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NOTES ON UNIPOTENT CLASSES*

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0. Introduction and statement of results.

0.1. Let G be a semisimple, almost simple algebraic group over $\mathbf{C}$. Let W be the Weyl group of G. For any unipotent class C of G, let ρ_C be the irreducible representation of W attached by Springer's correspondence to the pair consisting of C and the local system $\mathbf{C}$ on C. We say that C is *special* if ρ_C is a special representation of W (that is, a representation in the class $\mathcal{S}_W$ introduced in [L1]). It is known that $C \mapsto \rho_C$ is a bijection between the set of special unipotent classes of G and the set of special representations (up to isomorphism) of W.

The special unipotent classes play a key role in several problems in representation theory, such as the classification of irreducible complex representations of a reductive group over a finite field, and the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra. Unfortunately, their definition is totally ungeometrical. For this reason, special unipotent classes are often regarded as rather mysterious objects. To partially remedy this situation, we have felt the need to try to unveil some of the purely geometrical properties of special unipotent classes, or rather, of the closely connected special pieces (defined below); this has led to the present paper.

For any special unipotent class C in G, let $\tilde{C}$ be the subset of the unipotent variety $\mathcal{U}$ of G consisting of all elements in the closure $\bar{C}$ of C which are not in the closure of any special unipotent class $C' \neq C, C' \subset \bar{C}$; this definition appeared in [Sp]. Clearly, the sets $\tilde{C}$ (for various special unipotent classes C) are irreducible,locally closed subvarieties of $\mathcal{U}$; Spaltenstein [Sp] has shown that they in fact form a partition of $\mathcal{U}$. The subvarieties $\tilde{C}$ will be called *special pieces*. Note that each special piece is a union of a special class (which is open dense in the special piece) and of a certain number (possibly zero) of non-special classes. Let $\gamma(C)$ be the set of unipotent classes that are contained in the special piece $\tilde{C}$.

One of the results of this paper is:

THEOREM 0.2. *Two unipotent classes C_1, C_2 of G belong to the same special piece if and only if ρ_{C_1}, ρ_{C_2} belong to the same two-sided cell of W.*

0.3. Now let $\mathbf{c}$ be a two-sided cell of W and let $\hat{W}_\mathbf{c}$ be the set of irreducible representations of W (up to isomorphism) that belong to $\mathbf{c}$. To $\mathbf{c}$ we attach, as in [L3], a finite group $\mathcal{G}_\mathbf{c}$ and an imbedding

(a) $\hat{W}_\mathbf{c} \hookrightarrow M(\mathcal{G}_\mathbf{c})$;

here $M(\mathcal{G}_\mathbf{c})$ is the set of pairs (g, τ) where g is an element of $\mathcal{G}_\mathbf{c}$ defined up to conjugacy and τ is an irreducible representation of the centralizer of g in $\mathcal{G}_\mathbf{c}$ defined up to isomorphism.

THEOREM 0.4. *Let C be a special unipotent class of G and let $\mathbf{c}$ be the two sided cell of W such that ρ_C belongs to $\mathbf{c}$. For any $C_1 \in \gamma(C)$, the image under 0.3(a) of ρ_{C_1} (which belongs to $\hat{W}_\mathbf{c}$ by 0.2) is of the form $(g, 1)$ where g is an element of $\mathcal{G}_\mathbf{c}$ defined up to conjugacy. Let $\mathcal{G}'_\mathbf{c}$ be the subgroup of $\mathcal{G}_\mathbf{c}$ generated by the conjugacy classes of the*

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elements g attached to various $C_1 \in \gamma(C)$. Then $C_1 \mapsto g$ is an imbedding of $\gamma(C)$ into the set of conjugacy classes of $\mathcal{G}_{\mathbf{c}}$ and a bijection of $\gamma(C)$ onto the set of conjugacy classes of $\mathcal{G}'_{\mathbf{c}}$.

0.5. In [L2] it was conjectured that (in the setup of 0.4), $\tilde{C}$ is a rational homology manifold. This is now known to be true: it has been proved for the classical groups in [KP], and has been checked for groups of type E_n in [BS]; for type F_4 it can be checked from [Sh] and for G_2 it was already known at the time of [L2]. The argument in [KP] exhibits $\tilde{C}$ (for a classical group) as a quotient of a smooth irreducible variety by the action of a finite group, which we can now interpret as the group $\mathcal{G}'_{\mathbf{c}}$ in 0.4.

0.6. Assume now that G is adjoint of exceptional type. Since the group $\mathcal{G}'_{\mathbf{c}}$ is now defined in general, it seems likely that even in this case, $\tilde{C}$ should be the quotient of a smooth irreducible variety $C^\dagger$ by an action of $\mathcal{G}'_{\mathbf{c}}$.

If $\tilde{C} = C$, this is of course trivial: we take $C^\dagger = C$. Assume now that $\tilde{C} \neq C$. In this case, $\mathcal{G}'_{\mathbf{c}} = \mathcal{G}_{\mathbf{c}}$ is a symmetric group S_r where $2 \leq r \leq 5$. To each $C_1 \in \gamma(C)$ corresponds a conjugacy class $g \in S_r$ of cycle type given by the partition $n_1+n_2+\cdots = r$ and to this we associate a subgroup of S_r of the form $H_{C_1} = S_{n_1} \times S_{n_2} \times \dots$ which is well defined up to conjugacy.

We expect that the action of $\mathcal{G}_{\mathbf{c}} = S_r$ on $C^\dagger$ has the property that C_1 is precisely the set of orbits of points of $C^\dagger$ whose isotropy group in S_r is conjugate to H_{C_1}.

This is consistent with the following result on the intersection cohomology of $\tilde{C}$. (For a unipotent class C' of G we denote by $A_{C'}$ the group of connected components of the centralizer in G of an element of C'.)

PROPOSITION 0.7. *Assume that G is as in 0.6, $C, \mathbf{c}$ are as in 0.4, and $\tilde{C} \neq C$.*

(a) We have $A_C = \mathcal{G}_{\mathbf{c}}$ except if C is the class of type $E_8(b_6)$ (notation of [Ca]*) when $A_C = S_3, \mathcal{G}_{\mathbf{c}} = S_2$.*

(b) Let $C_1 \in \gamma(C), C_1 \neq C$. Then A_{C_1} may be identified with $N(H_{C_1})/H_{C_1}$ where $N(H_{C_1})$ is the normalizer of H_{C_1} in $\mathcal{G}_{\mathbf{c}}$.

(c) Let $\mathcal{L}$ be an irreducible local system on C coming from an irreducible representation E of $\mathcal{G}_{\mathbf{c}}$ (which is naturally a quotient of A_C, see (a).) Then the intersection cohomology complex $IC(\tilde{C}, \mathcal{L})$ is a constructible sheaf; its restriction to a unipotent class $C_1 \subset \tilde{C}, C_1 \neq C$ is the local system associated to the representation of $A_{C_1} = N(H_{C_1})/H_{C_1}$ obtained by taking the space of H_{C_1}-invariant vectors in E and regarding it as a representation of $N(H_{C_1})/H_{C_1}$ in a natural way.

(d) If $C_1, C_2 \in \gamma(C)$, then C_1 is contained in the closure of C_2 if and only if H_{C_2} is conjugate to a subgroup of H_{C_1}.

This is verified case by case; for (c), we make use of the tables of [BS],[Sp],[Sh].

The only G-equivariant local system on C not covered by (c) above is the local system $\mathcal{L}$ on the class C of type $E_8(b_6)$ coming from the two-dimensional irreducible representation of $A_C = S_3$. But in this case, again $IC(\tilde{C}, \mathcal{L})$ is a constructible sheaf (equal to $\mathbf{C}$) on $\tilde{C}$.

If we drop the assumption that G is adjoint, and assume that $\mathcal{L}$ is an irreducible G-equivariant local system on C, then it is very likely that $IC(\tilde{C}, \mathcal{L})$ is again a constructible sheaf on $\tilde{C}$. (I have verified this in type E_6 and it can be probably verified also in the only remaining case, E_7). Note that, in the case where $\tilde{C} = C$, the fact that $IC(\tilde{C}, \mathcal{L})$ is a constructible sheaf on $\tilde{C}$ is obvious.

