

## FANO MANIFOLDS WITH LONG EXTREMAL RAYS\*

MARCO ANDREATTA<sup>†</sup> AND GIANLUCA OCCHETTA<sup>†</sup>

**Abstract.** Let  $X$  be a Fano manifold of pseudoindex  $i_X$  whose Picard number is at least two and let  $R$  be an extremal ray of  $X$  with exceptional locus  $\text{Exc}(R)$ . We prove an inequality which bounds the length of  $R$  in terms of  $i_X$  and of the dimension of  $\text{Exc}(R)$  and we investigate the border cases.

In particular we classify Fano manifolds  $X$  of pseudoindex  $i_X$  obtained blowing up a smooth variety  $Y$  along a smooth subvariety  $T$  such that  $\dim T < i_X$ .

**Key words.** Fano manifolds, rational curves, extremal rays

**AMS subject classifications.** 14J45; 14J40, 14E30

**1. Introduction.** A smooth complex projective variety of dimension  $n$  is called Fano if its anticanonical bundle  $-K_X = \wedge^n TX$  is ample. The index of  $X$ ,  $r_X$ , is the largest natural number  $m$  such that  $-K_X = mH$  for some (ample) divisor  $H$  on  $X$ , while the pseudoindex  $i_X$  is defined as the minimum anticanonical degree of rational curves on  $X$  and it is an integral multiple of  $r_X$ .

The pseudoindex is related to the Picard number  $\rho_X$  of  $X$  by a conjecture which claims that  $\rho_X(i_X - 1) \leq n$ , with equality if and only if  $X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$ ; this conjecture appeared in [7] as a generalization of a similar one (with the index in place of the pseudoindex) proposed by Mukai in 1988.

A recent very important result, which can be considered a special case of the conjecture, states that  $i_X \geq n + 1$  if and only if  $X \simeq \mathbb{P}^n$  ([11, Corollary 0.4] or [15, Theorem 1.1]). A first step towards the proof of the conjecture was made by Wiśniewski in [24], where he proved that if  $i_X > \frac{n+2}{2}$  then  $\rho_X = 1$ . More recently several authors ([7], [21], [1], [9]) dealt with this problem but the general case is still open.

In this paper we investigate a related problem.

Let  $X$  be a smooth variety of dimension  $n$  and let  $R$  be an extremal ray of  $X$ . Let  $l(R) := \min\{-K_X \cdot C \mid C \text{ a rational curve in } R\}$  be the length of  $R$  and  $\text{Exc}(R) = \{x \in X \mid x \in C \text{ a rational curve in } R\}$  be its exceptional locus.

In general the length of an extremal ray is bounded above by  $n + 1$ , equality holding if and only if  $X \simeq \mathbb{P}^n$  (again by the results in [11] or [15]), while the length of an extremal ray whose associated contraction is birational is bounded above by  $n - 1$ , equality holding if and only if the associated contraction is the blow-up of a point in a smooth variety (see [2, Theorem 1.1]).

We first prove that, if  $X$  is a Fano manifold of pseudoindex  $i_X$  and  $\rho_X > 1$ , the following holds:

$$i_X + l(R) \leq \dim \text{Exc}(R) + 2. \quad (*)$$

Note that this is an improved statement of the conjecture in the case  $\rho_X = 2$ .

Then we investigate the cases in which equality holds. Equivalently we ask if on a Fano manifold of pseudoindex  $i_X$  an extremal ray  $R$  of maximal length does determine the structure of the variety, proving the following

---

\*Received March 22, 2005; accepted for publication October 27, 2005.

<sup>†</sup>Department of Mathematics, University of Trento, via Sommarive 14, I-38050 Povo, Italy (andreatt@science.unitn.it; occhetta@science.unitn.it).

**THEOREM 1.1.** *Let  $X$  be a Fano manifold of dimension  $n$ , pseudoindex  $i_X$  and Picard number  $\rho_X \geq 2$ , and let  $R$  be a fiber type or divisorial extremal ray such that*

$$i_X + l(R) = \dim \text{Exc}(R) + 2.$$

*Then  $X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$  or  $X \simeq \text{Bl}_{\mathbb{P}^t}(\mathbb{P}^n)$  with  $0 \leq t \leq \frac{n-3}{2}$ .*

We do not know how to prove a similar theorem if  $R$  is an extremal ray whose associated contraction is small (i.e.  $\dim \text{Exc}(R) \leq n - 2$ ). However if we replace in the assumptions the pseudoindex  $i_X$  with the index  $r_X$  then we have the following

**THEOREM 1.2.** *Let  $X$  be a Fano manifold of dimension  $n$ , index  $r_X$ , and Picard number  $\rho_X \geq 2$ , and let  $R$  be an extremal ray such that*

$$r_X + l(R) = \dim \text{Exc}(R) + 2.$$

*Then, denoted by  $e$  the dimension of  $\text{Exc}(R)$ , we have  $X = \mathbb{P}_{\mathbb{P}^k}(\mathcal{O}^{\oplus e-k+1} \oplus \mathcal{O}(1)^{\oplus n-e})$ , with  $k = n - r + 1$ .*

Finally we consider the next step, namely the case

$$i_X + l(R) = \dim \text{Exc}(R) + 1.$$

For a fiber type or divisorial extremal ray  $R$  we prove that  $\rho_X \leq 3$ , describing the Kleiman-Mori cone of  $X$  and classifying the varieties with  $\rho_X = 3$ , (Theorem 5.1).

If we assume moreover that  $R$  is the ray associated to a smooth blow-up, we have a complete classification:

**THEOREM 1.3.** *Let  $X$  be a Fano manifold and let  $R$  be an extremal ray whose associate contraction  $\varphi_R : X \rightarrow Y$  is the blow up of a smooth variety  $Y$  along a smooth subvariety  $T \subset Y$ , such that*

$$i_X + l(R) \geq n \quad \text{or equivalently} \quad i_X \geq \dim T + 1.$$

*Then  $X$  is one of the following*

- a)  $\text{Bl}_{\mathbb{P}^t}(\mathbb{P}^n)$ , with  $\mathbb{P}^t$  a linear subspace of dimension  $\leq \frac{n}{2} - 1$ ,
- b)  $\text{Bl}_{\mathbb{P}^t}(\mathbb{Q}^n)$ , with  $\mathbb{P}^t$  a linear subspace of dimension  $\leq \frac{n}{2} - 1$ ,
- c)  $\text{Bl}_{\mathbb{Q}^t}(\mathbb{Q}^n)$ , with  $\mathbb{Q}^t$  a smooth quadric of dimension  $\leq \frac{n}{2} - 1$  not contained in a linear subspace of  $\mathbb{Q}^n$ ,
- d)  $\text{Bl}_p(V)$  where  $V$  is  $\text{Bl}_Y(\mathbb{P}^n)$  and  $Y$  is a submanifold of dimension  $n - 2$  and degree  $\leq n$  contained in an hyperplane  $H$  such that  $p \notin H$ ,
- e)  $\text{Bl}_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ .

Note that if  $T$  is a point the condition  $i_X \geq \dim T + 1 = 1$  is empty. In this case the theorem is actually the main theorem of [6], where Fano manifolds which are the blow-up at a point of a smooth variety are classified (those varieties correspond to cases a) and b) with  $t = 0$  and d) of the above theorem). That paper has been for us a very important source of inspiration.

**Acknowledgements.** We would like to thank the referees for many useful remarks and improvements they suggested. This work was done in the framework of the National Research Project ‘‘Geometric properties of real and complex manifolds’’, supported by the MIUR of the Italian Government.

**2. Background material.** In (2.1) and (2.2) we recall basic definitions and facts concerning Fano-Mori contractions and families of rational curves; our notation is consistent with the one in [17] to which we refer the reader.

Afterwards, in (2.3), for the reader's convenience we recall some results of [1] and [10] which are frequently used in the rest of the paper.

**2.1. Fano-Mori contractions.** Let  $X$  be a smooth complex Fano variety of dimension  $n$  and let  $K_X$  be its canonical divisor. By the Cone Theorem the cone of effective 1-cycles which is contained in the  $\mathbb{R}$ -vector space of 1-cycles modulo numerical equivalence,  $NE(X) \subset N_1(X)$ , is polyhedral; a face of  $NE(X)$  is called an **extremal face** and an extremal face of dimension one is called an **extremal ray**.

From the structure of the cone it follows that

LEMMA 2.1. [6, Lemme 2.1] *Let  $X$  be a Fano manifold and  $D$  an effective divisor on  $X$ . Then there exists an extremal ray  $R \subset NE(X)$  such that  $D \cdot R > 0$ .*

To an extremal face  $\sigma$  is associated a morphism with connected fibers  $\varphi_\sigma : X \rightarrow W$  onto a normal variety, which contracts the curves whose numerical class is in  $\sigma$ ;  $\varphi_\sigma$  is called an **extremal contraction** or a **Fano-Mori contraction**.

A Cartier divisor  $H$  such that  $H = \varphi_\sigma^* A$  for an ample divisor  $A$  on  $W$  is called a **good supporting divisor** of the map  $\varphi_\sigma$  (or of the face  $\sigma$ ).

An extremal ray  $R$  (and the associated extremal contraction  $\varphi_R$ ) is called **numerically effective** (nef for short) or of **fiber type** if  $\dim W < \dim X$ , otherwise the ray (and the contraction) is **non nef** or **birational**. This terminology is due to the fact that there exists an effective divisor  $E$  such that  $E \cdot R < 0$  if and only if the ray is not nef. If the codimension of the exceptional locus of a birational ray  $R$  is equal to one the ray and the associated contraction are called **divisorial**, otherwise they are called **small**.

**2.2. Families of rational curves.** Let  $X$  be a normal projective variety and let  $\text{Hom}(\mathbb{P}^1, X)$  be the scheme parametrizing morphisms  $f : \mathbb{P}^1 \rightarrow X$ . We consider the open subscheme  $\text{Hom}_{bir}(\mathbb{P}^1, X) \subset \text{Hom}(\mathbb{P}^1, X)$ , corresponding to those morphisms which are birational onto their image, and its normalization  $\text{Hom}_{bir}^n(\mathbb{P}^1, X)$ . The group  $\text{Aut}(\mathbb{P}^1)$  acts on  $\text{Hom}_{bir}^n(\mathbb{P}^1, X)$  and the quotient exists.

DEFINITION 2.2. The space  $\text{Ratcurves}^n(X)$  is the quotient of  $\text{Hom}_{bir}^n(\mathbb{P}^1, X)$  by  $\text{Aut}(\mathbb{P}^1)$ , and the space  $\text{Univ}(X)$  is the quotient of the product action of  $\text{Aut}(\mathbb{P}^1)$  on  $\text{Hom}_{bir}^n(\mathbb{P}^1, X) \times \mathbb{P}^1$ .

DEFINITION 2.3. We define a **family of rational curves** to be an irreducible component  $V \subset \text{Ratcurves}^n(X)$ . Given a rational curve  $f : \mathbb{P}^1 \rightarrow X$  we will call a **family of deformations** of  $f$  any irreducible component  $V \subset \text{Ratcurves}^n(X)$  containing the equivalence class of  $f$ .

