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A SHARP ESTIMATE FOR MULTILINEAR MARCINKIEWICZ INTEGRAL OPERATOR*

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Abstract. A sharp estimate for multilinear Marcinkiewicz integral operator is obtained. By using this estimate, we obtain the weighted norm inequalities and $L\log L$ type estimate for the multilinear operator.

Key words. Multilinear operator; Marcinkiewicz integral operator; Sharp estimate; BMO

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1. Introduction. Let T be a singular integral operator. In[1][2][3], Cohen and Gosselin studied the $L^p(p>1)$ boundedness of the multilinear singular integral operator T^A defined by

$$T^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}K(x,y)f(y)dy.$$

In[8], Hu and Yang obtain a variant sharp estimate for the multilinear singular integral operator. The main purpose of this paper is to prove a sharp estimate for some multilinear operator related to Marcinkiewicz integral operator. As the applications, we obtain the weighted norm inequalities and $L\log L$ type estimate for the multilinear operator.

2. Notations and results. Suppose that S^{n-1} is the unit sphere of $R^n(n\geq 2)$ equipped with normalized Lebesgue measure $d\sigma=d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip$_\gamma$ condition on $S^{n-1}(0<\gamma\leq 1)$, i.e.

$$|\Omega(x')-\Omega(y')|\leq M|x'-y'|^\gamma, \quad x',y'\in S^{n-1};$$

(ii) $\int_{S^{n-1}}\Omega(x')dx'=0$.

Let m be a positive integer and A be a function on R^n. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\Omega^A(f)(x)=\left[\int_0^\infty |F_t^A(f)(x)|^2\frac{dt}{t^3}\right]^{1/2},$$

where

$$F_t^A(f)(x)=\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\frac{R_{m+1}(A;x,y)}{|x-y|^m}f(y)dy$$

and

$$R_{m+1}(A;x,y)=A(x)-\sum_{|\alpha|\leq m}\frac{1}{\alpha!}D^\alpha A(y)(x-y)^\alpha.$$

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Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz integral operator (see [12]).

Let H be the Hilbert space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3}\right)^{1/2} < \infty\right\}$, then for each fixed $x \in R^n$, $F_t^A(f)(x)$ and $F_t(f)(x)$ may be viewed as a mapping from $(0, +\infty)$ to H, and it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

Note that when $m = 0$, μ_Ω^A is just the commutator generated by Macinkiewicz integral and a function A(see [10][16]). while when $m > 0$, it is non-trivial generalizations of the commutator. It has been known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-5]).

First, let us introduce some notation(see[7][11][13]).

For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

where, and in what follows, Q will denote a cube with sides parallel to the axes, and $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that

$$f^\#(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$. For $0 < r < \infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$, for $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \quad \text{when} \quad k \ge 2.$$

Let B be a Young function and $\tilde{B}$ be the complementary associated to B, we denote that, for a function f,

$$||f||_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}$$

and the maximal function by

$$M_B(f)(x) = \sup_{x \in Q} ||f||_{B,Q}.$$

The main Young function to be using in this paper is $B(t) = t(1 + log^+ t)$ and its complementary $\tilde{B}(t) = exp\ t$, the corresponding maximal denoted by M_{LlogL} and M_{expL}. We have the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le ||f||_{B,Q}||g||_{\tilde{B},Q}$$

and the following inequality (in fact they are equivalent), for any $x \in R^n$,

$$M_{LlogL}(f)(x) \le CM^2(f)(x)$$

and the following inequalities, for all cube Q and any $b \in BMO(R^n)$,

$$||b - b_Q||_{\exp L,Q} \le C||b||_{BMO}$$

and

$$|b_{2^{k+1}Q} - b_{2Q}| \le 2k||b||_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$(see[7]).

Now we are in position to state our results.

