\frac{\varepsilon}{R}. \tag{9.18}$$

If furthermore α is R-perfect, then $k=\mathrm{Id}$. Similar results hold for β and γ, for different R-perfect triples.

In the next proofs, C_i will denote constants depending only on G.

Proof. Since W is an $(\mathbf{M}\varepsilon/R, R)$-almost closing pair of pants, let (S_0^*, T^*, S_1^*) be the perfect triple obtained by the first item in Proposition 9.8, and let α_* be such that $S_0^*=\alpha_* S_1$. Let then

$$\tau=\Psi(S_0, \alpha^-, \alpha^+), \qquad \sigma=\Psi(S_1, \alpha^-, \alpha^+), \qquad u=\Psi(T, \alpha^-, \alpha^+),$$
$$\tau^*=\Psi(S_0^*, \alpha_*^-, \alpha_*^+), \quad \sigma^*=\Psi(S_1^*, \alpha_*^-, \alpha_*^+), \quad u^*=\Psi(T^*, \alpha_*^-, \alpha_*^+).$$

Recall that $\alpha(\sigma)=\tau$ and $\alpha_*(\sigma^*)=\tau^*$. For ε small enough, and hence R large enough, by the structure theorem (Theorem 9.5), we obtain that the four points σ, σ^*, S_1 and S_1^* are all $C_1(\varepsilon/R)$-close, and thus d_σ, d_{σ^*}, d_{S_1} and $d_{S_1^*}$ are all 2-Lipschitz equivalent, for ε/R small enough. The same holds for τ, τ^*, S_0 and S_0^*, as well as for u, u^*, T and T^*.

Let g in G such that $\sigma=g\cdot\sigma^*$. Since σ and σ^* are $C_1(\varepsilon/R)$-close, it follows that

$$d_{S_1^*}(g, \mathrm{Id}) \leqslant 2d_\sigma(g, \mathrm{Id}) \leqslant C_2\frac{\varepsilon}{R}. \tag{9.19}$$

Applying the third item of Theorem 9.5 and then the first one, we obtain that, for some constant C_3 depending only on G,

$$d(\varphi_{2R}(\sigma), \alpha(\sigma)) \leqslant C_3\frac{\varepsilon}{R}, \qquad d(\varphi_{2R}(\sigma^*), \alpha_*(\sigma^*)) \leqslant C_3\frac{\varepsilon}{R},$$
$$d(\varphi_R(\sigma), u) \leqslant C_3\frac{\varepsilon}{R}, \qquad d(\varphi_R(\sigma^*), u^*) \leqslant C_3\frac{\varepsilon}{R}.$$

Since $\varphi_{2R}(\sigma)=g\cdot\varphi_{2R}(\sigma^*)$, we have

$$\begin{aligned} d(g\tau, \tau) &\leqslant d(g(\tau), g\varphi_{2R}(\sigma))+d(g\varphi_{2R}(\sigma), \tau^*)+d(\tau^*, \tau) \\ &\leqslant d(\tau, \varphi_{2R}(\sigma))+d(\varphi_{2R}(\sigma^*), \tau^*)+d(\tau^*, \tau) \leqslant (2C_3+C_1)\frac{\varepsilon}{R}, \end{aligned}$$

and thus, as above,

$$d_{S_0^*}(g,\mathrm{Id}) \leqslant 2d_\tau(g,\mathrm{Id}) \leqslant C_4\frac{\varepsilon}{R}. \tag{9.20}$$

Similarly

$$\begin{aligned}d(gu,u) &\leqslant d(g(u),g\varphi_R(\sigma))+d(g\varphi_R(\sigma),u^*)+d(u^*,u)\\ &\leqslant d(u,\varphi_R(\sigma))+d(\varphi_R(\sigma^*),u^*)+d(u^*,u) \leqslant (2C_3+C_1)\frac{\varepsilon}{R},\end{aligned}$$

and thus, as above,

$$d_{T^*}(g,\mathrm{Id}) \leqslant 2d_u(g,\mathrm{Id}) \leqslant C_4\frac{\varepsilon}{R}. \tag{9.21}$$

Inequalities (9.19)–(9.21) prove the inequality

$$\sup(d_{S_0^*}(g,\mathrm{Id}),d_{T^*}(g,\mathrm{Id}),d_{S_1^*}(g,\mathrm{Id})) \leqslant C\frac{\varepsilon}{R}.$$

We can thus replace S_0^*, T^* and S_1^* by $g^{-1}S_0^*$, $g^{-1}T^*$, $g^{-1}S_1^*$, respectively, so that, for this new perfect triple, one has $g=1$.

Let us write now $k=(\alpha_*)^{-1}\alpha$. Then,

$$d(k\sigma^*,\sigma^*)=d(\alpha(\sigma),\alpha_*(\sigma^*))=d(\tau,\tau^*) \leqslant (2C_3+2C_1)\frac{\varepsilon}{R}.$$

This implies that

$$d_{S_1^*}(k,\mathrm{Id}) \leqslant 2d_{\sigma^*}(k,\mathrm{Id}) \leqslant 2C_5\frac{\varepsilon}{R}. \tag{9.22}$$

Finally, since $\sigma^*=\sigma$, one has $\alpha_*^\pm=\alpha^\pm$. Thus $k(\alpha_*^\pm)=\alpha_*^\pm$, and in particular k commutes with α_*. We deduce that

$$\begin{aligned}d_{S_1^*}(k,\mathrm{Id}) &= d_{S_1^*}(k\alpha_*^{-1},\alpha_*^{-1}) = d_{S_1^*}(\alpha_*^{-1}k,\alpha_*^{-1})\\ &= d_{\alpha_*(S_1^*)}(k,\mathrm{Id}) = d_{S_0^*}(k,\mathrm{Id}) \leqslant 2C_5\frac{\varepsilon}{R}.\end{aligned} \tag{9.23}$$

We then deduce that

$$d_{\sigma^*}(k,\mathrm{Id}) \leqslant C_6\frac{\varepsilon}{R} \quad \text{and} \quad d_{\tau^*}(k,\mathrm{Id}) \leqslant C_6\frac{\varepsilon}{R}.$$

Since k belongs to L_{α_*}, k commutes with $\alpha_*^{1/2}$. But $u^*=\alpha_*^{1/2}(\tau^*)$, and thus the equation above yields

$$d_{u^*}(k,\mathrm{Id}) \leqslant C_6\frac{\varepsilon}{R}.$$

From this, we deduce that

$$d_T(k,\mathrm{Id}) \leqslant C_7\frac{\varepsilon}{R}. \tag{9.24}$$

Inequalities (9.22)–(9.24) prove inequality (9.18).

Assume finally that α is perfect, such that $\alpha=f\alpha_* f^{-1}$. Then,

$$\alpha_* k = h\alpha_* f^{-1}.$$

Since k is small, $\alpha_* k$ is P-loxodromic. As k fixes $\alpha_*^\pm$ and $\alpha_* k$ has α^+ as unique attracting fixed point and α^- as unique repulsive fixed point, $\alpha_* k$ also has α^+ as unique attracting fixed point and α^- as unique repulsive fixed point. It follows that $f(\alpha_*^\pm)=\alpha_*^\pm$. Thus, f belongs to L_{α_*}, and as such commutes with α_*. It follows that $k=\mathrm{Id}$. $\square$

9.6. Negatively almost closing pair of pants

In this section, we have only dealt with positively almost closing pair of pants. Perfectly symmetric results are obtained for negatively almost closing pair of pants, once they have been defined correctly — which we have not done yet. We postpone this discussion to §10.5 after the discussion of the "inversion".

10. Triconnected pairs of tripods and pair of pants

We define in this section triconnected pairs of tripods. These objects consist of a pair of tripods together with three homotopy classes of path between them. One may think of them as a very loosely almost closing pair of pants.

We then define weights for these tripods, and show that, when the weight of a triconnected pair of tripod is non-zero, this triconnected pair of tripods actually defines an almost closing pair of pants.

Apart from important definitions, and in particular the inversion of tripods discussed in the last section, the main result of this section is the closing up tripod theorem (Theorem 10.9).

This section will make use of a discrete subgroup Γ of G, with *non-zero injectivity radius* — or more precisely such that $\Gamma\backslash\mathrm{Sym}(\mathsf{G})$ has a non-zero injectivity radius. When Γ is a lattice, this is equivalent to the lattice being uniform.

10.1. Triconnected and biconnected pair of tripods and their lift

Definition 10.1. (Triconnected and biconnected pairs of tripods)

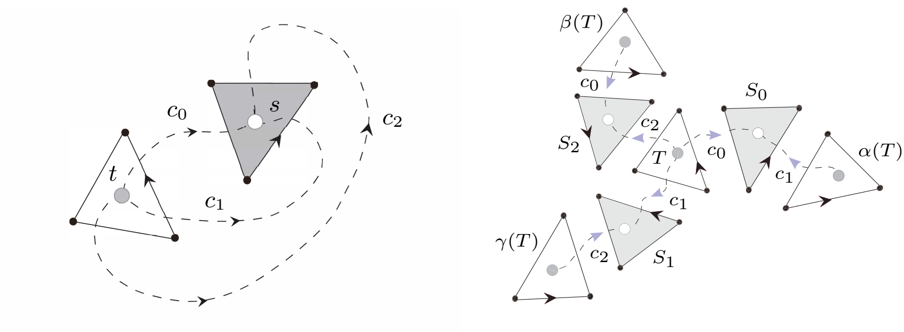
(i) A *triconnected pair of tripods* in $\Gamma\backslash\mathsf{G}$ (see Figure 10.1a) is a quintuple

$$W=(t,s,c_0,c_1,c_2),$$

where t and s are two tripods in $\Gamma\backslash\mathcal{G}$, and c_0, c_1 and c_2 are three homotopy classes of paths from t to s, from $\omega^2(t)$ to $\omega(s)$ and from $\omega(t)$ to $\omega^2(s)$, respectively, up to loops defined in a K_0-orbit. The associated *boundary loops* are the following elements of $\pi_1(\Gamma\backslash\mathcal{G}/\mathsf{K}_0,s)\simeq\Gamma$:

$$\alpha=c_0\bullet c_1^{-1}\quad \beta=c_2\bullet c_0^{-1}\quad\text{and}\quad \gamma=c_1\bullet c_2^{-1}.$$

The *associated pair of pants* is the triple $P=(\alpha,\beta,\gamma)$. Observe that $\alpha\;\gamma\;\beta=1$.



(a) A triconnected pair of tripods.

(b) Triconnected pairs of tripods in the universal cover.

Figure 10.1. Triconnected pairs of tripods and their lifts.

(ii) A *triconnected pair of tripods in the universal cover* is a quadruple (T, S_0, S_1, S_2) such that T, S_0, S_1 and S_2 are tripods in the same connected component of $\mathcal{G}$. The *boundary loops* of (T, S_0, S_1, S_2) are the elements α, β and γ of G such that

$$S_0 = \alpha(S_1), \quad S_2 = \beta(S_1) \quad \text{and} \quad S_1 = \gamma(S_2).$$

Similarly, we give the following definitions.

(i) A *biconnected pair of tripods* is a quadruple $b=(t, s, c_0, c_1)$, where t and s are tripods, and c_0 and c_1 are homotopy classes of paths from t to s and from $\omega^2 t$ to ωs, respectively, in $\Gamma\backslash\mathcal{G}$ (up to loops in K_0-orbits). Its *boundary loop* is $\alpha=c_0 \bullet c_1^{-1}$.

(ii) A *biconnected pair of tripods in the universal cover* is a triple (T, S_0, S_1) such that T, S_0 and S_1 are tripods in the same connected component of $\mathcal{G}$. The *boundary loop* of (T, S_0, S_1) is the element α of G such that $S_0=\alpha(S_1)$.

A triconnected pair of tripods $q=(t, s, c_0, c_1, c_2)$ defines a triconnected pair of tripods (T, S_0, S_1, S_2) in the universal cover, up to the diagonal action of Γ, called the *lift of a triconnected pair of tripods*, where T is a lift of t in $\mathcal{G}$, and S_0, S_1 and S_2 are the three lifts of s, such that S_0, ωS_1 and $\omega^2 S_2$ are the endpoints of the paths lifting respectively c_0, c_1 and c_2 starting respectively at T, $\omega^2 T$ and ωT, as in Figure 10.1b. Observe that

$$S_0 = \alpha(S_1), \quad S_1 = \gamma(S_2) \quad \text{and} \quad S_2 = \beta(S_0),$$

where α, β and γ are the three boundary loops of q.

Conversely, since $\mathcal{G}/\mathsf{K}_0$ is contractible, we may think of a triconnected pair of tripods as a quadruple of tripods (T, S_0, S_1, S_2) in the same connected component of $\mathcal{G}$ well defined, up to the diagonal action of Γ, such that S_i all lie in the same Γ-orbit. In particular, we define an action of ω on the space of triconnected pairs of tripods by

$$\omega(T, S_0, S_1, S_2) := (\omega T, \omega^2 S_2, \omega^2 S_0, \omega^2 S_1). \tag{10.1}$$

On triconnected pairs of tripods $q=(t,s,c_0,c_1,c_2)$, the corresponding rotation is

$$\omega(t,s,c_0,c_1,c_2)=(\omega(t),\omega^2(s),c_2,c_0,c_1).$$

10.2. Weight functions

We fix a positive ε_0 less than half the injectivity radius of $\Gamma\backslash\mathcal{G}$. Let us fix a smooth positive function Ξ from $\mathbb{R}$ to $\mathbb{R}$, with support in $]-1,1[$, and define for every tripod τ and real ε the *bell function* $\Theta_{\tau,x}$ by

$$\Theta_{\tau,\varepsilon}(x)=\frac{1}{\int_{B(\tau,\varepsilon)}\Xi(d(y,\tau)/\varepsilon)\,d\mu(y)}\Xi\left(\frac{1}{\varepsilon}d(x,\tau)\right).$$

The following proposition is straightforward.

Proposition 10.2. *The function $\Theta_{\tau,\varepsilon}$ has its support in an ε neighborhood of τ, is positive and of integral* 1. *For any isometry g of $\mathcal{G}$,*

$$\Theta_{g(\tau),\varepsilon}\circ g=\Theta_{\tau,\varepsilon}.$$

Finally, there exists a constant D, independent of ε and τ, such that

$$\|\Theta_{\tau,\varepsilon}\|_{C^k}\leqslant D\varepsilon^{-k-D}. \tag{10.2}$$

Proof. The technical part is to prove the inequality (10.2). Let $G_\lambda(x)=\Xi(\lambda d(x,\tau))$, for $\lambda>1$. An induction shows that, for an auxiliary connection ∇ on $\Gamma\backslash\mathcal{G}$, the kth derivative of G_λ, $\nabla^{(k)}G_\lambda$, is a polynomial of degree at most k in λ, the derivatives of $d(x,\tau)$ and the derivatives of Ξ, and whose coefficients only depend on $\Gamma\backslash\mathcal{G}$. Since, moreover, this polynomial vanishes when $d(x,\tau)>1$, we obtain that

$$\|\nabla^{(k)}G_\lambda\|\leqslant K\cdot\lambda^k, \tag{10.3}$$

where K only depends on $\Gamma\backslash\mathcal{G}$. Let us also consider the function

$$f(\lambda)=\int_{\Gamma\backslash\mathcal{G}}\Xi\left(\frac{1}{\varepsilon}d(y,\tau)\right)d\mu(y)=\int_{\Gamma\backslash\mathcal{G}}\Xi\left(\frac{1}{\varepsilon}d(y,\tau)\right)d\mu(y).$$

Taking a lower bound of the function Ξ to be a step function equal to A on the interval $]-C,C[$ and zero elsewhere. Then,

$$f(\lambda)\geqslant A\int_{B(\tau,C/\lambda)}d\mu(y)\geqslant K\left(\frac{1}{\lambda}\right)^{\dim(\mathcal{G})}, \tag{10.4}$$

where K only depends on $\mathcal{G}$ and Ξ. The proposition now follows at once from inequalities (10.3) and (10.4). □

As a consequence, the family of functions $\Theta_{T,\varepsilon}$ also makes sense on $\Gamma\backslash\mathcal{G}$, and the same property holds: notice that this is the point where we make use of the fact that the lattice is uniform.

Definition 10.3. (Weight functions) Let $\varepsilon<\varepsilon_0$ and let R be a positive real greater than 1. The *upstairs weight function* is defined on the space of pairs of tripods (T,S) in $\mathcal{G}$ by

$$\mathrm{A}_{\varepsilon,R}(T,S):=\int_{\mathcal{G}}\Theta_{T,\varepsilon/|R|}(x)\cdot\Theta_{S,\varepsilon/|R|}(K\circ\varphi_R(x))\,dx. \tag{10.5}$$

The *downstairs weight function* is defined on the space of pairs of tripods (t,s) in $\Gamma\backslash\mathcal{G}$ by

$$\mathrm{a}_{\varepsilon,R}(t,s):=\int_{\Gamma\backslash\mathcal{G}}\Theta_{t,\varepsilon/|R|}(x)\cdot\Theta_{s,\varepsilon/|R|}(K\circ\varphi_R(x))\,dx.$$

Let t and s be tripods in $\Gamma\backslash\mathcal{G}$. Let c_0 be a path from t to s. The *connected tripod weight function* is defined by

$$\mathrm{a}_{\varepsilon,R}(t,s,c_0):=\mathrm{A}_{\varepsilon,R}(T,S_0),$$

where T is any lift of t in $\mathcal{G}$, and S_0 is the lift of s which is the endpoint of the lift of c_0 starting at T.

Remarks. (i) For R negative, we define

$$\mathrm{A}_{\varepsilon,R}(T,S):=\int_{\mathcal{G}}\Theta_{T,\varepsilon/|R|}(x)\cdot\Theta_{S,\varepsilon/|R|}(K^{-}\circ\varphi_R(x))\,dx,$$

where $K^{-}(x)=\omega\circ K(x)=\omega^2(\bar{x})$. Similarly, we define $\mathrm{a}_{\varepsilon,\mathrm{R}}$.

(ii) By construction, for any $g\in\mathsf{G}$ we have

$$\mathrm{A}_{\varepsilon,R}(gT,gS)=\mathrm{A}_{\varepsilon,R}(T,S).$$

(iii) The value of $\mathrm{a}_{\varepsilon,R}(t,s,c)$ only depends on t, s and the homotopy class of c.

(iv) Let $\pi(t,s)$ be the set of homotopy classes of paths from t to s. Then,

$$\sum_{c\in\pi(t,s)}\mathrm{a}_{\varepsilon,R}(t,s,c)=\int_{\Gamma\backslash\mathcal{G}}\Theta_{t,\varepsilon/|R|}(x).\Theta_{s,\varepsilon/|R|}(K\circ\varphi_R(x))\,dx=\mathrm{a}_{\varepsilon,R}(t,s). \tag{10.6}$$

Definition 10.4. (Weight of a triconnected pair of tripods) Let R be a positive real. Let $W=(T,S_0,S_1,S_2)$ be a triconnected pair of tripods in the universal cover. The *weight* of W is defined by

$$\mathrm{B}_{\varepsilon,R}(W)=\mathrm{A}_{\varepsilon,R}(T,S_0)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2T,\omega S_1)\cdot\mathrm{A}_{\varepsilon,R}(\omega T,\omega^2S_2). \tag{10.7}$$

Similarly, the *weight* of a biconnected pair of tripods $B=(T,S_0,S_1)$ is defined by

$$\mathrm{D}_{\varepsilon,R}(B):=\mathrm{A}_{\varepsilon,R}(T,S_0)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2T,\omega S_1). \tag{10.8}$$

The functions $\mathrm{B}_{\varepsilon,R}$ and $\mathrm{D}_{\varepsilon,R}$ are Γ-invariant, and thus descend to functions $\mathrm{b}_{\varepsilon,R}$ and $\mathrm{d}_{\varepsilon,R}$ for triconnected pairs of tripods and biconnected pairs of tripods in $\Gamma\backslash\mathcal{G}$, respectively. Using the definition of $\mathrm{b}_{\varepsilon,R}$ and equation (10.1),

$$\mathrm{b}_{\varepsilon,R}\circ\omega=\mathrm{b}_{\varepsilon,R}, \tag{10.9}$$

$$\sum_{c_0,c_1,c_2}\mathrm{b}_{\varepsilon,R}(t,s,c_0,c_1,c_2)=\mathrm{a}_{\varepsilon,R}(t,s).\mathrm{a}_{\varepsilon,R}(\omega^2(t),\omega(s)).\mathrm{a}_{\varepsilon,R}(\omega(t),\omega^2(s)), \tag{10.10}$$

where the last equation used equations (10.7) and (10.6),

As an immediate consequence of the definitions of the weight functions, we have the following.

Proposition 10.5. *Let $(\alpha,\beta,\gamma,\tau_0,\tau_1)$ be an $(\varepsilon/R,R)$-almost closing pair of pants and let $W:=(\tau_0,\tau_1,\alpha^{-1}(\tau_1),\beta(\tau_1))$. Then, $\mathrm{B}_{\varepsilon,R}(W)$ is non-zero.*

One of our main goals is to prove the converse.

For $R<0$, for reasons that will become clear in Proposition 10.13, we define

$$\begin{aligned}\mathrm{B}_{\varepsilon,R}(W)&:=\mathrm{A}_{\varepsilon,R}(T,\omega^2S_1)\cdot\mathrm{A}_{\varepsilon,R}(\omega T,\omega S_0)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2T,S_2),\\ \mathrm{D}_{\varepsilon,R}(W)&:=\mathrm{A}_{\varepsilon,R}(T,\omega^2S_1)\cdot\mathrm{A}_{\varepsilon,R}(\omega T,\omega S_0).\end{aligned}$$

We have the following result.

Proposition 10.6. (Symmetry) *We have, for $R>0$,*

$$\begin{aligned}\mathrm{A}_{\varepsilon,R}(S,T)&=\mathrm{A}_{\varepsilon,R}(\omega^2T,\omega S),\\ \mathrm{D}_{\varepsilon,R}(B)&=\mathrm{A}_{\varepsilon,R}(T,S_0)\cdot\mathrm{A}_{\varepsilon,R}(S_1,T).\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}\mathrm{A}_{\varepsilon,R}(S,T)&=\int_{\mathcal{G}}\Theta_{S,\varepsilon/|R|}(x)\cdot\Theta_{T,\varepsilon/|R|}(K\circ\varphi_R(x))\,dx\\ &=\int_{\mathcal{G}}\Theta_{S,\varepsilon/|R|}(\varphi_{-R}\circ K^{-1}(x))\cdot\Theta_{T,\varepsilon/|R|}(x))\,dx\\ &=\int_{\mathcal{G}}\Theta_{S,\varepsilon/|R|}(\varphi_{-R}(\overline{\omega^2(x)}))\cdot\Theta_{T,\varepsilon/|R|}(x))\,dx\\ &=\int_{\mathcal{G}}\Theta_{S,\varepsilon/|R|}(\overline{\varphi_R\omega^2(x)})\cdot\Theta_{T,\varepsilon/|R|}(x))\,dx\\ &=\int_{\mathcal{G}}\Theta_{S,\varepsilon/|R|}(\overline{\varphi_R(x)})\cdot\Theta_{T,\varepsilon/|R|}(\omega(x))\,dx\\ &=\int_{\mathcal{G}}\Theta_{\omega S,\varepsilon/|R|}(K\circ\varphi_R(x))\cdot\Theta_{\omega^2T,\varepsilon/|R|}(x)\,dx\\ &=\mathrm{A}_{\varepsilon,R}(\omega^2T,\omega S),\end{aligned}$$

where we used the following facts:

(i) in the first equality, the fact that K and φ_R preserve the volume form;

(ii) in the second equality, the fact that $K(x)=\omega(\bar{x})$;

(iii) in the third equality, the fact that $\varphi_R(\bar{x})=\overline{\varphi_{-R}}$;

(iv) in the fourth equality, the change of variable $y=\omega^2(x)$;

(v) for the fifth equality, the fact that ω is an isometry.

The second equation in the proposition is an immediate consequence of the first. □

10.2.1. Weight functions and mixing

Recall that a flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is *exponentially mixing* if there exists some integer k and positive constants C and a such that, given two smooth C^k functions f and g, one has, for all positive t,

$$\left|\int_X f.g\circ\varphi_t\, d\mu - \left(\int_X f\, d\mu\right)\cdot\left(\int_X g\, d\mu\right)\right| \leqslant Ce^{-at}\|f\|_{C^k}\,\|g\|_{C^k}. \tag{10.11}$$

In Appendix B, we recall the fact that the action of $\{\varphi_t\}_{t\in\mathbb{R}}$ on $\Gamma\backslash\mathcal{G}$ is exponentially mixing when Γ is a lattice. As an immediate corollary, we have the following.

PROPOSITION 10.7. (Weight function and mixing) *Given a uniform lattice Γ, there exist a positive constant $q=q(\Gamma)$ depending only on Γ and a positive constant $\boldsymbol{K}=K(\varepsilon,\Gamma)$ depending only on ε and Γ such that, for R large enough and every $t,s\in\Gamma\backslash\mathcal{G}$, one has*

$$|\mathrm{a}_{\varepsilon,R}(t,s)-1| \leqslant \boldsymbol{K}\exp(-q|R|). \tag{10.12}$$

Proof. This follows from the definition of exponential mixing and the definition of the function $\mathrm{a}_{\varepsilon,R}$, by equations (10.6) and (10.2). □

Here is an easy corollary.

COROLLARY 10.8. *For any positive ε, for R large enough, the function $\mathrm{a}_{\varepsilon,R}$ never vanishes. Also, given any t and s, there exist (c_0,c_1,c_2) such that $\mathrm{b}_{\varepsilon,R}(t,s,c_0,c_1,c_2)$ is not zero.*

Proof. The first part follows from the previous Proposition 10.7, the second part from equation (10.10). □

10.3. Triconnected pair of tripods and almost closing pair of pants

The main theorem of this section is to relate triconnected pairs of tripods to an almost closing pair of pants, and to prove the converse of Proposition 10.5

THEOREM 10.9. (Closing up tripods) *There exists a constant* $\mathbf{M}$, *depending only on* G, *such that the following holds. For any positive* ε, *there exists* R_0 *such that, for any triconnected pair of tripods* $W=(T,S_0,S_1,S_2)$ *with boundary loops* α, β *and* γ *satisfying* $\mathrm{B}_{\varepsilon,R}(W)\neq 0$ *with* $R>R_0$, $(\alpha,\beta,\gamma,T,S_0)$ *is an* $(\mathbf{M}\varepsilon/R,R)$*-positively almost closing pair of pants.*

The proof of Theorem 10.9 is an immediate consequence of the following proposition.

