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INTEGRAL CANONICAL MODELS OF UNITARY SHIMURA VARIETIES*

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Abstract. We prove the existence of integral canonical models of unitary Shimura varieties in arbitrary unramified mixed characteristic. Errata to [Va1] are also included.

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AMS subject classifications. Primary 11G10, 11G18, 14F30, 14G35, 14G40, 14K10

1. Introduction. A *Shimura pair* (G, X) consists of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{R}}$ that satisfy Deligne's axioms of [De2, Subsubsect. 2.1.1]: the Hodge $\mathbb{Q}$–structure on $\mathrm{Lie}(G)$ defined by any $x \in X$ is of type $\{(-1,1),(0,0),(1,-1)\}$, no simple factor of the adjoint group G^{ad} of G becomes compact over $\mathbb{R}$, and $(\mathrm{Ad} \circ x)(i)$ defines a Cartan involution of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})$. Here $\mathrm{Ad} : G_{\mathbb{R}} \to \mathbf{GL}_{\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})}$ is the adjoint representation. These axioms imply that X has a natural structure of a hermitian symmetric domain, cf. [De2, Cor. 1.1.17]. We say (G, X) is *unitary* if the group G^{ad} is non-trivial and all simple factors of $G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}$ are of some A_n Lie type with $n \in \mathbb{N}$ (i.e., are groups isomorphic to $\mathbf{PGL}_{n,\overline{\mathbb{Q}}}$ for some $n \in \mathbb{N}$). Let X^0 be a connected component of X. Let $Z(G)$ be the center of G.

Let $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of finite adèles of $\mathbb{Q}$. Let $\overline{Z(G)(\mathbb{Q})}$ be the closure of $Z(G)(\mathbb{Q})$ in $Z(G)(\mathbb{A}_f)$. Let $\mathfrak{C}(G)$ be the set of compact, open subgroups of $G(\mathbb{A}_f)$ endowed with the inclusion relation. For $O \in \mathfrak{C}(G)$, the quotient complex space $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/O)$ is a finite disjoint union of quotients of X^0 by arithmetic subgroups of $G(\mathbb{Q})$. Each such quotient has a natural structure of a normal, quasi-projective complex variety, cf. [BB, Thm. 10.11]. By the complex Shimura variety $\mathrm{Sh}(G, X)_{\mathbb{C}}$ one means the $\mathbb{C}$-scheme

(1)
$$\mathrm{Sh}(G,X)_{\mathbb{C}} := \mathrm{proj.lim.}_{O\in\mathfrak{C}(G)} G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/O) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/\overline{Z(G)(\mathbb{Q})})$$

together with the natural right action of $G(\mathbb{A}_f)$ on it (see [De2, Cor. 2.1.11] for the equality part). This action is continuous in the sense of [De2, Subsubsect. 2.7.1].

Let $E(G, X)$ be the number field that is the *reflex field* of (G, X). Roughly speaking, $E(G, X)$ is the smallest subfield of $\mathbb{C}$ over which $\mathrm{Sh}(G, X)_{\mathbb{C}}$ has a (good) canonical model $\mathrm{Sh}(G, X)$ (see [De1] and [De2] for the case of Shimura pairs of abelian type; see [Mi1] and [Mi4] for the general case). One calls $\mathrm{Sh}(G, X)$ the *Shimura variety* defined by (G, X).

Let $p \in \mathbb{N}$ be a prime such that the group $G_{\mathbb{Q}_p}$ is *unramified* i.e., $G_{\mathbb{Q}_p}$ has a Borel subgroup and it splits over a finite, unramified extension of $\mathbb{Q}_p$. It is known that $G_{\mathbb{Q}_p}$ is unramified if and only if it is the generic fibre of a reductive group scheme $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, cf. [Ti, Subsubsects. 1.10.2 and 3.8.1]. Each subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ of

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the form $H := G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is called *hyperspecial*. We refer to the triple (G, X, H) as a *Shimura triple* (with respect to p). Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of $\mathbb{Q}$ with the p-component omitted; thus we have $\mathbb{A}_f = \mathbb{Q}_p \times \mathbb{A}_f^{(p)}$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal (p). For a subfield E of $\overline{\mathbb{Q}}$, let $E_{(p)}$ be the normalization of $\mathbb{Z}_{(p)}$ in E. As the group $G_{\mathbb{Q}_p}$ is unramified, the field $E(G, X)$ is unramified over p (cf. [Mi3, Cor. 4.7 (a)]) and thus $E(G, X)_{(p)}$ is a finite, étale $\mathbb{Z}_{(p)}$-algebra. We recall some basic definitions from [Va1] and [Mi2].