Note also that the constructibility of $IC(\tilde{C}, \mathcal{L})$ remains valid in the case of classical groups, where it can be deduced from the results of [KP].

0.8. Now let $\mathcal{G}$ be a linear algebraic group over $\mathbf{C}$. Let X be an algebraic variety over $\mathbf{C}$ with an algebraic action of $\mathcal{G}$. For $j \in \mathbf{Z}$, we write $H_j^{\mathcal{G}}(X)$ for the equivariant homology space $H_j^{\mathcal{G}}(X, \mathbf{C})$ defined in [L5, §1]. This is a finite dimensional $\mathbf{C}$-vector space and is 0 for $j < 0$.

THEOREM 0.9. *Let $G = Sp_{2r}(\mathbf{C}), G' = SO_{2r+1}(\mathbf{C})$. Let W be the Weyl group of G and of G'. Let C be a special unipotent class of G and let C' be a special unipotent class of G' such that $\rho_C = \rho_{C'}$. Then $\dim H_j^G(\tilde{C}) = \dim H_j^{G'}(\tilde{C}')$ for all j.*

The theorem above is equivalent to the last assertion of the Conjecture 3 in [L2].

0.10. Notation. For two integers x, y we write $x \ll y$ if $x \le y-2$ and $x \lll y$ if $x \le y-3$; we write $[x, y] = \{z \in \mathbf{Z} | x \le z \le y\}$.

If A, B are multisets of integers, we write $A \le B$ if each number in A is $\le$ than each number in B.

For a finite set X we denote by $|X|$ the cardinal of X. Let $\mathfrak{P}(X)$ be the set of subsets of X and let $\mathfrak{P}_{\text{ev}}$ be the set of subsets of even cardinal of X. Then $\mathfrak{P}(X)$ is naturally a vector space over F_2 and $\mathfrak{P}_{\text{ev}}$ is a vector subspace.

1. Combinatorics: type C_r.

1.1. We fix an integer $r \ge 2$. For any $n \ge 0$ we define $\Psi_{2r;n}$ to be the set of all sequences of integers $\mathbf{a} = (a_0 \le a_1 \le a_2 \le \cdots \le a_{2n})$ such that

$a_0 \ge 0; a_1 \ge 1;\ a_p \ll a_{p+2}$ for $p \in [0, 2n-2]$; $r = \sum_{p \in [0,2n]} a_p - (2n^2 + n)$.

There is a natural map $\Psi_{2r,n} \to \Psi_{2r,n+1}$ given by

$(a_0 \le a_1 \le a_2 \le \cdots \le a_{2n}) \mapsto (0 \le 1 \le a_0 + 2 \le a_1 + 2 \le a_2 + 2 \le \cdots \le a_{2n} + 2)$.

This is a bijection if n is large enough (compared to r). We will denote by Ψ_{2r} the limit of $\Psi_{2r,n}$ as $n \to \infty$ (with respect to the maps above); we will fix n large enough so that $\Psi_{2r,n} \xrightarrow{\sim} \Psi_{2r}$ and we identify the last two sets. We may also assume that $a_0 = 0, a_1 = 1$ for any $\mathbf{a} \in \Psi_{2r}$. Let

$B(\mathbf{a}) = \sum_{0 \le i < j \le 2n} \inf(a_i, a_j) - n(4n^2 - 1)/3$.

1.2. A *ladder* of $\mathbf{a} \in \Psi_{2r}$ is a non-empty subset $[k, l]$ of $[0, 2n]$ such that

$(a_k, a_{k+1}, a_{k+2}, \ldots, a_l) = (a, a+1, a+2, a+3, \ldots)$

and $a_{k-1} \ll a_k$ (if $k > 0$), $a_l \ll a_{l+1}$ (if $l < 2n$).

A *staircase* of $\mathbf{a} \in \Psi_{2r}$ is a non-empty subset $[k, l]$ of $[0, 2n]$ with $l - k + 1$ even such that

$(a_k, a_{k+1}, a_{k+2}, \ldots, a_l) = (a, a, a+2, a+2, a+4, a+4, \ldots)$

and $a_{k-2} \lll a_k$ (if $k \ge 2$), $a_l \lll a_{l+2}$ (if $l + 2 \le 2n$).

It is easy to see that $[0, 2n]$ is a disjoint union of subsets that are either ladders or staircases.

Let $\mathbf{z}$ be an indeterminate. For any integer $s \ge 0$ we set

$\phi(2s) = \phi(2s+1) = \prod_{j=1}^{s} (1 - \mathbf{z}^{2j})$.

Let

$\Pi(\mathbf{a}) = \mathbf{z}^{-2r^2 + 2B(\mathbf{a})} \prod_{\mathbf{i}} \phi(|\mathbf{i}|)^{-1}$.

Here $\mathbf{i}$ runs over the set of subsets of $[0, 2n]$ that are ladders (not containing 0) or staircases.

1.3. We say that $\mathbf{a}, \mathbf{a}'$ in Ψ_{2r} are *congruent* if

$$\begin{aligned} &\{a_0, a_1 - 1, a_2 - 1, a_3 - 2, a_4 - 2, \ldots, a_{2n} - n\} \\ &= \{a'_0, a'_1 - 1, a'_2 - 1, a'_3 - 2, a'_4 - 2, \ldots, a'_{2n} - n\} \end{aligned}$$

as multisets. We then write $\mathbf{a} \sim \mathbf{a}'$.

LEMMA 1.4. *For* $\mathbf{a}, \mathbf{a}' \in \Psi_{2r}$, *the conditions (a),(b), (c) below are equivalent:*

(a) $\mathbf{a} \sim \mathbf{a}'$;

(b) $\{a_{2p}-p, a_{2p+1}-p-1\} = \{a'_{2p}-p, a'_{2p+1}-p-1\}$ *for* $p \in [0, n-1]$ *as multisets and* $a_{2n} = a'_{2n}$;

(c) $a_{2n} = a'_{2n}$ *and, for any* $p \in [0, n-1]$, $\begin{pmatrix} a_{2p} & a_{2p+1} \\ a'_{2p} & a'_{2p+1} \end{pmatrix}$ *is of the form (i)* $\begin{pmatrix} s & t \\ s & t \end{pmatrix}$, *or (ii)* $\begin{pmatrix} s & s \\ s-1 & s+1 \end{pmatrix}$, *or (iii)* $\begin{pmatrix} s-1 & s+1 \\ s & s \end{pmatrix}$.

Assume that (a) holds. We show that (b) holds. This follows from

$\{a_0, a_1 - 1\} \leq \{a_2 - 1, a_3 - 2\} \leq \cdots \leq \{a_{2n-2} - (n-1), a_{2n-1} - n\} \leq \{a_{2n} - n\}$ and the analogous fact for $\mathbf{a}'$.

Assume now that (b) holds. We show that (c) holds. Fix $p \in [0, n-1]$. We have $\{a_{2p}, a_{2p+1} - 1\} = \{a'_{2p}, a'_{2p+1} - 1\}$. If $a_{2p} = a'_{2p}, a_{2p+1} = a'_{2p+1}$, then we are in case (i). Assume now that $a_{2p} = a'_{2p+1} - 1, a_{2p+1} = a'_{2p} + 1$. We have $a'_{2p+1} - 1 \leq a'_{2p} + 1 \leq a'_{2p+1} + 1$. Hence either $a'_{2p} = a'_{2p+1}$ and we are in case (iii) or $a'_{2p} = a'_{2p+1} - 1$ and we are in case (i) or $a'_{2p} = a'_{2p+1} - 2$ and we are in case (ii).

The implications (c) $\implies$ (b) $\implies$ (a) are obvious. The lemma is proved.

1.5. We say that $\mathbf{b} = (b_0 \leq b_1 \leq \cdots \leq b_{2n}) \in \Psi_{2r}$ is *special* if $b_{2p} < b_{2p+1}$ for all $p \in [0, n-1]$. Let Ψ^0_{2r} be the set of special elements of Ψ_{2r}.