PROPOSITION 10.10. *For* μ *small enough and then* R *large enough, if* $B=(T,S_0,S_1)$ *is a biconnected pair of tripod with boundary loop* α *such that* $\mathrm{D}_{\mu,R}(B)\neq 0$, *then* T *and* S_0 *are* $(\mu/R,R)$*-almost closing for* α.

Proof. We have $S_0=\alpha(S_1)$. Since $\mathrm{A}_{R,\mu}(T,S_0)\neq 0$, there exists u such that

$$\Theta_{T,\mu/R}(u).\Theta_{S_0,\mu/R}(K\circ\varphi_R(u))\neq 0.$$

Thus, from the definition of Θ,

$$d(u,T)\leqslant\frac{\mu}{R}\quad\text{and}\quad d(K\circ\varphi_R(u),S_0)\leqslant\frac{\mu}{R}. \tag{10.13}$$

Similarly, since $\mathrm{A}_{R,\mu}(\omega^2(T),\omega(S_1))\neq 0$, there exists a tripod z such that

$$d(\omega(z),T)\leqslant d(z,\omega^2(T))\leqslant\frac{\mu}{R}\quad\text{and}\quad d(K\circ\varphi_R(z),\omega(S_1))\leqslant\frac{\mu}{R}. \tag{10.14}$$

Here, we used that ω is an isometry for d (see the beginning of §3.4). Let

$$v:=\alpha(\omega^2\circ K\circ\varphi_R(z))=\alpha(\overline{\varphi_R(z)}).$$

Then, using the fact the metric on $\mathcal{G}$ is invariant by G and ω, and using Corollary 3.17,

$$d(v,S_0)=d(v,\alpha(S_1))=d(\omega^2\circ K\circ\varphi_R(z),S_1)=d(K\circ\varphi_R(z),\omega(S_1))\leqslant\frac{\mu}{R}. \tag{10.15}$$

Moreover, using the commutation properties (Proposition 3.10), we have

$$K\circ\varphi_R(v)=\omega(\overline{\varphi_R(\alpha(\overline{\varphi_R(z)}))})=\alpha\circ\omega(\varphi_{-R}(\varphi_R(z)))=\alpha(\omega(z)).$$

Thus, by the right inequality in (10.14), and combining with inequality (10.15), we have

$$d(K\circ\varphi_R(v),\alpha(T))\leqslant\frac{\mu}{R}\quad\text{and}\quad d(v,S_0)\leqslant\frac{\mu}{R}. \tag{10.16}$$

The result now follows from inequality (10.16) and (10.13). □

10.4. Reversing orientation on triconnected and biconnected pairs of tripods

We need an analogue of the transformation that reverses the orientation on pairs of pants. Let $\mathbf{J}_0$ in $\mathrm{Aut}(\mathsf{G}_0)$ be a reflection for $\mathfrak{s}_0$ (see §2.1). Let σ be the involution $x \mapsto \bar{x}$ defined in §3.3. For an even $\mathfrak{sl}_2$-triple, $\mathbf{J}_0$ and σ commute: this follows from a direct matrix computation. Recall also that the automorphism $\mathbf{J}_0$ fixes L_0 pointwise, since by definition $\mathbf{J}_0 \in \mathsf{Z}_{\mathsf{G}_0}(\mathsf{L}_0)$.

Definition 10.11. (Reversing orientation on $\mathcal{G}$) The *orientation-reversing involution* $\mathbf{I}_0$ is the automorphism of G_0 defined by $\mathbf{I}_0 := \mathbf{J}_0 \circ \sigma$.

We use the same notation to define its action on the space of tripods $\mathcal{G} = \mathrm{Hom}(\mathsf{G}_0, \mathsf{G})$ by precomposition.

Remarks. (i) $\mathbf{I}_0$ commutes with σ, and if $\mathfrak{s}_0 = (a_0, x_0, y_0)$ is the fundamental $\mathfrak{sl}_2$-triple, then

$$\mathbf{I}_0(a_0, x_0, y_0) = (-a_0, y_0, x_0). \tag{10.17}$$

(ii) We have $\mathbf{I}_0 \circ \varphi_R = \varphi_{-R} \circ \mathbf{I}_0$, and similarly $\omega \circ \mathbf{I}_0 = \mathbf{I}_0 \circ \omega^2$ and $\mathbf{I}_0 \circ K \circ \mathbf{I}_0 = \omega \circ K$.

(iii) For any tripod τ, we have $\delta^{\pm} \mathbf{I}_0(\tau) = \delta^{\mp} \tau$ and $\delta^0(\mathbf{I}_0(\tau)) = \delta^0(\tau)$.

(iv) Since the action of $\mathbf{I}_0$ commutes with the action of G and generates together with ω a finite group, we may assume that it preserves the left-invariant metric on $\mathcal{G}$. In particular, we may choose our foot projection such that it commutes with $\mathbf{I}_0$ according to assertion (4.1):

$$\Psi(\mathbf{I}_0(x), \mathbf{I}_0(y), \mathbf{I}_0(z)) = \mathbf{I}_0(\Psi(x, y, z)). \tag{10.18}$$

(v) When G is isomorphic to $\mathsf{PSL}_2(\mathbb{C})$, $\mathbf{I}_0$ corresponds to the symmetry J with respect to a geodesic.

Definition 10.12. (Reversing orientation and rotation on $\mathcal{Q}$) The *reversing involution* $\mathbf{I}_0$ (see Figure 10.2) on the set of triconnected pairs of tripods $\mathcal{Q}$, is given by

$$\mathbf{I}_0(t, s, c_0, c_1, c_2) := (\mathbf{I}_0(t), \omega \mathbf{I}_0(s)), \omega^2 \mathbf{I}_0(c_1), \omega^2 \mathbf{I}_0(c_0), \omega^2 \mathbf{I}_0(c_2)). \tag{10.19}$$

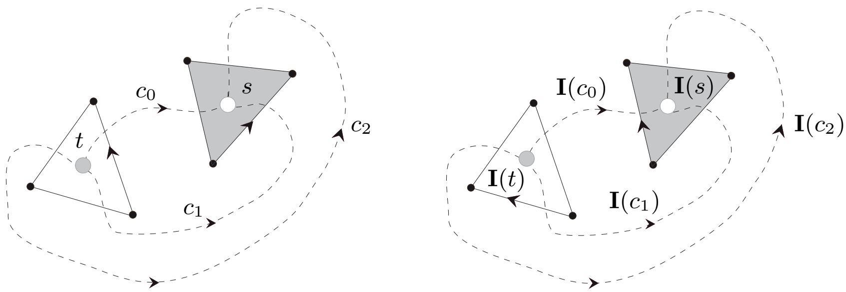
On the set $\mathcal{B}$ of biconnected pairs of tripods, it is given by

$$\mathbf{I}_0(t, s, c_0, c_1) := (\mathbf{I}_0(t), \omega \mathbf{I}_0(s), \omega^2 \mathbf{I}_0(c_1), \omega^2 \mathbf{I}_0(c_0)). \tag{10.20}$$

In order for this definition to make sense, we need to check that $\mathbf{I}_0$ sends a triconnected pair of tripods to a triconnected pair of tripods.

Proof. Recall that a triconnected pair of tripods is a quintuple (t, s, c_0, c_1, c_2), where

- c_0 goes from t to s,
- c_1 goes from $\omega^2 t$ to ωs,
- c_2 goes from ωt to $\omega^2 s$.



(a) Before. (b) After.

Figure 10.2. Reversing orientation on triconnected pairs of tripods. Here, $\mathbf{I}{=}\omega\mathbf{I}_0$.

Let us check that $(\mathbf{I}_0(t),\omega\mathbf{I}_0(s)),\omega^2\mathbf{I}_0(c_1),\omega^2\mathbf{I}_0(c_0),\omega\mathbf{I}_0(c_2))$ is a triconnected pair of tripods. Indeed, denoting $u{=}\mathbf{I}_0(t)$ and $v{=}\omega\mathbf{I}_0(s)$, we have

(i) $\omega^2\mathbf{I}_0(c_1)$ goes from $\omega^2\mathbf{I}_0\omega^2(t){=}\mathbf{I}_0(t){=}u$ to $\omega^2\mathbf{I}_0\omega(s){=}\omega\mathbf{I}_0(s){=}v$,

(ii) $\omega^2\mathbf{I}_0(c_0)$ goes from $\omega^2\mathbf{I}_0(t){=}\omega^2(u)$ to $\omega^2\mathbf{I}_0(s){=}\omega(v)$,

(iii) $\omega^2\mathbf{I}_0(c_2)$ goes from $\omega^2\mathbf{I}_0\omega(t){=}\omega\mathbf{I}_0(t){=}\omega(u)$ to $\omega^2\mathbf{I}_0\omega^2(s){=}\mathbf{I}_0(s){=}\omega^2(v)$,

Thus, the image by $\mathbf{I}_0$ of a triconnected pair of tripods is a triconnected pair of tripods. □

Reversing orientation plays well with the weight functions and boundary loops:

Proposition 10.13. *Let $q{=}(t,s,c_0,c_1,c_2)$ be a triconnected pair of tripods.*

(i) *If the lift of q is $W{=}(T,S_0,S_1,S_2)$, then the lift of $\mathbf{I}_0(q)$ is*

$$\mathbf{I}_0(W):=(\mathbf{I}_0(T),\omega\mathbf{I}_0(S_1),\omega\mathbf{I}_0(S_0),\omega\mathbf{I}_0(S_2)).$$

(ii) *If the boundary loops of q are (α,β,γ), then those of $\mathbf{I}_0(q)$ are $(\alpha^{-1},\gamma^{-1},\beta^{-1})$. In particular, $\mathbf{I}_0$ sends $\mathcal{Q}_{[\alpha]}$ to $\mathcal{Q}_{[\alpha^{-1}]}$.*

(iii) *Finally,*

$$\mathbf{I}_0\circ\omega=\omega^2\circ\mathbf{I}_0,\quad \mathrm{b}_{\varepsilon,R}\circ\mathbf{I}_0=\mathrm{b}_{\varepsilon,-R}\quad \textit{and}\quad \mathrm{d}_{\varepsilon,R}\circ\mathbf{I}_0=\mathrm{d}_{\varepsilon,-R}. \tag{10.21}$$

Proof. This follows either from squinting at symmetries in Figure 10.2 or from tedious computations that we officiate now.

For the first item, observe that

- the lift of $\omega^2\mathbf{I}_0(c_1)$ goes from $\omega^2\mathbf{I}_0\omega^2(T){=}\mathbf{I}_0(T)$ to $\omega^2\mathbf{I}_0\omega(S_1){=}\omega\mathbf{I}_0(S_1)$,
- the lift of $\omega^2\mathbf{I}_0(c_0)$ goes from $\omega^2\mathbf{I}_0(T)$ to $\omega^2\mathbf{I}_0(S_0){=}\omega(\omega\mathbf{I}_0(S_0))$,
- the lift of $\omega^2\mathbf{I}_0(c_2)$ goes from $\omega^2\mathbf{I}_0\omega(T){=}\mathbf{I}_0(T)$ to $\omega^2\mathbf{I}_0\omega^2(S_2){=}\omega^2(\omega\mathbf{I}_0(S_2))$.

The second item now follows from the first and the fact that Γ commutes with $\mathbf{I}_0$ and ω.

Let us now check the third item. For the sake of convenience, we use the abusive notation $\Theta_{t,\varepsilon/|R|}=\Theta_t$. Thus, for $R>0$,

$$\begin{aligned}
\mathrm{A}_{\varepsilon,R}(\mathbf{I}_0(T),\mathbf{I}_0(S)) &= \int_{\mathcal{G}} \Theta_{\mathbf{I}_0(T)}(x)\cdot\Theta_{\mathbf{I}_0(S)}(K\circ\varphi_R(x))\,dx \\
&= \int_{\mathcal{G}} \Theta_T(\mathbf{I}_0(x))\cdot\Theta_S(\mathbf{I}_0\circ K\circ\varphi_R(x))\,dx \\
&= \int_{\mathcal{G}} \Theta_T(x)\cdot\Theta_S(\mathbf{I}_0\circ K\circ\varphi_R\circ\mathbf{I}_0(x))\,dx \\
&= \int_{\mathcal{G}} \Theta_T(x)\cdot\Theta_S(\omega\circ K\circ\varphi_{-R}(x))\,dx \\
&= \mathrm{A}_{\varepsilon,-R}(T,S),
\end{aligned}$$

where, for the second equality we use the equivariance of Θ, for the third a change of variables in $\mathcal{G}$, and for the fourth the commuting relations (ii) following Definition 10.11. Let $q=(t,s,c_0,c_1,c_2)$, with lift $W=(T,S_0,S_1,S_2)$. Recall that

$$\begin{aligned}
\mathrm{B}_{\varepsilon,R}(W) &= \mathrm{A}_{\varepsilon,R}(T,S_0)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2T,\omega S_1)\cdot\mathrm{A}_{\varepsilon,R}(\omega T,\omega^2 S_2), \\
\mathrm{B}_{\varepsilon-R}(W) &= \mathrm{A}_{\varepsilon,R}(\omega T,\omega S_0)\cdot\mathrm{A}_{\varepsilon,R}(T,\omega^2 S_1)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2T,S_2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{b}_{\varepsilon,R}(\mathbf{I}_0(W)) &= \mathrm{A}_{\varepsilon,R}(\mathbf{I}_0(T),\omega\mathbf{I}_0(S_1)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2\mathbf{I}_0(T),\omega^2\mathbf{I}_0(S_0))\cdot\mathrm{A}_{\varepsilon,R}(\omega\mathbf{I}_0(T),\mathbf{I}_0(S_2)) \\
&= \mathrm{A}_{\varepsilon,R}(\mathbf{I}_0(T),\mathbf{I}_0(\omega^2S_1))\cdot\mathrm{A}_{\varepsilon,R}(\mathbf{I}_0(\omega T),\mathbf{I}_0(\omega S_0))\cdot\mathrm{A}_{\varepsilon,R}(\mathbf{I}_0(\omega^2T),\mathbf{I}_0(S_2)) \\
&= \mathrm{A}_{\varepsilon,-R}(T,\omega^2S_1)\cdot\mathrm{A}_{\varepsilon,-R}(\omega T,\omega S_0)\cdot\mathrm{A}_{\varepsilon,R}(\omega^2T,S_2) \\
&= \mathrm{B}_{\varepsilon,-R}(T,S_0,S_1,S_2) \\
&= \mathrm{b}_{\varepsilon,-R}(q).
\end{aligned}$$

The commutations properties between ω and $\mathbf{I}_0$ are straightforward. The relations for $\mathrm{d}_{\varepsilon,\mathrm{R}}$ are obtained in the same way. □

10.5. Definition of negatively almost closing pair of pants

Let ε and R be positive constants. If $p=(\alpha,\beta,\gamma,\tau_0,\tau_1)$ is an (ε,R)-(positively-)almost closing pair of pants. Then, by definition, we have that

$$\mathbf{I}_0(p):=(\alpha^{-1},\gamma^{-1},\beta^{-1},\mathbf{I}_0(\tau_0),\omega\mathbf{I}_0(\tau_1)),$$

is an $(\varepsilon,-R)$-(negatively-)almost closing pair of pants.

Then, Theorem 10.9 holds for $R<0$, as an immediate symmetry consequence of the properties of $\mathbf{I}_0$.

11. Spaces of biconnected pairs of tripods and triconnected pairs of tripods

We present in this section the spaces of biconnected and triconnected pairs of tripods that we shall discuss in the next sections. Our goals in this section are the following:

(i) the definition of the various spaces involved;

(ii) the equidistribution and mixing proposition (Proposition 11.7);

(iii) some local connectedness properties of the space of almost closing pair of pants.

Throughout this section, Γ will be a uniform lattice in G. Let $\alpha\in\Gamma$ be a P-loxodromic element. Recall that (see, for instance, [11, Proposition 3.5]) the centralizer

$$\Gamma_\alpha := \mathsf{Z}_\Gamma(\alpha)$$

of α in Γ is a uniform lattice in the centralizer $\mathsf{Z}_\mathsf{G}(\alpha)$ of α in G.

11.1. Biconnected pairs of tripods

Let α be a P-loxodromic element and Λ be a uniform lattice in $\mathsf{Z}_\mathsf{G}(\alpha)$. We define the *upstairs space of biconnected pairs of tripods* $\mathcal{B}_\alpha$ as

$$\mathcal{B}_\alpha := \{\text{biconnected pairs of tripods } (T, S_0, S_1) \text{ in the universal cover, with } S_0 = \alpha S_1\},$$

and the *downstairs space of biconnected pairs of tripods as*

$$\mathcal{B}_\alpha^\Lambda := \Lambda \backslash \mathcal{B}_\alpha.$$

We shall also denote by $[\Gamma]$ the set of conjugacy classes of elements in Γ, that we also interpret as the set of free homotopy classes in $\Gamma\backslash\mathsf{G}/\mathsf{K}_0$, where K_0 is the maximal compact of G_0.

11.1.1. An invariant measure

Observe that $\mathcal{B}_\alpha$ can be identified with $(\mathcal{G}\times\mathcal{G})^*$ (that is the space of pairs of tripods in the same connected component). From the covering map from $\mathcal{B}_\alpha^\Lambda$ to $(\Lambda\backslash\mathsf{G})\times(\Lambda\backslash\mathsf{G})$, we deduce an invariant form λ in the Lebesgue measure class of $\mathcal{B}_\alpha$ and $\mathcal{B}_\alpha^\Lambda$.

Let also $\mathrm{D}_{\varepsilon,R}$ and $\mathrm{d}_{\varepsilon,R}$ be the weight functions defined in Definition 10.4 (with respect to $\Gamma=\Lambda$). By construction, $\mathrm{D}_{\varepsilon,R}$ is a function on $\mathcal{B}_\alpha$, while $\mathrm{d}_{\varepsilon,R}$ is a function on $\mathcal{B}_\alpha^\Lambda$. We now consider the measures

$$\tilde{\nu}_{\varepsilon,R} = \mathrm{D}_{\varepsilon,R}\cdot\lambda \quad\text{and}\quad \nu_{\varepsilon,R} = \mathrm{d}_{\varepsilon,R}\cdot\lambda,$$

on $\mathcal{B}_\alpha^u$ and $\mathcal{B}_\alpha^\Lambda$, respectively. The following assertion is obvious.

PROPOSITION 11.1. *The measure $\nu_{\varepsilon,R}$ is locally finite and invariant under*

$$\mathsf{C}_\alpha := \mathbf{Z}^\circ_{\mathsf{G}}(\Lambda).$$

We finally consider $\mathcal{B}_{\varepsilon,R}(\alpha)$ and $\mathcal{B}^{\Lambda}_{\varepsilon,R}(\alpha)$, the supports of the functions $\mathrm{D}_{\varepsilon,R}$ and $\mathrm{d}_{\varepsilon,R}$, respectively. It will be convenient in the sequel to distinguish between positive and negative, and we introduce, for $R>0$,

$$\mathcal{B}^{+}_{\varepsilon,R}(\alpha)=\{B\in\mathcal{B}_\alpha : \mathrm{D}_{\varepsilon,R}(B)>0\} \qquad \mathcal{B}^{\Lambda,+}_{\varepsilon,R}(\alpha)=\{B\in\mathcal{B}^{\Lambda}_\alpha : \mathrm{d}_{\varepsilon,R}(B)>0\},$$
$$\mathcal{B}^{-}_{\varepsilon,R}(\alpha)=\{B\in\mathcal{B}_\alpha : \mathrm{D}_{\varepsilon,-R}(B)>0\}, \quad \mathcal{B}^{\Lambda,-}_{\varepsilon,R}(\alpha)=\{B\in\mathcal{B}^{\Lambda}_\alpha : \mathrm{d}_{\varepsilon,-R}(B)>0\}.$$

Recall that, by Proposition 10.10, if $(T,S_0,\alpha(S_0))$ belongs to $\mathcal{B}_{\varepsilon,R}(\alpha)$, then T and S_0 are $(\varepsilon/R,R)$-almost closing for α.

11.1.2. biconnected pairs of tripods and lattices

Let Γ be a uniform lattice in G and α be a P-loxodromic element in Γ. We may now consider the set of biconnected pairs of tripods in $\Gamma\backslash\mathsf{G}$ whose loop is in the homotopy class defined by α:

$$\mathcal{B}^{\Gamma}_{[\alpha]} := \{\text{biconnected pairs of tripods } (t,s,c_0,c_1) \text{ in } \Gamma\backslash\mathsf{G}, \text{ with } c_0\bullet c_1^{-1}\in[\alpha]\}.$$

We have the following interpretation.

PROPOSITION 11.2. *The projection from $\mathcal{B}^{\Gamma_\alpha}_\alpha$ to $\mathcal{B}^{\Gamma}_{[\alpha]}$ is an isomorphism.*

In the sequel, we will use the following abuse of language:

$$\mathcal{B}^{\Gamma}_\alpha := \mathcal{B}^{\Gamma_\alpha}_\alpha.$$

11.2. Triconnected pair of tripods

We need to give names to various spaces of triconnected pairs of tripods, including their "boundary related" versions. As above, let Γ be a uniform lattice, α be an element in Γ and Λ be a lattice in $\mathsf{Z}_{\mathsf{G}}(\alpha)$. We introduce the following spaces:

$$\begin{aligned}
\mathcal{Q} &:= \{(T,S_0,S_1,S_2)\in\mathcal{G}^4 : S_1,S_2\in\Gamma\cdot S_0\},\\
\mathcal{Q}^{\Gamma} &:= \{(t,s,c_0,c_1,c_2) \text{ triconnected pairs of tripods in } \Gamma\backslash\mathsf{G}\},\\
\mathcal{Q}_\alpha &:= \{(T,S_0,S_1,S_2)\in\mathcal{Q} : S_1=\alpha S_0\},\\
\mathcal{Q}^{\Lambda}_\alpha &:= \Lambda\backslash\mathcal{Q}_\alpha,\\
\mathcal{Q}^{\Gamma}_{[\alpha]} &:= \{(t,s,c_0,c_1,c_2)\in\mathcal{Q}^{\Gamma} : c_0\bullet c_1^{-1}\in[\alpha]\}.
\end{aligned}$$

The following identifications are obvious.

PROPOSITION 11.3. *The space* $\mathcal{Q}^\Gamma$ *is isomorphic to* $\Gamma\backslash\mathcal{Q}$. *Similarly, the space* $\mathcal{Q}^\Gamma_{[\alpha]}$ *is isomorphic to* $\mathcal{Q}^{\Gamma_\alpha}_\alpha$. *Finally,*

$$\mathcal{Q}^\Gamma = \bigsqcup_{[\alpha]\in[\Gamma]} \mathcal{Q}^\Gamma_{[\alpha]}.$$

By a slight abuse of language we shall write

$$\mathcal{Q}^\Gamma_\alpha := \mathcal{Q}^{\Gamma_\alpha}_\alpha.$$

11.2.1. Triconnected pairs of tripods in $\Gamma\backslash\mathsf{G}$

Parallel to what we did for biconnected pairs of tripods, let us introduce the following spaces. First, let $\mathcal{Q}^\Gamma$ be the set of triconnected pairs of tripods in $\Gamma\backslash\mathcal{G}$ and let

$$\mathcal{Q}^\Gamma_{\varepsilon,R} = \{w \in \mathcal{Q}^\Gamma : \mathrm{b}_{\varepsilon,R}(w) > 0\}.$$

We will assume $R>0$ and write, accordingly, $\mathcal{Q}^{\Gamma,+}_{\varepsilon,R} = \mathcal{Q}^\Gamma_{\varepsilon,R}$ and $\mathcal{Q}^{\Gamma,-}_{\varepsilon,R} = \mathcal{Q}^\Gamma_{\varepsilon,-R}$. Let

$$((\Gamma\backslash\mathcal{G})\times(\Gamma\backslash\mathcal{G}))^*$$

be the set of pairs of points in $\Gamma\backslash\mathcal{G}$ in the same connected component. We first observe the following result.

PROPOSITION 11.4. *The (forgetting) map p from* $\mathcal{Q}^\Gamma$ *to* $((\Gamma\backslash\mathcal{G})\times(\Gamma\backslash\mathcal{G}))^*$ *sending* (t, s, c_0, c_1, c_2) *to* (t, s) *is a covering.*

Proof. Let $\mathcal{Q}_u$ be the space of quadruples (T, S_0, S_1, S_2), where all the S_i lie in the same Γ-orbit. The map $\pi\colon (T, S_0, S_1, S_2)\mapsto(T, S_0)$ is a covering. Let $\Gamma\times\Gamma$ be acting on $\mathcal{Q}_u$ by

$$(\gamma, \eta)\cdot(T, S_0, S_1, S_2) = (\gamma T, \eta S_0, \eta S_1, \eta S_2).$$

Then,

$$(\Gamma\times\Gamma)\backslash\mathcal{Q}_u = \mathcal{Q}$$

and π, being equivariant, gives rise to p. Thus, p is a covering. □

Definition 11.5. (Measures) The *Lebesgue measure* Λ is the locally finite measure on $\mathcal{Q}$ associated to the pullback of the G-invariant volume form on $\Gamma\backslash\mathcal{G}$.