1.1. Definitions. **(a)** A $\mathbb{Z}_{(p)}$-scheme $\mathcal{Y}$ is called *healthy regular* if it is regular and faithfully flat and if for each open subscheme $\mathcal{U}$ of $\mathcal{Y}$ which contains $\mathcal{Y}_{\mathbb{Q}}$ and whose complement in $\mathcal{Y}$ is of codimension in $\mathcal{Y}$ at least 2, every abelian scheme over $\mathcal{U}$ extends to an abelian scheme over $\mathcal{Y}$. A flat $E_{(p)}$-scheme $\mathcal{Z}$ is said to have the *extension property* if for each $E_{(p)}$-scheme $\mathcal{Y}$ that is healthy regular, every morphism $\mathcal{Y}_E \to \mathcal{Z}_E$ extends uniquely to a morphism $\mathcal{Y} \to \mathcal{Z}$.

(b) By an *integral canonical model* $\mathcal{N}$ of (G, X, H) we mean a faithfully flat $E(G, X)_{(p)}$-scheme together with a continuous right action of $G(\mathbb{A}_f^{(p)})$ on it in the sense of [De2, Subsubsect. 2.7.1], such that the following three axioms hold:

(i) we have $\mathcal{N}_{E(G,X)} = \mathrm{Sh}(G, X)/H$ and the action of $G(\mathbb{A}_f^{(p)})$ on $\mathcal{N}_{E(G,X)}$ is canonically identified with the action of $G(\mathbb{A}_f^{(p)})$ on $\mathrm{Sh}(G, X)/H$;

(ii) there exists a compact, open subgroup H_p of $G(\mathbb{A}_f^{(p)})$ such that $\mathcal{N}/H_p$ is a smooth $E(G, X)_{(p)}$-scheme of finite type and $\mathcal{N}$ is a pro-étale cover of it;

(iii) the $E(G, X)_{(p)}$-scheme $\mathcal{N}$ has the extension property.

(c) If the integral canonical model $\mathcal{N}$ of (G, X, H) exists, then we say $\mathcal{N}$ is quasi-projective (resp. projective) if in the axiom (ii) of (b) we can choose H_p such that $\mathcal{N}/H_p$ is a quasi-projective (resp. projective), smooth $E(G, X)_{(p)}$-scheme.

1.2. The uniqueness of $\mathcal{N}$. Each regular scheme that is formally smooth and faithfully flat over either $\mathbb{Z}_{(p)}$ or $E(G, X)_{(p)}$ is healthy regular, cf. [Va2, Thm. 1.3]. Thus if the integral canonical model $\mathcal{N}$ of (G, X, H) exists, then it is a regular, formally smooth $\mathbb{Z}_{(p)}$-scheme (cf. axiom (ii) of Definition 1.1 (b)) and thus it is a healthy regular scheme. From this and the axiom (iii) of Definition 1.1 (b) we get (cf. Yoneda lemma): if the integral canonical model $\mathcal{N}$ of (G, X, H) exists, then it is uniquely determined up to a canonical isomorphism (cf. axiom (i) of Definition 1.1 (b)). The main goal of this paper is to prove the following basic result conjectured by Milne (see [Mi2, Conj. 2.7]).