Given a family  $V$  of rational curves, we have the following basic diagram:

$$\begin{array}{ccc} p^{-1}(V) =: U & \xrightarrow{i} & X \\ p \downarrow & & \\ & & V \end{array}$$

where  $i$  is the map induced by the evaluation  $ev : \text{Hom}_{bir}^n(\mathbb{P}^1, X) \times \mathbb{P}^1 \rightarrow X$  and  $p$  is the  $\mathbb{P}^1$ -bundle induced by the projection  $\text{Hom}_{bir}^n(\mathbb{P}^1, X) \times \mathbb{P}^1 \rightarrow \text{Hom}_{bir}^n(\mathbb{P}^1, X)$ .

We define  $\text{Locus}(V)$  to be the image of  $U$  in  $X$ ; we say that  $V$  is a covering family if  $\text{Locus}(V) = X$ . If  $L \in \text{Pic}(X)$  is a line bundle, we will denote by  $L \cdot V$  the intersection number of  $L$  and a general member of the family  $V$ ; moreover we will denote by  $[V]$  the numerical equivalence class in  $N_1(X)$  of a general member of the family  $V$ .

Given a family  $V \subseteq \text{Ratcurves}^n(X)$ , we denote by  $V_x$  the subscheme of  $V$  parametrizing rational curves passing through  $x$ .

DEFINITION 2.4. Let  $V$  be a family of rational curves on  $X$ . Then

- (a)  $V$  is **unsplit** if it is proper;
- (b)  $V$  is **locally unsplit** if for the general  $x \in \text{Locus}(V)$  every component of  $V_x$  is proper;
- (c)  $V$  is **generically unsplit** if there is at most a finite number of curves of  $V$  passing through two general points of  $\text{Locus}(V)$ .

PROPOSITION 2.5. [17, IV.2.6] Let  $X$  be a smooth projective variety,  $V$  a family of rational curves and  $x \in \text{Locus}(V)$  a point such that every component of  $V_x$  is proper. Then

- (a)  $\dim X - K_X \cdot V \leq \dim \text{Locus}(V) + \dim \text{Locus}(V_x) + 1$ ;
- (b)  $-K_X \cdot V \leq \dim \text{Locus}(V_x) + 1$ .

REMARK 2.6. The assumptions of the proposition are clearly satisfied for every  $x \in \text{Locus}(V)$  if  $V$  is an unsplit family; the same result holds for a general  $x \in \text{Locus}(V)$  assuming that  $V$  is generically unsplit, though this does not imply the existence of a point such that  $V_x$  is proper; for example, if  $X$  is the blow up at a point  $p$  of  $\mathbb{P}^2$ , then the family  $V$  of rational curves whose general member is the strict transform of a line which does not contain  $p$  is generically unsplit, but  $V_x$  is not proper for any  $x \in \text{Bl}_p(\mathbb{P}^2)$ .

Proposition 2.5, in case  $V$  is the unsplit family of deformations of a minimal extremal rational curve, i.e. a curve of minimal degree in an extremal face of  $X$ , gives the fiber locus inequality:

PROPOSITION 2.7. [14], [25] Let  $\varphi$  be a Fano-Mori contraction of  $X$  and let  $E = E(\varphi)$  be its exceptional locus; let  $S$  be an irreducible component of a (non trivial) fiber of  $\varphi$ . Then

$$\dim E + \dim S \geq \dim X + l - 1$$

where

$$l = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } S\}.$$

If  $\varphi$  is the contraction of a ray  $R$ , then  $l(R) := l$  is called the length of the ray.

DEFINITION 2.8. We define a Chow family of rational curves to be an irreducible component  $\mathcal{V} \subset \text{Chow}(X)$  parametrizing rational and connected 1-cycles. If  $V$  is a family of rational curves, the closure of the image of  $V$  in  $\text{Chow}(X)$  is called the Chow family associated to  $V$ .

We say that  $V$  is **quasi-unsplit** if every component of any reducible cycle in  $\mathcal{V}$  is numerically proportional to  $V$ .

Let  $X$  be a smooth variety,  $\mathcal{V}^1, \dots, \mathcal{V}^k$  Chow families of rational curves on  $X$  and  $Y$  a subset of  $X$ .

DEFINITION 2.9. We denote by  $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$  the set of points  $x \in X$  such that there exist cycles  $C_1, \dots, C_k$  with the following properties:

- $C_i$  belongs to the family  $\mathcal{V}^i$ ;
- $C_i \cap C_{i+1} \neq \emptyset$ ;
- $C_1 \cap Y \neq \emptyset$  and  $x \in C_k$ ,

i.e.  $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$  is the set of points that can be joined to  $Y$  by a connected chain of  $k$  cycles belonging respectively to the families  $\mathcal{V}^1, \dots, \mathcal{V}^k$ .

We denote by  $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$  the set of points  $x \in X$  such that there exist cycles  $C_1, \dots, C_m$  with the following properties:

- $C_i$  belongs to a family  $\mathcal{V}^j$ ;
- $C_i \cap C_{i+1} \neq \emptyset$ ;
- $C_1 \cap Y \neq \emptyset$  and  $x \in C_m$ ,

i.e.  $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$  is the set of points that can be joined to  $Y$  by a connected chain of at most  $m$  cycles belonging to the families  $\mathcal{V}^1, \dots, \mathcal{V}^k$ .

DEFINITION 2.10. We define a relation of rational connectedness with respect to  $\mathcal{V}^1, \dots, \mathcal{V}^k$  on  $X$  in the following way:  $x$  and  $y$  are in  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$  relation if there exists a chain of cycles in  $\mathcal{V}^1, \dots, \mathcal{V}^k$  which joins  $x$  and  $y$ , i.e. if  $y \in \text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_x$  for some  $m$ .

To the  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$  relation we can associate a fibration, at least on an open subset:

THEOREM 2.11. [8],[17, IV.4.16] *There exist an open subvariety  $X^0 \subset X$  and a proper morphism with connected fibers  $\pi : X^0 \rightarrow Z^0$  such that*

- (a) *the  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$  relation restricts to an equivalence relation on  $X^0$ ;*
- (b) *the fibers of  $\pi$  are equivalence classes for the  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$  relation;*
- (c) *for every  $z \in Z^0$  any two points in  $\pi^{-1}(z)$  can be connected by a chain of at most  $2^{\dim X - \dim Z^0} - 1$  cycles in  $\mathcal{V}^1, \dots, \mathcal{V}^k$ .*

The geometry of Fano manifolds is strongly related to the properties of families of rational curves of low degree. The following is a fundamental theorem, due to Mori:

THEOREM 2.12. [20] *Through every point of a Fano manifold  $X$  there exists a rational curve of anticanonical degree  $\leq \dim X + 1$ .*

REMARK 2.13. The families  $\{V^i \subset \text{Ratcurves}^n(X)\}$  containing rational curves with degree  $\leq \dim X + 1$  are only a finite number, so, for at least one index  $i$ , we have that  $\text{Locus}(V^i) = X$ . Among these families we choose one with minimal anticanonical degree, and we call it a **minimal covering family**.

Let  $X$  be a Fano manifold and  $\pi : X^0 \rightarrow Z^0$  a proper surjective morphism on a smooth quasiprojective variety  $Z^0$  of positive dimension.

A relative version of Mori's theorem, [18, Theorem 2.1], states that, for a general point  $z \in Z^0$ , there exists a rational curve  $C$  on  $X$  of anticanonical degree  $\leq \dim X + 1$  which meets  $\pi^{-1}(z)$  without being contained in it (an **horizontal curve**, for short).

As in remark 2.13 we can find a family  $V$  of horizontal curves such that  $\text{Locus}(V)$  dominates  $Z^0$  and  $-K_X \cdot V$  is minimal among the families with this property. Such a family is called a **minimal horizontal dominating family** for  $\pi$ .

LEMMA 2.14. [1, Lemma 6.5] *Let  $X$  be a Fano manifold, let  $X - \overset{\pi}{\succ} Z$  be the fibration associated to a  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$  relation and let  $V$  be a minimal horizontal dominating family for  $\pi$ . Then*

- (a) curves parametrized by  $V$  are numerically independent from curves contracted by  $\pi$ ;
- (b)  $V$  is locally unsplit;
- (c) if  $x$  is a general point in  $\text{Locus}(V)$  and  $F$  is the fiber containing  $x$ , then

$$\dim(F \cap \text{Locus}(V_x)) = 0.$$

**2.3. Chains of rational curves, numerical equivalence and cones.** In this subsection we present some results concerning the dimension, the maximum number of numerically independent curves and the cone of curves of subsets of the form  $\text{Locus}(V^1, \dots, V^k)_Y$  or  $\text{ChLocus}(V^1, \dots, V^k)_Y$  when  $V^1, \dots, V^k$  are unsplit families and  $Y$  is chosen in a suitable way.

**DEFINITION 2.15.** Let  $V^1, \dots, V^k$  be unsplit families on  $X$ . We will say that  $V^1, \dots, V^k$  are numerically independent if the numerical classes  $[V^1], \dots, [V^k]$  are linearly independent in the vector space  $N_1(X)$ . If moreover  $C \subset X$  is a curve we will say that  $V^1, \dots, V^k$  are numerically independent from  $C$  if in  $N_1(X)$  the class of  $C$  is not contained in the vector subspace generated by  $[V^1], \dots, [V^k]$ .

**Notation:** Let  $S$  be a subset of  $X$ . We write  $N_1(S) = \langle [V^1], \dots, [V^k] \rangle$  if the numerical class in  $X$  of every curve  $C \subset S$  can be written as  $[C] = \sum_i a_i [C_i]$ , with  $a_i \in \mathbb{Q}$  and  $C_i \in V^i$ . We write  $\text{NE}(S) = \langle [V^1], \dots, [V^k] \rangle$  (or  $\text{NE}(S) = \langle [R_1], \dots, [R_k] \rangle$ ) if the numerical class in  $X$  of every curve  $C \subset S$  can be written as  $[C] = \sum_i a_i [C_i]$ , with  $a_i \in \mathbb{Q}_{\geq 0}$  and  $C_i \in V^i$  (or  $[C_i]$  in  $R_i$ ).

The following lemma is a generalization of proposition 2.5 and of [7, Theoreme 5.2]

**LEMMA 2.16.** [1, Cfr. Lemma 5.4] *Let  $Y \subset X$  be a closed subset and  $V$  an unsplit family. Assume that  $[V] \notin \text{NE}(Y)$  and that  $Y \cap \text{Locus}(V) \neq \emptyset$ . Then for a general  $y \in Y \cap \text{Locus}(V)$*

- (a)  $\dim \text{Locus}(V)_Y \geq \dim(Y \cap \text{Locus}(V)) + \dim \text{Locus}(V_y)$ ;
- (b)  $\dim \text{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$ .

*Moreover, if  $V^1, \dots, V^k$  are numerically independent unsplit families such that curves contained in  $Y$  are numerically independent from curves in  $V^1, \dots, V^k$  then either  $\text{Locus}(V^1, \dots, V^k)_Y = \emptyset$  or*

$$(c) \dim \text{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k.$$

**REMARK 2.17.** In [1, Lemma 5.4] parts (a) and (b) were proved in the stronger assumption that curves contained in  $Y$  are numerically independent from curves in  $V$ . The proof is the same.