Given positive R and ε, the *weighted measure* $\mu_{\varepsilon,R}$ on $\mathcal{Q}$ is the measure supported on $\mathcal{Q}_{\varepsilon,R}$ given by

$$\mu_{\varepsilon,R} := \mathrm{b}_{\varepsilon,R}\Lambda.$$

For the sake of convenience, we will assume that $R>0$ and write

$$\mu^+_{\varepsilon,R} := \mu_{\varepsilon,R} \quad \text{and} \quad \mu^-_{\varepsilon,R} := \mu_{\varepsilon,-R}.$$

PROPOSITION 11.6. *For any positive ε, and for R large enough, $\mathcal{Q}^\Gamma_{\varepsilon,R}$ is non-empty, relatively compact and $\mu_{\varepsilon,R}$ is finite. Moreover,*

$$\omega_*\mu^\pm_{\varepsilon,R} = \mu^\pm_{\varepsilon,R} \quad \textit{and} \quad \mathbf{I}_{0*}\mu^\pm_{\varepsilon,R} = \mu^\mp_{\varepsilon,R}.$$

Proof. By Corollary 10.8, $\mathrm{b}_{\varepsilon,R}$ is not always zero, and thus $\mathcal{Q}^\Gamma_{\varepsilon,R}$ is non-empty. Let (T, S_0, S_1, S_2) be a lift of a triconnected pair of tripods $w=(t, s, c_0, c_1, c_2)$ satisfying

$$\mathrm{b}_{\varepsilon,R}(w) \neq 0.$$

Then, by Proposition 10.10, $d(T, S_i) \leqslant R+\varepsilon$. This implies that $\mathcal{Q}_{\varepsilon,R}$ is relatively compact, and thus $\mu_{\varepsilon,R}$ is finite. The invariance by ω comes from the invariance of $\mathrm{b}_{\varepsilon,R}$ by ω (equation (10.9)) and the invariance of the metric by ω. The last assertion comes from the fact that $\mathrm{b}_{\varepsilon,R} = \mathrm{b}_{\varepsilon,-R} \circ I$ by Proposition 10.13 and that Λ is invariant by I, since the invariant measure on $\mathcal{G}$ is invariant by $\mathrm{Aut}(\mathsf{G}_0)$ (see the beginning of §3.4). □

Let us finally define

$$\mathcal{Q}^\Gamma_{\varepsilon,R}(\alpha) := \mathcal{Q}^\Gamma_{\varepsilon,R} \cap \mathcal{Q}^\Gamma_\alpha.$$

11.3. Mixing: from triconnected pairs of tripods to biconnected pairs of tripods

We have a natural forgetful map π from $\mathcal{Q}^\Gamma_\alpha$ to $\mathcal{B}^\Gamma_\alpha$:

$$\pi(T, S_0, S_1, S_2) := (T, S_0, S_1).$$

We then have the following proposition which says that adding a third path is probabilistically independent for large R.

PROPOSITION 11.7. (Equidistribution and mixing) *We have the inclusion*

$$\pi(\mathcal{Q}_{\varepsilon,R}(\alpha)) \subset \mathcal{B}_{\varepsilon,R}(\alpha).$$

Moreover, there exists a function $C_{\varepsilon,R}$ depending on R and ε, a constant q and a constant $K(\varepsilon,\Gamma)$ such that $\pi_*(\mu_{\varepsilon,R}) = C_{\varepsilon,R}\cdot\nu_{\varepsilon,R}$. The function $\mathrm{C}_{\varepsilon,R}$ is a smooth almost constant function: there exists a constant q and a constant $K(\varepsilon,\Gamma)$ such that

$$\|\mathrm{C}_{\varepsilon,\mathrm{R}} - 1\|_{C^0} \leqslant K(\varepsilon,\Gamma)\exp(-q|R|). \tag{11.1}$$

In particular, given ε, for R large enough, the measure $\nu_{\varepsilon,R}$ is finite with relatively compact support.

Proof. By construction (for the second equality), and by assertion (10.6) (for the third one), we have

$$\pi_*(\mu_{\varepsilon,R})=\pi_*(\mathrm{b}_{\varepsilon,R}\Lambda)=\left(\sum_{c_2}\mathrm{a}_{\varepsilon,R}(\omega(t),\omega^2(s),c_2)1\right)\cdot\mathrm{d}_{\varepsilon,R}\Lambda=a_{\varepsilon,R}(\omega(t),\omega^2(s))\Lambda.$$

Thus, the result follows from exponential mixing: Proposition 10.7 and taking

$$\mathrm{C}_{\varepsilon,R}(t,s):=\sum_{c_2}\mathrm{a}_{\varepsilon,R}(\omega(t),\omega^2(s),c_2)=a_{\varepsilon,R}(\omega(t),\omega^2(s)).$$

Observe that $\mathrm{C}_{\varepsilon,R}$ is smooth, since only finitely many terms in the sum are non-zero: there are only finitely many homotopy classes of arcs of bounded length. □

11.4. Perfecting pants and varying the boundary holonomies

By a slight abuse of language, we will say that (T,S_0,S_1,S_2) with boundary loops (α,β,γ) is $(\varepsilon/R,R)$-almost closing if $(\alpha,\beta,\gamma,T,S_0)$, where $S_2=\beta(\omega^2S_0)$ and $S_0=\alpha(\omega^2S_1)$, is $(\varepsilon/R,R)$-almost closing.

Let us denote by $P_{\varepsilon,R}$ the space of (ε,R)-almost closing pair of pants.

We say that a boundary loop of a triconnected pair of pants is *T-perfect* if it is conjugate to $\exp(2Ta)$. Given R, recall that the boundary loops of an R-perfect pair of pants are R-perfect (see [23, Proposition 5.2.2]).

Recall that, if α is a P-loxodromic element, we denote by L_α the stabilizer in G of the attractive and repulsive points α^+ and α^-.

Our main result is the following.

Theorem 11.8. (Varying the boundary holonomies) *Let B be a positive number. Then, there exist positive constants ε_0 and C such that, given $\varepsilon<\varepsilon_0$, then for R large enough the following assertion holds. Let $W_0=(T,S_0,S_1,S_2)$ be a pair of pants in $P_{\varepsilon,R}$ with boundary loops α_0, β_0 and γ_0. Let $\{k_t\}_{t\in[0,1]}$ be a smooth path in L_{α_0} in the ball of radius of length less than $B\varepsilon/R$ with respect to d_T. Then, there exists a continuous family $\{W_t\}_{t\in[0,1]}$ in $P_{C\varepsilon,R}$ with boundary loops α_t, β_t and γ_t such that $W_0=W$ and the following conditions hold*:

(i) *for all t, we have that β_t and γ_t are conjugate to β_0 and γ_0, respectively*;

(ii) *α_t is conjugate to $\alpha_0\cdot k_t$.*

11.4.1. Lifting holonomies

Let $W=(T,S_0,S_1,S_2)$. For any h in G, let $W_h=(T,S_0,hS_1,S_2)$. Observe that, if h belongs to the ball of radius ε/R with respect to d_T, then W is $(M\varepsilon/R,R)$-almost closing for some uniform M.

The boundary loops of W_h are now $\alpha_h{:=}\alpha\cdot h^{-1}$, $\beta_h{=}\beta$ and $\gamma_h{:=}h\cdot\gamma$.

Let $\mathsf{G}_{\varepsilon,R}$ (resp. $\mathsf{L}_{\varepsilon,R}$) be the ball of radius ε/R in G (resp. L_0).

Let us consider, for this section, the following maps:

(i) the map A from the ball $\mathsf{G}_{\varepsilon,R}$ of radius ε/R in G to the ball $\mathsf{L}_{M\varepsilon,R}$ such that α_h is conjugate to $\exp(Ra_0)\cdot A(h)$;

(ii) the map C from $\mathsf{G}_{\varepsilon,R}$ to $\mathsf{L}_{M\varepsilon,R}$ such that γ_h is conjugate to $\exp(Ra_0)\cdot C(h)$.

For ε small enough and M large enough, these maps are well defined by the boundary loop proposition.

We need a succession of technical results.

LEMMA 11.9. (Horizontal distribution) *There exist constants K_0 and ε_0 such that, for ε less than ε_0 and for any h in $\mathsf{G}_{\varepsilon,R}$, there exists a linear subspace H_h in $\mathsf{T}_h\mathsf{G}$, depending smoothly on h, such that the following conditions hold*:

(i) *T_hC is zero restricted to H_h*;

(ii) *T_hA is uniformly K_0 biLipschitz from H_h to $T_{A(h)}\mathsf{L}_0$.*

We will refer to H as the *horizontal distribution.*

Proof. Let us linearize the problem. Given a deformation $\{k_t\}_{t\in[0,1]}$, we are looking for $\{h_t\}_{t\in[0,1]}$, $\{f_t\}_{t\in[0,1]}$ and $\{g_t\}_{t\in[0,1]}$ such that

$$\alpha\cdot h_t^{-1}=f_t\cdot\alpha\cdot k_t\cdot f_t^{-1}\quad\text{and}\quad h_t\cdot\gamma=g_t\cdot\gamma\cdot g_t^{-1}.$$

Writing

$$\dot{v}=\left.\frac{dv}{dt}\right|_{t=0},$$

we get the linearized equations

$$\dot{h}=(\mathrm{Ad}(\alpha)-1)\cdot\dot{f}+\dot{k}\quad\text{and}\quad\dot{h}=(1-\mathrm{Ad}(\gamma))\cdot\dot{g}.\tag{11.2}$$

Given $\dot{k}$ in the Lie algebra $\mathfrak{l}_\alpha$ of L_α, we want to find $\dot{h}$, $\dot{f}$ and $\dot{g}$, depending linearly on $\dot{k}$, with

$$\frac{1}{K}\|\dot{k}\|\leqslant\|\dot{h}\|\leqslant\mathrm{K}\ \|\dot{k}\|,$$

for some constant K depending only on G, ε and R, and where the norm comes from d_T such that furthermore the choice of $\dot{h}$ is smooth in h.

For $p{\in}\mathbf{F}$, let M_p be the stabilizer of p, and N_p be its nilpotent radical.

As a consequence of the boundary loop proposition (Proposition 9.9), for R large enough, $\mathrm{Ad}(\alpha)$ both contracts N_α^- by a factor less than $\frac{1}{2}$, and dilates N_α^- by a factor at least 2. A similar statement holds for $\mathrm{Ad}(\gamma)$, since the same holds for α_* and γ_*

Since, by the structure pant theorem, $(T,\alpha^-,\alpha^+,\gamma^-)$ is an (ε/R)-quasi-tripod, there exists a positive constant C, for ε small enough and R large enough, such that

$$d_T(\alpha^-,\alpha^+)>C,\quad d_T(\gamma^-,\alpha^+)>C\quad\text{and}\quad d_T(\gamma^-,\alpha^-)>C.$$

Indeed, noting that $D:=d_\tau(\partial^+\tau,\partial^-\tau)$ is a positive constant depending only on G, it follows that, if θ^- is a μ-quasi-tripod, then, for $i\neq j$,

$$d_{\dot\theta}(\partial^i\theta,\partial^j\theta)\geqslant d_{\dot\theta}(\partial^i\tau,\partial^j\tau)-d_{\dot\theta}(\partial^i\theta,\partial^i\tau)-d_{\dot\theta}(\partial^j\theta,\partial^j\tau)\geqslant D-2\mu.$$

Thus, α^-, α^+ and γ^- are $D-2\mu$ apart using d_{S_1}. Thus,

$$\mathfrak{g}=\mathfrak{n}_{\gamma^-}+\mathfrak{n}_{\alpha^-}+\mathfrak{n}_{\alpha^+}.$$

Then, denoting by $\mathfrak{n}^\circ_{\gamma^-}$ the orthogonal in $\mathfrak{n}_{\gamma^-}$ to $(\mathfrak{n}_{\alpha^-}\oplus\mathfrak{n}_{\alpha^+})\cap\mathfrak{n}_{\gamma^-}$, we have

$$\mathfrak{g}=\mathfrak{n}^\circ_{\gamma^-}\oplus\mathfrak{n}_{\alpha^-}\oplus\mathfrak{n}_{\alpha^+}. \tag{11.3}$$

Observe now that $\dim\mathsf{L}_\alpha=\dim\mathfrak{n}^\circ_{\gamma^-}$, and that the projection from L_α to $\mathfrak{n}^\circ_{\gamma^-}$ using the above projection is uniformly biLipschitz by a function that depends only on h.

We now claim that $H_h=\mathfrak{n}^\circ_{\gamma^-}$ solves our problem. Indeed, for $\dot k$ in G, let us consider the decomposition of $\dot k$ using the above decomposition (11.3) of $\mathfrak{g}$ as

$$\dot k=k^\circ_{\gamma^-}+k_{\alpha^-}+k_{\alpha^+},$$

where $k^\circ_{\gamma^-}$, k_{α^-} and k_{α^+} belong to $\mathfrak{n}^\circ_{\gamma^-}$, $\mathfrak{n}_{\alpha^-}$ and $\mathfrak{n}_{\alpha^+}$, respectively.

We now define $\dot f$, $\dot h$ and $\dot g$ by

$$\dot f=-(\mathrm{Ad}(\alpha)-1)^{-1}(k_{\alpha^-}+k_{\alpha^+}),\quad \dot h=k^\circ_{\gamma^-}\quad\text{and}\quad \dot g=(\mathrm{Ad}(\gamma)-1)^{-1}(k^\circ_{\gamma^-}).$$

Observe that $\dot h$ belongs to H_h and $\dot f$, $\dot g$ and $\dot h$ solve equation (11.2). In particular,

$$\mathsf{T}_hA(\dot h)=\dot k\quad\text{and}\quad T_hC(\dot h)=0. \qquad\square$$

An horizontal distribution (here H) for a submersion (here A) provides a way to lift paths from L_0 to G starting from any point in the fiber above the origin of the path. Using this classical idea form differential geometry, we now prove the following result that completes the proof of Theorem 11.8.

Lemma 11.10. (Existence of a section) *Let ε_0 and K_0 be as in the previous lemma. For ε less than $\varepsilon_0(2K_0)^{-2}$ and then R large enough, given h^0 in $\mathsf{G}_{\varepsilon,R}$ and $k^0:=A(h^0)$, there exists a continuous map Ξ from $\mathsf{L}_{(2K_0\varepsilon,R)}$ to $\mathsf{G}_{(4K_0^2\varepsilon,R)}$ such that the following conditions hold*:

(i) $\Xi(k^0)=h^0$;

(ii) *letting $h=\Xi(k)$, we have that α_h is conjugate to $\alpha\cdot k$, while β_h and γ_h are conjugate to β and γ, respectively.*

Proof. Let h^0 in $\mathsf{G}_{\varepsilon,R}$. Let us write $k^0 = \exp(u)$ and $k^0_t = \exp((1-t)u)$. Note that the path $\{{k^0}_t\}_{t\in[0,1]}$ has length less than $K_0\varepsilon/R$. Let $\{{h^0}_t\}_{t\in[0,1]}$ be the path lifting $\{{k^0}_t\}_{t\in[0,1]}$ using the horizontal distribution H and starting from h^0. In particular,

$$A(h^0_1) = k^0_1 = \mathrm{Id}\,.$$

Since $\{{h^0}_t\}_{t\in[0,1]}$ has length less than $K_0^2(\varepsilon/R)$, it follows that h_1 belongs to the ball of radius $2K_0^2\varepsilon/R$.

For any k in $\mathsf{L}_{(2K_0\varepsilon,R)}$, let us write $k = \exp(v)$ and $k_t = \exp(tv)$. Let us lift $\{k_t\}_{t\in[0,1]}$ to a path $\{h_t\}_{t\in[0,1]}$ starting from h^0_1, and define $\Xi(k) := h_1$. The map Ξ satisfies all of the conditions of the lemma:

(i) by uniqueness of the lift, we have $\Xi(k^0) = h^0$;

(ii) since all of the paths are tangent to the horizontal distribution, the holonomies around γ are all conjugate (since the horizontal distribution lies in the kernel of $\mathsf{T}C$); by construction, the holonomies around β are fixed.

Observe that the path $\{k_t\}_{t\in[0,1]}$ has length less than $2K_0\varepsilon/R$, thus $\{h_t\}_{t\in[0,1]}$ has length less than $2K_0^2\varepsilon/R$, and hence $\Xi(k)$ is in the ball of radius $4K_0^2\varepsilon/R$. □

THEOREM 11.11. (Deforming into a perfect pair of pants) *There exists a positive constant K such that, for ε small enough and then R large enough, the following holds. Let $W_0 = (T, S_0, S_1, S_2)$ be an (ε, R)-almost closing pair of pants. Then, there exists a deformation path $\{W_t\}_{t\in[0,1]}$ with $W_0 = W$ such that, for each t, W_t is a $(K\varepsilon, R)$-pair of pants, and W_1 is R-perfect.*

Moreover, if one of the boundaries is perfect, we may choose the deformation such that this boundary stays perfect.

Let $W = (T, S_0, S_1, S_2)$ be an (ε, R)-almost closing pair of pants. Let

$$W^* = (T, S_0^*, S_1^*, S_2^*)$$

be the R-perfect pair of pants based at T. Let ζ_i^* be the elements of G such that

$$S_0^* = \zeta_0^* T, \quad \omega^2 S_2^* = \zeta_2^* \omega T \quad \text{and} \quad \omega S_1 = \zeta_1^* \omega^2 T_2.$$

Then, the above theorem is the consequence of the following lemma.

LEMMA 11.12. *The quadruple W is an (ε, R)-almost closing pair of pants if and only if there exist f_i and g_i in G, for $i \in \{0,1,2\}$, which are $K\varepsilon/R$-close to the identity and such that*

$$S_0 = \zeta_0 T, \quad \omega^2 S_2 = \zeta_2 \omega T \quad \textit{and} \quad \omega S_1 = \zeta_1 \omega^2 T_2, \tag{11.4}$$

where $\zeta_i = f_i \zeta_i^* g_i$.

Proof. We say that a pair (T, S) is (μ, R)-*almost closing* if there exists a tripod u such that u is μ-close to T and $K\varphi_R(u)$ is close to S. Thus, $u{=}gT$ and $K\varphi_R(u){=}g'S$, where g and g' are μ-close to the identity with respect to d_T and d_S, respectively.

Let us consider ζ and ζ^* in G such that

$$S=\zeta T, S^* := K\varphi_R T = \zeta^* T.$$

Let us consider h such that $g'\zeta{=}\zeta h$. Then, h is μ-close to the identity with respect to d_T, as

$$d_T(h, \mathrm{Id}) = d(T, hT) = d(\zeta T, \zeta hT) = d(S, g'S) = d_S(g, g').$$

Since

$$\zeta h^{-1} T = g'^{-1} \zeta T = g'^{-1} S = K\varphi_R(u) = K\varphi(gT) = g\zeta^* T,$$

we thus have

$$\zeta = g\zeta^* h.$$

It follows (using the notation above) that (T, S) is almost closing if and only if

$$\zeta = g\zeta^* h,$$

with h and g μ-close to the identity with respect to d_T.

Repeating this argument for all of the pairs described in the lemma, the proof is completed. □

11.4.2. Proof of Theorem 11.11

Assume now that one of the boundaries of $W{=}(T, S_0, S_1, S_2)$—say α—is perfect. Let us use the boundary loop proposition (Proposition 9.9), and let (S_1^*, T^*, S_0^*) be the perfect triple associated to $\alpha^*{=}\alpha$.

Let g be such that $S_1{=}gS_1^*$. It follows that $S_0{=}\alpha g \alpha^{-1} S_0^*$. Then, g is close to the identity with respect to d_{S_1}. Let then $\{g_t\}_{t\in[0,1]}$ be a path from g to Id, such that g_t and is close to the identity with respect to d_{S_1}. Let

$$S_1^t = g_t S_1^* \quad \text{and} \quad S_0^t = \alpha g_t \alpha^{-1} S_0^* = \alpha S_1^t.$$

Observe that

$$d(S_0^t, S_0) = d(\alpha S_1^t, \alpha S_1) = d(S_1^t, S_1) = d_{S_1}(g_t, \mathrm{Id}).$$

As a first step in the deformation, we consider $W_t^{(1)}{=}(T, S_0^t, S_1^t, S_2)$ such that

$$W_1^{(1)} = (T, S_0^*, S_1^*, S_2) \quad \text{and} \quad W_0^{(1)} = (T, S_0, S_1, S_2).$$

We remark that the first boundary loop does not change.

As a second step, we choose a small path $\{T^t\}_{t\in[0,1]}$ joining T to T^*, and define $W_t^{(2)}=(T^t,S_0^*,S_1^*,S_2)$ such that

$$W_1^{(2)}=(T^*,S_0^*,S_1^*,S_2) \quad \text{and} \quad W_0^{(2)}=(T,S_0^*,S_1^*,S_2).$$

Again, we remark that the first boundary loop does not change.

Let S_2^* be such that (T^*,S_0^*,S_1^*,S_2^*) is perfect. Finally, as in Lemma 11.12, we write $S_2=f\zeta_2^*hT^*$, with f and h close to the identity with respect to d_{T^*}. We find paths $\{f^t\}_{t\in[0,1]}$ and $\{h^t\}_{t\in[0,1]}$ joining respectively f and h to Id. We finally write $S_2^t=f^t\zeta_2^*h^tT^*$, and choose the final step as $W_t^{(3)}=(T^*,S_0^*,S_1^*,S_2^t)$ such that

$$W_1^{(3)}=(T^*,S_0^*,S_1^*,S_2^*) \quad \text{and} \quad W_0^{(3)}=(T^*,S_0^*,S_1^*,S_2).$$

Again, we remark that the first boundary loop does not change.

12. Cores and foot projections

In this section we concentrate on discussing the analogues of the normal bundle to closed geodesics for hyperbolic 3-manifolds in our higher-rank situation. Ultimately, in the next section we want to show that pairs of pants with a common "boundary component" are nicely distributed in this "normal bundle". For now, we need to investigate and define the objects that we shall need for this study.

More precisely, we define the *feet space* which is a higher-rank version of the normal space to a geodesic in a 3-dimensional hyperbolic space. We also explain how biconnected pairs of tripods and triconnected pairs of tripods project to this feet space.

We will also introduce an important subspace of this feet space, called the *core*. The main result of this section is Theorem 12.4 about measures on the feet space.

In all this section α will be a semisimple P-loxodromic element in G, and Λ a uniform lattice in $\mathsf{Z}_{\mathsf{G}}(\alpha)$, the centralizer of α in G, such that $\alpha\in\Lambda$.

12.1. Feet spaces and their core

Definition 12.1. (Feet spaces of α) The *upstairs feet space* and the *downstairs feet space* of α, denoted respectively by $\mathcal{F}_\alpha$ are

$$\mathcal{F}_\alpha := \{\tau\in\mathcal{G} : \partial^\pm\tau=\alpha^\pm\}, \tag{12.1}$$

$$\mathcal{F}_\alpha^\Lambda := \Lambda\backslash\mathcal{F}_\alpha. \tag{12.2}$$

We denote by p the projection from $\mathcal{F}_\alpha$ to $\mathcal{F}_\alpha^\Lambda$.

If $g\in\mathsf{G}$, the map $F_g\colon\tau\mapsto g\tau$, defines a natural map f_g from $\mathcal{F}_\alpha$ to $\mathcal{F}_{g\alpha g^{-1}}$ which gives rise to a map

$$f_g\colon\mathcal{F}_\alpha^\Lambda\longrightarrow\mathcal{F}_{g\alpha g^{-1}}^{g\Lambda g^{-1}},$$

such that $p\circ F_g=f_g\circ p$, which is the identity if $g\in\Lambda$. We also introduce the groups

$$\mathsf{C}_\alpha:=\mathsf{Z}_{\mathsf{G}}{}^{\circ}(\Lambda), \tag{12.3}$$

$$\mathsf{L}_\alpha:=\{g\in\mathsf{G}:g(\alpha^\pm)=\alpha^\pm\}. \tag{12.4}$$

Let also consider K_α, the maximal compact factor of L_α. Below are some elementary remarks.

(i) Any tripod in $\mathcal{F}_\alpha$ gives an isomorphism of L_α with L_0, and the space $\mathcal{F}_\alpha$ is a principal left L_α torsor, as well as a principal right L_0 torsor. It follows that $\tau^{-1}(\exp(ta_0))$ does not depend on τ in $\mathcal{F}_\alpha$. Let then $a\in\mathfrak{g}$ be such that $\tau^{-1}(\exp(ta_0))=\exp(ta)$. As a consequence, for all τ in $\mathcal{F}_\alpha$, we have $\varphi_t(\tau)=\exp(ta)\tau$.

(ii) The group C_α acts by isometries on $\mathcal{F}_\alpha$ and $\mathsf{C}_\alpha\subset L_\alpha$.

12.1.1. The lattice case

When Γ is a lattice in G, we write by a slight abuse of language $\mathcal{F}_\alpha^\Gamma:=\mathcal{F}_\alpha^{\Gamma_\alpha}$, where we recall that $\Gamma_\alpha=\mathsf{Z}_\Gamma(\alpha)$. In that case, for a conjugacy class $[\alpha]$ in Γ, we let $\mathcal{F}_{[\alpha]}^\Gamma$ be the set of equivalence classes in

$$\bigsqcup_{\beta\in[\alpha]}\mathcal{F}_\beta^\Gamma,$$

under the action of Γ given by the maps f_g. Since, for $g\in\Gamma_\alpha$, f_g gives the identity on $\mathcal{F}_\alpha^\Gamma$, the space $\mathcal{F}_{[\alpha]}^\Gamma$ is canonically identified with $\mathcal{F}_\beta^\Gamma$ for any given β in $[\alpha]$.

12.1.2. The core of the feet space

A (possibly empty) special subset of the space of feet requires consideration.

Definition 12.2. (Core) Given (ε,R), the (ε,R)-*core of the space of feet* is the closed subset $\mathcal{X}_\alpha$ of $\mathcal{F}_\alpha$ defined by

$$\mathcal{X}_\alpha=\left\{\tau\in\mathcal{F}_\alpha:d(\varphi_{2R}(\tau),\alpha(\tau))\leqslant\frac{\varepsilon}{R}\right\}.$$

We denote by $\mathcal{X}_\alpha^\Lambda$ be the projection of $\mathcal{X}_\alpha$ on $\mathcal{F}_\alpha^\Lambda$.

We immediately have the following result.