THEOREM 1. Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then for any $0 < r < p < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,

$$(\mu_\Omega^A(f))_r^\#(x) \le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}(M_p(\mu_\Omega(f))(x) + M^2(f)(x)).$$

THEOREM 2. Let $1 < p < \infty$, $w \in A_p$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then μ_Ω^A is bounded on $L^p(w)$, that is

$$||\mu_\Omega^A(f)||_{L^p(w)} \le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^p(w)}.$$

THEOREM 3. Let $w \in A_1$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that for each $\lambda > 0$,

$$\begin{aligned} &w(\{x \in R^n : \mu_\Omega^A(f)(x) > \lambda\}) \\ &\le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}\int_{R^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx. \end{aligned}$$

REMARK. In Theorem 1, the sharp estimate for μ_Ω^A is given. As in [8][10], Theorem 2 and 3 follow from Theorem 1. So we only need to prove Theorem 1.

3. Some lemmas. We begin with some preliminary lemmas.

LEMMA 1. (Kolmogorov, [7, p.485]) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that

$$||f||_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q},$$
$$N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p} p / ||\chi_E||_{L^r}, (1/r = 1/p - 1/q),$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} ||f||_{WL^q}.$$

LEMMA 2. ([11, p.165]) Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function f and for all $\lambda > 0$,

$$w(\{y \in R^n : M^2 f(y) > \lambda\}) \leq C\lambda^{-1} \int_{R^n} |f(y)|(1 + \log^+(\lambda^{-1}|f(y)|))w(y)dy.$$

LEMMA 3. ([3, p.448]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

LEMMA 4. Let $1 < p < \infty$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$. Then μ_Ω^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is

$$||\mu_\Omega^A(f)||_{L^q} \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^p}.$$

Proof. By the Minkowski inequality and the condition on Ω, we have

$$\begin{aligned} \mu_\Omega^A(f)(x) \leq & \int_{R^n} \frac{|\Omega(x-y)||R_{m+1}(A; x, y)|}{|x-y|^{m+n-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \\ \leq & C \int_{R^n} \frac{|R_{m+1}(A; x, y)|}{|x-y|^{m+n}} |f(y)| dy. \end{aligned}$$

Thus, the lemma follows from [4][5].

4. Proof of Theorems. First, we prove Theorem 1.

Proof of Theorem 1. Fix $\tilde{x} \in R^n$. Let $Q = Q(x_0, l)$ be a cube centered at x_0 and having side length l such that $\tilde{x} \in Q$. It is suffice to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |\mu_\Omega^A(f)(x) - C_0|^r dx \right)^{1/r} \leq C(M_p(\mu_\Omega(f))(\tilde{x}) + M^2(f)(\tilde{x})).$$

Set $\tilde{Q} = 10\sqrt{n}Q$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. Let $f_1 = f\chi_{\tilde{Q}}$, $f_2 = f\chi_{R^n\setminus\tilde{Q}}$. We write, for $x \in Q$,

$$\begin{aligned} F_t^A(f)(x) = & \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{R_m(A;x,y)}{|x-y|^m} f(y)dy \\ & - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{(x-y)^\alpha D^\alpha \tilde{A}(y)}{|x-y|^m} f_1(y)dy \\ & - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{(x-y)^\alpha D^\alpha \tilde{A}(y)}{|x-y|^m} f_2(y)dy, \end{aligned}$$

then

$$\begin{aligned} & \left| \mu_\Omega^A(f)(x) - \mu_\Omega \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x_0-\cdot)^\alpha}{|x_0-\cdot|^m} D^\alpha \tilde{A} f_2 \right)(x_0) \right| \\ = & \left| ||F_t^A(f)(x)|| - || \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x_0-\cdot)^\alpha}{|x_0-\cdot|^m} D^\alpha \tilde{A} f_2 || \right| \\ \le & \left\| F_t \left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f \right)(x) \right\| + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| F_t \left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \right)(x) \right\| \\ & + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| F_t \left(\frac{(x-\cdot)^\alpha D^\alpha \tilde{A}}{|x-\cdot|^m} f_2 \right)(x) - F_t \left(\frac{(x_0-\cdot)^\alpha D^\alpha \tilde{A}}{|x_0-\cdot|^m} f_2 \right)(x_0) \right\| \\ \equiv & I(x) + II(x) + III(x), \end{aligned}$$