LEMMA 1.6. *Given* $\mathbf{a} \in \Psi_{2r}$, *there is a unique* $\mathbf{b} \in \Psi^0_{2r}$ *such that* $\mathbf{b} \sim \mathbf{a}$. *We have*

$b_{2p} = a_{2p}, b_{2p+1} = a_{2p+1}$, *if* $p \in [0, n-1], a_{2p} < a_{2p+1}$;

$b_{2p} = a_{2p} - 1, b_{2p+1} = a_{2p+1} + 1$, *if* $p \in [0, n-1], a_{2p} = a_{2p+1}$;

$b_{2n} = a_{2n}$.

The proof is immediate (using Lemma 1.4).

1.7. Let $\mathbf{b} \in \Psi^0_{2r}$. A *segment* of $\mathbf{b}$ is a non-empty subset $[k, l]$ of $[0, 2n]$ such that k is even, l is odd,

$$(b_k, b_{k+1}, b_{k+2}, \ldots, b_l) = (a, a+2, a+2, a+4, a+4, \ldots, a+l-k-1, a+l-k-1, a+l-k+1)$$

and such that $b_{k-1} < b_k$ (if $k \geq 2$), $b_l < b_{l+1}$. Let $S_\mathbf{b}$ be the set of segments of $\mathbf{b}$. Clearly, the segments of $\mathbf{b}$ are disjoint subsets of $[0, 2n]$.

PROPOSITION 1.8. *Let* $\mathbf{b} \in \Psi^0_{2r}$. *There is a 1-1 correspondence between* $\mathfrak{P}(S_\mathbf{b})$ *and the set* $\{\mathbf{a} \in \Psi_{2r} | \mathbf{a} \sim \mathbf{b}\}$: *to a subset* K *of* $S_\mathbf{b}$ *corresponds the sequence* $\mathbf{a} = \mathbf{a}(K) \in \Psi_{2r}$ *defined by*

$(a_k, a_{k+1}, a_{k+2}, \ldots, a_l) = (a+1, a+1, a+3, a+3, a+5, a+5, \ldots, a+l-k, a+l-k)$ *if* $[k, l] \in K$ *and* $a_t = b_t$ *if* $t \notin \cup_{[k,l] \in K} [k, l]$.

This follows easily from Lemmas 1.4, 1.6.

1.9. We fix $\mathbf{b} \in \Psi^0_{2r}$. An integer $k \in [0, 2n]$ is said to be *isolated* (for $\mathbf{b}$) if either:

(a) $k = 2n$ and $b_{2n-1} < b_{2n}$, or

(b) k is even, $0 < k < 2n$ and $b_{k-1} < b_k \ll b_{k+1}$, or

(c) k is odd and $b_{k-1} \ll b_k < b_{k+1}$.

LEMMA 1.10. *(a)* $\{k \in [0, 2n] | k \text{ isolated}\} = \{k_0 < k_1 < \cdots < k_{2f}\}$ *where* $k_t = t$ mod 2 *for* $t \in [0, 2f]$.

(b) Assume that $t \in [0, 2f-1]$ *is even. Then* $[k_t + 1, k_{t+1} - 1]$ *is a (possibly empty) union of staircases.*

(c) Assume that $t \in [0, 2f]$ *is odd. Then* $[k_t, k_{t+1}]$ *is a union of ladders.*

(d) The set $[k_{2f} + 1, k_{2n}]$ *is a (possibly empty) union of staircases.*

(e) The set $[0, k_0]$ *is a union of ladders.*

(f) Assume that $t \in [1, 2f]$. *Then either the unique ladder containing* k_t *has an odd cardinal, or else it equals* $[k_{t-1}, k_t]$ *(if t is even) or* $[k_t, k_{t+1}]$ *(if t is odd).*

(g) The unique ladder containing k_0 *has an odd cardinal.*

The proof is routine; it will be omitted.

1.11. Let $\mathbf{I} = \mathbf{I}_\mathbf{b}$ be the subset of $\mathfrak{P}([0, 2n])$ consisting of those subsets of form
$[k_0, k_1], [k_1, k_2], \dots, [k_{2f-1}, k_{2f}]$
that are either ladders or segments for $\mathbf{b}$. All segments of $\mathbf{b}$ appear in $\mathbf{I}$; they form the subset $S = S_\mathbf{b}$ of $\mathbf{I}$. The ladders in $\mathbf{I}$ form a subset $L = L_\mathbf{b}$ of $\mathbf{I}$. Note that $\mathbf{I} = S \sqcup L$. We say that $\mathbf{i}_1 \in S, \mathbf{i}_2 \in L$ are *adjacent* if they have a non-empty intersection (necessarily one of the k_t).

1.12. Let K be a subset of S and let $\mathbf{a} = \mathbf{a}(K)$ be as in 1.8. We want to compare the products defining $\Pi(\mathbf{b}), \Pi(\mathbf{a})$. From the definitions we see that
$\mathbf{z}^{-2r^2+2B(\mathbf{a})} = \mathbf{z}^{-2r^2+2B(\mathbf{b})} \prod_{\mathbf{i} \in K} \mathbf{z}^{|\mathbf{i}|}$.
For any $[k_t, k_{t+1}] \in L$, the factor $\phi(|[k_t+1, k_{t+1}-1]|)^{-1}$ in $\Pi(\mathbf{b})$ is replaced in $\Pi(\mathbf{a})$ by the factor $\phi(|[k_t, k_{t+1}]|)^{-1} = \phi(|[k_t+1, k_{t+1}-1]|)^{-1}(1-\mathbf{z}^{k_{t+1}-k_t+1})^{-1}$; moreover, if the ladder of $\mathbf{b}$ containing k_t (resp. k_{t+1}) has even cardinal, necessarily $k_t - k_{t-1} + 1$ (resp. $k_{t+2} - k_{t+1} + 1$) then in $\mathbf{a}$ it becomes a ladder of cardinal $k_t - k_{t-1}$ or $k_t - k_{t-1} - 1$ (resp. $k_{t+2} - k_{t+1}$ or $k_{t+2} - k_{t+1} - 1$) and the corresponding factor $\phi(k_t - k_{t-1} + 1)^{-1}$ (resp. $\phi(k_{t+2} - k_{t+1} + 1)^{-1}$) of $\Pi(\mathbf{b})$ would be replaced in $\Pi(\mathbf{a})$ by the factor
$\phi(k_t - k_{t-1})^{-1} = \phi(k_t - k_{t-1} - 1)^{-1} = \phi(k_t - k_{t-1} + 1)^{-1}(1 - \mathbf{z}^{k_t - k_{t-1}+1})$
(resp.
$\phi(k_{t+2} - k_{t+1})^{-1} = \phi(k_{t+2} - k_{t+1} - 1)^{-1} = \phi(k_{t+2} - k_{t+1} + 1)^{-1}(1 - \mathbf{z}^{k_{t+2}-k_{t+1}+1})$).
On the other hand, if the ladder of $\mathbf{b}$ containing k_t (resp. k_{t+1}) has odd cardinal, then it becomes a ladder for $\mathbf{b}$ shorter by one, and the contributions to $\Pi(\mathbf{b}), \Pi(\mathbf{a})$ would be the same (since $\phi(2s) = \phi(2s+1)$). The factors of $\Pi(\mathbf{b}), \Pi(\mathbf{a})$ other than those mentioned above are the same. We see that
$\Pi(\mathbf{a}) = \Pi(\mathbf{b}) \prod_{\mathbf{i}_1 \in K} \frac{\mathbf{z}^{|\mathbf{i}_1|}}{1-\mathbf{z}^{|\mathbf{i}_1|}} \prod_{\mathbf{i}_2 \in L - K^\sharp} (1 - \mathbf{z}^{|\mathbf{i}_2|})$.
Here $K^\sharp$ denotes the set of all elements of L that are not adjacent to any element of K. We can rewrite this as follows:

$$\Pi(\mathbf{a}) = \Pi(\mathbf{b}) \prod_{\mathbf{i}_2 \in L} (1 - \mathbf{z}^{|\mathbf{i}_2|}) \prod_{\mathbf{i}_1 \in K} \frac{\mathbf{z}^{|\mathbf{i}_1|}}{1 - \mathbf{z}^{|\mathbf{i}_1|}} \frac{1}{\prod_{\mathbf{i}_2 \in K^\sharp}(1 - \mathbf{z}^{|\mathbf{i}_2|})}$$
$$= \Pi(\mathbf{b}) \prod_{\mathbf{i}_2 \in L} (1 - \mathbf{z}^{|\mathbf{i}_2|}) \prod_{\mathbf{i}_1 \in K} \frac{1}{\mathbf{z}^{-|\mathbf{i}_1|} - 1} \prod_{\mathbf{i}_2 \in K^\sharp} \frac{\mathbf{z}^{-|\mathbf{i}_2|}}{\mathbf{z}^{-|\mathbf{i}_2|} - 1}.$$