PROPOSITION 12.3. *The sets* $\mathcal{X}_\alpha$ *and* $\mathcal{X}_\alpha^\Lambda$ *are invariant under the action of* C_α. *Moreover,* $\mathrm{p}^{-1}\mathcal{X}_\alpha^\Lambda = \mathcal{X}_\alpha$. *Finally, when non-empty,* $\mathcal{X}_\alpha^\Lambda$ *is compact.*

Proof. The first statement follows from the fact that $\mathsf{Z}_\mathsf{G}(\alpha)$ acts by isometries on $\mathcal{F}_\alpha$ commuting both with α and the flow $\{\varphi_t\}_{t\in\mathbb{R}}$. The second statement comes from the fact that $\mathcal{X}_\alpha$ is, in particular, invariant under the action of $\mathsf{Z}_\mathsf{G}(\alpha)$. Let us finally prove the compactness assertion, the action of the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on $\mathcal{F}_\alpha$ is given by the left action of the 1-parameter subgroup generated by a. Indeed, if τ is in $\mathcal{F}_\alpha$, then

$$d(\varphi_{2R}(\tau), \alpha(\tau)) = d(\tau, \exp(-2Ra)\alpha(\tau)).$$

Let $\beta := \exp(-2Ra)\alpha$. Let τ_0 be an element of $\mathcal{F}_\alpha$. Then, the core $\mathcal{X}_\alpha$ is the set of the elements $g\tau_0$, where $g \in \mathsf{L}_\alpha$ satisfies

$$d(\tau_0, g^{-1}\beta g\cdot\tau_0) \leqslant \frac{\varepsilon}{R}.$$

Since α is semisimple and centralizes a, β is semisimple as well. Thus, the orbit map

$$\begin{aligned} \mathsf{L}_\alpha/\mathsf{Z}_{\mathsf{L}_\alpha}(\beta) &\longrightarrow \mathcal{G}, \\ g &\longmapsto g^{-1}\beta g\cdot\tau_0, \end{aligned}$$

is proper. It follows that the set

$$\left\{ g \in \mathsf{L}_\alpha/\mathsf{Z}_{\mathsf{L}_\alpha}(\beta) : d(\tau_0, g^{-1}\beta g\cdot\tau_0) \leqslant \frac{\varepsilon}{R} \right\}$$

is compact. The result now follows from the facts that $\mathsf{Z}_\mathsf{G}(\alpha) = \mathsf{Z}_\mathsf{G}(\beta)$ and Λ is a uniform lattice in $\mathsf{Z}_\mathsf{G}(\alpha)$ by hypothesis. □

12.2. Main result

The result uses the Lévy–Prokhorov distance as described in Appendix A.

THEOREM 12.4. *There exists a constant* $\mathbf{M}$ *depending only on* G *such that, for* ε *small enough and then* R *large enough, the following holds. Let* α *be a loxodromic element. Let* $\boldsymbol{\mu}$ *be a measure supported on the core* $\mathcal{X}_\alpha^\Lambda$.

Let T_0 *be a compact torus in* $\mathsf{C}_\alpha \cap \mathsf{K}_\alpha$. *Let* $\boldsymbol{\nu}$ *be a measure on* $\mathcal{F}_\alpha^\Lambda$ *which is invariant under* C_α *and supported on* $\mathcal{X}_\alpha^\Lambda$. *Assume that we have a function* C *such that*

$$\boldsymbol{\mu} = C\boldsymbol{\nu}, \quad \text{with } \|C-1\|_\infty \leqslant \frac{\varepsilon}{R^2}. \tag{12.5}$$

Then, for all $\mathbf{j}$ *in* T_0, *denoting by* d_L *the Lévy–Prokhorov distance between measures on* $\mathcal{F}_\alpha$, *we have*

$$d_L(\mathbf{j}_*(\varphi_1)_*(\boldsymbol{\mu}), \boldsymbol{\mu}) \leqslant \mathbf{M}\frac{\varepsilon}{R}.$$

The difficulty here is that we want a relation involving the Lévy–Prokhorov distance. We realize our goal by exhibiting tori T_α containing T_0 of controlled diameters, compare $\boldsymbol{\mu}$ with its average along T_α and apply Theorem A.1 in Appendix A. An extra difficulty comes from the fact that we cannot guarantee in general that φ_1 (which acts on the feet space as an element of G) belongs to T_α; however, *on the core*, it behaves pretty much like an element of T_α (see Lemma 12.5). The fact that the core might not be connected adds extra technical difficulties: the torus T_α may depend on the connected component.

In the specific case of $\mathsf{SL}(n,\mathbb{C})$, or more generally complex groups, and the principal $\mathfrak{sl}_2$, this difficulty disappears: the feet space is a torus isomorphic to C_α and the action of the flow (which is a right action) is interpreted as an element of C_α. In this situation, this whole section becomes easier and the theorem follows from Theorem A.1.

After some preliminaries, we prove Theorem 12.4 in §12.2.3. We then describe an example where some of the hypothesis of the theorem are satisfied in the last subsection §12.3 of this section.

12.2.1. A 1-dimensional torus

A critical point in the proof is to find a 1-parameter subgroup containing α.

Lemma 12.5. *There exists a constant $\mathbf{M}_1$ depending only on G such that, for ε small enough and then R large enough, the following holds. Let α be a loxodromic element. Let $\mathcal{A}$ be a non-empty connected component of $\mathcal{X}_\alpha^\Lambda$. Then, there exists a 1-parameter subgroup $\mathsf{T}_\alpha\subset\mathsf{Z}_\mathsf{G}(\mathsf{Z}_\mathsf{G}(\alpha))$ containing α, as well as an element $f\in\mathsf{T}_\alpha$, such that, for any $\tau\in\mathcal{A}$,*

$$\operatorname{diam}(\mathsf{T}_\alpha.\tau)\leqslant \mathbf{M}_1\cdot R, \tag{12.6}$$

$$d(\varphi_1(\tau),f(\tau))\leqslant \mathbf{M}_1\frac{\varepsilon}{R}. \tag{12.7}$$

A first step in the proof of this lemma is the following proposition where we use the same notation.

Proposition 12.6. *There exists a constant $\mathbf{M}_2$ such that, for ε small enough and then R large enough, the following holds. Let α be a P-loxodromic element. Let $\mathcal{A}^u$ be a non-empty connected component of $\mathcal{X}_\alpha$. Then, there exists u_α in $\mathfrak{g}$ invariant by $\mathsf{Z}_\mathsf{G}(\alpha)$, with $\exp(2Ru_\alpha)=\alpha$, such that, for all $\tau\in\mathcal{A}^u$,*

$$d(\varphi_t(\tau),\exp(tu_\alpha)(\tau))\leqslant \mathbf{M}_2\frac{\varepsilon}{R}\quad \textit{for all } 0\leqslant t\leqslant 2R, \tag{12.8}$$

$$d(\tau,\exp(tu_\alpha)(\tau))\leqslant \mathbf{M}_2 R\quad \textit{for all } 0\leqslant t\leqslant 2R. \tag{12.9}$$

Proof. If τ belongs to the (ε, R)-core of α, then

$$d_0(\tau^{-1}(\alpha), \exp(2Ra_0)) \leqslant \frac{\varepsilon}{R}.$$

Since d_0 is right invariant, we obtain that, letting $b{:=}\tau^{-1}(\alpha)\exp(-Ra_0)$,

$$d_0(b, \mathrm{Id}) \leqslant \frac{\varepsilon}{R}.$$

Thus, for ε/R small enough, there exists a unique v_α (of smallest norm) in $\mathfrak{l}_0$ such that

$$b = \exp(2Rv_\alpha) \quad \text{and} \quad d_0(\exp(tv_\alpha), \mathrm{Id}) \leqslant \frac{\varepsilon}{R} \text{ for all } t \in [0, 2R]. \tag{12.10}$$

Let $u_\alpha{:=}\mathsf{T}\xi_\tau(a_0+v_\alpha)$. Since a_0 is in the center of $\mathfrak{l}_0$, we get from the first equation that

$$\alpha = \xi_\tau(\exp(2R(a_0+v_\alpha)) = \exp(2Ru_\alpha).$$

The second inequality in assertion (12.10) now yields that, for all τ in $\mathcal{X}_\alpha$,

$$d(\varphi_t(\tau), \exp(tu_\alpha)\tau) = d_0(\exp(tv_\alpha), \mathrm{Id}) \leqslant \frac{\varepsilon}{R}.$$

This proves inequality (12.8). Finally, inequality (12.9) follows from the fact that there exists a constant A depending only on G such that $d(\varphi_t(\tau), \tau) \leqslant At$ for all t and τ.

If ε/R is small enough, exp is a diffeomorphism in the neighborhood of v_α, and hence of u_α. It follows that u_α only depends on the connected component of $\mathcal{X}_\alpha^u$ containing τ.

Similarly, since u_α is a regular point of exp, it commutes with the Lie algebra $\mathfrak{z}(\alpha)$ of $\mathsf{Z}_\mathsf{G}(\alpha)$. After complexification, it commutes with $\mathfrak{z}_\mathbb{C}(\alpha)$, and hence is fixed by

$$\mathsf{Z}_{\mathsf{G}^\mathbb{C}}(\alpha) = \exp(\mathfrak{z}_\mathbb{C}(\alpha))$$

(since centralizers are connected in complex semisimple groups) and in particular with $\mathsf{Z}_\mathsf{G}(\alpha)$. □

We now prove Lemma 12.5 as an application.

Proof of Lemma 12.5. Let $\mathcal{A}^u$ be a connected component of the lift of $\mathcal{A}$ to $\mathcal{F}_\alpha$. The hypotheses of Proposition 12.6 are satisfied for $\mathcal{A}$, and let $u_\alpha \in \mathfrak{l}_\alpha$ be as in the conclusion of this proposition. Let $\mathsf{T}_\alpha{:=}\{\exp(tu_\alpha)\}_t$. Since u_α is fixed by $\mathsf{Z}_\mathsf{G}(\alpha)$, we have

$$\mathsf{T}_\alpha \subset \mathsf{Z}_\mathsf{G}(\mathsf{Z}_\mathsf{G}(\alpha)).$$

Let $V_\alpha{=}\exp([0, 2R]u_\alpha)$ be a fundamental domain for the action of α on V_α. By inequality (12.9), for all $\tau \in \mathcal{A}$ we have

$$\mathrm{diam}(V_\alpha \tau) \leqslant \mathbf{M}_2 \cdot R$$

for some constant $\mathbf{M}_3$ depending only on G. Since α acts trivially on $\mathcal{F}_\alpha^\Lambda$, we obtain that

$$V_\alpha \tau = \mathsf{T}_\alpha \tau.$$

This concludes the proof of the first assertion of Proposition 12.5. The second assertion follows at once from inequality (12.8). □

12.2.2. Averaging measures

Let μ and ν be as in the hypotheses of Theorem 12.4.

Let $\{\mathcal{X}^i_\alpha\}_{i\in I}$ be the collection of connected components of $\mathcal{X}^\Lambda_\alpha$. Let us denote by $\mathbf{1}_A$ the characteristic function of a subset A. Let

$$\boldsymbol{\mu}_i := \mathbf{1}_{\mathcal{X}^i_\alpha}\boldsymbol{\mu} \quad \text{and} \quad \boldsymbol{\nu}_i := \mathbf{1}_{\mathcal{X}^i_\alpha}\boldsymbol{\nu}$$

be such that

$$\boldsymbol{\mu} = \sum_{i\in I}\boldsymbol{\mu}_i \quad \text{and} \quad \boldsymbol{\nu} = \sum_{i\in I}\boldsymbol{\nu}_i.$$

Let $\mathsf{T}^i_\alpha := \mathsf{T}^0_{\alpha,\mathcal{X}^i_\alpha}$ be associated to $\mathcal{A}_0 = \mathcal{X}^i_\alpha$ as a consequence of Lemma 12.5. Let, finally, consider the tori $\mathsf{Q}^i_\alpha = \mathsf{T}_0 \times \mathsf{T}^i_\alpha$.

We first state and prove the following.

Proposition 12.7. *For a constant* $\mathbf{M}_5$ *depending only on* G, *and* R *large enough,*

$$d_L(\boldsymbol{\mu}_i, g_*\boldsymbol{\mu}_i) \leqslant \mathbf{M}_5\frac{\varepsilon}{R} \quad \textit{for all } g\in\mathsf{T}_0, \tag{12.11}$$

$$d_L(\boldsymbol{\mu}_i, \varphi_{1*}\boldsymbol{\mu}_i) \leqslant \mathbf{M}_5\frac{\varepsilon}{R}. \tag{12.12}$$

Proof. In the proof, M_i will be constants depending only on G. Let $\hat{\boldsymbol{\mu}}_i$ be the average of $\boldsymbol{\mu}_i$ with respect to Q^i_α. By hypothesis, $\boldsymbol{\mu}_i = C\boldsymbol{\nu}_i$, where $\|C-1\| \leqslant \varepsilon/R^2$. Since $\mathsf{Q}^i_\alpha \subset \mathsf{C}_\alpha$, and since C_α preserves $\boldsymbol{\nu}_i$, it follows that

$$\boldsymbol{\mu}_i = D\cdot\hat{\boldsymbol{\mu}}_i,$$

where $\|D-1\| \leqslant 2\varepsilon/R^2$. We now apply Theorem A.1 to get that

$$d_L(\boldsymbol{\mu}_i, \hat{\boldsymbol{\mu}}_i) \leqslant B\cdot M_1\frac{\varepsilon}{R^2},$$

where M_1 only depends on the dimension of T_0 and

$$B := \sup(\operatorname{diam}(\mathsf{Q}^i_\alpha.\tau : \tau\in\mathcal{X}^i_\alpha)).$$

By inequality (12.7), $\operatorname{diam}(\mathsf{T}^i_\alpha\tau) \leqslant \mathbf{M}_1\cdot R$, for $\tau\in\mathcal{X}_\alpha$. Moreover, since $\mathsf{T}_0\subset\mathsf{K}_0$, we have

$$\operatorname{diam}(\mathsf{T}_0.\tau) \leqslant \operatorname{diam}\mathsf{K}_0,$$

and thus $B \leqslant M_2 R$. Therefore,

$$d_L(\boldsymbol{\mu}_i, \hat{\boldsymbol{\mu}}_i) \leqslant M_3\frac{\varepsilon}{R}.$$

Observe now that $g\in\mathsf{Q}^i_\alpha$ acts by isometry on $\mathcal{F}_\alpha$ and thus, for any measure λ_1 and λ_0,

$$d_L(g_*\lambda_0, g_*\lambda_1) = d_L(\lambda_0, \lambda_1).$$

Using the fact that $g_*\hat{\boldsymbol{\mu}}_i = \hat{\boldsymbol{\mu}}_i$, it then follows that

$$d_L(\boldsymbol{\mu}_i, g_*\boldsymbol{\mu}_i) \leqslant d_L(\boldsymbol{\mu}_i, \hat{\boldsymbol{\mu}}_i) + d_L(\hat{\boldsymbol{\mu}}_i, g_*\boldsymbol{\mu}_i) = 2d_L(\boldsymbol{\mu}_i, \hat{\boldsymbol{\mu}}_i) \leqslant 2M_3\frac{\varepsilon}{R}.$$

This proves the first assertion.

For the second inequality, by inequality (12.8), there exists $f\in\mathsf{Q}^i_\alpha$ such that for any τ in $\mathcal{X}^i_\alpha$ one has

$$d(f(\tau), \varphi_1(\tau)) \leqslant \mathbf{M}_1\frac{\varepsilon}{R}.$$

Thus, from Proposition A.4,

$$d_L(f_*\boldsymbol{\mu}_i, (\varphi_1)_*\boldsymbol{\mu}_i) \leqslant \mathbf{M}_0\frac{\varepsilon}{R}.$$

Thus,

$$d_L(\boldsymbol{\mu}_i, (\varphi_1)_*\boldsymbol{\mu}_i) \leqslant d_L(f_*\boldsymbol{\mu}_i, (\varphi_1)_*\boldsymbol{\mu}_i) + d_L(f_*\boldsymbol{\mu}_i, \boldsymbol{\mu}_i) \leqslant \mathbf{M}_5\frac{\varepsilon}{R}.$$

The last assertion of Proposition 12.7 now follows. □

12.2.3. Proof of Theorem 12.4

From Proposition 12.7

$$d_L(\boldsymbol{\mu}_i, \varphi_{1*}\boldsymbol{\mu}_i) \leqslant \mathbf{M}_5\cdot\frac{\varepsilon}{R} \quad \text{and} \quad d_L(\boldsymbol{\mu}_i, g_*\boldsymbol{\mu}_i) \leqslant \mathbf{M}_5\frac{\varepsilon}{R} \text{ for all } g\in\mathsf{T}_0.$$

Thus, by Proposition A.2

$$d_L(\boldsymbol{\mu}, \varphi_{1*}\boldsymbol{\mu}) \leqslant \mathbf{M}_5\frac{\varepsilon}{R} \quad \text{and} \quad d_L(\boldsymbol{\mu}, g_*\boldsymbol{\mu}) \leqslant \mathbf{M}_5\frac{\varepsilon}{R} \text{ for all } g\in\mathsf{T}_0.$$

Then, if $g\in\mathsf{T}_0$, using again that g acts by isometry on $\mathcal{F}^\Lambda_\alpha$, and hence on its space of measures, we have

$$d_L(\boldsymbol{\mu}, g_*\varphi_{1*}\boldsymbol{\mu}) \leqslant d_L(g_*\boldsymbol{\mu}, g_*\varphi_{1*}\boldsymbol{\mu}) + d_L(g_*\boldsymbol{\mu}, \boldsymbol{\mu}) = d_L(\boldsymbol{\mu}, \varphi_{1*}\boldsymbol{\mu}) + d_L(g_*\boldsymbol{\mu}, \boldsymbol{\mu}) \leqslant 2\mathbf{M}_5\frac{\varepsilon}{R}.$$

12.3. Foot projection of biconnected and triconnected pairs of tripods

For ε small enough and then R large enough, thanks to item (ii) of Lemma 9.4, we can define the foot projection $\boldsymbol{\Psi}$ from $\mathcal{B}_{\varepsilon,R}(\alpha)$ to $\mathcal{F}_\alpha$ by

$$\boldsymbol{\Psi}(T, S_0, \alpha(S_0)) = \Psi(T, \alpha^+; \alpha^-),$$

Similarly, we define the foot projection $\mathbf{\Psi}$ from $\mathcal{Q}_{\varepsilon,R}(\alpha)$ to $\mathcal{F}_\alpha$ by

$$\mathbf{\Psi}(T, S_0, S_1, S_2)) = \Psi(T, \alpha^+; \alpha^-),$$

Let then $\nu_{\varepsilon,R}$ and $\mu_{\varepsilon,R}$ be as defined in §11.5 and §11.1.1, respectively:

$$\boldsymbol{\nu}_{\varepsilon,R} = \mathbf{\Psi}^* \nu_{\varepsilon,R} \quad \text{and} \quad \boldsymbol{\mu}_{\varepsilon,R} = \mathbf{\Psi}^* \mu_{\varepsilon,R},$$

We now summarize some properties of the projection.

Proposition 12.8. *First, we have*

$$\mathbf{\Psi} \circ \mathbf{I}_0 = \mathbf{I}_0 \circ \mathbf{\Psi}.$$

Moreover, assume ε small enough and then R large enough. The foot projection $\mathbf{\Psi}$ is proper. The measure $\boldsymbol{\nu}_{\varepsilon,R}$ is supported on $\mathcal{X}_\alpha^\Lambda$ and is finite. Finally,

$$\boldsymbol{\mu}_{\varepsilon,R} = C_{\varepsilon,R} \boldsymbol{\nu}_{\varepsilon,R},$$

with

$$\|1 - C_{\varepsilon,R}\|_\infty \leqslant \frac{\varepsilon}{R^2}.$$

Proof. By construction, we have

$$\begin{aligned} \mathbf{\Psi} \circ \mathbf{I}_0(t, s, c_0, c_1, c_2) &= \mathbf{\Psi}(\mathbf{I}_0(t), \mathbf{I}_0(s), \mathbf{I}_0(c_1), \mathbf{I}_0(c_0), \mathbf{I}_0(c_2)) \\ &= \Psi(\mathbf{I}_0(T), \alpha^-, \alpha^+) \\ &= \mathbf{I}_0(\Psi(T, \alpha^+, \alpha^-)), \end{aligned}$$

where the last equality comes from the fact that $\mathbf{I}_0$ exchanges the vertices $\partial^+\tau$ and $\partial^+\tau$ of the tripod and assertion (10.18).

By Lemma 9.4 and Proposition 10.10, if $B := (T, S_0, \alpha(S_0))$ is in the support of $\mathrm{D}_{\varepsilon,R}$ and $\tau_\alpha := \mathbf{\Psi}(B)$, then

$$d(T, \tau_\alpha) + d(S_0, \tau_\alpha) \leqslant M(\varepsilon + R)$$

for some universal constant M. This implies the properness of $\mathbf{\Psi}$.

Moreover, by Proposition 10.10, (T, S_0) is almost closing for α. Thus, by the last assertion of the closing lemma (Lemma 9.4), τ_α belongs to the $(M\varepsilon, R)$-core $\mathcal{X}_\alpha$. Thus, $\boldsymbol{\nu}_{\varepsilon,R}$ is supported on the core. Since $\mathcal{X}_\alpha^\Lambda$ is compact (Proposition 12.3), $\boldsymbol{\nu}_{\varepsilon,R}$ is finite.

The last assertion of this proposition now follows from Proposition 11.7. □

13. Pairs of pants are evenly distributed

We will want to glue pairs of pants along their boundary components if their "foot projections" differ by approximately a "Kahn–Marković" twist. Given a pair of pants, the existence of other pairs of pants which can admissibly be glued along a given boundary component will be obtained by an equidistribution theorem.

Since we need to glue pairs of pants along boundary data, a whole part of this section is to explain the boundary data which in this higher-rank situation is more subtle than for the hyperbolic 3-space. We also need to explain what does reversing the orientation mean in this context.

The main result is the even distribution theorem (Theorem 13.2), which requires many definitions before being stated. Its proof relies on a Margulis-type argument using mixing, as well as the presence of some large centralizers of elements of Γ. This is the only part where the flip assumption — revisited in this section — is used. This is of course structurally modeled on the corresponding section in [14]. Let us sketch the construction.

(i) The space of triconnected pairs of tripods carries a *measure* μ^+ coming from the weight functions defined above. Similarly, we have a measure μ^- obtained while using the orientation-reversing diffeomorphism on the space of tripods (see Definition 11.5).

(ii) The *boundary data* associated to a boundary geodesic α with loxodromic holonomy and endpoints α^+ and α^- will be the set of tripods with endpoints α^+ and α^-, up to the action of the centralizer of α. In the simplest case of the principal $\mathfrak{sl}_2$ in a complex simple group, this space of feet is a compact torus.

We have now a projection $\mathbf{\Psi}$ from the space of triconnected pairs of tripods to the space of feet $\mathcal{F}{:=}\bigsqcup_\alpha \mathcal{F}^\Gamma_\alpha$, just by taking the projection of one of the defining tripods (and using Theorem 9.5). Our goal is to establish the *even distribution theorem* (Theore 13.2), which says that the projected measures $\mathbf{\Psi}_*\mu^+$ and $\mathbf{\Psi}_*\mu^-$ do not differ by much after a Kahn–Marković twist. Roughly speaking the proof goes as follows.

(i) This projection $\mathbf{\Psi}$ factors through the space of "biconnected pairs of tripods" (by forgetting one of the path connecting the tripods) which carries itself a weight and a measure. The mixing argument then tells us that the projected measure from triconnected pairs of tripods to biconnected pairs of tripods are approximately the same, or in other words the forgotten path is roughly probabilistically independent form the others.

(ii) It is then enough to show that the projected measures from the space of biconnected pairs of tripods is evenly distributed. In the simplest case of the principal $\mathfrak{sl}_2$ in a complex simple group, this comes from the fact these measures are invariant under the centralizer of α which, in that case, acts transitively on the boundary data. The general case is more subtle (and involves the Flip assumption) since the action of the centralizer of α on space of feet is not transitive anymore.

In this section Γ will be a uniform lattice in G, α a P-loxodromic element in Γ, Γ_α the centralizer of α in Γ, which is a uniform lattice in $\mathsf{Z}_{\mathsf{G}}(\alpha)$ (see [11, Proposition 3.5]).

13.1. The main result of this section: even distribution

We can now state the main result of this section. This is the only part of the paper that makes uses of the flip assumption. The theorem uses the notion of Lévy–Prokhorov distance for measures on a metric space which is discussed in Appendix A. We first need the following definition.

Definition 13.1. (Kahn–Marković twist) For any $\alpha\in\Gamma$, the *Kahn–Marković twist* $\mathbf{T}_\alpha$ is the element $\varphi_1\circ\sigma$ that we see as a diffeomorphism of the space of feet $\mathcal{F}_\alpha^\Gamma$. Similarly, we consider the (global Kahn–Marković twist) product map $\mathbf{T}=\prod_{\alpha\in\Gamma}\mathbf{T}_\alpha$ from $\mathcal{F}$ to itself.

Our main result is then the following.

Theorem 13.2. (Even distribution) *For any small enough positive ε, there exists a positive R_0 such that, if $R>R_0$ then $\mu_{\varepsilon,R}^\pm$ are finite non-zero, and furthermore*

$$d_L(\mathbf{\Psi}_*^+\mu_{\varepsilon,R}^+,\mathbf{T}_*\mathbf{\Psi}_*^-\mu_{\varepsilon,R}^-)\leqslant M\frac{\varepsilon}{R}, \tag{13.1}$$

where $\mathbf{T}$ is the Kahn–Marković twist, the constant M only depends on G, and d_L is the Lévy–Prokhorov distance.

The notation $\mathbf{\Psi}^+$ and $\mathbf{\Psi}^-$ in this theorem and the sequel of this section is for redundancy: it means $\mathbf{\Psi}$ as considered from $\mathcal{B}_{\varepsilon,R}^+(\alpha)$ and $\mathcal{B}_{\varepsilon,R}^-(\alpha)$, respectively.

The metric on $\mathcal{F}$ is the metric coming from its description as a disjoint union, not the induced metric from $\mathcal{G}$. This whole section is devoted to the proof of this theorem. We shall use the flip assumption.

13.2. Revisiting the flip assumption

We fix, in all this section, a reflection $\mathbf{J}_0$. We will explain in this section the consequence of the *flip assumption* that we shall use, as well as give examples of groups satisfying the flip assumptions. Recall that, for an element α in Γ, we write $\Gamma_\alpha=\mathsf{Z}_\Gamma(\alpha)$.

Let α be a P-loxodromic element. Let

$$\mathsf{L}_\alpha:=\{g\in\mathsf{G}:g(\alpha^\pm)=\alpha^\pm\}.$$

Observe first that, since $\mathbf{J}_0\in\mathsf{Z}_{\mathsf{G}}(\mathsf{L}_0)$, $\mathbf{J}_\alpha:=\tau(\mathbf{J}_0)$ does not depend on τ, for all τ with $\partial^\pm\tau=\alpha^\pm$, and belongs to $\mathsf{Z}_{\mathsf{G}}(\mathsf{L}_\alpha)$. Thus, for all τ in $\mathcal{F}_\alpha$,

$$\mathbf{J}_\alpha\cdot\tau=\tau\cdot\mathbf{J}_0.$$

The element $\mathbf{J}_\alpha$ of $\mathcal{G}$ is called the *reflection* of axis α.

Let also K_α be the maximal compact factor of L_α.