**LEMMA 2.18.** [21, Lemma 1] *Let  $Y \subset X$  be a closed subset and  $V$  an unsplit family of rational curves. Then every curve contained in  $\text{Locus}(V)_Y$  is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_Y + \mu C_V,$$

where  $C_Y$  is a curve in  $Y$ ,  $C_V$  belongs to the family  $V$  and  $\lambda \geq 0$ .

**COROLLARY 2.19.** [1, Corollary 4.4] *If  $X$  is rationally connected with respect to some (quasi) unsplit families  $V^1, \dots, V^k$  then  $N_1(X) = \langle [V^1], \dots, [V^k] \rangle$ .*

PROPOSITION 2.20. [1, Corollary 4.2], [10, Corollary 2.23]

- (a) Let  $V$  be a quasi-unsplit family of rational curves and  $x$  a point in  $\text{Locus}(V)$ . Then  $NE(\text{ChLocus}_m(V)_x) = \langle [V] \rangle$  for every  $m \geq 1$ .
- (b) Let  $V$  be a family of rational curves and  $x$  a point in  $X$  such that  $V_x$  is proper. Then  $NE(\text{Locus}(V_x)) = \langle [V] \rangle$ .
- (c) Let  $\sigma$  be an extremal face of  $NE(X)$ ,  $F$  a fiber of the associated contraction and  $V$  an unsplit family independent from  $\sigma$ . Then  $NE(\text{ChLocus}_m(V)_F) = \langle \sigma, [V] \rangle$  for every  $m \geq 1$ .

COROLLARY 2.21. Let  $D \subset X$  be an effective divisor and  $V$  an unsplit family such that  $[V] \notin NE(D)$  and that  $D \cdot V > 0$ ; then, for every  $x \in \text{Locus}(V)$  we have  $\dim \text{Locus}(V_x) = 1$ ; in particular, if  $V$  is the family of deformations of a minimal extremal rational curve in a ray  $R$  then every non trivial fiber of  $\varphi_R$  is one dimensional.

*Proof.* Since  $D \cdot V > 0$ , for every  $x \in \text{Locus}(V)$  we have  $D \cap \text{Locus}(V_x) \neq \emptyset$ , and so  $\dim(D \cap \text{Locus}(V_x)) \geq \dim \text{Locus}(V_x) - 1$ . It follows that  $\dim \text{Locus}(V_x) = 1$ , since a curve in the intersection would be a curve in  $D$  whose numerical class is proportional to  $[V]$ .  $\square$

**3. Some technical results.** In order to make the exposition clearer, we collect in this section two technical lemmata we will use in the proofs of the main theorems.

LEMMA 3.1. Let  $X$  be a Fano manifold whose cone of curves is generated by a divisorial extremal ray  $R_1$  with exceptional locus  $E$  and a fiber type extremal ray  $R_2$ , and let  $V$  be a quasi unsplit covering family of rational curves. Then  $[V] \in R_2$ ; in particular  $E \cdot V > 0$ .

*Proof.* Consider the  $\text{rc}\mathcal{V}$  fibration  $X - \overset{\pi}{\succ} Z$ ; by proposition 2.20 we have  $\dim Z > 0$  since  $V$  is quasi unsplit and  $\rho_X = 2$  and  $V$  is extremal by [10, Lemma 2.28] since  $X$  has not small contractions.

The last assertion follows from lemma 2.1, since  $E \cdot R_1 < 0$ .  $\square$

LEMMA 3.2. Let  $X$  be a Fano manifold of dimension  $n$ , pseudoindex  $i_X \geq 2$  and Picard number  $\rho_X = 2$  whose extremal contractions are:

1. the blow up  $\varphi : X \rightarrow Y$  of a smooth variety along a smooth subvariety  $T \subset Y$ , associated to an extremal ray  $R_1$ , with exceptional locus  $E = \text{Exc}(R_1)$ ;
2. a fiber type contraction associated to a ray  $R_2$ .

Suppose that  $i_X + l(R_1) \geq n$  and that there exists a covering family  $V$  of rational curves of degree  $\leq n + 1$  such that  $E \cdot V = 0$ . Then  $V$  is not quasi unsplit and all the reducible cycles in the associated Chow family  $\mathcal{V}$  have two irreducible components,  $C_1$  and  $C_2$ , where  $C_1$  and  $C_2$  are curves in the rays  $R_1$  and  $R_2$  respectively.

*Proof.* First of all we note that, since  $E \cdot V = 0$ , by lemma 3.1,  $V$  is not a quasi unsplit family.

Let  $C = \sum C_i$  be a reducible cycle in  $\mathcal{V}$ . At least one of the components of  $C$ , let it be  $C_1$ , has negative intersection with  $E$ ; in fact, if  $E \cdot C_i = 0$  for every  $i$  the effective divisor  $E$  would be numerically trivial on the whole  $NE(X)$  since  $\rho_X = 2$ .

Denote by  $V^1$  a family of deformations of  $C_1$ ; if  $V^1$  is not unsplit then there exists a reducible cycle  $\sum C_{1j}$  in  $\mathcal{V}^1$ , and for at least one of the components, call it  $C_{11}$ , we have  $E \cdot C_{11} < 0$ .

Denote by  $V^{11}$  a family of deformations of  $C_{11}$ . If  $V^{11}$  is not unsplit, we repeat the argument, and the procedure terminates because  $-K_X \cdot V > -K_X \cdot V^1 > -K_X \cdot V^{11} >$

$\dots > 0$ .

Therefore every reducible cycle  $\sum C_i$  in  $\mathcal{V}$  has an irreducible component on which  $E$  is negative and such that its family of deformations is unsplit.

Let  $\Gamma$  be one of these components and  $W$  a family of deformations of  $\Gamma$ ; since  $E \cdot \Gamma < 0$  we have  $\text{Locus}(W) \subset E$ . We claim that  $[W] \in R_1$ . Assume by contradiction that  $W$  is independent from  $R_1$ .

Denote by  $F$  a non trivial fiber of  $\varphi$  meeting  $\text{Locus}(W)$ ; by proposition 2.7 we have  $\dim F \geq \dim X - \dim E + l(R_1) - 1 \geq l(R_1)$ , and by lemma 2.16 b) we have  $\dim \text{Locus}(W)_F \geq -K_X \cdot W + \dim F - 1$ . Combining the inequalities we have

$$n - 1 \geq \dim \text{Locus}(W)_F \geq -K_X \cdot W + \dim F - 1 \geq i_X + l(R_1) - 1 \geq n - 1.$$

This forces  $\text{Locus}(W) = E$ , so  $F \subset \text{Locus}(W)$  and we can apply part a) of lemma 2.16 and get

$$i_X - 1 + \dim F = \dim \text{Locus}(W)_F \geq \dim F + \dim \text{Locus}(W_y) \geq \dim F + i_X - 1,$$

the last equality following from proposition 2.5. Therefore  $\dim \text{Locus}(W_y) = i_X - 1$  so  $W$  is covering, by proposition 2.5, a contradiction.

It follows that  $[W] \in R_1$  and, for every reducible cycle  $\sum_{i=1}^k C_i$  in  $\mathcal{V}$ , we have

$$(1) \quad n + 1 \geq -K_X \cdot V = -K_X \cdot \sum_{i=1}^k C_i \geq l(R_1) + (k - 1)i_X \geq n.$$

Hence  $k = 2$  and every reducible cycle has two components,  $C_1$ , which belongs to  $R_1$  and  $C_2$ .

Let us assume, and prove it later, that  $V$  is not locally unsplit. Since  $V$  is also covering and  $\text{Locus}(V_1) = E$ , the family  $V_2$  of deformations of  $C_2$  must be a covering family. We have  $-K_X \cdot V_2 \leq i_X + 1 < 2i_X$ , hence  $V_2$  is an unsplit family; therefore, by lemma 3.1, its numerical class belongs to the ray  $R_2$ .

To conclude we have therefore to show that  $V$  is not locally unsplit. Assume that the contrary holds.

From (1) it follows that  $-K_X \cdot V \geq n$ . On the other hand we can not have  $-K_X \cdot V = n + 1$ , otherwise  $\rho_X = 1$  by proposition 2.20 b). So we have  $-K_X \cdot V = n$  and, again by inequality (1), that  $l(R_1) + i_X = n$ , so  $l(R_1) \leq n - 2$  and therefore the center of the blow up,  $T$ , has dimension  $\dim T = n - 1 - l(R_1) \geq 1$ .

By proposition 2.5, for a general  $x \in X$ ,  $D_x = \text{Locus}(V_x)$  is a divisor; we claim that this divisor is zero on  $R_1$ . To prove it recall that  $R_1$  is a divisorial ray, so, by proposition 2.7, for every nontrivial fiber  $F$  of  $\varphi_{R_1}$  we have  $\dim F \geq \dim X - \dim E + l(R) - 1 \geq l(R) \geq i_X \geq 2$ . Moreover, by proposition 2.20 b),  $\text{NE}(D_x) = \langle [V] \rangle$ , so we can apply corollary 2.21 to the divisor  $D_x$  and to the unsplit family associated to  $R_1$  to get  $D_x \cdot R_1 = 0$ .

On the other hand  $\varphi(D_x)$  is an effective, hence ample divisor on  $Y$ , so it meets  $T$ . It follows that  $D_x \cap E \neq \emptyset$ ; this, together with  $D_x \cdot R_1 = 0$  implies that  $D_x$  contains fibers of  $\varphi_{R_1}$ , a contradiction with  $\text{NE}(D_x) = \langle [V] \rangle$ .  $\square$

**4. A bound on the length.**

LEMMA 4.1. *Let  $X$  be a Fano manifold with  $\rho_X \geq 2$ , let  $R$  be an extremal ray of  $X$  and denote by  $\text{Exc}(R)$  its exceptional locus. Then there exists a family of rational*



curves  $V$  whose general member does not belong to  $R$  such that  $\text{Exc}(R) \cap \text{Locus}(V) \neq \emptyset$  and, for some  $x \in \text{Exc}(R) \cap \text{Locus}(V)$ ,  $V_x$  is proper.

Moreover, if  $R$  is not nef and  $W$  is a minimal covering family, then, among the families of deformations of irreducible components of cycles in the associated Chow family  $\mathcal{W}$ , there is a family  $V$  as above and one of the following happens

- a)  $\text{Exc}(R) \subset \overline{\text{Locus}}(V)$ .
- b) There exists a reducible cycle  $C_R + C_V + \sum_{i=1}^k C_i$  in  $\mathcal{W}$  with  $[C_R] \in R$  and  $C_V$  a curve in  $V$ .

*Proof.* If  $R$  is a nef ray it's enough to choose  $V$  as the family of deformation of a minimal extremal rational curve in any ray  $R_1 \neq R$ , so we can assume that  $R$  is not nef.

Let  $W$  be a minimal covering family for  $X$ . Note that, since  $W$  is covering, it is certainly independent from  $R$ : indeed, since  $R$  is birational, curves whose numerical classes are in  $R$  are contained in  $\text{Exc}(R)$ .