The last product can be expanded into a sum:

$$\prod_{\mathbf{i}_2 \in K^\sharp} \frac{\mathbf{z}^{-|\mathbf{i}_2|}}{\mathbf{z}^{-|\mathbf{i}_2|} - 1} = \frac{\prod_{\mathbf{i}_2 \in K^\sharp} \mathbf{z}^{-|\mathbf{i}_2|}}{\prod_{\mathbf{i}_2 \in K^\sharp}(\mathbf{z}^{-|\mathbf{i}_2|} - 1)} = \frac{\sum_{Z; Z \subset K^\sharp} \prod_{\mathbf{i}_2 \in Z}(\mathbf{z}^{-|\mathbf{i}_2|} - 1)}{\prod_{\mathbf{i}_2 \in K^\sharp}(\mathbf{z}^{-|\mathbf{i}_2|} - 1)}$$
$$= \sum_{Z; Z \subset K^\sharp} \frac{1}{\prod_{\mathbf{i}_2 \in K^\sharp - Z}(\mathbf{z}^{-|\mathbf{i}_2|} - 1)} = \sum_{Y; Y \subset K^\sharp} \frac{1}{\prod_{\mathbf{i}_2 \in Y}(\mathbf{z}^{-|\mathbf{i}_2|} - 1)}.$$

Hence

$$\Pi(\mathbf{a}) = \Pi(\mathbf{b}) \prod_{\mathbf{i}_2 \in L} (1 - \mathbf{z}^{|\mathbf{i}_2|}) \sum_{Y; Y \subset K^\sharp} \frac{1}{\prod_{\mathbf{i} \in Y \cup K}(\mathbf{z}^{-|\mathbf{i}|} - 1)}.$$

We now compute

$$\sum_{\mathbf{a};\mathbf{a}\sim\mathbf{b}} \Pi(\mathbf{a}) = \sum_{K;K\subset S} \Pi(\mathbf{a}(K)) = \Pi(\mathbf{b}) \prod_{\mathbf{i}_2\in L} (1-\mathbf{z}^{|\mathbf{i}_2|}) \sum_{\substack{Y,K;Y\subset L\\ K\subset S;Y\subset K^\natural}} \frac{1}{\prod_{\mathbf{i}\in Y\cup K}(\mathbf{z}^{-|\mathbf{i}|}-1)}.$$

Using the definitions and Lemma 1.10, we see that

$\Pi(\mathbf{b})\prod_{\mathbf{i}_2\in L}(1-\mathbf{z}^{|\mathbf{i}_2|}) = \mathbf{z}^{-2r^2+2B(\mathbf{a})}\prod_{\mathbf{i}\in\mathbf{I}}\phi(|\mathbf{i}|-2)^{-1}\prod_{\mathbf{j}}\phi(|\mathbf{j}|)^{-1}$

where $\mathbf{j}$ runs over the subsets of $[0,2n]-\cup_{[k,l]\in\mathbf{I}}[k+1,l-1]$ that are ladders of $\mathbf{b}$ (not containing 0) or staircases of $\mathbf{b}$.

1.13. Let $\Lambda_{\mathbf{b},1}$ be the subspace of $\mathfrak{P}_{\mathrm{ev}}(\{k_0,k_1,\dots,k_{2f}\})$ spanned by the sets $\{k_{2g},k_{2g+1}\}$ with $g\in[0,f-1]$. (These form a basis $B_{\mathbf{b},1}$.) Let $\Lambda_{\mathbf{b},2}$ be the subspace of $\mathfrak{P}_{\mathrm{ev}}(\{k_0,k_1,\dots,k_{2f}\})$ spanned by the sets $\{k_{2g-1},k_{2g}\}$ with $g\in[1,f]$. (These form a basis $B_{\mathbf{b},2}$.) We define an (injective) map $S_{\mathbf{b}}\hookrightarrow B_{\mathbf{b},1}$ by $(b_k,b_{k+1},b_{k+2},\dots,b_l)\mapsto\{k,l\}$. This extends uniquely to an (injective) linear map of F_2-vector spaces $T:\mathfrak{P}(S_{\mathbf{b}})\hookrightarrow\Lambda_{\mathbf{b},1}$.

2. Combinatorics: type B_r.

2.1. We fix an integer $r\geq 2$. For any $n\geq 0$ we define $\Psi'_{2r+1;n}$ to be the set of all sequences of integers $\mathbf{a}=(a_0\leq a_1\leq a_2\leq\cdots\leq a_{2n})$ such that

$a_0\geq 0; a_p\ll a_{p+2}$ for $p\in[0,2n-2]$; $r=\sum_{p\in[0,2n]}a_p-2n^2$.

There is a natural map $\Psi'_{2r+1,n}\to\Psi'_{2r+1,n+1}$ given by

$(a_0\leq a_1\leq a_2\leq\cdots\leq a_{2n})\mapsto(0\leq 0\leq a_0+2\leq a_1+2\leq a_2+2\leq\cdots\leq a_{2n}+2)$.

This is a bijection if n is large enough (compared to r). We will denote by Ψ'_{2r+1} the limit of $\Psi'_{2r+1,n}$ as $n\to\infty$ (with respect to the maps above); we will fix n large enough so that $\Psi'_{2r+1,n}\xrightarrow{\sim}\Psi'_{2r+1}$ and we identify the last two sets. We may also assume that $a_0=0,a_1=0$ for any $\mathbf{a}\in\Psi'_{2r+1}$. Let

$B(\mathbf{a})=\sum_{0\leq i<j\leq 2n}\inf(a_i,a_j)-n(n-1)(4n+1)/3$.

2.2. The ladders and staircases of of $\mathbf{a}\in\Psi'_{2r+1}$ are defined exactly as in 1.2. Again, $[0,2n]$ is a disjoint union of subsets that are either ladders or staircases. We set

$\Pi(\mathbf{a})=\mathbf{z}^{-2r^2+2B(\mathbf{a})}\prod_{\mathbf{i}}\phi(|\mathbf{i}|)^{-1}$.

Here $\mathbf{i}$ runs over the set of subsets of $[0,2n]$ that are staircases (not containing 0) or ladders; ϕ is as in 1.2.

2.3. We say that $\mathbf{a},\mathbf{a}'$ in Ψ'_{2r+1} are *congruent* if

$$\begin{aligned}&\{a_0,a_1,a_2-1,a_3-1,a_4-2,\dots,a_{2n}-n\}\\ &=\{a'_0,a'_1,a'_2-1,a'_3-1,a'_4-2,\dots,a'_{2n}-n\}\end{aligned}$$

as multisets. We then write $\mathbf{a}\sim\mathbf{a}'$.

LEMMA 2.4. *For $\mathbf{a},\mathbf{a}'\in\Psi'_{2r+1}$, the conditions (a),(b),(c) below are equivalent:*

(a) $\mathbf{a}\sim\mathbf{a}'$;

(b) $\{a_{2p-1}-(p-1),a_{2p}-p\}=\{a'_{2p-1}-(p-1),a'_{2p}-p\}$ *for* $p\in[1,n]$ *as multisets and* $a_0=a'_0$;

(c) $a_0=a'_0$ *and, for any* $p\in[1,n]$, $\begin{pmatrix}a_{2p-1}&a_{2p}\\a'_{2p-1}&a'_{2p}\end{pmatrix}$ *is of the form (i)* $\begin{pmatrix}s&t\\s&t\end{pmatrix}$, *or (ii)* $\begin{pmatrix}s&s\\s-1&s+1\end{pmatrix}$, *or (iii)* $\begin{pmatrix}s-1&s+1\\s&s\end{pmatrix}$.