Definition 13.3. (Weak flip assumption) We say that the lattice Γ in G satisfies the *weak flip assumption* if there is some integer M depending only on G such that, given a P-loxodromic element α in Γ, then the following conditions hold:

- there exists a subgroup Λ_α of $\Gamma_\alpha \cap \mathsf{Z_G}^\circ(\alpha)$, normalized by Γ_α, with $[\Gamma_\alpha:\Lambda_\alpha]\leqslant \mathrm{M}$;
- moreover, $\mathbf{J}_\alpha$ belongs to a connected compact torus $\mathsf{T}^0_\alpha \subset \mathsf{Z_G}(\Lambda_\alpha)\cap\mathsf{K}_\alpha$.

Denoting by $\mathsf{Z_F}(B)$ the centralizer in the group F of the set B, and by H° the connected component of the identity of the group H, we now introduce the following group for a P-loxodromic element α in Γ satisfying the weak flip assumption

$$\mathsf{C}_\alpha := (\mathsf{Z_G}(\Lambda_\alpha))^\circ < \mathsf{L}_\alpha. \tag{13.2}$$

13.2.1. Relating the flip assumptions

We first relate the flip assumptions (Definitions 2.2 and 2.3) to the weak flip assumption (Definition 13.3).

PROPOSITION 13.4. *If* G *and* $\mathfrak{s}_0$ *satisfy the flip assumption* (*Definition* 2.2), *or the regular flip assumption* (*Definition* 2.3), *then* G, $\mathfrak{s}_0$ *and* Γ *satisfy the weak flip assumption* (*Definition* 13.3).

Proof. Let us first make the following preliminary remark: as an easy consequence of a general result by John Milnor [28], the following holds: *Given a center-free semisimple Lie group* G, *there exists a constant* $\mathbf{N}$ *such that, for every semisimple* $g\in\mathsf{G}$, *the number of connected components of* $\mathsf{Z_G}(g)$ *is less than* $\mathbf{N}$. In particular, $[\Gamma_\alpha:\Gamma_\alpha\cap\mathsf{Z_G}^\circ(\alpha)])\leqslant\mathbf{N}$.

We have to study the two cases of the flip and regular flip assumptions. Assume first that G and $\mathfrak{s}_0$ satisfy the flip assumption with reflection $\mathbf{J}_0$. Let α be an element of Γ which is P-loxodromic. Then, any element β commuting with α preserves α^+ and α^-, thus $\Gamma_\alpha<\mathsf{L}_\alpha$. The flip assumption hypothesis thus implies that, taking $\Lambda_\alpha=\Gamma_\alpha\cap\mathsf{Z_G}^\circ(\alpha)$,

$$\mathbf{J}_\alpha \in (\mathsf{Z_G}(\mathsf{L}_\alpha))^\circ \subset (\mathsf{Z_G}(\Gamma_\alpha))^\circ \subset (\mathsf{Z_G}(\Lambda_\alpha))^\circ.$$

Moreover, $\mathbf{J}_\alpha$ is an involution that belongs to the center of L_α, and thus to its compact factor. Since, by [20, Corollary IV.4.47], the center of a connected compact subgroup is included in any maximal torus, we may choose for T^0_α a maximal torus in $(\mathsf{Z_G}(\Lambda_\alpha))^\circ\cap\mathsf{K}_\alpha$ containing $\mathbf{J}_\alpha$. This concludes this case.

Let us move to the regular flip assumption. In that case $\mathsf{L}_0=\mathsf{A}_0\times\mathsf{K}_0$, where A_0 is a torus without compact factor, and K_0 is a compact factor, accordingly $\mathsf{L}_\alpha=\mathsf{A}_\alpha\times\mathsf{K}_\alpha$ with the same convention. Let α be a P-loxodromic element in Γ, as above we notice that $\Gamma_\alpha \subset \mathsf{L}_\alpha$. Since Γ_α is discrete torsion-free, $\Gamma_\alpha \cap \mathsf{K}_\alpha = \{e\}$. Thus, the projection of Γ_α on A_α is injective, and Γ_α is Abelian. Let π be the projection of L_α on K_α, $\mathsf{B} = \pi(\Gamma_\alpha)$, and B_1 be a maximal Abelian containing B in K_α. Using again [28], there is a constant M depending only on G such that, if C is maximal Abelian in K_0, then $[\mathsf{C} : \mathsf{C}^\circ] \leqslant \mathrm{M}$. Let

$$\Lambda_\alpha := \pi^{-1}(\mathsf{B}_1^\circ) \cap \Gamma_\alpha \subset \mathsf{Z}_{\mathsf{G}}{}^\circ(\alpha).$$

Then, $[\Gamma_\alpha : \Lambda_\alpha] \leqslant \mathrm{M}$. Moreover, setting T_α^0 to be a maximal torus containing B_1°, we have (since $\mathbf{J}$ is central in K_0)

$$\mathbf{J}_\alpha \in \mathsf{T}_\alpha^0 \subset \mathsf{Z}_{\mathsf{G}}(\mathsf{B}_1^\circ) \cap \mathsf{K}_\alpha \subset \mathsf{Z}_{\mathsf{G}}(\Lambda_\alpha) \cap \mathsf{K}_\alpha.$$

This concludes the proof of the proposition. □

13.2.2. Groups satisfying (or not) the flip assumptions

Let us show a list of group satisfying the flip assumptions. Examples (iv) below were pointed out to us by Fanny Kassel, who also pointed out earlier mistakes.

(i) If G is a complex semi-simple Lie group. Let $\mathfrak{s} = (a, x, y)$ be an even $\mathfrak{s}$-triple. Then, $\mathbf{J}_0 = \exp\left(\frac{1}{2} i\zeta a\right)$, for the smallest $\zeta > 0$ such that $\exp(i\zeta a) = 1$ is a reflection that satisfies the flip assumption: indeed, $\exp(ita)$ for t real lies in $\mathsf{Z}_{\mathsf{G}}(\mathsf{Z}_{\mathsf{G}}(a))$.

(ii) The isometry groups of the hyperbolic space, $\mathsf{SO}(1,p)$, do not satisfy the weak flip assumption when p is even, while they satisfy it for p odd.

(iii) The groups $\mathsf{SO}^+(2,4)$ satisfy the flip assumption for some $\mathfrak{sl}_2$-triple with compact centralizer. More precisely, embed $\mathsf{SO}^+(1,2) \times \mathsf{SO}^+(1,2)$ in $\mathsf{SO}^+(2,4)$ and consider Δ, which is an even $\mathfrak{s}$-triple in $\mathsf{SO}^+(1,2)$ embedded diagonally in $\mathsf{SO}^+(1,2) \times \mathsf{SO}^+(1,2)$. Then, one can check that $\mathsf{Z}_{\mathsf{G}}(a) = \mathsf{GL}(2,\mathbb{R}) \times \mathsf{SO}(2)$ and, in this decomposition,

$$J_0 = (\mathrm{Id}, -\mathrm{Id}).$$

Thus, although a is not regular, J_0 belongs to the connected component of the identity of $\mathsf{Z}_{\mathsf{G}}(\mathsf{Z}_{\mathsf{G}}(a))$. Finally, one notices that $\mathsf{Z}_{\mathsf{G}}(\Delta)$ is compact.

(iv) The groups $\mathsf{SU}(p,q)$, with $q > p > 0$, satisfy the flip assumption. We consider H to be the irreducible $\mathsf{SL}_2(\mathbb{R})$ in $\mathsf{SO}(p, p+1)$. Then, we see $\mathsf{SO}(p, p+1)$ as subgroup of $\mathsf{SU}(p,q)$ in $\mathsf{GL}(p+q)$. Then, the centralizer of a in $\mathsf{GL}(p+q)$ is $\mathbb{C}^{2p} \times \mathsf{GL}(q-p)$. It follows that the centralizer $\mathsf{Z}_{\mathsf{G}}(a)$ in $\mathsf{SU}(p,q)$ is $\mathbb{C}^p \times \mathsf{SU}(q-p)$. Thus, a is regular and $\mathsf{Z}_{\mathsf{G}}(a)$ is connected. It flows that H satisfies the regular flip assumption.

On the other hand, one easily checks that the groups $\mathsf{SL}(n,\mathbb{R})$ do not satisfy the flip assumption for the irreducible $\mathsf{SL}(2,\mathbb{R})$.

13.3. Proof of the even distribution theorem (Theorem 13.2)

Let Λ_α the subgroup of Γ_α of index at most $\mathbf{M}$ appearing in Definition 13.3.

Recall that

$$\mathbf{T}_\alpha = \varphi_1 \circ \sigma = \varphi_1 \circ \mathbf{J}_0 \circ \mathbf{I}_0 = \varphi_1 \circ \mathbf{J}_\alpha \circ \mathbf{I}_0,$$

where $\mathbf{J}_\alpha$ is defined in the beginning of §13.2.

The second equality comes from the fact that, as seen in the beginning of §13.2, the right action of $\mathbf{J}_0$ and the left action of $\mathbf{J}_\alpha$ coincide on $\mathcal{F}_\alpha$. Using Propositions 10.13, 11.6 and 12.8, we have

$$\mathbf{\Psi}^-_* \mu^-_{\varepsilon,R} = \mathbf{\Psi}^-_* \mathbf{I}_{0*} \mu^+_{\varepsilon,R} = \mathbf{I}_{0*} \mathbf{\Psi}^+_* \mu^+_{\varepsilon,R}.$$

Let then

$$\boldsymbol{\mu} = \mathbf{\Psi}^+_* \mu^+_{\varepsilon,R} \quad \text{and} \quad \boldsymbol{\nu} = \mathbf{\Psi}^+_* \nu^+_{\varepsilon,R}.$$

Our goal is thus to prove that there exists a constant $\mathbf{M}_5$ depending only on G, such that

$$d_L(\varphi_{1*} \mathbf{J}_{\alpha*} \boldsymbol{\mu}, \boldsymbol{\mu}) \leqslant \mathbf{M}_5 \frac{\varepsilon}{R}, \tag{13.3}$$

where we consider $\boldsymbol{\mu}$ as a measure on $\mathcal{F}^{\Gamma_\alpha}_\alpha$. We perform a further reduction. Let p the covering map from $\mathcal{F}^{\Lambda_\alpha}_\alpha$ to $\mathcal{F}^{\Gamma_\alpha}_\alpha$. let $\boldsymbol{\mu}^0$ and $\boldsymbol{\nu}^0$ be the preimages of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ respectively on $\mathcal{F}^{\Lambda_\alpha}_\alpha$. Since, $\mathrm{p}_* \boldsymbol{\mu}^0 = q\boldsymbol{\mu}$ and $\mathrm{p}_* \boldsymbol{\nu}^0 = q\boldsymbol{\nu}$ where q is the degree — less than $\mathbf{M}$ — of the covering p, it is enough by Proposition A.5 to prove that there exists a constant $\mathbf{M}_6$ depending only on G, such that

$$d_L(\varphi_{1*} \mathbf{J}_* \boldsymbol{\mu}^0, \boldsymbol{\mu}^0) \leqslant \mathbf{M}_6 \frac{\varepsilon}{R}. \tag{13.4}$$

But now this is a consequence of Theorem 12.4, taking $\boldsymbol{\nu} := \Psi_* \nu_{\varepsilon,R}$, where $\nu_{\varepsilon,R}$ is defined in paragraph 11.1.1 and its invariance under C_α is checked in the same paragraph. The main hypothesis (12.5) of Theorem 12.4 — for instance the fact that $\boldsymbol{\nu}$ is supported on the core — is a consequence of Proposition 11.7.

14. Building straight surfaces and gluing

Our aim in this section is to define *straight surfaces* and prove their existence in Theorem 14.2. Loosely speaking, a straight surface is obtained by gluing almost Fuchsian pairs of pants using Kahn–Marković twists. We also explain that a straight surface comes with a fundamental group.

This section is just a rephrasing of a similar argument in [14] and uses as a central argument the Even Distribution Theorem 13.2.

14.1. Straight surfaces

Recall that in a graph, a *flag* adjacent to vertex v a is a pair (v,e) such that the edge e is adjacent to the vertex v. The *link* $L(v)$ of a vertex v is the set of flags adjacent to v. A trivalent ribbon graph is a graph with a cyclic permutation ω of order 3, without fixed points, on flags such that $\omega(v,e)=(v,f)$ and such that every link $L(v)$ is equipped with a cyclic permutation ω_v of order 3. If a graph is bipartite, so that we can write its set of vertices as $V^-\sqcup V^+$, we denote by $e^\pm$ the vertices of an edge e that belong to $V^\pm$, respectively.

Let Γ be a discrete subgroup of G.

Definition 14.1. (Straight surface) Let ε and R be positive numbers. An (ε,R) *straight surface* for Γ is a pair $\Sigma=(\mathcal{R},W)$, where $\mathcal{R}$ is a finite bipartite trivalent ribbon graph whose set of vertices is $V^-\sqcup V^+$, and W is labeling of flags in $\mathcal{R}$ such that the following conditions hold:

(i) for every flag (v,e) with $v\in V^\pm$, $W(v,e)$ belongs to $Q^{\Gamma,\pm}_{\varepsilon,R}$;

(ii) the labeling map is equivariant:

$$W(\omega_v(v,e))=\omega(W(v,e));$$

(iii) for any edge e,

$$d(\mathbf{\Psi}^+(W(e^+,e)),\mathbf{T}\mathbf{\Psi}^-((W(e^-,e)))\leqslant\frac{\varepsilon}{R}. \tag{14.1}$$

We may now associate to a straight surface Σ a topological surface given by the gluing of pairs of pants (labeled by vertices) along their boundaries (labeled by edges), surface whose fundamental group is denoted by $\pi_1(\Sigma)$. The labeling of vertices of edges will then give rise to a representation of $\pi_1(\Sigma)$ into Γ (see §16.1). The main theorem of this section is the following.

THEOREM 14.2. (Existence of straight surfaces) *Let $\mathfrak{s}$ be an $\mathfrak{sl}_2$-triple in the Lie algebra of a semisimple group G-satisfying the flip assumption, and let Γ be a uniform lattice in G.*

Then, for every ε, there exists R_0 such that, for any $R\geqslant R_0$, there exists an (ε,R)-straight surface for Γ.

14.2. Marriage and equidistribution

We want to prove the following lemma.

LEMMA 14.3. (Trivalent graph) *Let Y be a compact metric space. Let ω be an order-3 symmetry acting freely on Y. Let μ be a ω-invariant finite (non-zero) measure on Y. Let α be a real number. Let f^0 and f^1 be two biLipschitz maps from Y to a metric space Z such that*

$$d_L(f^0_*\mu, f^1_*\mu) < \alpha.$$

Then, there exists a non-empty finite trivalent bipartite ribbon graph $\mathcal{R}$, whose vertex set is $V_0 \sqcup V_1$, such that

- *we have an ω-equivariant labeling W of flags by elements of Y,*
- *if e is an edge from v_0 to v_1 such that $v_i \in V_i$, then*

$$d(f^0 \circ W(v_0, e), f^1 \circ W(v_1, e)) \leqslant \alpha.$$

This will be an easy consequence of the following theorem.

THEOREM 14.4. (Measured marriage theorem) *Let Y be a compact metric space equipped with a finite (non-zero) measure μ. Let f and g be two uniformly Lipschitz maps from Y to a metric space Z such that*

$$d_L(f_*\mu, g_*\mu) < \beta.$$

Then, there exist a non-empty finite set $\overline{Y}$, a map p from $\overline{Y}$ in Y and a bijection ϕ from $\overline{Y}$ to itself such that

$$d(f \circ p, g \circ p \circ \phi) \leqslant 2\beta.$$

Assume now that we have a free action of an order-3 symmetry σ on Y preserving the measure. Then, there exists $\overline{Y}$, ϕ and p as before, where $\overline{Y}$ is equipped with an order-3 symmetry $\tilde{\sigma}$ and p is $\tilde{\sigma}$-equivariant.

Proof. If μ is the counting measure and Y is finite, this theorem is a rephrasing of Hall marriage theorem (see [14, Theorem 3.2]). We reduce to this case by the following trick: by approximation (See Proposition A.3), we may approximate (with respect to the Lévy Prokhorov metric) μ by a finitely supported atomic measure ν.

Then, by Proposition A.5, $f_*\nu$ and $f_*\mu$ are very close and the same holds for g. Thus,

$$d_L(f_*(\nu), g_*(\nu)) < 2\beta. \tag{14.2}$$

Since ν is atomic, we can replace Y with the finite set $\mathrm{Supp}(\nu)$ of cardinality N. Then, ν become an element of $\mathbb{R}^N$. The inequality (14.2) now turns to be equivalent to the following statement. Let $\mathcal{E}$ be the set of pair of subsets (Z_0, Z_1) of Y such that

$$Z_0 = f^{-1}(B) \quad \text{and} \quad Z_0 = g^{-1}(B_{2\beta})$$

for some B in X. Then,

$$\sum_{x\in Z_0}\nu(\{x\})\leqslant\sum_{x\in Z_1}\nu(\{x\})\quad\text{for all } (Z_0,Z_1) \text{ in } \mathcal{E}.$$

In other words, it is equivalent to finitely many inequalities with integer coefficients and linear in the weights of ν.

Since we have a solution with real positive coefficients of this set of inequalities, we also have a solution with positive rational coefficients, or in other words a measure λ with rational weights such that

$$d_L(f_*(\lambda),g_*(\lambda))<2\beta.$$

After multiplication we may assume that all weights are integers. Note that multiplying both measures by the same $c>0$ does not change the Lévy–Prokhorov distance. Then, finally we let $\overline{Y}$ to be the set Y counted with the multiplicity given by λ, and we can conclude using the observation at the beginning of the paragraph. Finally, the procedure can be made equivariant with respect to finite-order symmetries. □

14.2.1. Proof of Lemma 14.3

Let $\overline{Y}$, $\tilde{\sigma}$ and h be as in Theorem 14.4. Let us write $V=\overline{Y}/\langle\sigma\rangle$ and let π be the projection from $\overline{Y}$ to V. Let now $\mathcal{R}=V_0\sqcup V_1$ be the disjoint union of two copies of V; this will be the set of vertices of the graph. An edge is given by a point y in $\widetilde{Y}$, that we consider joining the vertex $v_0:=\pi(y)$ to $v_1:=\pi(\phi(y))$. The labeling is given by $W=p$.

14.3. Existence of straight surfaces: Proof of Theorem 14.2

We apply Lemma 14.3 to the following data:

(i) the set $Y:=\mathcal{Q}^{\Gamma,+}_{\varepsilon,R}$ (which is non-empty by Proposition 11.6) and the set $Z:=\mathcal{F}$;

(ii) the measure $\mu:=\mu^+_{\varepsilon,R}$ (which is ω-invariant and satisfy $I_*\mu^{\pm}_{\varepsilon,R}=\mu^{\mp}_{\varepsilon,R}$, by Proposition 11.6);

(iii) the functions $f^0:=\mathbf{\Psi}^+$ and $f^1:=\mathbf{T}\circ\mathbf{\Psi}^+=\mathbf{T}\circ\mathbf{\Psi}^-\circ\mathbf{I}_0$;

(iv) $\alpha:=M\varepsilon/R$.

Observe that we label vertices of V_0 by $W(v,e)$, while we label vertices of V_1 by $I(W(v,e))$. For ε small enough and then R large enough (depending on ε), the following statements hold:

- the set $\mathcal{Q}^+_{\varepsilon,R}$ is non-empty, by Proposition 11.6;
- by Theorem 13.2, we have the inequality $d(f^0_*\mu,f^1_*\mu)\leqslant M\varepsilon/R$, using the fact that

$$I_*\mu^{\pm}_{\varepsilon,R}=\mu^{\mp}_{\varepsilon,R}.$$

Theorem 14.2 is now a rephrasing of the trivalent graph lemma (Lemma 14.3). Namely, according to this lemma, we have a trivalent bipartite graph $\mathcal{R}$ such that every flag (e^+,e) is labeled by an element $W(e^+,e)$ of $\mathcal{Q}^{\Gamma,+}_{\varepsilon,R}$ and every flag (e^-,e) is labeled by an element $W(e^-,e)$ of $\mathcal{Q}^{\Gamma,-}_{\varepsilon,R}$, satisfying

$$d(\mathbf{\Psi}^+(W(e^+,e)),\mathbf{T}\mathbf{\Psi}^-((W(e^-,e)))\leqslant M\frac{\varepsilon}{R}. \tag{14.3}$$

In other words, $(\mathcal{R},W)$ is an $(M\varepsilon,R)$-straight surface.

15. The perfect lamination

In this section, we concentrate on plane hyperbolic geometry. We present some results of [14] concerning the R-perfect lamination. This perfect lamination is associated to a tiling by hexagons.

We also introduce a new concept: *accessible points* from a given hexagon. Apart from the definition, the most important result is the accessibility lemma (Lemma 15.9), which guarantees that accessible points are almost (in a quantitative way) dense.

15.1. The R-perfect lamination and the hexagonal tiling

Let us consider two ideal triangles in the (oriented) hyperbolic plane, glued to each other by a swish of length R (with $R>0$) to obtain a pair of pants P_0, called the *positive R-perfect pair of pants*. Symmetrically, the *negative R-perfect pair of pants* P_1 is obtained by a swish of length $-R$. Both perfect pairs of pants come by construction with ideal triangulations and orientations.

The *R-perfect surface* S_R is the genus-2 oriented surface obtained by gluing the two pairs of pants P_0 and P_1 with a swish of value 1. The surface S_R possesses three *cuffs* which are the three geodesic boundaries of the initial pairs of pants. These cuffs are oriented, where the orientation comes from the orientation on P_0.

Let Λ_R be the Fuchsian group such that $\Lambda_R\backslash\mathbf{H}^2=S_R$.

The *R-perfect lamination* $\mathcal{L}_R$ of $\mathbf{H}^2$ is the lift of the cuffs of S_R in $\mathbf{H}^2$.

Observe that each leaf of $\mathcal{L}_R$ carries a natural orientation. Connected components of the complement of $\mathcal{L}_R$ are *even* or *odd* whenever they cover respectively a copy of P_0 or P_1.

We denote by $\mathcal{L}_R^\infty$ the set of endpoints of $\mathcal{L}_R$ in $\partial_\infty\mathbf{H}^2$.

15.1.1. Length, intersection and diameter

We collect here important facts about the R-perfect lamination from Kahn–Marković paper [14].

LEMMA 15.1. (Length control [14, Lemma 2.3]) *There exists a constant K such that, for R large enough and for all geodesic segments γ in $\mathbf{H}^2$ of length ℓ, we have*

$$\sharp(\gamma \cap \mathcal{L}_r) \leqslant K \cdot R \cdot \ell.$$

LEMMA 15.2. (Uniformly bounded diameter [14, Lemma 2.7]) *There exists a constant M independent of R such that, for all R,* $\operatorname{diam}(S_R) \leqslant M$.

As a corollary of the first lemma, using the language of §7.1, we have the following.

COROLLARY 15.3. *There exists a constant K such that, for R large enough, any coplanar sequence of cuffs whose underlying geodesic lamination is a subset of $\mathcal{L}_R$ is a KR-sequence of cuffs.*

15.1.2. Tilings: connected components, tiling hexagons and tripods

Let C be a connected component of $\mathbf{H}^2 \setminus \mathcal{L}_R$.

Observe that C is tiled by right-angled *tiling hexagons* coming from the decomposition in pairs of pants of S_R. Each such hexagon H is described by a triple of geodesics (a, b, c) in $\mathcal{L}_R$, whose endpoints (with respect to the orientation) are respectively (a^-, a^+), (b^-, b^+) and (c^-, c^+), such that the sextuple $(a^-, a^+, b^-, b^+, c^-, c^+)$ is positively oriented. Let us then define three disjoint intervals, called *sides at infinity* in $\partial_\infty \mathbf{H}^2$, by $\partial_a H := [b^+, c^-]$, $\partial_c H := [a^+, b^-]$ and $\partial_b H := [c^+, a^-]$. Each such side corresponds to the edge of the hexagon connecting the two corresponding cuffs.

Definition 15.4. (i) The *successor* of an hexagon $H = (a, b, c)$ is the unique hexagon of the form $\operatorname{Suc}(H) = (a, d, b)$.

(ii) The *opposite* of an hexagon $H = (a, b, c)$ is the hexagon $\operatorname{Opp}(H) = (a, b', c')$, such that H and $\operatorname{Opp}(H)$ meet along a geodesic segment of length $R - 1$.

(iii) Given a tiling hexagon H, an *admissible tripod* with respect to H is given by three points (x, y, z) in $\partial_a H \times \partial_b H \times \partial_c H$.

We remark that

$$\operatorname{Opp} \circ \operatorname{Suc} \circ \operatorname{Opp} \circ \operatorname{Suc} = \operatorname{Id}.$$

We can furthermore color hexagons.

PROPOSITION 15.5. *There exists a labeling of hexagons by two colors (black and white) such that H and* $\mathrm{Opp}(H)$ *have the same color, while H and* $\mathrm{Suc}(H)$ *have different colors.*

We denote by $T_R(H)$ the set of admissible tripods with respect to a given hexagon H and T_R the set of all admissible tripods. Elementary hyperbolic geometry yields the following result.

PROPOSITION 15.6. *There exists a universal constant K, such that for R large enough, the following conditions hold:*

(i) *the diameter of each tiling hexagon is less than $R+K$;*

(ii) *each hexagon has long edges (along cuffs) of length R, and short edges of length ℓ, where*

$$\frac{e^{\ell}+1}{e^{\ell}-1}=\sqrt{\frac{1+e^{3R}}{e^{R}+e^{2R}}} \quad \textit{and} \quad \lim_{R\to\infty} e^{R\ell/2}=1;$$

(iii) *the distance between any two admissible tripods with respect to the same hexagon is at most $2e^{-R/2}$.*

15.1.3. Cuff groups and graphs

The *cuff elements* are those elements of the Fuchsian group Λ_R whose axis are cuffs, a *cuff group* Λ is a finite-index subgroup of Λ_R containing all the primitive cuff elements: equivalently, $\Lambda\backslash\mathbf{H}^2$ is obtained by gluing R-perfect pairs of pants by swishes of length 1. We will identify oriented cuffs with primitive cuff elements.

To a cuff group Λ, we can associate a ribbon graph $\mathcal{R}$. Observe that $S:=\Lambda\backslash\mathbf{H}^2$ is tiled by hexagons. We consider the graph $\mathcal{R}$ whose vertices are hexagons in the above tiling of S, up to cyclic symmetry, and edges corresponding to pair of hexagons who lift to opposite hexagons.