If there exists  $x \in \text{Exc}(R)$  such that  $W_x$  is unsplit then we are done, otherwise for every  $x \in \text{Exc}(R)$  there exists in  $\mathcal{W}$  a reducible cycle  $\sum_{i=1}^{m_x} C_{i,x}$ , with rational components, passing through  $x$ .

As the point  $x$  varies in  $\text{Exc}(R)$  the families of deformations of the curves  $C_{i,x}$  are a finite number, since their anticanonical degree is bounded by  $-K_X \cdot W \leq \dim X + 1$  so, calling these families  $T^1, \dots, T^l$  for at least one index  $j$  we have  $\text{Exc}(R) \subset \overline{\text{Locus}}(T^j)$ . If  $T^j$  is independent from  $R$  then let  $W^1 = T^j$ , otherwise take a reducible cycle in  $\mathcal{W}$  of the form  $C_j + \sum_{i \neq j} C_i$ , with  $C_j$  in the family  $T^j$ , passing through a point  $x \in \text{Exc}(R)$ . Since  $[W] = [C_j + \sum_{i \neq j} C_i]$  is independent from  $R$  and every component which is proportional to  $R$  is contained in  $\text{Exc}(R)$  there exists an irreducible component  $C_k$  independent from  $R$  which meets  $\text{Exc}(R)$ . In this case denote by  $W^1$  the family of deformations of  $C_k$ .

We have thus found a family  $W^1$  which is independent from  $R$  such that  $\overline{\text{Locus}}(W^1) \cap \text{Exc}(R) \neq \emptyset$ . Moreover either we can choose  $W^1$  such that  $\text{Exc}(R) \subset \overline{\text{Locus}}(W^1)$  or there exists a reducible cycle in  $\mathcal{W}$  with one component belonging to  $R$ . Let  $x_1 \in \text{Locus}(W^1) \cap \text{Exc}(R)$ . If  $W_{x_1}^1$  is unsplit we set  $V = W^1$  and we are done, otherwise we repeat the argument, replacing  $W$  with  $W^1$  and  $\text{Exc}(R)$  with  $\text{Locus}(W^1) \cap \text{Exc}(R)$ . Since  $n + 1 > \deg W > \deg W^1 > \dots > 0$  the procedure terminates.  $\square$

REMARK 4.2. Both cases a) and b) of the lemma are possible; we give an example of case b), which is less intuitive. Let  $X$  be the blow up of  $\mathbb{P}^5$  along a Veronese surface  $S$ ;  $X$  is a Fano manifold with  $\rho_X = 2$  and pseudoindex  $i_X = 2$  whose other contraction is the blow up of the dual  $\mathbb{P}^5$  with center a Veronese surface. Let  $W$  be the family of rational curves in  $X$  whose general member is the strict transform of a line meeting  $S$  in one point; it is not difficult to prove that this family is a minimal covering family. If  $x$  is a point outside the two exceptional divisors  $E_1$  and  $E_2$  then  $W_x$  is unsplit, while, if  $x$  is contained in  $E_1 \cup E_2$  curves in  $W_x$  degenerate to reducible cycles  $C_1 + C_2$ , with  $C_i$  a line in a fiber of the contraction of  $E_i$ , so we are in case b). Note that the family  $V$  given by the lemma is the family of curves of minimal degree in  $E_2$ , so  $\text{Exc}(R_1) \not\subset \overline{\text{Locus}}(V)$  and we are not in case a).

*Proof of inequality (\*).* Let  $V$  and  $x \in \text{Exc}(R) \cap \text{Locus}(V)$  be as in lemma 4.1. Let  $\varphi_R : X \rightarrow Y$  be the extremal contraction associated to  $R$  and let  $F_x$  be the fiber of  $\varphi_R$  which contains  $x$ . The numerical class of every curve in  $F_x$  is in  $R$  and, by proposition 2.20, b) the numerical class of every curve in  $\text{Locus}(V_x)$  is

proportional to  $[V]$  so, since  $V$  is independent from  $R$ , we have  $\dim \text{Locus}(V_x) \cap F_x = 0$ . Moreover, by inequalities 2.5 and 2.7, we have  $\dim \text{Locus}(V_x) \geq i_X - 1$  and  $\dim F_x \geq \dim X - \dim \text{Exc}(R) + l(R) - 1$ . Combining these inequalities we get

$$(2) \quad \begin{aligned} \dim X &\geq \dim \text{Locus}(V_x) + \dim F_x \\ &\geq i_X + \dim X - \dim \text{Exc}(R) + l(R) - 2 \end{aligned}$$

which gives

$$i_X + l(R) \leq \dim \text{Exc}(R) + 2,$$

and the proposition is proved.  $\square$

**5. The border cases.**

*Proof of 1.1.* First of all note that, since the length of a fiber type extremal ray is  $\leq n + 1$ , equality holding if and only if  $X \simeq \mathbb{P}^n$ , and the length of a birational extremal ray is  $\leq n - 1$ , the assumptions of the theorem imply  $i_X \geq 2$ .

Let  $V$  be the family given by lemma 4.1, let  $x \in \text{Exc}(R)$  be a point such that  $V_x$  is proper and let  $F_x$  be the fiber of  $\varphi_R$  containing  $x$ . If equality holds in (\*), then equality holds everywhere in (2); in particular we have

$$(3) \quad \dim F_x = l(R) + \dim X - \dim \text{Exc}(R) - 1$$

$$(4) \quad \dim \text{Locus}(V_x) = i_X - 1.$$

Let  $V_R$  be a family of curves in  $R$  such that  $-K_X \cdot V_R = l(R)$ ; in particular note that this family is unsplit.

We clearly have  $\text{Locus}(V_R) \subset \text{Exc}(R)$  and, for every  $x \in \text{Locus}(V_R)$ , denoting by  $F_x$  the fiber of  $\varphi_R$  containing  $x$ ,  $\text{Locus}((V_R)_x) \subset F_x$ ; combining equation 3 with inequality 2.5 for the family  $V_R$  we get

$$\dim X + l(R) - 1 = \dim \text{Exc}(R) + \dim F_x =$$

$$\geq \dim \text{Locus}(V_R) + \dim \text{Locus}((V_R)_x) \geq \dim X + l(R) - 1,$$

hence  $\text{Locus}(V_R) = \text{Exc}(R)$  (and  $\text{Locus}((V_R)_x) = F_x$ ).

Equality 4, together with inequality 2.5 yields that  $\dim \text{Locus}(V) = n$ , so  $V$  is a covering family, and that  $-K_X \cdot V = i_X$ , so  $V$  is unsplit.

If  $\varphi_R : X \rightarrow Y$  is of fiber type then we can apply [21, Theorem 1] to  $V$  and  $V_R$  to get that  $X \simeq \mathbb{P}^{i_X-1} \times \mathbb{P}^{l(R)-1}$ .

Suppose now that  $\varphi_R : X \rightarrow Y$  is divisorial and call  $E$  the divisor  $\text{Exc}(R)$ .

By (3) we have  $\dim F_x = l(R)$ ; note that, since  $V$  is covering and unsplit, this equality holds for every non trivial fiber  $F$ , hence we can apply [2, Theorem 5.1] and we obtain that  $Y$  is smooth and  $\varphi_R$  is the blow up of  $Y$  along a smooth subvariety  $T$ .

Let  $F$  be any non trivial fiber of  $\varphi_R$ ; by lemma 2.16 b) we have

$$\dim \text{Locus}(V)_F \geq \dim F + i_X - 1 \geq n,$$

so, by proposition 2.20 c) we have  $NE(X) = \langle [R], [R_V] \rangle$ , where  $R_V$  is the ray spanned by the numerical class of  $V$ .

The target  $Y$  of  $\varphi_R$  is a smooth variety with  $\rho_Y = 1$  covered by rational curves, hence a Fano manifold; let  $V_Y$  be a minimal dominating family of rational curves for  $Y$  and let  $V^*$  be the family of deformations of the strict transform of a general curve in  $V_Y$ . By [17, Proposition 3.7] a general member of  $V_Y$  does not meet  $T$ , which has codimension at least two in  $Y$ , hence  $E \cdot V^* = 0$ . Therefore, by lemma 3.2, the family  $V^*$  is not quasi unsplit and all the reducible cycles in the associated Chow family  $\mathcal{V}^*$  have two irreducible components,  $C_R$  and  $C_V$ , where  $C_R$  and  $C_V$  are curves in the rays  $R$  and  $R_V$  respectively. In particular

$$(5) \quad n + 1 \geq -K_Y \cdot V_Y = -K_X \cdot V^* = -K_X \cdot (C_R + C_V) \geq l(R) + i_X = n + 1,$$

and  $Y \simeq \mathbb{P}^n$  by the proof of [15, Theorem 1.1]. (Note that the assumptions of the quoted result are different, but the proof actually works in our case, since for a very general  $y$  the pointed family  $V_{Y_y}$  has the properties 1-3 in [15, Theorem 2.1]).

By equation 5 we also have  $-K_X \cdot C_R = l(R)$  and  $-K_X \cdot C_V = i_X$ , so  $C_R$  and  $C_V$  are minimal extremal rational curves; in particular  $E \cdot C_R = -1$  and therefore, since  $E \cdot V^* = 0$  we have  $E \cdot C_V = 1$ .

Let  $\psi : X \rightarrow Z$  be the contraction of the ray  $R_V$ ; we know that  $E \cdot C_V > 0$ , so every fiber of  $\psi$  meets a non trivial fiber  $F$  of  $\varphi_R$  and therefore its dimension is  $n - \dim F = i_X - 1$ , since fibers of different extremal ray contractions can meet only in points.

Let now  $G$  be a general fiber of  $\psi$ ;  $G$  is smooth, and, by adjunction

$$K_G + (\dim G + 1)E|_G = \mathcal{O}_G,$$

so  $G$  is a projective space and  $E \cap G$  is an hyperplane which dominates  $T$ . Therefore  $T$  is a projective space by [19, Theorem 4.1].

The bound on the dimension of  $T$  follows from the fact that  $\dim T = n - l(R) - 1 = i_X - 2$  and  $2i_X \leq l(R) + i_X = n + 1$ .  $\square$

**THEOREM 5.1.** *Let  $X$  be a Fano manifold of Picard number  $\rho_X \geq 2$ , and let  $R$  be a fiber type or divisorial extremal ray such that*

$$i_X + l(R) = \dim \text{Exc}(R) + 1.$$

*Then  $\rho_X \leq 3$  and  $\rho_X = 3$  if and only if  $X$  is*

- a)  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2}$ ,
- b)  $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ ,
- c)  $Bl_p(V)$  where  $V$  is  $Bl_Y(\mathbb{P}^n)$  and  $Y$  is a submanifold of dimension  $n - 2$  and degree  $\leq n$  contained in an hyperplane  $H$  such that  $p \notin H$ .

*If  $\rho_X = 2$ , except for the cases*

- d)  $Bl_p(\mathbb{Q}^n)$ ,
- e)  $Bl_{\mathbb{P}^{l(R)-2}}(\mathbb{P}^n)$ .