thus,

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q \left| \mu_\Omega^A(f)(x) - \mu_\Omega \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x_0-\cdot)^\alpha}{|x_0-\cdot|^m} D^\alpha \tilde{A} f_2 \right)(x_0) \right|^r dx \right)^{1/r} \\ \le & \left(\frac{C}{|Q|} \int_Q I(x)^r dx \right)^{1/r} + \left(\frac{C}{|Q|} \int_Q II(x)^r dx \right)^{1/r} + \left(\frac{C}{|Q|} \int_Q III(x)^r dx \right)^{1/r} \\ \equiv & I + II + III. \end{aligned}$$

Now, let us estimate I, II and III, respectively. First, using Lemma 3, we have

$$\begin{aligned} I \le & \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \left(\frac{1}{|Q|} \int_Q (\mu_\Omega(f)(x))^p dx \right)^{1/p} \\ \le & C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M_p(\mu_\Omega(f))(\tilde{x}); \end{aligned}$$

For II, by Lemma 1 and the weak type (1,1) of μ_Ω (see[6][14]), we have

$$\begin{aligned}
II \le & C\sum_{|\alpha|=m}|Q|^{-1}\frac{||\mu_\Omega(D^\alpha\tilde{A}f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}} \le C\sum_{|\alpha|=m}|Q|^{-1}||\mu_\Omega(D^\alpha\tilde{A}f_1)||_{WL^1}\\
\le & C\sum_{|\alpha|=m}|Q|^{-1}\int_{\tilde{Q}}|D^\alpha\tilde{A}(y)||f(y)|dy \le C\sum_{|\alpha|=m}||D^\alpha A||_{\exp L,\tilde{Q}}||f||_{LlogL,\tilde{Q}}\\
\le & C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M_{L\log L}f(\tilde{x}) \le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x});
\end{aligned}$$

To estimate III, we write, for $|\alpha|=m$,

$$\begin{aligned}
&\left|\left|F_t\left(\frac{(x-\cdot)^\alpha D^\alpha\tilde{A}}{|x-\cdot|^m}f_2\right)(x)-F_t\left(\frac{(x_0-\cdot)^\alpha D^\alpha\tilde{A}}{|x_0-\cdot|^m}f_2\right)(x_0)\right|\right|\\
=&\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\frac{(x-y)^\alpha D^\alpha\tilde{A}(y)}{|x-y|^m}f_2(y)dy\right.\right.\\
&\left.\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\frac{(x_0-y)^\alpha D^\alpha\tilde{A}(y)}{|x_0-y|^m}f_2(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
\le&\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f_2(y)||D^\alpha\tilde{A}(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}|f_2(y)||D^\alpha\tilde{A}(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{(x-y)^\alpha\Omega(x-y)}{|x-y|^{m+n-1}}\right.\right.\right.\\
&\qquad\left.\left.\left.-\frac{(x_0-y)^\alpha\Omega(x_0-y)}{|x_0-y|^{m+n-1}}\right||f(y)||D^\alpha\tilde{A}(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
=&III_1+III_2+III_3.
\end{aligned}$$

Note that $|x-y|\sim|x_0-y|$ for $x\in\tilde{Q}$ and $y\in R^n\setminus\tilde{Q}$. By the condition on Ω, and similar to the proof of Lemma 4, we obtain