Observe that $\mathcal{R}$ is the covering of the corresponding graph for S_R and has thus two connected components which correspond respectively to the 2-coloring in black and white hexagons. The distinction between odd and even components (and thus between odd and even hexagons) gives to $\mathcal{R}$ the structure of a bipartite graph.

Hexagons in S correspond to links of $\mathcal{R}$. By construction, each hexagon H is associated to a perfect triconnected pair of tripods $W_0(H)$ with respect to $\mathsf{SL}_2(\mathbb{R})$, in other words an element in $\mathcal{Q}_{0,R}$. We have thus associated to each cuff group Λ a $(0,R)$-straight surface $\Sigma(\Lambda):=(\mathcal{R},W_0)$, which actually has two connected components. One easily checks that every connected $(0,R)$-straight surface Σ is obtained from a well-defined cuff group Λ, as a connected component of $\Sigma(\Lambda)$.

15.2. Good sequence of cuffs and accessible points

Let us start with a definition associated to a positive number K.

Definition 15.7. A pair (c_1, c_2) of cuffs is *K-acceptable* if the following conditions hold:

(i) there is no cuffs between c_1 and c_2;

(ii) $d(c_1, c_2) \leqslant K$.

A triple (c_1, c_2, c_3) of cuffs is *K-acceptable* if the following conditions hold:

(i) c_2 is the unique cuff between c_1 and c_3;

(ii) $d(c_1, c_3) \leqslant K$.

Observe that, if a triple (c_1, c_2, c_3) of cuffs is K-acceptable, then both the pairs (c_1, c_2) and (c_2, c_3) are K-acceptable

Definition 15.8. (i) A *K-good sequence of cuffs* is a sequence of cuffs $\{c_m\}_{m=1}^p$ such that, for every m, whenever it makes sense, (c_m, c_{m+1}, c_{m+2}) is K-acceptable.

(ii) A *K-accessible point* with respect to a tiling hexagon H is a point in $\partial_\infty \mathbf{H}^2$ which is a limit of subsequences of endpoints of the cuffs of a K-good sequence of cuffs, where c_1 and c_2 contain long segments of the boundary of H.

Observe that we have an associated nested sequence of chords, where the chord is defined by the geodesic c_n and the half-space containing c_{n+1} and not containing c_{n-1}.

For a hexagon H (with respect to the lamination $\mathcal{L}_R$), we denote by $W_H^R(K)$ the set of K-accessible points from H.

The main result of this section is the following lemma.

Lemma 15.9. (Accessibility) *Let K_0 be a positive constant large enough. There exists some function $a: R \mapsto a(R)$ converging to zero as $R \to \infty$, such that $W_H^R(K_0)$ is $a(R)$-dense for any tiling hexagon H in $\mathbf{H}^2$, with respect to the visual distance from the center of mass of H.*

We also observe that, if R is greater than R_0, then there exists a positive real C_0 such that the visual distance from the center of mass of H is C_0-equivalent to the distance associated to any admissible tripod with respect to H (see Definition 15.4).

15.3. Preliminary on acceptable pairs and triples

We need first to understand K-acceptable pairs.

Proposition 15.10. *For R large enough, let (c_1, c_2) be a K-acceptable pair. Then, the following statements hold.*

(i) *We have* $\frac{1}{2} e^{-R/2} \leqslant d(c_1, c_2) \leqslant 2 e^{-R/2}$.

(ii) *There exist exactly two hexagons* H_1 *and* H_2 *whose sides are* c_1 *and* c_2*, respectively; moreover,* $H_2=\mathrm{Suc}(H_1)$.

(iii) *If* (c_1,η) *is* K*-acceptable, and furthermore* η *and* c_2 *lie in the same connected component of* $\mathbf{H}^2\setminus c_1$*, then there exists* $\gamma\in\Lambda_R$ *preserving* c_1 *such that* $\eta=\gamma\cdot c_2$.

We have also a proposition on K-acceptable triples.

PROPOSITION 15.11. *There exists* K_0 *such that, if* (c_1,c_2) *is a* K*-acceptable pair with* $K>K_0$*, then the following statements hold.*

(1) *There exist exactly three* K*-acceptable triples starting with* c_1 *and* c_2*. Fixing an orientation of* c_2*, we can describe the last geodesic in the triple as* $c_3^+:=\langle c_1,c_2\rangle^+$*, and similarly* c_3^0 *and* c_3^-*, where, if* x^i *is the projection of* c_3^i *on* c_2*, then* (x^-,x^0,x^+) *is oriented.*

(2) *If* (c_1,c_2,c_3) *is a* K*-acceptable triple, then* $d(c_1,c_3)\leqslant K_0$*, and moreover, if* x_i *is the point in* c_2 *closest to* c_i*, then* $d(x_1,x_2)\leqslant 3R$.

(3) *Moreover, if* $(H_1,\mathrm{Suc}(H_1))$ *and* $(H_2,\mathrm{Suc}(H_2))$ *are the pairs of hexagons bounded by* (c_1,c_2) *and* (c_2,c_3)*, respectively, then*

$$H_2=\gamma^p\,\mathrm{Opp}(H_1),$$

where γ *is the cuff element associated to* c_2 *and* $p\in\{-1,0,1\}$.

(4) *if* c *is a geodesic not intersecting* c_1 *and* c_2*, such that* c_2 *is between* c_1 *and* c *and* $d(c,c_1)<K$*, then there is a cuff* c_3 *such that* (c_1,c_2,c_3) *is a* K*-acceptable triple and*

- *either* c_3 *does not intersect* c *and*
  - c_3 *lies between* c *and* c_2*,*
  - *or* c *lies between* c_3 *and* c_2*,*
- *or* c_3 *intersects* c.

These two propositions have immediate consequences summarized in the following corollary.

COROLLARY 15.12. (i) *For all positive* K_1 *and* K_2 *greater than* K_0*, there exists* R_0 *such that, for all* $R>R_0$ *and all hexagons* H*, one has* $W_H^R(K_1)=W_H^R(K_2)$.

(ii) *Any finite* K*-good sequence of cuffs* $\{c_m\}_{m=1}^p$ *can be extended to an infinite* K*-good sequence* $\{c_m\}_{m\in\mathbb{N}}$.

15.3.1. Proof of Proposition 15.10

If there is no cuffs between c_1 and η, then c_1 and η are common bounds of the universal cover of a pair of pants. Then, for R large enough, the following holds:

- either $d(c_1,\eta)>\frac{1}{2}R$,
- or they bound two hexagons with a common short edge that joins c_1 to c_2.

By construction of the shear coordinates, the pair of pants obtained by gluing two ideal triangles using an R-swish has $2R$ as length of its boundaries. Thus, the two hexagons have opposite long sides of length R and short side of length approximately $e^{-R/2}$ by the last item of Proposition 15.6. The result now follows.

This shows the first assertion.

Finally, all K-acceptable pairs (c_1, η)—if η and c_2 are in the same connected component of $\mathbf{H}^2 \setminus c_1$—are equivalent under the action of Λ_R, and the first item follows.

15.3.2. Controlling distances to geodesics

We will denote in general by $[c, d]$ the geodesic arc passing between c and d, where c and d could be at infinity. We first need a statement from elementary hyperbolic geometry.

Proposition 15.13. *If a and b are two non-intersecting geodesics, if x is the point on a closest to b, and if y is a point on a such that $d(x,y)>R_0$, then*

$$d(y,b) \geqslant \inf\left(\tfrac{1}{10} d(a,b) e^{3d(x,y)/4}, \tfrac{1}{4} d(x,y) - d(a,b)\right).$$

Proof. Let w and z be the projections on b of x and y, respectively. Let $A := d(x,y)$.

(i) Assume first $d(z,w) \leqslant \frac{3}{4}A$. Then,

$$d(y,z) \geqslant d(y,x) - d(x,w) - d(w,z) \geqslant A - d(a,b) - \tfrac{3}{4}A \geqslant \tfrac{1}{4}A - d(a,b),$$

(ii) If now $d(z,w) \geqslant \frac{3}{4}A$, then

$$d(y,z) \geqslant \frac{1}{10} d(x,w) e^{3A/4}.$$

This concludes the proof of the inequalities. □

15.3.3. Proof of Proposition 15.11

Let (c_1, c_2) be a K-acceptable pair. Let c_3^0 be the unique cuff such that (c_2, c_3^0) is a K-acceptable pair and, if z is the projection of c_3^0 on c_2 and y is the projection of c_1 on c_2, then $d(z,y)=1$.

Let γ be the primitive element of Λ_R preserving c_2, and for $p \in \mathbb{Z}$ we set

$$c_3^p = \gamma^{p/2}(c_3^0) \quad \text{and} \quad z^p = \gamma^{p/2}(z).$$

Observe that z^p is the projection of c_3^p on c_2 and that $d(z, z^p) = pR$.

Obviously, (c_1, c_2, c_3^0) is K-acceptable, since $d(c_1, c_3^0) \leqslant 2$ for R large enough.

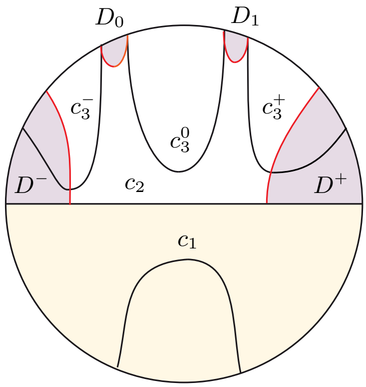


Figure 15.1. K-acceptable triples.

Observe now that the configuration of five geodesics given by c_3^0, c_3^1, c_2, c_1 and $\gamma(c_1)$ converges to a pair of ideal triangles swished by 1. Thus, there exists a universal constant K_0 such that, for R-large enough,

$$d(c_1, c_3^1) \leqslant K_0, \tag{15.1}$$

$$d(c_1, c_3^{-1}) \leqslant K_0, \tag{15.2}$$

where the second inequality is obtained by a similar argument.

As a consequence, for $K>2$, (c_1, c_2, c_3^p) is K-acceptable for $p\in\{+1, 0, -1\}$ and $K \geqslant K_0$. We want to show that these are the only ones. Let us write to simplify $c_3^{\pm}=c_3^{\pm 1}$ and $z^{\pm}=z^{\pm 1}$. We introduce the following notation:

- let D_2 be the connected component of $\mathbf{H}^2\setminus c_2$ not containing c_1;
- let $\eta^{\pm}$ be the geodesic arc orthogonal to c_2 passing though $z^{\pm}$ and lying inside D_2;
- let $D^{\pm}$ be the convex set bounded by $\eta^{\pm}$ and the geodesic arc $[z^{\pm}, c_2(\pm\infty)]$,

Observe that the following conditions hold:

(i) $c_3^p \subset D^+$ for all $p>3$;

(ii) the closest point m to c_1 in $D^{\pm}$ lies on c_2 (geodesic arcs orthogonal to $\eta^{\pm}$ never intersect c_2 and c_1).

It follows that, for all $p>1$, one has

$$d(c_3^p, c_1) \geqslant d(D^{\pm}, c_1) = d(m, c_1) \geqslant \inf\left(\tfrac{1}{10} d(c_1, c_2) e^{3A/4}, \tfrac{1}{4} A - d(c_1, c_2)\right),$$

where $A=d(m, y)$ and the last inequality comes from Proposition 15.13. Observe that

$$d(m, y) \geqslant d(z^{\pm}, y) \geqslant d(z^{\pm}, z) - d(z, y) = R-1.$$

Since $d(c_1,c_2)\geqslant\frac{1}{2}e^{-R/2}\frac{1}{2}$, we obtain from the previous inequality that

$$d(c_3^p,c_1)\geqslant d(D^\pm,c_1)\geqslant \inf\left(\tfrac{1}{1000}e^{r/4-1},\tfrac{1}{4}R-2\right).$$

Thus, for R large enough,

$$d(c_3^p,c_1)\geqslant d(D^\pm,c_1)\geqslant\tfrac{1}{8}R.$$

It follows that (c_1,c_2,c_3^p) is not K-acceptable for R large enough and $p>1$ (and a symmetric argument yields the case $p<1$). This finishes the proof of the first point.

The second point follows from inequalities (15.1) and (15.2). The third point is an immediate consequence of the previous construction.

We use the notation of the previous paragraph to prove the last point. Let c be such that $d(c_1,c)\leqslant K$. Since $d(D^\pm,c_1)\geqslant\frac{1}{8}R$, it follows that

$$c\not\subset D^+\sqcup D^-.$$

Let furthermore D_0 (resp. D_1) be the hyperbolic half-plane not containing c_1 bounded by $[c_3^-(+\infty),c_3^0(-\infty)]$ (resp. $[c_3^0(+\infty),c_3^+(-\infty)]$). Observe that

$$d(D_0,c_2)\geqslant R\quad\text{and}\quad d(D_1,c_2)\geqslant R.$$

Thus,

$$c\not\subset D_0\sqcup D_1.$$

Thus, the result now follows from the examination of Figure 15.1.

15.4. Preliminary on accessible points

The following proposition is obvious and summarizes some properties of accessible points.

Proposition 15.14. *A K-good sequence of cuffs $\{\gamma_m\}_{m\in\mathbb{N}}$ admits a unique accessible point which is also the Hausdorff limit of $\{\gamma_m\}_{m\in\mathbb{N}}$ in the compactification of $\mathbf{H}^2$, as well as the limit of the nested sequence of associated chords.*

We can explain our first construction of accessible points.

Proposition 15.15. *There exists a function $\alpha(R)$ converging to zero as R goes to infinity with the following property. Given K, there exists R_0 such that, for all $R>R_0$, the following holds: let (c_1,c_2) be a K-acceptable pair, let a be an extremity at infinity of c_2. Then, there exists an accessible point β in $\partial_\infty\mathbf{H}^2$ such that, for all x on c_1,*

$$d_x(\beta,a)\leqslant\alpha(R).$$

Proof. It is enough to prove this inequality whenever x is the projection of a on c_1. Let us consider the K-good sequence $\{c_m\}_{m\in\mathbb{N}}$, starting with c_1 and c_2, characterized by the following induction procedure.

First we choose an orientation on c_2 such that $a{=}c_2(+\infty)$, let also $b{=}c_1(-\infty)$ when c_1 inherits the orientation form c_2.

Assume $\{c_1, ..., c_p\}$ is defined. We choose the orientation on c_i compatible with c_2. Then, we choose $c_{p+1}{:=}\langle c_{p-1}, c_p\rangle^+$, where the notation is from Proposition 15.11.

Let β be the accessible point from this sequence. We will now show that

$$\lim_{R\to\infty} d_x(\beta, a) = 0.$$

This will prove the result setting $\alpha(R){=}{:}d_x(\beta, a)$. Let us start by the following construction and observations:

- let z the projection of c_3 on c_2;
- let η the geodesic arc orthogonal to c_2 starting at z and intersecting c_3;
- let D be the convex set bounded by η and $[y, a]$.

Observe that, for all $p{>}3$, $c_p{\subset}D$. It is therefore enough to prove that D converges to $\{a\}$, whenever R goes to infinity. Since

$$d(x, D) = d(x, z),$$

it will be enough to prove that $d(x, z)$ converges to ∞. Then, let y be the projection of c_1 on c_2. We know that

$$A := d(y, z) \geqslant R - 1.$$

It then follows from Proposition 15.13 that

$$d(x, z) \geqslant d(c_1, z) \geqslant \inf\big(\tfrac{1}{2}d(c_1, c_2)e^{3A/4}, \tfrac{1}{4}A - d(c_1, c_2)\big).$$

Since $d(c_1, c_2){\geqslant}\frac{1}{2}e^{-R/2}$ for R large enough, it follows, again for R large enough, that

$$d(x, z) \geqslant \tfrac{1}{8}R.$$

In particular,

$$\lim_{R\to\infty} d(x, z) = \infty.$$

This concludes the proof. □

15.5. Proof of the accessibility lemma (Lemma 15.9)

Let x be the center of mass of H.

Let us work by contradiction. Then, there exist $\beta>0$ and, for all R, an interval I_R in $\partial_\infty \mathbf{H}^2$ of visual length with respect to x greater than 2β, such that W_H^R does not intersect I_R. As a consequence, there exist a non-empty closed interval I of length β and a subsequence $\{R_m\}_{m\in\mathbb{N}}$ going to infinity such that

$$W_m(K) := W_H^{R_m}(K)$$

never intersects I.

Let γ be the geodesic connecting the extremity of I, and D_0 be the closed geodesic half-plane whose boundary is γ and boundary at infinity I. We may as well assume — at the price of taking a smaller β — that D_0 does not intersect H.

Let then K be the distance form γ to x. Assume that m is large enough (that is, R_m is large enough) so that

$$W_m(K) = W_m(K_0).$$

Let also η be a geodesic inside D_0 such that $d(\eta, x)=2K$ and $d(\eta,\gamma)=K$. Let $D_1\subset D_0$ be bounded by η. Let also ζ be the geodesic segment joining x to η. This segment intersects finitely many cuffs, and let c be the closest cuff to η, non-intersecting D_0. Let us consider all the cuffs $\{c_1, ..., c_p\}$ intersecting ζ between x and $c=c_p$. Then, $\{c_1, ..., c_p\}$ is a K-good sequence of cuffs.

We can now work out the contradiction. According to the last item of Proposition 15.11, there exists a cuff c_{p+1} such that (c_{p-1}, c_p, c_{p+1}) is a K-acceptable triple and either

- γ intersects c_{p+1},
- or γ is between c_p and c_{p+1}.

Indeed, c_{p+1} cannot be between c_p and γ, by the construction of c_p.

In both cases, c_{p+1} has an extremity — call it a — inside D_1. Then, according to Proposition 15.15, we can find an accessible point with respect to a sequence starting with (c_p, c_{p+1}) — hence starting with (c_1, c_2) — such that the corresponding accessible point y satisfies, for any ε and R large enough,

$$d_z(y, a) \leqslant \varepsilon,$$

where z is the intersection of c_p with ζ. Hence, since a lies in D_0,

$$d_x(y, a) \leqslant \varepsilon.$$

But this implies that $y\in D_0$ for ε small enough, and thus the contradiction.

16. Straight surfaces and limit maps

We finally make the connection with the first part of the paper and the path of quasi-tripods. Our starting object in this section will be a straight surface as discussed in the previous section, or more generally an *equivariant straight surface* (see Definition 16.2). Such an equivariant straight surface comes with a monodromy ρ and our main result, Theorem 16.7, shows that there exists a ρ-equivariant limit curve which is furthermore Sullivan. This implies the Anosov property, and in particular the fact that the representation is faithful.

The proof involves introducing another object: unfolding a straight surface gives rise to a labeling of each hexagon of the fundamental tiling of the hyperbolic plane by tripods, satisfying some coherence relations (see Proposition 16.8).

Then, we show that accessible points with respect to a given hexagon can be reached through nice paths of tripods. The labeling of hexagons gives deformations of these paths into paths of quasi-tripods. We can now use the limit point theorem (Theorem 7.2), and thus associate to an accessible point a point in $\mathbf{F}$: the limit point of the sequence of quasi-tripods.

Using finally the improvement theorem (Theorem 8.14) and the explicit control on limit points in Theorem 7.2, we show that we can define an actual Sullivan limit map.

16.1. Equivariant straight surfaces

We extend the definition of straight surfaces (which require a discrete subgroup of G) to that of an equivariant straight surface that you may think as of a "local system" in our setting, similar in spirit to the definition of positive representations in [10].

Recall that an almost closing pair of pants for G is a quintuple

$$T=(\alpha,\beta,\gamma,T_0,T_1)$$

such that α, β and γ are P-loxodromic elements in G, and T_0 and T_1 are tripods satisfying the conditions of Definition 9.1. We denoted by $P^{\pm}_{\varepsilon,R}$ the space of $(\varepsilon/R,\pm R)$-almost closing pairs of pants.

Also, if $T=(\alpha,\beta,\gamma,T_0,T_1)$ is an $(\varepsilon/R,R)$-almost closing pair of pants, we defined

$$\mathbf{\Psi}(T)=\Psi(T_0,\alpha^+,\alpha^-),$$

where Ψ is the foot map for quasi-tripods defined in Definition 4.4. We now give the following.

Definition 16.1. (Configuration spaces and gluing)

(i) The *configuration space of pairs of pants* is defined as

$$\mathcal{P}^{\pm}_{\varepsilon,R} := \mathsf{G}\backslash P^{\pm}_{\varepsilon,R}.$$

(ii) The *configuration space of gluing* is $\mathcal{Z}_{\varepsilon,R} := \mathsf{G}\backslash Z_{\varepsilon,R}$ where $Z_{\varepsilon,R}$ is the set of pairs $(T^+, T^-) \in P^+_{\varepsilon,R} \times P^-_{\varepsilon,R}$ such that

$$T^{\pm} = (\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}, T^{\pm}_0, T^{\pm}_1), \quad \alpha_+ = (\alpha_-)^{-1} \quad \text{and} \quad d(\mathbf{\Psi}(T^+), \mathbf{T}\mathbf{\Psi}(T^-)) \leqslant \frac{\varepsilon}{R}.$$

Observe that we have obvious G-equivariant projections $\pi^{\pm}: Z_{\varepsilon,R} \mapsto P^{\pm}_{\varepsilon,R}$, and also denote by $\pi^{\pm}$ the resulting projection $\mathcal{Z}_{\varepsilon,R} \mapsto \mathcal{P}^{\pm}_{\varepsilon,R}$. An element of $\mathcal{Z}_{\varepsilon,R}$ is called a *gluing*.

(iii) A *perfect gluing* based at an element q of $\mathcal{P}^{\pm}_{\varepsilon,R}$ is given by the class of a pair $(w, \varphi_1 \mathbf{J}_{\alpha} \mathbf{I}_0 w)$, where $w = (\alpha, \beta, \gamma, t, s)$ is a representative in $P^{\pm}_{\varepsilon,R}$ of the class q, and $\mathbf{J}_{\alpha}$ is the reflection of axis α defined in the beginning of §13.2.

With this definition at hand, we can now introduce the main object of this section.

Definition 16.2. (Equivariant straight surfaces) Let ε and R be positive numbers. An (ε, R)-*equivariant straight surface* is a pair $(\mathcal{R}, Z)$, where $\mathcal{R}$ is a bipartite trivalent ribbon graph whose set of vertices is $V^- \sqcup V^-$, such that the following conditions hold:

(i) Every edge e is labeled by an element $Z(e)$ of $\mathcal{Z}_{\varepsilon,R}$. For convenience, we define the corresponding label of flag

$$W(e^{\pm}, e) := \pi^{\pm} Z(e) \in \mathcal{P}^{\pm}_{\varepsilon,R}. \tag{16.1}$$

By an abuse of notation, we will talk about equivariant straight surfaces as triples $(\mathcal{R}, Z, W)$, even though W is redundant.

(ii) The labeling map from the link of a vertex v is equivariant with respect to the order-3 symmetries:

$$W(\omega_v(v, e)) = \omega(W(v, e)).$$

Given a discrete subgroup Γ, and given ε small enough and then R large enough, a straight surface for Γ gives rise to an equivariant straight surface whose underlying graph is finite.

Observe that, given a bipartite trivalent graph $\mathcal{R}$, there is just one $(0, R)$-equivariant straight surface, that we call the *perfect surface* for $\mathcal{R}$.

16.1.1. Monodromy of an equivariant straight surface

The *fundamental group* (as a graph of groups) $\pi_1(\Sigma)$ of $\Sigma=(\mathcal{R},Z,W)$ is described as follows.

First let $\mathcal{R}^u$ be the universal cover of the trivalent graph $\mathcal{R}$ and $\pi_1(\Sigma^u)$ be the group with the following presentation:

(i) one generator for every oriented edge, such that the element associated to the opposite edge is the inverse;

(ii) one relation for every vertex: the product of the three generators corresponding to the edges is 1, using the orientation at each vertex.

Observe that the fundamental group of $\mathcal{R}$ acts by automorphisms on $\pi_1(\Sigma^u)$. We now define

$$\pi_1(\Sigma):=\pi_1(\Sigma^u)\rtimes\pi_1(\mathcal{R}).$$

Then, we have the following result.

Proposition 16.3. *When the underlying graph is finite, the group $\pi_1(\Sigma)$ is isomorphic to the fundamental group of a surface whose Euler characteristic is the* (*opposite*) *of the number of vertices of $\mathcal{R}$.*

Let us denote by $[(v,e)]$ the flag in $\mathcal{R}$ which is the projection of the flag (v,e) in $\mathcal{R}^u$. The following follows at once from the fact that G acts freely on the space of almost closing pair of pants.

Proposition 16.4. *There exist a map Z^u from the set of oriented of edges of $\mathcal{R}^u$ into $Z^{\pm}_{\varepsilon,R}$, and a map W^u from the set of flags edges of $\mathcal{R}^u$ with values in $P^{\pm}_{\varepsilon,R}$, such that*

$$[W^u(v,e)]=W([v,e])\quad\textit{and}\quad \pi^{\pm}Z^u(e)=W^u(e^{\pm},e).$$

Moreover, (W^u,Z^u) is unique up to the action of G.

A pair $(\mathcal{R}^u,Z^u)$ is called a *lift* of $(\mathcal{R},Z)$. As a corollary of this construction, we obtain from W^u a representation ρ^u of $\pi_1(\Sigma^u)$ in G, where the image by ρ^u of the element represented by the flag (v,e) is α, when $W^u(v,e)=(\alpha,\beta,\gamma,T_0,T_1)$. Moreover, by uniqueness of W^u up to the action of G, we obtain also a representation ρ_0 of $\pi_1(\mathcal{R})$ into G such that, if $a\in\pi_1(\Sigma^u)$ and $b\in\pi_1(\mathcal{R})$, then

$$\rho^u(b\cdot a\cdot b^{-1})=\rho_0(b)\cdot\rho^u(a)\cdot\rho_0(b^{-1}).$$

Definition 16.5. (Monodromy–Cuff elements) The *monodromy* of $\Sigma=(\mathcal{R},Z,W)$ is the unique morphism ρ from $\pi_1(\Sigma)$ to G extending both ρ^u and ρ_0. The *cuff limit map* is the map ξ' which, for every cuff element a, associates to the attractive point a^+ of a the attracting fixed point $\xi'(a^+):=\rho(a)^+$ in $\mathbf{F}$ of the P-loxodromic element $\rho(a)$.