The proof is similar to that of 1.4.

2.5. We say that $\mathbf{b} = (b_0 \le b_1 \le \dots b_{2n}) \in \Psi'_{2r+1}$ is *special* if $b_{2p-1} < b_{2p}$ for all $p \in [1, n]$. Let Ψ'^0_{2r+1} be the set of special elements of Ψ'_{2r+1}.

LEMMA 2.6. *Given* $\mathbf{a} \in \Psi'_{2r+1}$, *there is a unique* $\mathbf{b} \in \Psi'^0_{2r+1}$ *such that* $\mathbf{b} \sim \mathbf{a}$. *We have*

$b_{2p-1} = a_{2p-1}, b_{2p} = a_{2p}$, *if* $p \in [1, n], a_{2p-1} < a_{2p}$;

$b_{2p-1} = a_{2p-1} - 1, b_{2p} = a_{2p} + 1$, *if* $p \in [1, n], a_{2p-1} = a_{2p}$;

$b_0 = a_0$.

The proof is immediate (using Lemma 2.4).

2.7. Let $\mathbf{b} \in \Psi'^0_{2r+1}$. A *segment* of $\mathbf{b}$ is a non-empty subset $[k, l]$ of $[0, 2n]$ such that k is odd, l is even,

$$(b_k, b_{k+1}, b_{k+2}, \dots, b_l)$$
$$= (a, a+2, a+2, a+4, a+4, \dots, a+l-k-1, a+l-k-1, a+l-k+1)$$

and such that $b_{k-1} < b_k$, $b_l < b_{l+1}$ (if $l < 2n$). Let $S_{\mathbf{b}}$ be the set of segments of $\mathbf{b}$. Clearly, the segments of $\mathbf{b}$ are disjoint subsets of $[0, 2n]$.

PROPOSITION 2.8. *Let* $\mathbf{b} \in \Psi'^0_{2r+1}$. *There is a 1-1 correspondence between* $\mathfrak{P}(S_{\mathbf{b}})$ *and the set* $\{\mathbf{a} \in \Psi'_{2r+1} | \mathbf{a} \sim \mathbf{b}\}$: *to a subset* K *of* $S_{\mathbf{b}}$ *corresponds the sequence* $\mathbf{a} = \mathbf{a}(K) \in \Psi'_{2r+1}$ *defined by*

$(a_k, a_{k+1}, a_{k+2}, \dots, a_l) = (a+1, a+1, a+3, a+3, a+5, a+5, \dots, a+l-k, a+l-k)$ *if* $[k, l] \in K$ *and* $a_t = b_t$ *if* $t \notin \cup_{[k,l] \in K} [k, l]$.

This follows easily from Lemmas 2.4, 2.6.

2.9. We fix $\mathbf{b} \in \Psi'^0_{2r+1}$. An integer $k \in [0, 2n]$ is said to be *isolated* (for $\mathbf{b}$) if either:

(a) $k = 2n$ and $b_{2n-1} \ll b_{2n}$, or

(b) k is even, $0 < k < 2n$ and $b_{k-1} \ll b_k < b_{k+1}$, or

(c) k is odd and $b_{k-1} < b_k \ll b_{k+1}$.

LEMMA 2.10. *(a)* $\{k \in [0, 2n] | k$ *isolated*$\} = \{k_0 < k_1 < \dots < k_{2f}\}$ *where* $k_t = t$ mod 2 *for* $t \in [0, 2f]$.

(b) Assume that $t \in [0, 2f-1]$ *is odd. Then* $[k_t + 1, k_{t+1} - 1]$ *is a (possibly empty) union of staircases.*

(c) Assume that $t \in [0, 2f]$ *is even. Then* $[k_t, k_{t+1}]$ *is a union of ladders.*

(d) The set $[k_{2f} + 1, k_{2n}]$ *is a union of ladders.*

(e) The set $[0, k_0]$ *is a (possibly empty) union of staircases.*

(f) Assume that $t \in [0, 2f-1]$. *Then either the unique ladder containing* k_t *has an odd cardinal, or else it equals* $[k_{t-1}, k_t]$ *(if* t *is odd) or* $[k_t, k_{t+1}]$ *(if* t *is even).*

(g) The unique ladder containing k_{2f} *has an odd cardinal.*

The proof is routine; it will be omitted.

2.11. Let $\mathbf{I} = \mathbf{I_b}$ be the subset of $\mathfrak{P}([0, 2n])$ consisting of those subsets of form

$[k_0, k_1], [k_1, k_2], \dots, [k_{2f-1}, k_{2f}]$

that are either ladders or segments for $\mathbf{b}$. All segments of $\mathbf{b}$ appear in $\mathbf{I}$; they form the subset $S = S_{\mathbf{b}}$ of $\mathbf{I}$. The ladders in $\mathbf{I}$ form a subset $L = L_{\mathbf{b}}$ of $\mathbf{I}$. Note that $\mathbf{I} = S \sqcup L$. We say that $\mathbf{i}_1 \in S, \mathbf{i}_2 \in L$ are *adjacent* if they have a non-empty intersection (necessarily one of the k_t).

2.12. Let K be a subset of S and let $\mathbf{a} = \mathbf{a}(K)$ be as in 2.8. As in 1.12, we see that

$$\sum_{\mathbf{a};\mathbf{a}\sim\mathbf{b}} \Pi(\mathbf{a}) = \Pi(\mathbf{b}) \prod_{\mathbf{i}_2\in L} (1 - \mathbf{z}^{|\mathbf{i}_2|}) \sum_{Y,K;Y\subset L;K\subset S;Y\subset K^\sharp} \frac{1}{\prod_{\mathbf{i}\in Y\cup K}(\mathbf{z}^{-|\mathbf{i}|} - 1)}.$$

and that

$\Pi(\mathbf{b})\prod_{\mathbf{i}_2\in L}(1-\mathbf{z}^{|\mathbf{i}_2|}) = \mathbf{z}^{-2r^2+2B(\mathbf{a})}\prod_{\mathbf{i}\in\mathbf{I}}\phi(|\mathbf{i}|-2)^{-1}\prod_{\mathbf{j}}\phi(|\mathbf{j}|)^{-1}$
where $\mathbf{j}$ runs over the subsets of $[0,2n] - \cup_{[k,l]\in\mathbf{I}}[k+1,l-1]$ that are staircases of $\mathbf{b}$ (not containing 0) or ladders of $\mathbf{b}$. ($K^\sharp$ is defined as in 1.12.)

2.13. Let $\Lambda_{\mathbf{b},1}, B_{\mathbf{b},1}, \Lambda_{\mathbf{b},2}, B_{\mathbf{b},2}$ be defined as in 1.13. We define an (injective) map $S_{\mathbf{b}} \hookrightarrow B_{\mathbf{b},2}$ by $(b_k, b_{k+1}, b_{k+2}, \dots, b_l) \mapsto \{k,l\}$. This extends uniquely to an (injective) linear map of F_2-vector spaces $T : \mathfrak{P}(S_{\mathbf{b}}) \hookrightarrow \Lambda_{\mathbf{b}_2}$.

3. Combinatorics: type D_r.

3.1. We fix an integer $r \geq 4$. For any $n \geq 0$ we define $\Psi'_{2r;n}$ to be the set of all sequences of integers $\mathbf{a} = (a_0 \leq a_1 \leq a_2 \leq \dots \leq a_{2n+1})$ such that
$a_0 \geq 0; a_p \ll a_{p+2}$ for $p \in [0, 2n-1]$; $r = \sum_{p\in[0,2n+1]} a_p - (2n^2+2n)$.
There is a natural map $\Psi'_{2r,n} \to \Psi'_{2r,n+1}$ given by

$$(a_0 \leq a_1 \leq a_2 \leq \dots \leq a_{2n+1}) \mapsto$$
$$(0 \leq 0 \leq a_0 + 2 \leq a_1 + 2 \leq a_2 + 2 \leq \dots \leq a_{2n+1} + 2.)$$

This is a bijection if n is large enough (compared to r). We will denote by Ψ'_{2r} the limit of $\Psi'_{2r,n}$ as $n \to \infty$ (with respect to the maps above); we will fix n large enough so that $\Psi'_{2r,n} \xrightarrow{\sim} \Psi'_{2r}$ and we identify the last two sets. We may also assume that $a_0 = 0, a_1 = 0$ for any $\mathbf{a} \in \Psi'_{2r}$.