*the cone of curves  $NE(X)$  is generated by  $R$  and by a fiber type extremal ray and moreover  $i_X \geq 2$ .*

*Proof.* If  $i_X = 1$  and  $R$  is divisorial we have  $l(R) \geq n - 1$  so, by [2, Theorem 1.1],  $X$  is the blow up at a point of a smooth variety  $X'$ ; by [6, Theorem 1.1] we are in case c) or in case d).

If  $i_X = 1$  and  $R$  is of fiber type then  $l(R) = n$ ; in particular  $\varphi_R : X \rightarrow B$  is equidimensional with  $(n - 1)$ -dimensional fibers over a smooth curve  $B$ . The general

fiber of  $\varphi_R$  is a projective space by [11, Corollary 0.4] or [15, Theorem 1.1]. Over an open Zariski subset  $U$  of  $B$  the morphism  $p$  is a projective bundle. By taking the closure in  $X$  of a hyperplane section of  $p$  defined over the open set  $U$  we get a global relative hyperplane section divisor (we use  $\rho(X/B) = 1$ ) hence  $p$  is a projective bundle globally by [13, Lemma 2.12]. Since  $X$  is a Fano manifold  $B \simeq \mathbb{P}^1$ . Write  $X = \mathbb{P}_{\mathbb{P}^1}(\oplus_{i=0}^{n-1} \mathcal{O}(a_i))$  with  $0 \leq a_0 \leq \dots \leq a_{n-1}$ . A straightforward computation shows that  $X$  is Fano if and only if either all the  $a_i$  are zero or all the  $a_i$  but  $a_{n-1}$  are zero and  $a_{n-1} = 1$ . In the first case  $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$  and  $i_X = 2$ , in the second case  $X = Bl_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$  and we are in case e).

From now on we can assume  $i_X \geq 2$ .

Let  $V$  be the family given by lemma 4.1, let  $x \in \text{Exc}(R)$  be a point such that  $V_x$  is proper and let  $F_x$  be the fiber of  $\varphi_R$  containing  $x$ . First of all we prove that  $V$  is an unsplit family. In fact, if  $V$  were not unsplit then  $-K_X \cdot V \geq 2i_X$  and  $\dim \text{Locus}(V_x) \geq 2i_X - 1$ .

In this case we would have

$$\begin{aligned} \dim \text{Locus}(V_x) + \dim F_x &\geq 2i_X - 1 + n + l(R) - \dim \text{Exc}(R) - 1 = \\ &\geq n + i_X - 1 > n \end{aligned}$$

and so  $\dim \text{Locus}(V_x) \cap F_x \geq 1$ , a contradiction, since  $V$  is independent from  $R$ .

Now we divide the proof in two cases, according to the type of  $R$ .

**Case 1:**  $R$  is nef.

Let  $\varphi_R : X \rightarrow Z$  be the contraction associated to  $R$ , and let  $V_R$  be a minimal dominating family of curves in  $R$ ; we claim that  $V_R$  is unsplit. If this were not the case then  $-K_X \cdot V_R \geq 2l(R)$ , so, for a general  $x \in X$  we would have  $\dim \text{Locus}(V_R)_x \geq 2l(R) - 1 \geq i_X + l(R) - 1 = n$  and  $\rho_X$  would be one by proposition 2.20 b).

Recall that, according to the proof of lemma 4.1, in this case  $V$  is the family of deformations of a minimal extremal rational curve in a ray  $R_1$  different from  $R$ .

Suppose that  $R_1$  is not nef; by inequality 2.7, if  $G$  is a nontrivial fiber of the associated contraction we have  $\dim G \geq i_X$  and, by lemma 2.16

$$\dim \text{Locus}(R)_G \geq \dim G + l(R) - 1 \geq i_X + l(R) - 1 = n.$$

It follows that  $\dim G = i_X = l(R_1)$  and  $X = \text{Locus}(R)_G$ , so  $\text{NE}(X) = \langle [R], [R_1] \rangle$  by proposition 2.20 c).

Since  $\dim G = i_X = l(R_1)$  for every nontrivial fiber of the contraction associated to  $R_1$ , this contraction is a smooth blow up by [2, Theorem 5.1].

We can repeat the second part of the proof of theorem 1.1, replacing  $R$  with  $R_1$  and  $R_V$  with  $R$  and obtain that  $X = Bl_{\mathbb{P}^{l(R)-2}}(\mathbb{P}^n)$ , so we are in case e).

Suppose now that  $R_1$  is nef and consider the  $\text{rc}(R, R_1)$  fibration  $\pi_{R, R_1} : X \rightarrow Z$ . Let  $F$  be a general fiber of  $\pi_{R, R_1}$  and  $x \in F$  a point;  $F$  contains  $\text{Locus}(R, R_1)_x$  which has dimension  $\geq i_X + l(R) - 2 = n - 1$  by lemma 2.16, so  $\dim Z \leq 1$ .

Suppose that  $\dim Z = 1$  and let  $V'$  be a minimal horizontal dominating family for  $\pi_{R, R_1}$ ; by lemma 2.14 c), if  $x'$  is a general point in  $\text{Locus}(V')$ , we have

$\dim \text{Locus}(V')_{x'} = 1$  and so  $-K_X \cdot V' = 2 = i_X$ .

In particular  $V'$  is unsplit and, by 2.5, covering. We can apply [21, Theorem 1] to  $V$ ,  $V_R$  and  $V'$  to conclude that  $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2}$  and we are in case a).

If  $\dim Z = 0$  then  $X$  is  $\text{rc}(R, R_1)$ -connected and  $\rho_X = 2$  by corollary 2.19; in this case we clearly have  $\text{NE}(X) = \langle [R], [R_1] \rangle$ .

**Case 2:**  $R$  is not nef.

Let  $W$  be a minimal covering family for  $X$  and let  $V$  be a family as in lemma 4.1, chosen among the families of deformations of irreducible components of cycles in  $\mathcal{W}$ .

**Step 1**  $V$  is an unsplit covering family.

Suppose that  $V$  is not a covering family. Then, by inequality 2.5,  $\dim \text{Locus}(V_x) \geq i_X$  and therefore  $E := \text{Exc}(R)$  is not contained in  $\text{Locus}(V)$ . Indeed, in this case, by lemma 2.16 a), denoted by  $F$  a nontrivial fiber of  $\varphi_R$ , we would have  $\dim \text{Locus}(V)_F \geq \dim F + \dim \text{Locus}(V_x) = n$ , a contradiction.

So we are in case b) of lemma 4.1 and there exists a reducible cycle  $C_R + C_V + \sum_{i=1}^k C_i$  in  $\mathcal{W}$  with  $[C_R] \in R$  and  $C_V$  in  $V$ . Hence we have

$$n \geq -K_X \cdot W \geq -K_X \cdot (C_R + C_V + \sum_{i=1}^k C_i) \geq l(R) + i_X + ki_X = n + ki_X$$

forcing  $-K_X \cdot W = n$  and  $k = 0$ .

We have thus proved that in  $\mathcal{W}$  there exists a reducible cycle  $C_R + C_V$ , with  $[C_R]$  in  $R$  and  $C_V$  in  $V$ .

Let  $D = \text{Locus}(W_x)$  for a general  $x \in X$ ; by proposition 2.20 b)  $\text{NE}(D) = \langle [W] \rangle$ .

By corollary 2.21, since the fibers of  $\varphi_R$  are at least two dimensional we have  $D \cdot R = 0$ ; by the same corollary, since  $\dim \text{Locus}(V_x) \geq 2$  we have  $D \cdot V = 0$ . This implies also that  $D \cdot W = D \cdot (C_R + C_V) = 0$ .

By lemma 2.1 there exists an extremal ray  $R_1$  such that  $D \cdot R_1 > 0$ ; let  $V^1$  be a family of deformations of a minimal curve in  $R_1$ . By lemma 2.16 b) we have  $\dim \text{Locus}(V^1)_D \geq \dim D + i_X - 1 \geq n$ , hence  $X = \text{Locus}(V^1)_D$  and  $\rho_X = 2$ .

This is a contradiction, since  $D$  is zero on  $R$  and  $V$  and so, if  $\rho_X = 2$  it would be zero on the entire cone. Therefore  $V$  is a covering family as claimed.

**Step 2**  $\rho_X \leq 3$ .

Let  $F$  be a nontrivial fiber of  $\varphi_R$ ; by lemma 2.16 b) we have

$$\dim \text{Locus}(V)_F \geq \dim F + i_X - 1 \geq n - 1$$

If  $X = \text{Locus}(V)_F$  then, by proposition 2.20 c)  $\text{NE}(X) = \langle [R], [V] \rangle$  and we are done. Note that this is always the case if  $\dim F > l(R)$ , so we assume from now on that  $\varphi_R$  is equidimensional with fibers of dimension  $l(R)$ , hence it is a smooth blow up by [2, Theorem 5.1].

An irreducible component of  $\text{Locus}(V)_F$  is thus a divisor  $D \subset X$  such that  $\text{NE}(D) = \langle [R], [V] \rangle$ . If  $D \cdot V > 0$  then  $X = \text{ChLocus}_2(V)_F$  and  $\text{NE}(X) = \langle [R], [V] \rangle$  again by proposition 2.20 c), so we can assume  $D \cdot V = 0$ .

By lemma 2.1 there exists an extremal ray  $R_1$  such that  $D \cdot R_1 > 0$ .

If  $R_1 \not\subset \text{NE}(D)$  then, by lemma 2.16 b), denoted by  $V^1$  a family of deformations of a minimal extremal rational curve in  $R_1$ , we have  $\dim \text{Locus}(V^1)_D = n$ . By lemma

2.18  $N_1(X) = \langle [R], [V], [V^1] \rangle$ , so  $\rho_X \leq 3$ , equality holding if and only if  $R_1$  is not contained in the vector subspace of  $N_1(X)$  spanned by  $R$  and  $[V]$ .

If  $R_1 \subset \text{NE}(D)$  then  $R_1 = R$  because  $D \cdot V = 0$ . It follows that  $\text{Locus}(R)_D = E$ , so  $N_1(E) = \langle [R], [V] \rangle$ .

If  $E \cdot V > 0$  then  $\text{Locus}(V)_E = X$  and  $N_1(X) = \langle [R], [V] \rangle$  by lemma 2.18, so  $\rho_X = 2$ . We claim that we cannot have  $E \cdot V = 0$ ; in fact, in this case every curve of  $V$  which meets  $E$  is entirely contained in  $E$ , so  $E = \text{Locus}(V)_F = D$  and we have  $D \cdot R < 0$ , a contradiction.

**Step 3**  $\rho_X = 2$ , description of the cone.

We have to prove that  $\text{NE}(X) = \langle [R], [R_1] \rangle$  where  $R_1$  is a fiber type extremal ray. By step two this is the case if for a nontrivial fiber  $F$  of  $\varphi_R$  either we have  $X = \text{Locus}(V)_F$  or an irreducible component of  $\text{Locus}(V)_F$  is a divisor  $D$  such that  $D \cdot V > 0$ . We can therefore assume that an irreducible component of  $\text{Locus}(V)_F$  is a divisor  $D$  such that  $D \cdot V = 0$ ; moreover we know that there exists an extremal ray  $R_1$  of  $X$  on which  $D$  is positive.