$$
\begin{aligned}
III_1 \leq & C\int_{R^n} \frac{|f_2(y)||D^\alpha \tilde{A}(y)|}{|x-y|^{n-1}} \left(\int_{|x-y|\leq t<|x_0-y|} \frac{dt}{t^3}\right)^{1/2} dy \\
\leq & C\int_{R^n} \frac{|f_2(y)||D^\alpha \tilde{A}(y)|}{|x-y|^{n-1}} \left(\frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2}\right)^{1/2} dy \\
\leq & C\int_{R^n} \frac{|f_2(y)||D^\alpha \tilde{A}(y)|}{|x-y|^{n-1}} \frac{|x_0-x|^{1/2}}{|x-y|^{3/2}} dy \\
\leq & C\sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|Q|^{1/(2n)}}{|x_0-y|^{n+1/2}} |D^\alpha \tilde{A}(y)||f(y)|dy \\
\leq & C\sum_{k=0}^\infty 2^{-k/2} \frac{1}{|2^{k+1}\tilde{Q}|} \int_{2^{k+1}\tilde{Q}} (|D^\alpha \tilde{A}(y) - (D^\alpha A)_{2^{k+1}\tilde{Q}}| \\
& \qquad + |(D^\alpha A)_{2^{k+1}\tilde{Q}} - (D^\alpha A)_{\tilde{Q}}|)|f(y)|dy \\
\leq & C\sum_{k=1}^\infty k2^{-k/2} (||D^\alpha A||_{\exp L, 2^k\tilde{Q}} ||f||_{LlogL, 2^k\tilde{Q}} + ||D^\alpha A||_{BMO} Mf(\tilde{x})) \\
\leq & C\sum_{k=1}^\infty k2^{-k/2} ||D^\alpha A||_{BMO} M_{L\log L}(f)(\tilde{x}) \\
\leq & C||D^\alpha A||_{BMO} M^2(f)(\tilde{x});
\end{aligned}
$$

Similarly, we have $III_2 \leq C||D^\alpha A||_{BMO} M^2(f)(\tilde{x})$.

For III_3, by the following inequality (see [14]):

$$
\left|\frac{(x-y)^\alpha \Omega(x-y)}{|x-y|^{m+n-1}} - \frac{(x_0-y)^\alpha \Omega(x_0-y)}{|x_0-y|^{m+n-1}}\right| \leq C\left(\frac{|x-x_0|}{|x_0-y|^n} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),
$$

we gain

$$
\begin{aligned}
III_3 \leq & C|Q|^{1/n} \int_{R^n} \frac{|f_2(y)||D^\alpha \tilde{A}(y)|}{|x_0-y|^n} \left(\int_{|x_0-y|\leq t, |x-y|\leq t} \frac{dt}{t^3}\right)^{1/2} dy \\
& + C|Q|^{\gamma/n} \int_{R^n} \frac{|f_2(y)||D^\alpha \tilde{A}(y)|}{|x_0-y|^{n-1+\gamma}} \left(\int_{|x_0-y|\leq t, |x-y|\leq t} \frac{dt}{t^3}\right)^{1/2} dy \\
\leq & C\left(\int_{R^n} \frac{|Q|^{1/n}|D^\alpha \tilde{A}(y)|}{|x_0-y|^{n+1}} |f(y)|dy + \int_{R^n} \frac{|Q|^{\gamma/n}|D^\alpha \tilde{A}(y)|}{|x_0-y|^{n+\gamma}} |f(y)|dy\right) \\
\leq & C\sum_{k=1}^\infty k2^{-k} (||D^\alpha A||_{\exp L, 2^k\tilde{Q}} ||f||_{LlogL, 2^k\tilde{Q}} + ||D^\alpha A||_{BMO} M(f)(\tilde{x})) \\
& + C\sum_{k=1}^\infty k2^{-\gamma k} (||D^\alpha A||_{\exp L, 2^k\tilde{Q}} ||f||_{LlogL, 2^k\tilde{Q}} + ||D^\alpha A||_{BMO} M(f)(\tilde{x})) \\
\leq & C||D^\alpha A||_{BMO} M^2(f)(\tilde{x});
\end{aligned}
$$

Thus,

$$III \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

From Theorem 1 and the weighted boundedness of μ_Ω and M, we may obtain the conclusion of Theorem 2.

From Theorem 1 and Lemma 2, we may obtain the conclusion of Theorem 3.

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