3.2. The ladders and staircases of $\mathbf{a} \in \Psi'_{2r}$ are defined as in 2.2 (replacing $2n$ by $2n+1$). Again, $[0, 2n+1]$ is a disjoint union of subsets that are either ladders or staircases.

3.3. We say that $\mathbf{a}, \mathbf{a}'$ in Ψ'_{2r} are *congruent* if

$$\{a_0, a_1, a_2 - 1, a_3 - 1, a_4 - 2, \dots, a_{2n+1} - n\}$$
$$= \{a'_0, a'_1, a'_2 - 1, a'_3 - 1, a'_4 - 2, \dots, a'_{2n+1} - n\}$$

as multisets. We then write $\mathbf{a} \sim \mathbf{a}'$.

LEMMA 3.4. *For $\mathbf{a}, \mathbf{a}' \in \Psi'_{2r}$, the conditions (a),(b),(c) below are equivalent:*
(a) $\mathbf{a} \sim \mathbf{a}'$;
(b) $\{a_{2p-1} - (p-1), a_{2p} - p\} = \{a'_{2p-1} - (p-1), a'_{2p} - p\}$ for $p \in [1,n]$ as multisets and $a_0 = a'_0, a_{2n+1} = a'_{2n+1}$;
(c) $a_0 = a'_0, a_{2n+1} = a'_{2n+1}$ and, for any $p \in [1,n]$, $\begin{pmatrix} a_{2p-1} & a_{2p} \\ a'_{2p-1} & a'_{2p} \end{pmatrix}$ is of the form (i) $\begin{pmatrix} s & t \\ s & t \end{pmatrix}$, or (ii) $\begin{pmatrix} s & s \\ s-1 & s+1 \end{pmatrix}$, or (iii) $\begin{pmatrix} s-1 & s+1 \\ s & s \end{pmatrix}$.
The proof is similar to that of 1.4.

3.5. We say that $\mathbf{b} = (b_0 \leq b_1 \leq \dots b_{2n+1}) \in \Psi'_{2r}$ is *special* if $b_{2p-1} < b_{2p}$ for all $p \in [1,n]$. Let Ψ'^0_{2r} be the set of special elements of Ψ'_{2r}.

LEMMA 3.6. *Given $\mathbf{a} \in \Psi'_{2r}$, there is a unique $\mathbf{b} \in \Psi'^0_{2r}$ such that $\mathbf{b} \sim \mathbf{a}$. We have*
$b_{2p-1} = a_{2p-1}, b_{2p} = a_{2p}$, if $p \in [1,n], a_{2p-1} < a_{2p}$;
$b_{2p-1} = a_{2p-1} - 1, b_{2p} = a_{2p} + 1$, if $p \in [1,n], a_{2p-1} = a_{2p}$;

$b_0 = a_0$;
$b_{2n+1} = a_{2n+1}$.
The proof is immediate (using Lemma 3.4).

3.7. The *segments* of $\mathbf{b} \in \Psi'^0_{2r}$ are defined just as in 2.7 (replacing $2n$ by $2n+1$). Let $S_\mathbf{b}$ be the set of segments of $\mathbf{b}$. Clearly, the segments of $\mathbf{b}$ are disjoint subsets of $[0, 2n+1]$.

PROPOSITION 3.8. *Let $\mathbf{b} \in \Psi'^0_{2r}$. There is a 1-1 correspondence between $\mathfrak{P}(S_\mathbf{b})$ and the set $\{\mathbf{a} \in \Psi'_{2r} | \mathbf{a} \sim \mathbf{b}\}$: to a subset K of $S_\mathbf{b}$ corresponds the sequence $\mathbf{a} = \mathbf{a}(K) \in \Psi'_{2r}$ defined by*
$(a_k, a_{k+1}, a_{k+2}, \ldots, a_l) = (a+1, a+1, a+3, a+3, a+5, a+5, \ldots, a+l-k, a+l-k)$
if $[k,l] \in K$ and $a_t = b_t$ if $t \notin \cup_{[k,l]\in K}[k,l]$.

This follows easily from Lemmas 3.4, 3.6.

3.9. We fix $\mathbf{b} \in \Psi'^0_{2r}$. An integer $k \in [0, 2n+1]$ is said to be *isolated* (for $\mathbf{b}$) if either:

(a) $k = 2n+1$ and $b_{2n} < b_{2n+1}$, or
(b) k is even, $0 < k$ and $b_{k-1} \ll b_k < b_{k+1}$, or
(c) k is odd, $k < 2n+1$ and $b_{k-1} < b_k \ll b_{k+1}$.

LEMMA 3.10. *$\{k \in [0, 2n+1] | k$ isolated$\} = \{k_0 < k_1 < k_2 < \cdots < k_{2f+1}\}$ where $k_t = t \mod 2$ for $t \in [0, 2f+1]$.*

The proof is routine; it will be omitted.

3.11. Assume that $f \geq 0$ in the previous lemma. Let $\Lambda_\mathbf{b}$ be the quotient of the F_2-vector space $\mathfrak{P}_{\text{ev}}(\{k_0, k_1, \ldots, k_{2f+1}\})$ by the line spanned by $\{k_0, k_1, \ldots, k_{2f+1}\}$. Let $\Lambda_{\mathbf{b},2}$ be the subspace of $\Lambda_\mathbf{b}$ spanned by the images of the sets $\{k_{2g-1}, k_{2g}\}$ with $g \in [1, f]$. (These form a basis $B_{\mathbf{b},2}$.) We define an (injective) map $S_\mathbf{b} \hookrightarrow B_{\mathbf{b},2}$ by $(b_k, b_{k+1}, b_{k+2}, \ldots, b_l) \mapsto \{k, l\}$. This extends uniquely to an (injective) linear map of F_2-vector spaces $T : \mathfrak{P}(S_\mathbf{b}) \hookrightarrow \Lambda_{\mathbf{b}_2}$.

4. Comparison of types C_r and B_r.

4.1. We fix $r \geq 2$. We will choose the same (large) n in 1.1, 2.1. There is a 1-1 correspondence $\Psi^0_{2r} \leftrightarrow \Psi'^0_{2r+1}$ given by
$\mathbf{b} = (b_0, b_1, \ldots, b_{2n}) \leftrightarrow \mathbf{b}' = (b'_0, b'_1, \ldots, b'_{2n})$
where $b_k = b'_k$, if k is even, $b_k = b'_k + 1$, if k is odd.

PROPOSITION 4.2. *If $\mathbf{b}, \mathbf{b}'$ are as above, then*

(a)
$$\sum_{\mathbf{a}\in\Psi_{2r};\mathbf{a}\sim\mathbf{b}} \Pi(\mathbf{a}) = \sum_{\mathbf{a}'\in\Psi'_{2r+1};\mathbf{a}'\sim\mathbf{b}'} \Pi(\mathbf{a}').$$

First note that the integers $k_0 < k_1 < \cdots < k_{2f}$ in $[0, 2n]$ that are isolated for $\mathbf{b}$ (see 1.9) are the same as the corresponding integers for $\mathbf{b}'$ (see 2.9). Let us write L', S' for the sets denoted L, S in 1.11 (to distinguish them from the sets L, S in 2.11). It is clear that $L' = S$ and $S' = L$. Moreover, if $K \subset S = L'$ and $Y \subset L = S'$ then the conditions $Y \subset K^\sharp$ and $K \subset Y^\sharp$ are equivalent: they both are equivalent to the condition that any set in K is disjoint from any set in Y. Hence we have

$$\sum_{Y,K;Y\subset L;K\subset S;Y\subset K^\sharp} \frac{1}{\prod_{\mathbf{i}\in Y\cup K}(\mathbf{z}^{-|\mathbf{i}|} - 1)}$$
$$= \sum_{Y',K';Y'\subset L';K'\subset S';Y'\subset K'^\sharp} \frac{1}{\prod_{\mathbf{i}\in Y'\cup K'}(\mathbf{z}^{-|\mathbf{i}|} - 1)}.$$

Let $P = [0, 2n] - \cup_{[k,l]\in L\cup S}[k+1, l-1]$. We note that there is 1-1 correspondence between the set of ladders of cardinal ≥ 2 of $\mathbf{b}$ contained in P (but not containing 0) and the set of staircases of $\mathbf{b}'$ contained in P (but not containing 0); in this correspondence a ladder has cardinal equal or bigger by one than that of the corresponding staircase. Similarly, there is 1-1 correspondence between the set of staircases of $\mathbf{b}$ contained in P and the set of ladders of cardinal ≥ 2 of $\mathbf{b}'$ contained in P; again, in this correspondence a ladder has cardinal equal or bigger by one than that of the corresponding staircase.