If  $R_1 \not\subset \text{NE}(D)$  then  $\text{NE}(X) = \langle [R], [R_1] \rangle$  and moreover, by corollary 2.21 the contraction associated to  $R_1$  has nontrivial one dimensional fibers, and so it is of fiber type, since  $i_X \geq 2$  by proposition 2.7.

If  $R_1 \subset \text{NE}(D)$  then  $R_1 = R$  thus, if  $V$  is not extremal,  $D$  is negative on an extremal ray  $R_2$ , and so  $\text{Exc}(R_2) \subset D$ , against  $\text{NE}(D) = \langle [R], [V] \rangle$ . Therefore  $V$  is extremal and  $\text{NE}(X) = \langle [R], [V] \rangle$ .

**Step 4**  $\rho_X = 3$ , description of the cone.

By step two, if  $\rho_X = 3$ , then  $\text{Locus}(V)_F$  has dimension  $n - 1$ ; moreover, denoted by  $D$  one irreducible component of  $\text{Locus}(V)_F$  we have  $D \cdot V = 0$  and  $D \cdot R_1 > 0$  for a ray  $R_1$  not contained in the vector subspace of  $N_1(X)$  spanned by  $R$  and  $[V]$ .

Since  $\text{NE}(D) = \langle [R], [V] \rangle$ , by corollary 2.21, every nontrivial fiber of the contraction associated to  $R_1$  is one dimensional. Combining this with  $i_X \geq 2$ , by inequality 2.7, we have that  $V^1$  is a covering unsplit family.

By lemma 2.16, denoting again by  $F$  a nontrivial fiber of  $\varphi_R$ , we have  $\dim \text{Locus}(V, V^1)_F = \dim \text{Locus}(V^1, V)_F = n$  (and  $-K_X \cdot V = -K_X \cdot V^1 = i_X = 2$ ), so  $X = \text{Locus}(V, V^1)_F = \text{Locus}(V^1, V)_F$ .

We can write  $X = \text{Locus}(V, V^1)_F = \text{Locus}(V)_{\text{Locus}(V^1)_F}$  and therefore, by lemma 2.18 and proposition 2.20, the numerical class of every curve in  $X$  can be written as a linear combination  $a[V] + b[V^1] + c[R]$  with  $b, c \geq 0$ .

On the other hand  $X = \text{Locus}(V^1, V)_F = \text{Locus}(V^1)_{\text{Locus}(V)_F}$ , so the numerical class of every curve in  $X$  can be written as a linear combination  $a[V] + b[V^1] + c[R]$  with  $a, c \geq 0$ . By the uniqueness of the decomposition it follows that  $\text{NE}(X) = \langle [V], [V^1], [R] \rangle$ .

**Step 5** If  $\rho_X = 3, i_X \geq 2$  and  $R$  is not nef then  $X \simeq \text{Bl}_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ .

We have thus proved that the cone of curves of  $X$  is generated by  $R$ , which is the ray associated to a smooth blow up  $\varphi_R : X \rightarrow Y$ , and by other two fiber type extremal rays, call them  $R_1$  and  $R_2$ , which both have length two. In particular we have  $i_X = 2$ , so  $l(R) = n - 2$  and  $\dim F_R = n - 2$  for every nontrivial fiber of  $\varphi_R$ . Moreover, since  $E = \text{Exc}(R)$  is non negative on  $R_1$  and  $R_2$ , by [25, Proposition 3.4]  $Y$  is a Fano manifold.

The effective divisor  $E$  is positive on at least one of the rays  $R_i$  by lemma 2.1; let us assume that  $E \cdot R_1 > 0$ . Let  $\sigma$  be the extremal face spanned by  $R$  and  $R_1$  and consider the associated contraction  $\varphi_\sigma$ .

Let  $x \in X$  be a point, let  $\Gamma_1$  be a curve in  $R_1$  through  $x$  and let  $F$  be a nontrivial fiber of  $\varphi_R$  meeting  $\Gamma_1$ . The fiber of  $\varphi_\sigma$  through  $x$  contains  $\text{Locus}(R_1)_F$ , which has dimension  $n - 1$  by lemma 2.16, so the target of  $\varphi_\sigma$  is a smooth curve, which has to be rational since  $X$  is Fano. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_R} & Y \\ & \searrow \varphi_\sigma & \downarrow \psi_\sigma \\ & & \mathbb{P}^1 \end{array}$$

The general fiber  $F_\sigma$  of  $\varphi_\sigma$  is, by adjunction, a Fano manifold of index  $\geq 2$  which has a divisorial extremal ray of length  $\dim F_\sigma - 1$ , so, by theorem 1.1,  $F_\sigma \simeq \text{Bl}_p \mathbb{P}^{n-1}$ . It follows that the general fiber of  $\psi_\sigma$  is  $\mathbb{P}^{n-1}$ . The Fano manifold  $Y$  has a fiber type extremal ray  $\psi_\sigma$  of length  $\dim Y$  while the other ray is of fiber type, since the associated contraction contracts the images of curves in  $R_2$ . Therefore  $i_Y \geq 2$ .

We can thus apply theorem 1.1 to conclude that  $Y \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$ . Let  $T \simeq \mathbb{P}^1$  be the center of the blow up; we claim that  $T$  is a fiber of the projection  $Y \rightarrow \mathbb{P}^{n-1}$ . By contradiction, assume that this is not the case. Let  $C \simeq \mathbb{P}^1$  be a fiber of the projection  $Y \rightarrow \mathbb{P}^{n-1}$  meeting  $T$  and let  $\tilde{C}$  be the strict transform of  $C$ .

By the canonical bundle formula we have

$$-K_X \cdot \tilde{C} = -K_Y \cdot C - l(R)E \cdot \tilde{C} \leq 2 - l(R) \leq 0,$$

and so  $X$  is not a Fano manifold, a contradiction.  $\square$

**6. Blow ups.**

*Proof of 1.3.* If  $i_X + l(R) = n + 1$ , by theorem 1.1 we have that  $X = \text{Bl}_{\mathbb{P}^t}(\mathbb{P}^n)$ , with  $t \leq \frac{n-3}{2}$ .

We can thus assume that  $i_X + l(R) = n$ . By theorem 5.1, if  $\rho_X \geq 3$ , then  $X$  is either  $\text{Bl}_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$  or  $\text{Bl}_p(V_d)$  where  $V_d$  is  $\text{Bl}_Y(\mathbb{P}^n)$  and  $Y$  is a submanifold of dimension  $n - 2$  and degree  $\leq n$  contained in an hyperplane which does not contain  $p$ . Note that case a) of theorem 5.1 has been excluded since it is not a blow up.

We can thus assume, from now on, that  $\rho_X = 2$ ; again by theorem 5.1 either  $X \simeq \text{Bl}_p(\mathbb{Q}^n)$  or  $\rho_X = 2$ ,  $i_X \geq 2$  and the cone of curves of  $X$  is generated by  $R$  and by a fiber type extremal ray  $R_V$ . (Case e) of theorem 5.1 has been excluded since in that case  $R$  is a fiber type ray).

The target  $Y$  of  $\varphi_R$  is a smooth variety with  $\rho_Y = 1$  covered by rational curves, hence a Fano manifold; let  $V_Y$  be a minimal dominating family of rational curves for  $Y$  and let  $V^*$  be the family of deformations of the strict transform of a general curve in  $V_Y$ . By [17, Proposition 3.7] a general member of  $V_Y$  does not meet  $T$ , which has codimension at least two in  $Y$ , hence  $E \cdot V^* = 0$ .

Since  $E \cdot V^* = 0$ , by lemma 3.2, the family  $V^*$  is not quasi unsplit and all the reducible cycles in the associated Chow family  $\mathcal{V}^*$  have two irreducible components,  $C_1$  and  $C_2$ , where  $C_1$  and  $C_2$  are curves in the rays  $R$  and  $R_V$  respectively. Let  $\Gamma_R$  and  $\Gamma_V$  be curves in  $R$  and  $R_V$  respectively with minimal anticanonical degree.

Since  $\varphi_R$  is a smooth blow up  $E \cdot \Gamma_R = -1$ , hence the numerical class of every curve in  $R$  is an integral multiple of  $[\Gamma_R]$ ; in particular we can write  $[C_1] = m_1[\Gamma_R]$  with  $m_1$  a positive integer. By the canonical bundle formula

$$(6) \quad \begin{aligned} n + 1 &\geq -K_Y \cdot V_Y = -K_X \cdot V^* = -K_X \cdot (C_1 + C_2) \geq \\ &\geq m_1 l(R) + i_X \geq (m_1 - 1)l(R) + n. \end{aligned}$$

Recalling that  $l(R) \geq i_X \geq 2$  we have  $m_1 = 1$ , i.e.  $[C_1] = [\Gamma_R]$ . It follows that  $E \cdot C_2 = 1$ , so  $[C_2] = [\Gamma_V]$  and  $[V^*] = [\Gamma_R + \Gamma_V]$ .

Consider now the contraction of  $R_V$ ,  $\psi : X \rightarrow Z$  and let  $F$  be any fiber of  $\psi$ . Since  $E \cdot \Gamma_V > 0$  the fiber  $F$  meets a nontrivial fiber  $F_R$  of  $\varphi_R$  and therefore  $\dim F \leq n - \dim F_R = i_X$ .

On the other hand, by inequality 2.7  $\dim F \geq l(R_V) - 1 \geq i_X - 1$ , so the length of  $R_V$  is either  $i_X$  or  $i_X + 1$ . In the first case, by equation 6 we have  $-K_Y \cdot V_Y = n$ , while in the second we have  $-K_Y \cdot V_Y = n + 1$ .

The contraction  $\psi$  is supported by  $K_X + i_X E$  in the first case and by  $K_X + (i_X + 1)E$  in the second; in both cases, since for every fiber of  $\psi$  we have  $i_X - 1 \leq \dim F \leq i_X$ , the target variety  $Z$  is smooth by [3, Theorem 5.1].

The general fiber of  $\psi$  has dimension either  $i_X - 1$  or  $i_X$ , so the dimension of  $Z$  is either  $l(R) + 1$  or  $l(R)$ . We divide the proof in two cases, accordingly.

**Case 1**  $\dim Z = l(R) + 1$ .

In this case  $\psi$  is supported by  $K_X + i_X E$ , its general fiber has dimension  $i_X - 1$  and it is a projective space  $\mathbb{P}^{i_X - 1}$  by [3, Theorem 5.1], while jumping fibers, if they exist, have dimension  $i_X$  and are projective spaces  $\mathbb{P}^{i_X}$ , again by [3, Theorem 5.1].

We claim that, for at least one fiber  $F$  of  $\psi$ , we have  $E \cap F = \mathbb{P}^{i_X - 1}$ . The claim is clearly true if either  $E$  contains a fiber of dimension  $i_X - 1$  or, being  $E \cdot \Gamma_V = 1$ , if  $\psi$  has a jumping fiber ( $E$  cannot contain a jumping fiber  $F$ , otherwise, by lemma 2.16 a) we will have  $\dim E \geq \dim \text{Locus}(R)_F \geq i_X + l(R) \geq n$ ).