16.2. Doubling and deforming equivariant straight surfaces

Let $\mathcal{R}$ be a bipartite trivalent tree, v be a vertex in $\mathcal{R}$ and N be an integer. The N-*doubling graph at* v is the finite bipartite trivalent graph $\mathcal{R}^{(2)}$ obtained by the following procedure. We consider the ball B of radius N based at a lift of v in the universal cover of $\mathcal{R}$. Then, $\mathcal{R}^{(2)}$ is the graph obtained by taking two copies of B^0 and B^1, and adding two edges between the identified extreme points of B. Moreover, the cyclic order on the edges of B^1 is reversed from the cyclic order for B^1.

Let now $\Sigma=(\mathcal{R},Z)$ be an (ε,R)-equivariant straight surface over a tree, and v be a vertex of $\mathcal{R}$. The N-*doubling equivariant straight surface at* v is the surface

$$\Sigma^{(2)}=(\mathcal{R}^{(2)},Z^{(2)})$$

whose underlying graph is the N-doubling graph at v. Let us now describe the labeling $Z^{(2)}$. Let us denote by F the bijection between $B^0(v,N)$ and $B^1(v,N)$, and by π the projection from $B^0(v,N)$ to $\mathcal{R}$.

(i) If w is a vertex of $B^0(N,v)$, then we define

$$Z^{(2)}(w)=Z(\pi(w).$$

(ii) If w is a vertex of $B^1(N,v)$, then we define

$$Z^{(2)}(w)=\mathbf{I}_0Z(\pi(F(w))).$$

(iii) If e is an edge of $B^0(N,v)$, then we define

$$Z^{(2)}(e)=Z(\pi(e).$$

(iv) If e is an edge of $B^1(N,v)$, then we define

$$Z^{(2)}(e)=Z(\pi(e).$$

(v) If e is an edge joining w in B^0 to $F(w)$ in B^1, such that $e=\omega(f)$ where f is an edge joining w to a point in B^0, then we define $Z^{(2)}(e)$ to be the perfect gluing based at $\omega W(w,f)$,

(vi) Finally, if e is an edge joining w in B^0 to $F(w)$ in B^1, such that $e=\omega^2(f)$ where f is an edge joining w to a point in B^0, then we define $Z^{(2)}(e)$ to be the perfect gluing based at $\omega^2W(w,f)$.

We may deform equivariant straight surfaces. Let us say that a family of (ε,R)-equivariant straight surfaces $\Sigma_t=(\mathcal{R},Z_t,W_t)$, with $t\in[0,1]$, is *continuous* if W_t is continuous in t. The corresponding family of representations is then continuous as well. Our main result in this section is the following.

PROPOSITION 16.6. (Deforming the double) *There exist ε_0, R_0 and a constant* C *depending only on* G *such that, if $\varepsilon<\varepsilon_0$ and $R>R_0$, the following holds. Let $\Sigma=(\mathcal{R},Z)$ be an equivariant (ε,R)-straight surface, whose underlying graph is a tree. Let v be a vertex of $\mathcal{R}$ and N be a positive integer. Then, there exists a continuous family $\Sigma_t^{(2)}$ of $(C\varepsilon,R)$-equivariant straight surfaces, with $t\in[0,1]$, such that $\Sigma_1^{(2)}$ is the (v,N)-doubling of Σ and $\Sigma_0^{(2)}$ is the perfect surface for $\mathcal{R}^{(2)}$.*

Proof. Thanks to the doubling procedure, it is equivalent to show that we can have a deformation of any labeling of the ball of radius N in a trivalent tree.

In this proof, B_i will denote constants depending only on G.

We first prove that the fibers of the projection

$$\pi:=(\pi^+,\pi^-)\colon \mathcal{Z}_{\varepsilon,R}\longrightarrow \mathcal{P}^+_{\varepsilon,R}\times\mathcal{P}^-_{\varepsilon,R}$$

are connected, for ε small enough and R large enough. These fibers are described in the following way: given $p=(p^0,p^1)\in P^+\times P^-$, where $p^*=(\alpha_*,\beta_*,\gamma_*,\tau_0^*,\tau_1^*)$, then there exist a unique element $g\in\mathsf{Z}_{\mathsf{G}}(\alpha)$ such that

$$\mathbf{\Psi}(\tau_0^0,\alpha_0^+,\alpha_0^-)=g\mathbf{T}\mathbf{\Psi}(\tau_0^1,\alpha_1^+,\alpha_1^-).$$

This element g which is uniquely defined will be referred to as the *gluing parameter*.

Observe now that g is $(B_1\varepsilon/R)$-close to the identity with respect to the metric $d_{\tau_0^+}$. This follows from the definition of $Z_{\varepsilon,R}$ and assertion (3.4) in Proposition 3.16.

Step 1. Our first step is to deform the gluing parameters to the identity such that all gluing are perfect.

Let us use the tripod τ_0^+ as an identification of G with G_0. Let then $g_0=\tau_0^+(g)$ and $\alpha_0=\tau_0^+(\alpha)$. Then, g_0 is $(B_2\varepsilon/R)$-close to the identity with respect to d_0. Thus, for ε small enough, write $g=\exp(u)$ with $u\in\mathfrak{l}_0$ of norm less than $B_3\varepsilon/R$. Moreover, u is the unique such vector of norm less than B_4. We now prove that $u\in\mathfrak{z}_{\mathsf{G}}(\alpha_0)$. Observe that

$$\exp(\mathrm{ad}(\alpha_0)\cdot u)=\alpha_0 g_0\alpha_0^{-1}=g_0.$$

By assertion (9.5), we have $\alpha_0=\exp(2R\cdot a_0)h$, where h is $(\mathbf{M}\varepsilon/R)$-close to the identity for some constant $\mathbf{M}$. Thus, for ε small, the linear operator $\mathrm{ad}(\alpha_0)$ — acting on $\mathfrak{l}_0$ — is close to the identity and has norm less than 2. Thus,

$$\|\mathrm{ad}(\alpha_0)\cdot u\|\leqslant B_5\frac{\varepsilon}{R}.$$

It follows, by the uniqueness of u, that $\mathrm{ad}(\alpha_0)\cdot u=u$. Thus, $u\in\mathfrak{z}_{\mathsf{G}}(\alpha_0)$. Then, we can deform g to the identity through elements $(B_6\varepsilon/R)$-close to identity with respect to $d_{\tau_0^+}$.

Then, as a first step of our deformation, we deform each gluing parameter for every edge to the identity.

Observe now that an equivariant straight surface with trivial gluing parameters is the same thing as a labeling of each vertex with an element of $\mathcal{P}^{\pm}_{\varepsilon,R}$, with the constraints that boundary loops corresponding to opposite edges are conjugate. More formally, if (v,e) is a flag corresponding to the oriented edge e in the graph, and $\bar{e}$ is the edge with the opposite orientation, if $W(v,e)=(\alpha,\beta,\gamma,\tau_0,\tau_1)$ and we write $\alpha(e)=[\alpha]$, the constraint is that $\alpha(e)=\alpha(\bar{e})$. From now on, we keep the gluing parameters trivial.

Step 2. We now describe the second step of the deformation. Let $V=(T,S_0,S_1,S_2)$ be the pair of pants labeling v with boundary loops $(\alpha_0,\beta_0,\gamma_0)$. Using Theorem 11.11, we obtain a deformation of $(K\varepsilon,R)$ pair of pants $V_t=(T^t,S_0^t,S_1^t,S_2^t)$ with boundary loops $(\alpha_t,\beta_t,\gamma_t)$ to an R-perfect pair of pants; observe that the conjugacy classes $(\alpha_t,\beta_t,\gamma_t)$ of the boundaries may change. However, since V_t is a $(K\varepsilon,R)$-pair of pants we may write that $\alpha_t=g_t\alpha_0k_tg_t^{-1}$, where k_t belongs to the ball of radius $K\varepsilon/R$ in L_{α_0}.

Step 3. For the third step, let w be a vertex at distance 1 from v whose common boundary with v is, say, α. We use Theorem 11.8 to deform the pair of pants associated to w, following the deformation of the (unique) common boundary between v and w, while keeping the other boundary loops of w in the same conjugacy class. As a result, the "inner" boundary loop of w is now R-perfect.

Step 4. As a fourth step, we deform the pair of pants labeling w to a perfect pair of pants keeping the inner boundary loop of w R-perfect.

Then, we repeat inductively this procedure, namely third and Steps 3 and 4.

We have thus deformed our doubled equivariant straight surfaces to an equivariant straight surface with perfect pair of pants for each vertex and perfect gluing parameters, or in other words a perfect surface. □

16.3. Main result

Our main result is the following theorem that shows the existence of Sullivan curves.

THEOREM 16.7. (Sullivan and straight surfaces) *We assume that $\mathfrak{s}$ has a compact centralizer.*

For any positive ζ, there exist positive numbers ε_0 such that, for $\varepsilon<\varepsilon_0$, there exists R_0 such that if $R>R_0$ and Σ is an (ε,R)-equivariant straight surface with monodromy ρ and cuff limit map ξ', then there exists a unique ρ-equivariant ζ-Sullivan map ξ from $\partial_\infty\pi_1(S_R)$ to $\mathbf{F}$ extending ξ'.

In this section, we will always precise when we use the hypothesis that $\mathfrak{s}$ has a compact centralizer.

16.4. Hexagons and tripods

We need to connect our notion of equivariant straight surfaces to the picture of tiling by hexagons.

16.4.1. Labeling hexagons by tripods

Let us consider Σ_0 the perfect surface for $\mathcal{R}$, that is the unique $(0,R)$-straight surface of the form $(\mathcal{R},Z_0)$. Gluing perfect R-pairs of pants associated to the vertices of $\mathcal{R}$ along sides corresponding to edges of $\mathcal{R}$ by a 1-swish, we obtain a covering S of the perfect surface S_R. We now consider ρ as a representation of the cuff group Λ, which is such that $\Lambda\backslash\mathbf{H}^2{=}S$.

We recall (see §15.1.3) that, conversely, $\mathcal{R}$ is obtained as the adjacency graph of the tiling of S by (let us say) white hexagons.

Taking the universal cover of this perfect surface, one obtains a map ϖ from the set of tiling hexagons to the flags of $\mathcal{R}$, such that $\varpi(\mathrm{Suc}(H)){=}\varpi(H)$, if $\varpi(\mathrm{Opp}(H))$ is the opposite flag to $\varpi(H)$.

Proposition 16.8. (Straight surfaces and equivariant labeling) *Let $\Sigma{=}(\mathcal{R},Z,W)$ be an equivariant (ε,R)-straight surface, with monodromy ρ and cuff limit map ξ'. Then, there exists a labeling τ of tiling hexagons by tripods such that the following conditions hold*:

(i) $\tau(a,b,c){=}\omega(\tau(b,c,a))$;

(ii) *if* $H{=}(a,b,c)$, *then*

$$P(H):=(\tau(H),\tau(\mathrm{Suc}(H)),\rho(a),\rho(b),\rho(c))$$

is an (ε,R)-almost closing pair of pants;

(iii) *for a white hexagon*, $W(\varpi(H)){=}[P(H)]$;

(iv) *for all* $\gamma{\in}\Lambda$, $\tau(\gamma(H)){=}\rho(\gamma){\cdot}\tau(H)$.

We will refer to τ and P as *equivariant labelings* associated to the straight surface Σ.

Proof. From the definition of $\Sigma{=}(\mathcal{R},Z,W)$, we have a map from the set of white hexagons to $\mathcal{P}^{+}_{\varepsilon,R}$ given by $H{\mapsto}W(\varpi(H))$.

We are now going to lift $W\circ\varpi$ to a map P with values in $P^{\pm}_{\varepsilon,R}$: Let us choose a white hexagon H_0 and fix a lift

$$P(H_0)=(\alpha_0,\beta_0,\gamma_0,\tau(H_0),\tau_1(H_0))$$

of $W\circ\varpi$. For any white hexagon H, let us lift $W\circ\varpi(H)$ to

$$P(H)=(\alpha_H,\beta_H,\Gamma_H,\tau(H),\tau_1(H))$$

by using the following rules:

(i) if $H'=\mathrm{Opp}(H)$, then $P(H')$ is uniquely defined from $P(H)$ by requiring that $(P(H),P(H'))$ is a lift of $Z(e)$ in $Z_{\varepsilon,R}$, where e is the edge in $\mathcal{R}$ associated to the pair (H,H');

(ii) if $H'=\mathrm{Suc}^2(H)$, then $P(H')=\alpha_H P(H)$.

We leave to the reader to check that these rules are coherent. We finally choose a labeling of the black hexagons using the following rule: if $H'=\mathrm{Suc}(H)$ and H is labeled by

$$P(H)=(\alpha_H,\beta_H,\Gamma_H,\tau(H),\tau_1(H)),$$

then the labeling of H' is given by

$$P(H')=(\alpha_H,\beta_H^{-1}\gamma_H\beta_H,\beta_H,\tau_1(H),\alpha_H\tau(H)).$$

Our labeling by tripods is finally given by the maps $\tau: H\mapsto\tau(H)$, where

$$P(H)=(\alpha_H,\beta_h,\gamma_H,\tau(H),\tau_1(H)).\qquad\square$$

16.5. A first step: extending to accessible points

Our first step will not use the assumption that $\mathfrak{s}$ has a compact centralizer and will be used to show a weaker version of the surface subgroup theorem in that context.

Let us denote by W_H^R the set of accessible points from a tiling hexagon H and let us define the set of accessible points as

$$W^R:=\bigcup_H W_H^R,$$

the union set of all accessible points with respect to any hexagon. Observe that W^R is $\pi_1(S)$-invariant, and thus dense.

Our main result in this subsection is the next lemma that unlike Theorem 16.7 will not use the compact stabilizer hypothesis.

LEMMA 16.9. (Extension) *For any positive* ζ, *there exists a positive number* ε_0 *such that, for* $\varepsilon<\varepsilon_0$, *there exists* R_ε *such that for* $R>R_\varepsilon$, *the following holds.*

Let Σ *be an* (ε,R)*-equivariant straight surface with monodromy* ρ *and cuff limit map* ξ'. *Then, there exists a unique* ρ*-equivariant map* ξ *from the set of accessible points* W^R *to* $\mathbf{F}$ *such that, if* $\{c_m\}_{m\in\mathbb{N}}$ *is a nested sequence of cuffs converging to an accessible point* $y\in W^R_H$, *then*

$$\lim_{m\to\infty}(\xi'(c_m^\pm))=\xi(y).$$

Moreover, if η *is the circle map associated to* $\tau_0=\tau(H)$, *then, for any* τ *coplanar to* $\tau(H)$ *such that* $\tau^\pm=\tau_0^\pm$, *we have*

$$d_\tau(\xi(y),\eta(y))\leqslant\zeta.$$

We furthermore show that the dependence of ξ on the straight surface is continuous.

COROLLARY 16.10. *Let* $\{\Sigma_t\}_{t\in\mathbb{R}}$ *be a continuous family of* (ε,R)*-equivariant straight surfaces, and* $\{\xi_t\}_{t\in\mathbb{R}}$ *be the family of maps produced as above. Then, for every accessible point* z, *the map* $\xi_t(z)$ *is continuous as a function of* t.

We first construct a sequence of quasi-tripods associated to an accessible point and an equivariant labeling, then show that this sequence of quasi-tripods converges and complete the proof of the extension lemma (Lemma 16.9).

16.5.1. A sequence of quasi-tripods for an accessible point

Let Σ be an equivariant straight surface with monodromy ρ and cuff limit map ξ', let τ be an equivariant labeling obtained by Proposition 16.8.

Given K, choose R_0 such that Proposition 15.11 holds and $R>R_0$. Let y be an accessible point which is the limit of a sequence of cuffs $\{c_m\}_{m\in\mathbb{N}}$.

As a first step, we associate to $\{c_m\}_{m\in\mathbb{N}}$ a sequence of coplanar tripods $\{T_m\}_{m\in\mathbb{N}}$ *associated* to a K-good sequence of cuffs $\{c_m\}_{m\in\mathbb{N}}$: first we orient each cuff such that c_{m+1} is on the right of c_m, then we associate to every K-acceptable pair (c_m,c_{m+1}) the pair of tripods (T_{2m-1},T_{2m}) defined by

$$T_{2m-1}=(c_m^-,c_m^+,c_{m+1}^-,)\quad\text{and}\quad T_{2m}=(c_{m+1}^-,c_m^+,c_{m+1}^+,).$$

Let then A_m be the swish between T_m and T_{m+1}.

Our second step is to associate to our data a sequence of quasi-tripods. Recall that c_m and c_{m+1} are the common edges of exactly two hexagons

$$H_{2m-1}=(c_{m+1},\bar{c}_m,b_m)\quad\text{and}\quad H_{2m}=(c_{m+1},d_m,\bar{c}_m)=\mathrm{Suc}(H_{2m-1}),$$

where we denote by $\bar{c}$ the cuff c with the opposite orientation.

Let us consider the sequence $\{\theta_m\}_{m\in\mathbb{N}}$ of quadruples given by $\theta_m=(\tau(H_m),\xi'(T_m))$. Then, it follows by the second item of Theorem 9.5 that, for R_ε depending only on ε and $R>R_\varepsilon$, θ_m is an $\mathbf{M}_0(\varepsilon/R)$-quasi-tripod, for some constant $\mathbf{M}_0$ depending only on G.

We can now prove the following result.

Proposition 16.11. *There exists a positive constant $\mathbf{M}_1$ depending only on G such that the sequence $\{\theta_m\}_{m\in\mathbb{N}}$ is an $(\{A_m\}_{m\in\mathbb{N}},\mathbf{M}_1\varepsilon/R)$-swished sequence of quasi-tripods whose model is $\{T_m\}_{m\in\mathbb{N}}$.*

Proof. Let us first consider the pair $(\theta_{2m-1},\theta_{2m})$. From the definition, the (ε/R)-quasi-tripod

$$\beta_{2m-1}:=(\tau_{2m-1},\xi'(c_{m+1}^-),\xi'(c_m^+),\xi'(b_m^-))$$

is $(R,\varepsilon/R)$-swished from

$$\omega(\beta_{2m}):=(\omega(\tau_{2m}),\xi'(c_m^+),\xi'(c_{m+1}^-),\xi'(d_m^-))$$

for m odd, and $(-R,\varepsilon/R)$-swished for m even. Since, by construction,

$$\beta_{2m}^{\pm}=\theta_{2m}^{\pm},\quad \omega(\beta_{2m-1})^{\pm}=\omega(\theta_{2m-1})^{\pm}\quad\text{and}\quad \dot{\beta}_m=\dot{\theta}_m.$$

it follows that θ_{2m} is $(R,\varepsilon/R)$-swished from $\omega(\theta_{2m-1})$ for m odd, and $(-R,\varepsilon/R)$-swished for m even. Then, since

- $\omega(T_{2m})$ is $(2\varepsilon/R)$-close to $t_{2m}=(c_m^+,c_{m+1}^-,d_m^-)$ by Proposition 15.6, and similarly
- T_{2m-1} is $(2\varepsilon/R)$ close to $t_{2m-1}=(c_{m+1}^-,c_m^+,b_m^-)$,

it follows that A_m is $(2\varepsilon/R)$-close to R, for m odd, and to $-R$ for m-even. Thus, θ_{2m} is $(A_m,\mathbf{M}_2\varepsilon/R)$-swished from $\omega(\theta_{2m-1})$ for some constant $\mathbf{M}_2$.

Let us consider now the pair $(\theta_{2m-1},\theta_{2m})$. Since (c_{m-1},c_m,c_{m+1}) is a K-acceptable triple, it follows by item (3) of Proposition 15.11 that

$$H_{2m+1}=\eta_m\,\mathrm{Opp}(H_{2m-1}),$$

where $\eta_m=\gamma_m^p$, γ_m is the cuff element associated to c_m and $p\in\{-1,0,1\}$.

By the definition of a labeling, $\eta_m^{-1}(\theta_{2m})$ is $(1,\varepsilon/R)$-swished from θ_{2m-1}. By construction (see Proposition 16.8) $P(H_{2m-1})$ is an $(\varepsilon/R,R)$-almost closing pair of pants, and thus, by the last item of Theorem 9.5,

$$d(\eta(\theta_{2m}),\varphi_{2R}(\theta_{2m}))\leqslant\mathbf{M}_3\frac{\varepsilon}{R}$$

for some constant $\mathbf{M}_3$ depending only on G.

It follows that θ_{2m} is $(1+pR,\mathbf{M}_4\varepsilon/R)$-swished from θ_{2m-1} for a constant $\mathbf{M}_4$ depending only on G. Since $A_m=1+pR$, the quasi-tripod θ_{2m} is $(A_m,\mathbf{M}_4\varepsilon/R)$-swished from θ_{2m-1} for a constant $\mathbf{M}_4$ depending only on G.

This concludes the proof of the proposition. $\square$

16.5.2. Proof of Lemma 16.9 and its corollary

We first prove the following result which is the key argument in the proof.

PROPOSITION 16.12. (Extension) *For any positive ζ and K, there exist positive numbers ε_0, $q<1$, β and L such that, for $\varepsilon<\varepsilon_0$, there exists R_ε such that, for $R>R_\varepsilon$, we have the following. Suppose that we are given*

- *an (ε, R)-straight surface Σ,*
- *a nested sequence $\{c_m\}_{m\in\mathbb{N}}$ of cuffs,*

and assume furthermore that $\{c_m\}_{m\in\mathbb{N}}$ converges to an accessible point y with respect to a tiling hexagon H for Σ. Then, $\{\xi(c_m^\pm)\}_{\in\mathbb{N}}$ converges to a point Y such that, for any τ coplanar to $\tau(H)$ such that $\tau^\pm=\tau_0^\pm$ and $m>L$,

$$d_\tau(Y, \xi(c_m^\pm)) \leqslant q^m \beta. \tag{16.2}$$

Moreover, if η is the circle map associated to $\tau_0=\tau(H)$, then, for any τ coplanar to $\tau(H)$ such that $\tau^\pm=\tau_0^\pm$,

$$d_\tau(Y, \eta(y)) \leqslant \zeta. \tag{16.3}$$

Proof. Let ζ be a positive constant. The sequence of tripods $\{T_m\}_{m\in\mathbb{N}}$ is a $2KR$-sequence of tripods, by Corollary 15.3. From Proposition 16.11, it follows that $\{\theta_m\}_{m\in\mathbb{N}}$ is a $(KR, \varepsilon/R)$-deformed sequence of quasi-tripods. In particular, using Theorem 7.2 with $\beta=\zeta$, both $\{\xi(c_m^+)\}_{m\in\mathbb{N}}$ and $\{\xi(c_m^-)\}_{m\in\mathbb{N}}$ converge to a point $y(\theta)=:Y$ in $\mathbf{F}$.

Then, inequality (16.2) is a consequence of (7.1).

Since $y(\tau)=\eta(y)$, inequality (16.3) also follows from Theorem 7.2. □

The proof of Lemma 16.9 now follows immediately. The proof of Corollary 16.10 follows from the fact that, due to inequality (16.2), the convergence of $\{\xi(\theta_m^j)\}_{m\in\mathbb{N}}$ is uniform.

16.6. Proof of Theorem 16.7

We now make use of the compact stabilizer hypothesis using in particular the improvement theorem (Theorem 8.14).

It is enough (by eventually passing to the universal covering) to prove the theorem when the underlying graph of Σ is a tree and that is what we do now.

Let us start with an observation. Let τ be any tripod in $\mathbf{H}^2$. Since the diameter of the hyperbolic surfaces S_R is bounded independently of R (Lemma 15.2). It follows that there exists some constant C_0 such that, given any tripod τ, we can find a tiling hexagon H such that

$$d(\tau,\tau_H)\leqslant C_0, \tag{16.4}$$

where τ_H is an admissible tripod in $\mathbf{H}^2$ for H. It follows that there exists a universal constant C_1 such that, for any extended circle map ν,

$$d_{\nu(\tau)}\leqslant C_1\cdot d_{\nu(\tau_H)}. \tag{16.5}$$

Given a positive number ζ_1, let us fix ε and $R_0{=}R_\varepsilon$ such that Lemma 16.9 holds for $\zeta{=}\zeta_1/C_1$. Let then $R{>}R_0$.

Let $\Sigma{=}(\mathcal{R},Z)$ be an (ε,R)-equivariant straight surface with monodromy ρ and cuff limit map ξ' such that $\mathcal{R}$ is a tree.

Then, according to Proposition 16.6, for any vertex v in $\mathcal{R}$ and integer N, we can find a continuous family $\{\Sigma^{(2)}_t\}_{t\in[0,1]}$ of (ε,R)-equivariant labeling under $\{\rho^{(2)}_t\}_{t\in[0,1]}$ deforming the (v,N) double $\Sigma^{(2)}$ of Σ.

It follows by the extension lemma (Lemma 16.9) and Corollary 16.10 that we can find a continuous family $\{\xi^{(2)}_t\}_{t\in[0,1]}$ defined on the dense set of accessible points W^R such that the following conditions hold:

(i) $\xi^{(2)}_t$ is equivariant under $\rho^{(2)}_t$;

(ii) $\xi^{(2)}_0$ is a circle map;

(iii) for any tiling hexagon H and all y in W^R_H,

$$d_{\tau_t}(\nu^H_t(y),\xi^{(2)}_t(y))\leqslant\zeta=\frac{\zeta_1}{C_1}, \tag{16.6}$$

where ν^H_t is the circle map such that

$$\nu^H_t(\tau_H)=\tau_t:=\tau_t(H).$$

We now make the following remarks.

(i) By Lemma 16.9, $\xi^{(2)}_t$ is attractively continuous: for all $y{\in}W$ which is the limit of elements c^+_m, $\xi^{(2)}_t(y)$ is the limit of $\xi^{(2)}_t(c^+_m)$ as m goes to infinity, where $\xi^{(2)}_t(c^+_m)$ is the attractive element of the cuff element $\rho^{(2)}_t(c_m)$. Applying this for $t{=}1$, we get that $\xi^{(2)}$ extends the cuff maps $(\xi')^{(2)}$.

(ii) By the accessibility lemma (Lemma 15.9), and the remark following, there exists some positive $a(R)$, where $a(R)$ goes to zero when R goes to infinity, such that W^R_H is $a(R)$-dense with respect d_{τ_t}.

We thus now choose R_0 such that, for all R greater than R_0, one has $a(R)<a_0$, where a_0 is given from ζ_1 by Theorem 8.14.