Finally, one checks from the definitions that $B(\mathbf{b}) = B(\mathbf{b}')$ (where the left hand side is as in 1.1 and the right hand side is as in 2.1).

We now use the results in 1.12 and 2.12, taking into account the arguments above. The proposition follows.

5. Equivariant homology.

5.1. Assume that X is an algebraic variety over $\mathbf{C}$ with an algebraic action of a linear algebraic group $\mathcal{G}$ over $\mathbf{C}$, such that X is a union of finitely many $\mathcal{G}$-orbits $X_1, X_2, \dots, X_k$. We can assume that the numbering is chosen so that $X_1 \cup X_2 \cup \dots \cup X_s$ is closed in X and has dimension equal to $\dim X_s$ for $s = 1, \dots, k$. Let $x_s \in X_s$ and let $\mathcal{G}_s$ be the stabilizer of x_s in $\mathcal{G}$ for $s = 1, \dots, k$. Let $\bar{\mathcal{G}}_s$ be the reductive quotient of $\mathcal{G}_s$ and let $\bar{\mathfrak{g}}_s$ be its Lie algebra. Let $S^{j'}(\bar{\mathfrak{g}}_s)^{\bar{\mathcal{G}}_s}$ be the space of $\bar{\mathcal{G}}_s$-invariant elements on the j'-component of the symmetric algebra of $\bar{\mathfrak{g}}_s$ if $j' \in \mathbf{N}$ and 0, if $j' \notin \mathbf{N}$.

LEMMA 5.2. *For j odd we have $H_j^{\mathcal{G}}(X) = 0$. If $\mathbf{z}$ is an indeterminate, we have*

$$\sum_{j\geq 0} \dim H_{2j}^{\mathcal{G}}(X)\mathbf{z}^j = \sum_{s=1}^{k}\sum_{j\geq 0} \dim S^j(\bar{\mathfrak{g}}_s)^{\bar{\mathcal{G}}_s}\mathbf{z}^{j+\dim X-\dim X_s}.$$

By [L5, 1.6,1.8(c),1.11], we have

(a) $H_j^{\mathcal{G}}(X_s) = H_j^{\mathcal{G}_s}(pt) = H^j_{\bar{\mathcal{G}}_s}(pt) = S^{j/2}(\bar{\mathfrak{g}}_s)^{\bar{\mathcal{G}}_s}$.

Here *pt* denotes a point. Using (a) and the long exact sequence [L5, 1.5] we see by induction on s that $H^{\mathcal{G}}_{j+2\dim X_s}(X_1 \cup X_2 \cup \dots \cup X_s)$ is zero for odd j and is isomorphic to

$H^{\mathcal{G}}_{j+2\dim X_{s-1}}(X_1 \cup X_2 \cup \dots \cup X_{s-1}) \oplus H^{\mathcal{G}}_{j+2\dim X_s}(X_s)$

for $s = 2, \dots, k$. Applying this repeatedly, we see that

$\dim H^{\mathcal{G}}_{j+2\dim X}(X) = \sum_{s=1}^{k} \dim H^{\mathcal{G}}_{j+2\dim X_s}(X_s)$.

This, together with (a), implies the lemma.

LEMMA 5.3. *Let us fix $s \in [1, k]$. Assume that $\bar{\mathcal{G}}_s = \prod_{p=1}^{p_0} H_p$ where each H_p is one of the groups $SO_{N_p}(\mathbf{C})$ (N_p odd), $Sp_{N_p}(\mathbf{C})$ (N_p even), or $O_{N_p}(\mathbf{C})$. Then*

$$\sum_{j\geq 0} \dim S^j(\bar{\mathfrak{g}}_s)^{\bar{\mathcal{G}}_s}\mathbf{z}^j = \prod_{p=1}^{p_0} \phi(N_p)^{-1}$$

where $\phi(N_p)$ is as in 1.2.

This is well known.

6. Comments on Theorems 0.2, 0.4 and 0.9.

6.1. Let G be as in 0.1. The set of unipotent classes in G, assumed to be of type C_r, B_r or D_r, is in a natural 1-1 correspondence with the set $\Psi_{2r}, \Psi'_{2r+1}, \Psi'_{2r}$ (respectively) except that in the case of D_r each sequence of Ψ'_{2r} without isolated elements should be considered twice. (See [L4, Sec.11].) That description is particularly well suited to calculating the Springer correspondence in those cases. In this language, the problem of deciding when a unipotent classes C_1 in G satisfy the condition that ρ_{C_1} belongs to a given two-sided cell becomes the problem of deciding when an element of Ψ_{2r} (or Ψ'_{2r+1} or Ψ'_{2r}) is congruent to a given element of Ψ^0_{2r} (or Ψ'^0_{2r+1} or Ψ'^0_{2r}). This is described quite explicitly by 1.8, 2.8, 3.8 and yields Theorem 0.2 in these cases, since the condition that C_1 belongs to a given special piece has been decribed explicitly in [Sp]. The maps $C_1 \mapsto g$ in Theorem 0.4 can in these cases be identified with the maps T in 1.13, 2.13, 3.11. Hence Theorem 0.4 also holds in our cases. Of course, when G is of type A, both 0.2 and 0.4 are trivial.

In the case where G is of exceptional type, the proof of 0.2, 0.4 consists simply in analyzing existing tables. In the following five subsections we will indicate for each of the exceptional groups the map $C_1 \mapsto g$ of 0.4 (from $\gamma(C)$ to the set of conjugacy classes in a symmetric group S_r). We only consider the case where $\tilde{C} \neq C$. We shall use the notation of [Ca] for unipotent classes. We group together the unipotent classes in a fixed $\gamma(C)$ and for each C_1 in the group we specify g by a partition of r. (This determines in each case the value of r where $\mathcal{G}_{\mathbf{c}} = \mathcal{G}'_{\mathbf{c}} = S_r$.) The special class appears first in each group.

6.2. Type E_8.

$$A_2 \quad (1,1); 3A_1 \quad (2).$$

$$A_2 + A_1 \quad (1,1); 4A_1 \quad (2).$$

$$2A_2 \quad (1,1); A_2 + 3A_1 \quad (2).$$

$$D_4(a_1) \quad (1,1,1); A_3 + A_1 \quad (2,1); 2A_2 + A_1 \quad (3).$$

$$D_4(a_1) + A_1 \quad (1,1,1); \quad A_3 + 2A_1 \quad (2,1); 2A_2 + 2A_1 \quad (3).$$

$$D_4(a_1) + A_2 \quad (1,1); A_3 + A_2 + A_1 \quad (2).$$

$$D_5(a_1) \quad (1,1); D_4 + A_1 \quad (2).$$

$$A_4 + 2A_1 \quad (1,1); 2A_3 \quad (2).$$

$$E_6(a_3) \quad (1,1); A_5 \quad (2).$$

$$\begin{cases} E_8(a_7) \quad (1,1,1,1,1); E_7(a_5) \quad (2,1,1,1); E_6(a_3) + A_1 \quad (3,1,1); D_6(a_2) \quad (2,2,1); \\ D_5(a_1) + A_2 \quad (4,1); A_5 + A_1 \quad (3,2); A_4 + A_3 \quad (5). \end{cases}$$