Suppose by contradiction that neither of these two possibilities happens. The restriction of  $\psi$  to  $E$  is thus an equidimensional morphism with general fiber a projective space, such that  $E$  restricted to the general fiber is  $\mathcal{O}_{\mathbb{P}}(1)$ , so  $\psi$  makes  $E$  a projective bundle over  $Z$ .

Therefore  $E$ , which is also a projective bundle over  $T$ , has two projective bundle structures and  $\rho_E = 2$  so, by [22, Theorem 2],  $E$  is the projectivization of the tangent bundle of a projective space, but this is impossible since the two fibrations of  $E$  have fibers of dimension  $i_X - 2$  and  $l(R)$  and these two dimensions are different, being  $l(R) \geq i_X$ , so the claim is proved.

It follows that either  $\psi$  has a jumping fiber or  $E$  contains a fiber of  $\psi$ ; in both cases  $T$ , the center of the blow up, is dominated by the intersection of  $E$  with this fiber, and so it is a projective space of dimension  $i_X - 1$  by [19, Theorem 4.1].

To finish the proof, we have to show that  $Y \simeq \mathbb{Q}^n$ , and we will do this proving the existence of a line bundle  $L_Y \in \text{Pic}(Y)$  such that  $-K_Y = nL_Y$  and applying the Kobayashi-Ochiai theorem [16].

Take a line  $l$  in  $T$  and denote by  $Y_l$  the inverse image  $\varphi_R^{-1}(l)$ ;  $Y_l$  is a projective bundle over a smooth rational curve, so a toric variety. The restriction  $\psi|_{Y_l} : Y_l \rightarrow Z$  is thus a surjective morphism from a toric variety to a smooth variety with Picard number



one, so  $Z$  is a projective space by [22, Theorem 1].

Let  $L$  be the line bundle  $\psi^*\mathcal{O}_{\mathbb{P}}(1) + E$ ; we have  $L \cdot R = 0$  and therefore there exists  $L_Y \in \text{Pic}(Y)$  such that  $\varphi_R^*L_Y = L$ .

Moreover, since  $L \cdot V^* = 1$  we have  $L_Y \cdot V_Y = 1$ , so, recalling that  $-K_Y \cdot V_Y = \dim Y$  we get  $-K_Y = nL_Y$  and we conclude that  $Y \simeq \mathbb{Q}^n$  by the Kobayashi-Ochiai theorem [16].

**Case 2**  $\dim Z = l(R)$ .

In this case, as noted above, every fiber of  $\psi$  has dimension  $i_X$ . The contraction  $\psi$  is supported either by  $K_X + (i_X + 1)E$  and it is a projective bundle or by  $K_X + i_X E$  and it is a quadric bundle, by [3, Theorem 5.1].

Every nontrivial fiber of  $\varphi_R$  dominates  $Z$  so, by [19, Theorem 4.1]  $Z$  is a projective space.

Let  $L$  be the line bundle  $\psi^*\mathcal{O}_{\mathbb{P}}(1) + E$ ; we have  $L \cdot R = 0$  so there exists  $L_Y \in \text{Pic}(Y)$  such that  $\varphi_R^*L_Y = L$ .

Moreover, since  $L \cdot V^* = 1$  we have  $L_Y \cdot V_Y = 1$ .

**Case 2a**  $\psi : X \rightarrow Z$  is a projective bundle.

In this case  $-K_Y \cdot V_Y = n + 1$ , so  $-K_Y = (n + 1)L_Y$  and  $Y$  is a projective space by the Kobayashi-Ochiai theorem [16]. The intersection of  $E$  with the general fiber of  $\psi$  is thus a projective space and therefore the center  $T$  of the blow up is a linear space by [19, Theorem 4.1].

**Case 2b**  $\psi : X \rightarrow Z$  is a quadric bundle.

In this case  $-K_Y \cdot V_Y = n$ , so  $-K_Y = nL_Y$  and  $Y$  is a smooth quadric by the Kobayashi-Ochiai theorem.

The intersection of  $E$  with the general fiber of  $\psi$  is thus a smooth quadric, so the center  $T$  of the blow up is either a linear space or a smooth quadric by [23, Proposition 8].

Actually the first case can be excluded by direct computation, since the blow up of a quadric along a linear subspace is not a quadric bundle over  $\mathbb{P}^{l(R)}$ .

In the second case let  $\Pi \simeq \mathbb{P}^{i_X}$  be the linear subspace of dimension  $i_X$  of  $\mathbb{P}^{n+1}$  which contains  $T \simeq \mathbb{Q}^{i_X-1}$ .

Two cases are possible: either  $Y \supseteq \Pi$  or  $Y \cap \Pi = T$ . The first case has to be excluded because, if  $Y \supseteq \Pi$  the blow up of  $\mathbb{Q}^n$  along  $T$  does not give rise to a Fano manifold.

To see this, take a line  $l \subset \Pi$  not contained in  $T$ ; by the canonical bundle formula, if  $X = Bl_T \mathbb{Q}^n$  we have

$$-K_X \cdot \tilde{l} = -K_Y \cdot l - l(R)E \cdot \tilde{l} \leq n - 2l(R) \leq 0.$$

Finally note that in both cases the bound on the dimension of the center follows from the fact that  $i_X \leq l(R)$  and so  $2i_X \leq l(R) + i_X \leq n$ .  $\square$

**7. Varieties with a polarization.**

*Proof of 1.2.* Let  $V$  be the family given by lemma 4.1, let  $x \in \text{Exc}(R)$  be a point such that  $V_x$  is unsplit and let  $F_x$  be the fiber of  $\varphi_R$  containing  $x$ .

First of all we prove that  $\rho_X = 2$  and that the cone of curves of  $X$  is generated by  $R$  and by the ray spanned by  $[V]$ .

We are assuming that equality holds in (\*), so equality holds everywhere in (2); in particular we have

$$(7) \quad \dim F_x = l(R) + \dim X - \dim \text{Exc}(R) - 1 = \dim X - r_X + 1$$

$$(8) \quad \dim \text{Locus}(V_x) = r_X - 1.$$

This forces  $-K_X \cdot V = r_X$ , so the family  $V$  is unsplit. Moreover, by inequality 2.5  $V$  is a covering family.

Therefore, by lemma 2.16 we have  $\dim \text{Locus}(V)_{F_x} \geq \dim F_x + r_X - 1 = \dim X$ , so, by proposition 2.20 c), we have  $NE(X) = \langle [V], [R] \rangle$ .

Let  $\psi : X \rightarrow Z$  be the contraction of the ray  $R_V$  spanned by  $[V]$ , which is of fiber type since  $V$  is a covering family; curves parametrized by  $V$  have anticanonical degree  $r_X$ , so they are minimal extremal curves in  $R_V$  which has length  $r_X$ .

By inequality 2.7, every fiber of  $\psi$  has dimension  $\geq l(R_V) - 1 = r_X - 1$ , so  $\dim Z \leq n - r_X + 1$ . Again by inequality 2.7 the fibers of  $\varphi_R$  have dimension  $\geq n - e + l - 1 = n - r_X + 1$ , so they dominate  $Z$ . In particular every fiber of  $\psi$  meets a fiber  $F_R$  of  $\varphi_R$  and so its dimension is  $\leq \dim X - \dim F_R = r_X - 1$ ; therefore the contraction  $\psi : X \rightarrow Z$  is equidimensional.

Moreover we also have that the dimension of every fiber of  $\varphi_R$  is  $\leq \dim Z \leq n - r_X + 1$ , so  $\varphi_R$  is equidimensional with fibers of dimension  $n - r_X + 1$  and  $\dim Z = n - r_X + 1$ .

Denote by  $H$  the divisor such that  $-K_X = r_X H$ . The general fiber  $G$  of  $\psi$  is, by generic smoothness and adjunction, a projective space  $\mathbb{P}^{r_X-1}$  and  $H_G \simeq \mathcal{O}(1)$ , so, by [13, Lemma 2.12],  $\psi$  is a projective bundle over  $Z$ ,  $X = \mathbb{P}_Z(\mathcal{E})$ , with  $\mathcal{E} = \varphi_R^* H$ . In particular  $Z$  is a smooth Fano variety of Picard number one.

The canonical bundle formula yields

$$\psi^*(K_Z + \det \mathcal{E}) = K_X + r_X H = \mathcal{O}_X,$$

and so  $-K_Z = \det \mathcal{E}$ . Note also that, if  $C_R$  is a curve in  $R$  then

$$(9) \quad H \cdot C_R = \frac{-K_X \cdot C_R}{r_X} \geq \frac{l(R)}{r_X}.$$

Let  $V_Z$  be a minimal covering family for  $Z$  and  $C$  a curve in  $V_Z$ ; Let  $\nu : \mathbb{P}^1 \rightarrow C \subset Z$  be the normalization of  $C$  and let  $Z_C$  be the fiber product  $Z_C = \mathbb{P}^1 \times_C X$ .

$$\begin{array}{ccc} Z_C & \xrightarrow{\bar{\nu}} & X \\ p \downarrow & & \downarrow \psi \\ \mathbb{P}^1 & \xrightarrow{\nu} & Z \end{array}$$

The variety  $Z_C$  is a projective bundle over  $\mathbb{P}^1$ ,  $Z_C = \mathbb{P}_{\mathbb{P}^1}^1(\nu^* \mathcal{E})$ ; the vector bundle  $\nu^* \mathcal{E}$  is ample, so we can write  $\nu^* \mathcal{E} \simeq \bigoplus_{i=0}^{r_X-1} \mathcal{O}(a_i)$  with  $a_i > 0$  and  $a_i \leq a_{i+1} \quad \forall i$ .

Denote by  $m$  the maximum index  $i$  such that  $a_i = a_0$  and rewrite  $\nu^* \mathcal{E}$  in the following way

$$\nu^* \mathcal{E} \simeq \bigoplus^{m+1} \mathcal{O}(a_0) \oplus_{i=m+1}^{r_X-1} \mathcal{O}(a_i).$$

The cone of curves  $NE(Z_C)$  is generated by the class of a line in a fiber of  $p$  and by the class of a section  $C_0$  corresponding to a surjection  $\nu^* \mathcal{E} \rightarrow \mathcal{O}(a_0)$ .

The cone of curves  $NE(X)$  is generated by the class  $[V]$  of a line in a fiber of  $\varphi_V$  and by the class of  $\Gamma_R$ , a minimal extremal curve in  $R$ .

The morphism  $\bar{\nu}$  induces a map of spaces of cycles  $N_1(Z_C) \rightarrow N_1(X)$  which allows us to identify  $NE(Z_C)$  with a subcone of  $NE(X)$ .

Since  $\bar{\nu}(Z_C)$  contains lines in the fibers of  $\psi$  and contains curves in the fibers of

$\varphi_R$  (since for dimensional reasons  $\dim(\bar{\nu}(Z_C) \cap F_R) \geq 1$ ), we have an identification  $\text{NE}(Z_C) \simeq \text{NE}(X)$ .