Using the initial observation (inequality (16.5)), we now have that, for any tripod τ and any $t\in[0,1]$, we can find a circle map $\nu_t=\nu_t^H$ such that, for any y in some a_0-dense set, one has

$$d_{\nu_t(\tau)}(\nu_t(y),\xi_t^{(2)}(y))\leqslant C_1\zeta=\zeta_1,$$

where we have used both inequalities (9.7) and (16.2). In other words, we have that $\xi_t^{(2)}$ is (a_0,ζ_1)-Sullivan.

We are now in a position to apply the improvement theorem (Theorem 8.14). This shows that $\xi_t^{(2)}$, and in particular $\xi^{(2)}$, is $2\zeta_1$-Sullivan. By construction, we have that $\xi^{(2)}$ extends $(\xi')^{(2)}$.

Remember now that the doubling construction depends on the choice of a parameter N, and we now write $\xi_N^{(2)}$ and $(\xi')_N^{(2)}$ to mark the dependency in N.

Let us consider the universal cover of both the original surface Σ and the double $\Sigma^{(2)}$. By construction, the labeling are identical on the large ball $B(v,N)$. This large ball corresponds to a free subgroup of $\pi_1(\Sigma_R)$ with limit set $\Lambda(N)$. Thus, for any cuff element c_m with endpoints in $\Lambda(N)$,

$$(\xi')_N^{(2)}(c_m^+)=\xi'(c_m^+). \tag{16.7}$$

It follows that, if $M>N$,

$$\xi_M^{(2)}|_{\Lambda(N)}=\xi_N^{(2)}|_{\Lambda(N)}. \tag{16.8}$$

Recall now that all limit maps $\xi_M^{(2)}$ being ζ_1-Sullivan admits a modulus of continuity by Theorem 8.11. We may thus extract form the sequence $\{\xi_M^{(2)}\}_{M\in\mathbb{N}}$ a uniformly converging subsequence to a ζ_1-Sullivan map ξ.

Let finally $\Lambda:=\bigcup_N\Lambda(N)$ and observe that Λ is dense and contains the endpoints of all cuff elements. Then, the ζ_1-Sullivan map ξ coincide with ξ' on Λ, by equations (16.7) and (16.8)

This completes the proof of Theorem 16.7 (reverting to $\zeta=\zeta_1$).

17. Wrap up: proof of the main results

This section is just the wrap up of the proof of the main theorems obtained by combing the various theorems obtained in this paper.

THEOREM 17.1. *Let* G *be a semisimple Lie group of Lie algebra* $\mathfrak{g}$ *without compact factors. Let* $\mathfrak{s}=(a,x,y)$ *be an* $\mathfrak{sl}_2$*-triple in* $\mathfrak{g}$*. Assume that* $\mathfrak{s}$ *satisfies the flip assumption and that* $\mathfrak{s}$ *has a compact centralizer.*

Let Γ be a uniform lattice in G. Let ε be a positive real number. Then, there exist a closed hyperbolic surface S_ε and a faithful (G,P)-Anosov representation ρ_ε of $\pi_1(S_\varepsilon)$ in Γ, whose limit curve is ε-Sullivan with respect to $\mathfrak{s}$, where P is the parabolic associated to a.

As a corollary, considering the case of the principal $\mathsf{SL}_2(\mathbb{R})$ in a complex semisimple Lie group, we obtain the following result.

THEOREM 17.2. *Let G be a complex semisimple group and let Γ be a uniform lattice in G. Then, there exists a closed Anosov surface subgroup in Γ.*

Proof of Theorem 17.1. From Theorem 14.2, for any positive ε there exists R_0 such that, for any $R>R_0$, there exists an (ε,R)- straight surface Σ in Γ associated to $\mathfrak{s}$. This straight surface is equivariant under a representation ρ of a surface group Γ_0 in Γ.

By Theorem 16.7, for any ζ, for ε small enough, there exists R_0 such that, for $R>R_0$, an (ε,R)-straight surface equivariant under a representation ρ of a surface group Γ_0 in Γ is such that we can find a ζ-Sullivan ρ-equivariant Sullivan map from $\partial_\infty\Gamma_0$ to $\mathbf{F}$. By Theorem 8.3, for ζ-small enough, the corresponding representation is Anosov, and in particular faithful. ☐

17.1. The case of the non-compact stabilizer

In that context, we obtain a less satisfying result. Recall that we denote by c^+ the attractive point in $\partial_\infty\pi_1(S)$ of a non-trivial element c of $\pi_1(S)$.

Let $\mathsf{G}_1,\ldots,\mathsf{G}_n$ be semisimple Lie groups without compact factors. Let $\mathsf{G}:=\prod_{i=1}^n\mathsf{G}_i$, with Lie algebra $\mathfrak{g}$. Let Γ be a uniform lattice in G such that (up to finite cover) its projection on G_i is an irreducible lattice. Let (a,x,y) be an $\mathsf{SL}_2(\mathbb{R})$-triple in $\mathfrak{g}$ such that the following conditions hold:

- $\mathfrak{s}$ satisfies the flip assumption;
- the projections on all factors $\mathfrak{g}_i$ are non-trivial.

Let P be the parabolic associated to a.

THEOREM 17.3. *Let ε be a positive real number. Then, there exists some R and the following objects.*

- *A faithful representation $\rho\colon\Gamma_R\to\Gamma$, where $\Gamma_R=\pi_1(\widehat{S}_R)$ is the fundamental group of a finite covering $\widehat{S}_R$ of the surface S_R defined in §15.1, such that the image of every cuff element of Γ_R has an attractive fixed point in $\mathbf{F}$.*
- *A ρ-equivariant ξ from $\partial_\infty\Gamma_R$ to $\mathbf{F}$ such that the following conditions hold:*
  - *for a cuff element c, $\xi(c^+)$ is the attractive fixed point of $\rho(c)$;*

– *if* $\{c_m\}_{m\in\mathbb{N}}$ *is a sequence of nested cuff elements such that* $\{c_m^+\}_{m\in\mathbb{N}}$ *converges to* y, *then* $\{\xi(c_m^+)\}_{m\in\mathbb{N}}$ *converges to* $\xi(y)$.

Proof. The proof runs as before, except that we replace the use of the Theorem 16.7 by Lemma 16.9, from which we obtain the existence and properties of the application ξ which is equivariant under a representation ρ. Let us now show that ρ is injective. We already know that the image of any cuff element in Γ_R is non-trivial. Let γ be such that $\rho(\gamma)$ is the identity and assume by contradiction that γ is not the identity. Let γ^+ and γ^- be respectively the attracting and repelling points of γ in $\partial_\infty\Gamma_R$. Let c_m^+ be the endpoint of a cuff. Then, $\xi(\gamma^n c_m^+)=\xi(c_m^+)$. It follows, by taking the limit when n goes to infinity, that $\xi(c_m^+)=\xi(\gamma^+)$, since c_m^+ is different from γ^-. Thus, ξ would be constant, but this is a contradiction: $\xi(c_m^+)$ is different from $\xi(c_m^-)$. □

Appendix A. Lévy–Prokhorov distance

Let μ and ν be two finite measures of the same mass on a metric space X with metric d. For any subset A in X, let A_ε be its ε-neighborhood. Then, we define

$$d_L(\mu,\nu)=\inf\{\varepsilon>0:\nu(A_\varepsilon)\geqslant\mu(A)\text{ for all }A\subset X\}.$$

This function d_L is actually a distance (see [14, §3.3]) related to both the *Lévy–Prokhorov distance* and the *Wasserstein-∞ distance*. By a slight abuse of language, we will still call this distance the *Lévy–Prokhorov distance*.

We want to prove the following result which is an extension of a result proved in [14] for connected 2-dimensional tori. The proof uses different ideas.

THEOREM A.1. *Let X be a manifold. Assume that a connected compact torus T — with Haar measure ν — of dimension n acts freely on X preserving a bi-invariant Riemannian metric d and measure μ. Let ϕ be a positive function on X. Let*

$$\bar\phi:=\int_{\mathsf{T}}(\phi\circ g)\,d\nu(g)$$

be its T-average. Assume that $e^{-\kappa}\bar\phi\leqslant\phi\leqslant e^{\kappa}\bar\phi$. Then,

$$d_L(\phi\cdot\mu,\bar\phi\cdot\mu)\leqslant 2e^n\kappa\sup_{x\in X}\operatorname{diam}(\mathsf{T}\,x).$$

A.1. Elementary properties

The following properties of the Lévy–Prokhorov distance will be used in the proof.

PROPOSITION A.2. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ be two families of measures such that*

$$\mu=\sum_{n=1}^{\infty}\mu_n \quad \textit{and} \quad \nu=\sum_{n=1}^{\infty}\nu_n$$

are finite measures. Assume that $d_L(\mu_i,\nu_i)<\varepsilon$ for all i. Then, $d_L(\mu,\nu)<\varepsilon$.

Proof. Assume that $\varepsilon>d_L(\mu_i,v_i)$. Then, $\nu_i(A_\varepsilon)\geqslant\mu_i(A)$ for all i and all $A\subset X$. Thus, $\nu(A_\varepsilon)\geqslant\mu(A)$. It follows that $\nu\geqslant d(\mu,\nu)$. □

PROPOSITION A.3. *Let μ be a finite measure on a compact metric space X. Then, for all positive ε, there exists an atomic measure μ_ε with finite support such that*

$$d_L(\mu,\mu_\varepsilon)\leqslant\varepsilon.$$

If μ is invariant by a finite group of isometries H, then we may choose μ_ε which is invariant by H.

Proof. One can find a finite partition of X by (measurable) sets $U^1,...,U^n$ together with a finite set of points $x_1,...,x_n$ such that $x_i\in U^i\subset B(x_i,\varepsilon)$.

We then choose the atomic measure

$$\mu_\varepsilon:=\sum_{i=1}^{n}\mu(U^i)\delta_{x_i}$$

such that $\mu_\varepsilon(U^i)=\mu(U^i)$. Let $A\subset X$ and $A^i=A\cap U^i$. Let I be the set of indices i such that A^i is non-empty. Then, for $i\in I$,

$$U^i\subset B(x_i,\varepsilon)\subset A^i_{2\varepsilon}.$$

Thus,

$$A\subset\bigsqcup_{i\in I}U^i=\bigsqcup_{i\in I}(U^i\cap A^i_{2\varepsilon})\subset A_{2\varepsilon}.$$

It follows that, for all (measurable) subset A,

$$\mu(A)\leqslant\mu\Big(\bigsqcup_{i\in I}U^i\Big)=\mu_\varepsilon\Big(\bigsqcup_{i\in I}U^i\Big)=\mu_\varepsilon\Big(\bigsqcup_{i\in I}(U^i\cap A^i_{2\varepsilon})\Big)\leqslant\mu_\varepsilon(A_{2\varepsilon}).$$

In particular, $d(\mu,\mu_\varepsilon)\leqslant 2\varepsilon$.

To obtain the invariance by the finite group H, one just averages by H, using Proposition A.2. □

PROPOSITION A.4. *Let f and g be two maps from a measured space (Y,ν) to a metric space X. Assume that, for all y in Y, one has $d(f(y),g(y))\leqslant\kappa$. Then,*

$$d_L(f_*\nu, g_*\nu)\leqslant\kappa.$$

Proof. Observe that, by hypothesis, $f(B)\subset(g(B))_\kappa$ for any subset B of Y. Let A be a subset of X, and let $C=f^{-1}(A)$ and $D=g^{-1}(A)$. Then, $f(D)\subset A_\kappa$. It follows that

$$f_*\nu(A_\kappa)\geqslant f_*\nu(f(D))=\nu(f^{-1}(f(D))\geqslant\nu(D)=g_*\nu(A).$$

The assertion follows. □

PROPOSITION A.5. *Let π be a K-Lipschitz map from X to Y, and let μ and ν be measures on X. Then,*

$$d_L(\pi_*(\mu),\pi_*(\nu))\leqslant K.d(\mu,\nu).$$

We will actually apply this proposition when $\pi: X\to Y$ is a finite covering.

Proof. By renormalizing the distance, we may assume that the map π is contracting. Let $\varepsilon\geqslant d(\mu,\nu)$ and let $B\subset Y$. Observe that $\pi^{-1}(B)_\varepsilon\subset\pi^{-1}(B_\varepsilon)$. Then,

$$\pi_*\mu(B_\varepsilon)=\mu(\pi^{-1}(B_\varepsilon))\geqslant\mu(\pi^{-1}(B)_\varepsilon)\geqslant\nu(\pi^{-1}(B))=\pi_*\nu(B).$$

Then, by definition, $\varepsilon\geqslant d(\pi_*(\mu),\pi_*(\nu))$ and the result follows. □

A.2. Some lemmas

We need the following lemmas.

LEMMA A.6. *Let X be a metric space equipped some metric d. Let $\pi: X\to X_0$ be a fibration. Let d_x be the restriction of d to the fiber $\pi^{-1}(x)$. Let ν^0 and ν^1 be two measures on X such that $\pi_*\nu^0=\pi_*\nu^1=:\lambda$, where λ is a measure on X_0. For every $x\in X_0$, let ν_x^0 (resp. ν_x^1) be the disintegrated measure on $\pi^{-1}(x)$ coming from μ (resp. ν). Then,*

$$d_L(\nu^0,\nu^1)\leqslant\sup_{x\in X_0} d_x(\nu_x^0,\nu_x^1). \tag{A.1}$$

Proof. Let A be a subset of X and let $A^x:=A\cap\pi^{-1}(x)$. Let $(A^x)_\kappa$ be the κ-neighborhood of A^x in $\pi^{-1}(x)$. By construction, $(A^x)_\kappa\subset(A_\kappa)^x$. Thus, for any set A, if $\kappa\geqslant d_x(\nu_x^0,\nu_x^1)$ for all x, then

$$\nu^0(A_\kappa)=\int_{X_0}\nu_x^0((A_\kappa)^x)\,d\lambda(x)\geqslant\int_{X_0}\nu_x^0((A^x)_\kappa)\,d\lambda(x)\geqslant\int_{X_0}\nu_x^1(A^x)\,d\lambda(x)\geqslant\nu^1(A).$$

Thus, $\kappa\geqslant d(\nu^0,\nu^1)$. Inequality (A.1) follows. □

LEMMA A.7. *Let* T^1 *be the connected compact torus of dimension* 1 *equipped with a bi-invariant metric* d *and Haar measure* μ. *Let* ϕ *be a positive function on* T^1. *Let*

$$\bar{\phi} := \frac{1}{\mu(\mathsf{T}^1)} \int_{\mathsf{T}^1} \phi(g)\, d\mu(g)$$

be its T^1*-average, that we see as a constant function. Assume that* $\bar{\phi} \leqslant e^{\kappa}\phi$. *Then,*

$$d(\phi\mu, \bar{\phi}\mu) \leqslant \kappa \operatorname{diam}(\mathsf{T}^1),$$

where $\phi\mu$ *is the measure with density* ϕ *with respect to* μ.

Proof. We may as well assume after multiplying the distance by a constant that $\operatorname{diam}(\mathsf{T}^1) = \frac{1}{2}$. We may also reduce to the case where μ is the probability measure on T^1. Let A be a non-empty subset of T^1. Assume first that $A_{\kappa/2}$ is a strict subset of T^1. It follows that $\mathsf{T}^1 \setminus A$ contains an interval of diameter greater than κ, and thus

$$\mu(A_\kappa) \geqslant \mu(A) + \kappa \quad \text{and} \quad \mu(A) \leqslant 1 - \kappa.$$

Then,

$$\begin{aligned}(\phi\mu)(A_\kappa) &\geqslant e^{-\kappa} \int_{A_\kappa} \bar{\phi}\, d\mu \geqslant e^{-\kappa}(\mu(A)+\kappa)\bar{\phi} \\ &\geqslant e^{-\kappa}\bar{\phi}\mu(A)\left(1 + \frac{\kappa}{1-\kappa}\right) = \left(\frac{e^{-\kappa}}{1-\kappa}\right)\bar{\phi}\mu(A) \geqslant \bar{\phi}\mu(A).\end{aligned}$$

Finally if $A_\kappa = \mathsf{T}^1$,

$$(\phi\mu)(A_\kappa) = \int_{\mathsf{T}^1} \phi\, d\mu = \bar{\phi} \geqslant \bar{\phi}\mu(A).$$

This concludes the proof of the statement. □

These two lemmas have the following immediate consequence.

COROLLARY A.8. *Let* $X := \mathsf{T}^1 \times X_0$. *Let* d (*resp.* μ) *be a* ℓ_1 *product metric* (*resp. a measure*) *on* X *invariant by* T^1. *Let* ϕ *be a function on* X. *Let* $\bar{\phi} := \int_{\mathsf{T}^1} \phi \circ g\, d\mu(g)$ *be its* T^1*-average. Assume that* $\bar{\phi} \leqslant e^{\kappa}\phi$. *Then,*

$$d(\phi\mu, \bar{\phi}\mu) \leqslant \kappa \operatorname{diam}(\mathsf{T}^1).$$

Proof. Observe that, by hypothesis, for every $x \in X_0$, μ_x is the Haar measure on T^1. Let $\nu^0 = \phi\mu$ and $\nu^1 = \bar{\phi}\mu$. By Lemma A.7, we know that, for every x in X_0,

$$d(\nu_x^0, \nu_x^1) \leqslant \kappa \operatorname{diam}(T^1).$$

Let π the projection on the second factor, and let x a point in X_0. Let

$$F(x) := \int_{T_1} \phi(g,x)\, d\mu_x(g) = \int_{T_1} \bar{\phi}(g,x)\mu_x.$$

Then, we have

$$\pi_*\nu^0 = \pi_*\nu^1 = F\cdot\pi_*\mu.$$

Thus, we may apply Lemma A.6 to get

$$d(\phi\mu, \bar{\phi}\mu) = d(\nu^0, \nu^1) \leqslant \kappa\, \mathrm{diam}(\mathsf{T}^1). \qquad \square$$

Proof of Theorem A.1. Observe that the hypothesis implies that

$$e^{-\kappa}\bar{\phi} \leqslant \phi \leqslant e^{\kappa}\bar{\phi}. \tag{A.2}$$

We first treat the case of $X = \mathsf{T}\times X_0$ with a product metric, where $\mathsf{T} = (\mathsf{T}^1)^n$ with the ℓ_1 product metric d_1 which is of diameter 1 on each factor.

In this case, we prove, by induction on n, that

$$d_L(\phi\cdot\mu, \bar{\phi}\cdot\mu) \leqslant 2e^n\kappa.$$

We know that the result is true for $n=0$. Assume that the result is true for $n-1$. Let $\tilde{\phi}$ be the average of ϕ along T_0, which is the product of $n-1$ first T^1 factors. Then, averaging the inequality (A.2) by T_0, we get

$$e^{\kappa}\bar{\phi} \leqslant \tilde{\phi} \leqslant e^{\kappa}\bar{\phi}. \tag{A.3}$$

Then, applying Corollary A.8 using the action of the last T^1 factor, we get

$$d(\tilde{\phi}\mu, \bar{\phi}\mu) \leqslant \kappa. \tag{A.4}$$

Now, reusing inequality (A.2) again, and combining with inequality (A.3), we get

$$e^{-2\kappa}\tilde{\phi} \leqslant \phi \leqslant e^{2\kappa}\tilde{\phi}. \tag{A.5}$$

By the induction hypothesis, we obtain

$$d(\tilde{\phi}\mu, \phi\mu) \leqslant \kappa 4e^{n-1}. \tag{A.6}$$

Using inequalities (A.6) and (A.4), the triangle inequality for the Lévy–Prokhorov distance yields

$$d(\phi\mu, \bar{\phi}\mu) \leqslant d(\tilde{\phi}\mu, \bar{\phi}\mu) + d(\tilde{\phi}\mu, \phi\mu) \leqslant \kappa(4e^{n-1}+1) \leqslant \kappa 2e^n.$$

This proves the theorem in this initial case.

We now still consider the case $X = \mathsf{T}^n$, but now equipped with a bi-invariant Riemannian metric d. Observe that $\pi_1(X)$ can be generated by translations of length smaller than $2\mathrm{diam}(X)$. Thus, there exists a bi-invariant ℓ_1 product metric d_1 on this torus, whose factors have diameter 1, such that

$$d \leqslant 2\,\mathrm{diam}(\mathsf{T}) d_1.$$

The statement in that case follows from the following observation: let d_1 and d_2 be two metrics whose corresponding Lévy–Prokhorov distances are, respectively, δ_1 and δ_2. Assume that $d_2 \leqslant K \cdot d_1$. Then, $\delta_2 \leqslant K \cdot \delta_1$. Combining with inequality (A.5), in this second case we obtain

$$d(\phi\mu, \bar{\phi}\mu) \leqslant 2\kappa e^n \,\mathrm{diam}(\mathsf{T}).$$

Finally, we apply Lemma A.6 to conclude for the general case. □

Appendix B. Exponential mixing

The following lemma is well known to experts as a combination of various deep results. However, it is difficult to track it precisely in the literature. We thank Bachir Bekka and Nicolas Bergeron for their help on that matter.

Lemma B.1. *Let* G *be a semi-simple Lie group without compact factor, and* Γ *be an irreducible lattice in* G. *Then, the action of any non-trivial hyperbolic element is exponentially mixing.*

When the lattice is not irreducible, we have to impose furthermore that the projection of the hyperbolic element to all irreducible factors is non-trivial.

Proof. The extension to non-irreducible factors follow from simple considerations. Thus, let just prove the first statement. Let $\mathsf{G}_1, ..., \mathsf{G}_n$ be the simple factors of G. Let π be the *regular representation*, that is the unitary representation of G in $L^2_0(\mathsf{G}/\Gamma)$, the orthogonal to the constant function in $L^2(\mathsf{G}/\Gamma)$.

By Kleinbock–Margulis [19, Corollary 4.5], we have to show that the restriction π_i of π on G_i has a spectral gap (see also Katok–Spatzier [18, Corollary 3.2]).

In the simplest case when G is simple and Γ uniform, this follows by standard arguments; for instance, see Bekka's survey [2, Proposition 8.1].

When G is simple but Γ non-uniform, this now follows from Bekka [1, Lemma 4.1].

When finally G is actually a product, by Margulis arithmeticity theorem [27], Γ is arithmetic. For Γ uniform, the spectral gap follows from Burger–Sarnak [7] and Clozel [8]. For Γ non-uniform, this is due to Kleinbock–Margulis [19, Theorem 1.12]. □

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References

[1] Bekka, B., On uniqueness of invariant means. *Proc. Amer. Math. Soc.*, 126 (1998), 507–514.

[2] — Spectral rigidity of group actions on homogeneous spaces, in *Handbook of Group Actions.* Vol. IV, Adv. Lect. Math. (ALM), 41.

[3] Bergeron, N., La conjecture des sous-groupes de surfaces (d'après Jeremy Kahn et Vladimir Marković). *Astérisque*, 352 (2013), Exp. No. 1055, 429–458. *Séminaire Bourbaki.* Vol. 2011/2012. Exposés 1043–1058.

[4] Bochi, J., Potrie, R. & Sambarino, A., Anosov representations and dominated splittings. *J. Eur. Math. Soc.* (*JEMS*), 21 (2019), 3343–3414.

[5] Bourbaki, N., *Lie Groups and Lie Algebras.* Chapters 7–9. Elements of Mathematics (Berlin). Springer, Berlin–Heidelberg, 2005.

[6] Bridgeman, M., Canary, R., Labourie, F. & Sambarino, A., The pressure metric for Anosov representations. *Geom. Funct. Anal.*, 25 (2015), 1089–1179.

[7] Burger, M. & Sarnak, P., Ramanujan duals. II. *Invent. Math.*, 106 (1991), 1–11.

[8] Clozel, L., Démonstration de la conjecture τ. *Invent. Math.*, 151 (2003), 297–328.

[9] Cooper, D., Long, D. D. & Reid, A. W., Essential closed surfaces in bounded 3-manifolds. *J. Amer. Math. Soc.*, 10 (1997), 553–563.

[10] Fock, V. & Goncharov, A., Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, 103 (2006), 1–211.

[11] Gelander, T., Lectures on lattices and locally symmetric spaces, in *Geometric Group Theory*, IAS/Park City Math. Ser., 21, pp. 249–282. Amer. Math. Soc., Providence, RI, 2014.

[12] Guichard, O. & Wienhard, A., Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190 (2012), 357–438.

[13] Hamenstädt, U., Incompressible surfaces in rank one locally symmetric spaces. *Geom. Funct. Anal.*, 25 (2015), 815–859.

[14] Kahn, J. & Markovic, V., Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *Ann. of Math.*, 175 (2012), 1127–1190.

[15] — The good pants homology and the Ehrenpreis conjecture. *Ann. of Math.*, 182 (2015), 1–72.

[16] Kahn, J. & Wright, A., Nearly Fuchsian surface subgroups of finite covolume Kleinian groups. *Duke Math. J.*, 170 (2021), 503–573.

[17] Kapovich, M., Leeb, B. & Porti, J., A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings. *Geom. Topol.*, 22 (2018), 3827–3923.

[18] Katok, A. & Spatzier, R. J., First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity. *Inst. Hautes Études Sci. Publ. Math.*, 79 (1994), 131–156.

[19] Kleinbock, D. Y. & Margulis, G. A., Logarithm laws for flows on homogeneous spaces. *Invent. Math.*, 138 (1999), 451–494. Correction in *Invent. Math.*, 211 (2018), 855–862.

[20] Knapp, A. W., *Lie groups beyond an introduction*, second edition. Progr. Math., 140. Birkhäuser, Boston, MA, 2002.

[21] Kostant, B., The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.*, 81 (1959), 973–1032.

[22] Labourie, F., Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165 (2006), 51–114.

[23] Labourie, F. & McShane, G., Cross ratios and identities for higher Teichmüller–Thurston theory. *Duke Math. J.*, 149 (2009), 279–345.

[24] Lackenby, M., Surface subgroups of Kleinian groups with torsion. *Invent. Math.*, 179 (2010), 175–190.

[25] Long, D. D. & Reid, A. W., Thin surface subgroups in cocompact lattices in $\mathrm{SL}(3, \mathbf{R})$. *Illinois J. Math.*, 60 (2016), 39–53.

[26] Long, D. D., Reid, A. W. & Thistlethwaite, M., Zariski dense surface subgroups in $\mathrm{SL}(3, \mathbf{Z})$. *Geom. Topol.*, 15 (2011), 1–9.

[27] Margulis, G. A., Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1. *Invent. Math.*, 76 (1984), 93–120.

[28] Milnor, J., On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.*, 15 (1964), 275–280.

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