$$D_6(a_1) \quad (1,1); D_5 + A_1 \quad (2).$$

$$E_7(a_3) \quad (1,1); D_6 \quad (2).$$

$$E_8(b_6) \quad (1,1); A_7 \quad (2).$$

$$E_8(b_5) \quad (1,1,1); E_7(a_2) \quad (2,1); E_6 + A_1 \quad (3).$$

$$E_8(a_5) \quad (1,1); D_7 \quad (2).$$

$$E_8(a_3) \quad (1,1); E_7 \quad (2).$$

6.3. Type E_7.

$$A_2 \quad (1,1);(3A_1)' \quad (2).$$

$$A_2 \times A_1 \quad (1,1);4A_1 \quad (2).$$

$$D_4(a_1) \quad (1,1,1);(A_3+A_1)' \quad (2,1);2A_2+A_1 \quad (3).$$

$$D_4(a_1)+A_1 \quad (1,1);A_3+2A_1 \quad (2).$$

$$D_5(a_1) \quad (1,1);D_4+A_1 \quad (2).$$

$$E_6(a_3) \quad (1,1);(A_5)' \quad (2).$$

$$E_7(a_5) \quad (1,1,1);D_6(a_2) \quad (2,1);A_5+A_1 \quad (3).$$

$$E_7(a_3) \quad (1,1);D_6 \quad (2).$$

6.4. Type E_6.

$$A_2 \quad (1,1);3A_1 \quad (2).$$

$$D_4(a_1) \quad (1,1,1);A_3+A_1 \quad (2,1);2A_2+A_1 \quad (3).$$

$$E_6(a_3) \quad (1,1);A_5 \quad (2).$$

6.5. Type F_4.

$$\tilde{A}_1 \quad (1,1);A_1 \quad (2).$$

$$F_4(a_3) \quad (1,1,1,1);C_3(a_1) \quad (2,1,1);\tilde{A}_2+A_1 \quad (3,1);B_2 \quad (2,2);A_2+\tilde{A}_1 \quad (4).$$

6.6. Type G_2.

$$G_2(a_1) \quad (1,1,1);\tilde{A}_1 \quad (2,1);A_1 \quad (3).$$

6.7. Proof of Theorem 0.9. Using 5.2, we see that $H_j^G(\tilde{C}) = H_j^{G'}(\tilde{C}') = 0$ for j odd. Using 5.2 and 5.3, we see that $\sum_{j\geq 0} \dim H_{2j}^G(\tilde{C})$ is equal to $\mathbf{z}^{2r^2-2B(\mathbf{b})}$ times the left hand side of 4.2(a), where $\mathbf{b} \in \Psi_{2r}$ corresponds to the unipotent class $C \subset G$ and $B(\mathbf{b})$ is as in 1.1. Similarly, using 5.2 and 5.3, we see that $\sum_{j\geq 0} \dim H_{2j}^{G'}(\tilde{C}')$ is equal to $\mathbf{z}^{2r^2-2B(\mathbf{b}')}$ times the left hand side of 4.2(a), where $\mathbf{b}' \in \Psi'_{2r+1}$ corresponds to the unipotent class $C' \subset G'$ and $B(\mathbf{b}')$ is as in 2.1. Using now 4.2 and the equality $B(\mathbf{b}) = B(\mathbf{b}')$ we see that

$\sum_{j\geq 0} \dim H_{2j}^G(\tilde{C}) = \sum_{j\geq 0} \dim H_{2j}^{G'}(\tilde{C}')$.

This proves Theorem 0.9.

Theorem 0.9 shows that in the setup of 0.1, the equivariant homology Betti numbers of the pieces $\tilde{C}$ (with respect to the conjugation action of G) depend only on the Weyl group.

6.8. Let us now replace the ground field $\mathbf{C}$ by the algebraic closure of a finite field F_q and assume that G has a fixed split structure over F_q. Then the special pieces $\tilde{C} \subset G$ are again defined as in 0.1 (in small characteristics we must use the definition [L2] of the Springer representations). Now, following Mizuno [Mi] and Spaltenstein [Sp, III.5.2] there is a natural order preserving imbedding from the set of unipotent classes in the corresponding group over $\mathbf{C}$, to the set of unipotent classes in G. Let us call *MS-classes* the classes in the image of this map. For any MS-class C of G let $\hat{C}$ be the subset of the unipotent variety of G consisting of elements in the closure of C which are not in the closure of any MS-class distinct from C and contained in the closure of C. From the results in [Mi,Sp] one checks that the sets $\hat{C}$ form a partition of the unipotent variety (into locally closed subvarieties). (In good characteristic, the MS-pieces are the same as the unipotent classes.) One can also check that each special piece is a union of MS-pieces. We have the following result:

(a) *the number of F_q-rational points of an MS-piece is a polynomial in q that is independent of the characteristic (that is, it depends only on the corresponding unipotent class in the group over* $\mathbf{C}$*).*

We will indicate the necessary calculations (based on Table 10 in [Mi]) in the case of E_8. If we restrict ourselves to characteristic $\neq 2,3$, there is nothing to prove. In characteristic $p=2$ or $p=3$ the MS-pieces consist of one or two unipotent classes and we use the identities:

$$\frac{1}{q^{26}}+\frac{1}{(q^2-1)q^{26}}=\frac{1}{2(q+1)q^{25}}+\frac{1}{2(q-1)q^{25}},$$
$$\frac{1}{q^{34}}+\frac{1}{(q^2-1)q^{34}}=\frac{1}{2(q+1)q^{33}}+\frac{1}{2(q-1)q^{33}},$$
$$\frac{1}{2(q+1)(q^2-1)q^{35}}+\frac{1}{2(q-1)(q^2-1)q^{35}}=\frac{1}{(q^2-1)^2q^{34}},$$
$$\frac{1}{(q^2-1)q^{48}}+\frac{1}{(q^2-1)(q^6-1)q^{48}}=\frac{1}{2(q^2-1)(q^3-1)q^{45}}+\frac{1}{2(q^2-1)(q^3+1)q^{45}},$$
$$\frac{1}{(q^2-1)(q^4-1)q^{64}}+\frac{1}{(q^2-1)^2(q^4-1)q^{64}}=\frac{1}{2(q+1)(q^2-1)(q^4-1)q^{63}}+\frac{1}{2(q-1)(q^2-1)(q^4-1)q^{63}},$$

for $p=2$, and

$$\frac{1}{q^{30}}+\frac{1}{(q^2-1)q^{30}}=\frac{1}{(q^2-1)q^{28}}$$

for $p=3$. Similar arguments apply for the other types. (For classical types in characteristic 2 an MS-piece has a power of 2 unipotent classes.)

6.9. In [L2, Conj. 3] it was conjectured that

(a) *the number of F_q-rational points of a special piece $\tilde{C}$ is a polynomial in q that depends only on the Weyl group.*

We can now prove this as follows. Using 6.8(a), we are reduced to the case where the characteristic is large. In that case, we only have to compare the number of points in corresponding special pieces in type B_r and C_r. But this follows from exactly the same computation as the one in 6.7.

6.10. In view of 6.9(a), one can expect that the polynomials $|\tilde{C}(F_q)|$ have a meaning also when W is replaced by a finite non-crystallographic Coxeter group. (There should be one such polynomial for each two-sided cell of W.) There are obvious candidates for the polynomials attached to the two-sided cells of the trivial element or of the longest element w_0, namely

$$q^{e_1+e_2+\dots+e_l-l}(q^{e_1+1}-1)(q^{e_2+1}-1)\dots(q^{e_l+1}-1)$$

or 1. Here $e_1,e_2,\dots,e_l$ are the exponents of W. On the other hand, the polynomial attached to the cell containing w_0 times a simple reflection should be

$$(q^h-1)(q^{e_1-1}+q^{e_2-1}+\dots+q^{e_l-1})$$

where h is the Coxeter number. (This is suggested by [L2, (3.1)].) In the case when W is a dihedral group, the three polynomials above are all we need since there are only three two-sided cells. The sum of the three polynomials is equal to q^{hl}, as it should be.

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