In particular  $F_R \cap \bar{\nu}(Z_C)$ , which is a curve whose numerical class in  $X$  is a multiple of  $[\Gamma_R]$ , is the image of a curve  $\Gamma$  whose numerical class in  $Z_C$  is a multiple of  $[C_0]$ .

By [21, Lemma 3] the curve  $\Gamma$  is the union of disjoint minimal sections, so  $\bar{\nu}(Z_C) \cap F_R$  consists of the images via  $\bar{\nu}$  of disjoint minimal sections.

On the other hand, if  $C_0$  is a minimal section, then  $\bar{\nu}(C_0)$  is a curve whose numerical class is in  $R$ , so it is contained in a fiber of  $\varphi_R$ .

It follows that the dimension of  $\varphi_R(\text{Exc}(R))$  is the dimension of the space parametrizing minimal sections, which is  $m$ . Therefore

$$m = \dim \text{Exc}(R) - \dim F_R = l(R) + r_X - 2 - \dim F_R.$$

Moreover, since  $[C_0] \in R$  we have, by equation 9

$$a_0 = H \cdot C_0 \geq \frac{l(R)}{r_X},$$

hence  $a_i \geq a_0 + 1 \geq \frac{l(R)}{r_X} + 1$  for  $i = m + 1, \dots, r_X - 1$ .

It follows that

$$\begin{aligned} \dim Z + 1 &\geq -K_Z \cdot C = \det \mathcal{E} \cdot C = \\ &= (m + 1)a_0 + \sum_{m+1}^{r_X-1} a_i \geq (m + 1)\frac{l(R)}{r_X} + (r_X - m - 1)\left(\frac{l(R)}{r_X} + 1\right) = \\ &= l(R) + r_X - m - 1 = \dim F_R + 1 = \dim Z + 1. \end{aligned}$$

Therefore  $Z$  admits a minimal dominating family of degree  $\dim Z + 1$ , hence  $Z$  is a projective space of dimension  $n - r_X + 1$  by the proof of [15, Theorem 1.1].

Since equality holds everywhere we also have  $a_0 = 1, a_i = 2 \quad i = m + 1, \dots, r_X - 1$ , so the splitting type of  $\mathcal{E}$  on lines of  $Z$  is uniform.

If  $\dim \text{Exc}(R) \leq \dim X - 2$  then  $\text{rk } \mathcal{E} = r_X \leq l(R) < n - r + 1 = \dim Z$ , therefore  $\mathcal{E}$  is decomposable by [12] and  $\mathcal{E} \simeq \oplus^{m+1} \mathcal{O}(1) \oplus^{r_X-1-m} \mathcal{O}(2)$ .

If  $\dim \text{Exc}(R) = \dim X - 1$  then  $\text{rk } \mathcal{E} = \dim Z$  and the splitting type of  $\mathcal{E}$  is  $(1, \dots, 1, 2)$ , so, by [12], either  $\mathcal{E}$  is decomposable or  $\mathcal{E}$  is the tangent bundle of  $Z = \mathbb{P}^{\dim Z}$ , but the second case has to be excluded since  $X$  has a divisorial contraction.

Finally, if  $\text{Exc}(R) = X$  then the splitting type of  $\mathcal{E}$  is  $(1, \dots, 1)$ , so  $\mathcal{E}$  is decomposable by [4, Proposition 1.2] and  $X$  is a product of projective spaces.  $\square$

**PROPOSITION 7.1.** *Let  $X$  be a Fano manifold of Picard number  $\rho_X = 2$ , index  $r_X \geq 2$ , and let  $R$  be a fiber type or divisorial extremal ray such that  $r_X + l(R) = \dim \text{Exc}(R) + 1$ . Then, if  $R$  is divisorial either  $X$  is as in theorem 1.3 or  $X$  has the structure of a projective bundle over a smooth variety.*

*If  $R$  is of fiber type then  $X$  is a projective bundle or a quadric bundle or the projectivization of a Bănică sheaf over a smooth variety  $Y$ .*

*Proof.* By theorem 5.1 either  $X \simeq Bl_{\mathbb{P}^{l(R)-2}}(\mathbb{P}^n)$  or the cone of curves  $\text{NE}(X)$  is generated by  $R$  and by a fiber type extremal ray; let  $\psi : X \rightarrow Z$  be the contraction

of this ray.

Let  $H$  be the line bundle such that  $-K_X = r_X H$ , let  $A \in \text{Pic}(Z)$  be an ample divisor and let  $H' = H + \psi^* A$ . The contraction  $\psi$  is supported by  $K_X + r_X H'$ .

If  $R$  is divisorial then every fiber of  $\varphi_R$  has dimension  $\geq l(R)$ . If equality holds for every fiber,  $\varphi_R$  is a smooth blow up by [2, Theorem 5.1], so  $X$  is as in theorem 1.3.

We can therefore assume that there exists a fiber  $F$  of  $\varphi_R$  of dimension  $\geq l(R) + 1$ . The contraction  $\psi : X \rightarrow Z$  has fibers of dimension  $\geq r_X - 1 \geq n - l(R) - 1$ , so  $\dim Z \leq l(R) + 1$ . It follows that  $F$  dominates  $Z$  and meets every fiber of  $\psi$ , forcing the equidimensionality of  $\psi$ .

We can now conclude that  $X$  is a projective bundle over  $Z$  by [13, Lemma 2.12] since  $H \cdot V = 1$ .

If  $R$  is of fiber type then every fiber of  $\varphi_R$  has dimension  $\geq l(R) - 1$  and so the contraction  $\psi : X \rightarrow Z$  has fibers of dimension  $\leq n - l(R) + 1 \leq r_X$ , so we can conclude by [3, Theorem 5.1] and [5, Proposition 2.5].  $\square$

#### REFERENCES

- [1] MARCO ANDREATTA, ELENA CHIERICI, AND GIANLUCA OCCHETTA, *Generalized Mukai conjecture for special Fano varieties*, Centr. Eur. J. Math., 2:2 (2004), pp. 272–293.
- [2] MARCO ANDREATTA AND GIANLUCA OCCHETTA, *Special rays in the Mori cone of a projective variety*, Nagoya Math. J., 168 (2002), pp. 127–137.
- [3] MARCO ANDREATTA AND JAROSLAW A. WIŚNIEWSKI, *A view on contractions of higher dimensional varieties*, In Algebraic Geometry – Santa Cruz 1995, volume 62 of Proc. Sympos. Pure Math., pp. 153–183. Amer. Math. Soc., Providence, RI, 1997.
- [4] MARCO ANDREATTA AND JAROSLAW A. WIŚNIEWSKI, *On manifolds whose tangent bundle contains an ample subbundle*, Invent. Math., 146:1 (2001), pp. 209–217.
- [5] EDOARDO BALLICO AND JAROSLAW A. WIŚNIEWSKI, *On Bănică sheaves and Fano manifolds*, Compositio Math., 102:3 (1996), pp. 313–335.
- [6] LAURENT BONAVERO, FRÉDÉRIC CAMPANA, AND JAROSLAW A. WIŚNIEWSKI, *Variétés complexes dont l'éclatée en un point est de Fano*, C. R. Math. Acad. Sci. Paris, 334 (2002), pp. 463–468.
- [7] LAURENT BONAVERO, CINZIA CASAGRANDE, OLIVIER DEBARRE, AND STEPHANE DRUEL, *Sur une conjecture de Mukai*, Comment. Math. Helv., 78 (2003), pp. 601–626.
- [8] FRÉDÉRIC CAMPANA, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup., 25 (1992), pp. 539–545.
- [9] CINZIA CASAGRANDE, *The number of vertices of a Fano polytope*, math.AG/0411073, November 2004, to appear on Ann. Inst. Fourier.
- [10] ELENA CHIERICI AND GIANLUCA OCCHETTA, *The cone of curves of Fano varieties of coindex four*, math.AG/0401429, february 2004.
- [11] KOJI CHO, YOICHI MIYAOKA, AND NICK SHEPHERD-BARRON, *Characterizations of projective space and applications to complex symplectic manifolds*, in Higher dimensional birational geometry (Kyoto, 1997), volume 35 of Adv. Studies in Pure Math., pp. 1–88. Math. Soc. Japan, Tokyo, 2002.
- [12] GEORGES ELENCAWJG, ANDRÉ HIRSCHOWITZ, AND MICHAEL SCHNEIDER, *Les fibres uniformes de rang au plus  $n$  sur  $\mathbb{P}_n(\mathbb{C})$  sont ceux qu'on croit*, in Vector bundles and differential equations (Proc. Conf., Nice, 1979), volume 7 of Progr. Math., pp. 37–63. Birkhäuser Boston, 1980.
- [13] TAKAO FUJITA, *On polarized manifold whose adjoint bundles are not semipositive*, in Algebraic geometry, Sendai, volume 10 of Adv. Studies in Pure Math., pp. 167–178. Kinokuniya-North-Holland, 1987.
- [14] PALTIN IONESCU, *Generalized adjunction and applications*, Math. Proc. Camb. Phil. Soc., 99 (1986), pp. 457–472.
- [15] STEFAN KEBEKUS, *Characterizing the projective space after Cho, Miyaoka and Sheperd-Barron*, in Complex Geometry, Göttingen 2000, ppp. 147–155. Springer Berlin, 2002.
- [16] S KOBAYASHI AND T OCHIAI, *Characterization of complex projective space and hyperquadrics*, J. Math. Kyoto U., 13 (1972), pp. 31–47.

- [17] JÁNOS KOLLÁR, *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Math.*, Springer Verlag, Berlin, Heidelberg, New York, Tokio, 1996.
- [18] JÁNOS KOLLÁR, YOICHI MIYAOKA, AND SHIGEFUWI MORI, *Rational connectedness and boundedness of Fano manifolds*, *J. Diff. Geom.*, 36 (1992), pp. 765–779.
- [19] ROBERT LAZARSFELD, *Some applications of the theory of positive vector bundles*, in *Complete intersections (Acireale 1983)*, volume 1092 of *Lecture Notes in Math.*, pp. 29–61. Springer-Verlag, 1984.
- [20] SHIGEFUWI MORI, *Projective manifolds with ample tangent bundle*, *Ann. Math.*, 110 (1979), pp. 595–606.
- [21] GIANLUCA OCCHETTA, *A characterization of products of projective spaces*, to appear on *Canad. Math. Bull.*, February 2003.
- [22] GIANLUCA OCCHETTA AND JAROSLAW A. WIŚNIEWSKI, *On Euler-Jaczewski sequence and Remmert-Van de Ven problem for toric varieties*, *Math. Z.*, 241 (2002), pp. 35–44.
- [23] KAPIL HARI PARANJPE AND VASUDEVAN SRINIVAS, *Self maps of homogeneous spaces*, *Invent. Math.*, 98 (1989), pp. 425–444.
- [24] JAROSLAW A. WIŚNIEWSKI, *On a conjecture of Mukai*, *Manuscripta Math.*, 68 (1990), pp. 135–141.
- [25] JAROSLAW A. WIŚNIEWSKI, *On contractions of extremal rays of Fano manifolds*, *J. Reine und Angew. Math.*, 417 (1991), pp. 141–157.