1.3. Basic Theorem. *We assume that the Shimura pair (G, X) is unitary and that the group $G_{\mathbb{Q}_p}$ is unramified. Then every Shimura triple (G, X, H) with respect to p has a unique integral canonical model $\mathcal{N}$. Moreover, $\mathcal{N}$ is quasi-projective.*

The resulting smooth, quasi-projective $E(G, X)_{(p)}$-schemes $\mathcal{N}/H_p$ are the unitary equivalent of Mumford's moduli $\mathbb{Z}_{(p)}$-schemes $\mathcal{A}_{g,1,N}$ that parametrize isomorphism classes of principally polarized abelian schemes which are of relative dimension g and are endowed with level-N symplectic similitude structures (see [MFK, Thms. 7.9 and 7.10]). Here g, $N \in \mathbb{N}$, with N at least 3 and relatively prime to p. We recall from loc. cit. that $\mathcal{A}_{g,1,N}$ is a quasi-projective, smooth $\mathbb{Z}_{(p)}$-scheme.

1.4. On the proof of the Basic Theorem and literature. To explain the three main steps of the proof of the Basic Theorem and the relevant literature that pertains to them and to the Basic Theorem, in this Subsection we will assume that (G, X) is a simple, adjoint, unitary Shimura pair of isotypic A_n Dynkin type. In [De2, Prop. 2.3.10] it is proved the existence of an injective map $f_1 : (G_1, X_1) \hookrightarrow (\mathbf{GSp}(W, \psi), S)$ of Shimura pairs such that we have $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (G, X)$, where $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ is the adjoint Shimura variety of (G_1, X_1) (see [Va1, Subsubsect. 2.4.1]) and where $(\mathbf{GSp}(W, \psi), S)$ is a Shimura pair that defines a Siegel modular variety (thus (W, ψ) is a symplectic space over $\mathbb{Q}$).

The first step uses a modification of the proof of [De2, Prop. 2.3.10] to show that we can choose f_1 such that G_1 is the subgroup of $\mathbf{GSp}(W, \psi)$ that fixes a semisimple $\mathbb{Q}$–subalgebra $\mathcal{B}$ of $\mathrm{End}(W)$ cf. Proposition 3.2. Thus the injective map f_1 is a unitary embedding of PEL type and therefore it allows us to view $\mathrm{Sh}(G_1, X_1)$ naturally as a moduli $E(G_1, X_1)$-scheme of principally polarized abelian schemes endowed with symplectic similitude structures and with a suitable $\mathbb{Z}$-algebra of endomorphisms that is an order of $\mathcal{B}$. Following [Va1, Subsects. 6.5 and 6.6], Proposition 3.2 is worked out in the context of embeddings between reductive group schemes over $\mathbb{Z}_{(p)}$: we can choose f_1 such that moreover there exists a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W which is self dual with respect to ψ and which has the property that the Zariski closure $G_{1,\mathbb{Z}_{(p)}}$ of G_1 in $\mathbf{GL}_{L_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$ whose extension to $\mathbb{Z}_p$ has $G_{\mathbb{Z}_p}$ as its adjoint group scheme.

The second step only recalls the classical works [Zi], [LR], and [Ko] to get that the integral canonical model $\mathcal{N}_1$ of the Shimura triple $(G_1, X_1, G_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_p))$ exists and is a moduli scheme of principally polarized abelian schemes endowed with compatible level-N symplectic similitude structures for every $N \in \mathbb{N} \setminus p\mathbb{N}$ and with a suitable $\mathbb{Z}_{(p)}$-algebra of $\mathbb{Z}_{(p)}$-endomorphisms which is an order of $\mathcal{B}$ (see Subsections 4.1 to 4.3).

The third step uses the standard moduli interpretation of $\mathcal{N}_1$ to show that $\mathcal{N}$ exists as well (see Theorem 4.3 and Corollary 4.4). If $W(\mathbb{F})$ is the ring of Witt vectors with coefficients in an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and if we fix a $\mathbb{Z}_{(p)}$-monomorphism $E(G, X)_{(p)} \hookrightarrow W(\mathbb{F})$, then every connected component $\mathcal{C}$ of $\mathcal{N}_{W(\mathbb{F})}$ will be isomorphic to the quotient of a connected component $\mathcal{C}_1$ of $\mathcal{N}_{1,W(\mathbb{F})}$ by a suitable group action $\mathfrak{T}$ whose generic fibre is free and which involves a torsion group. The key point is to show that the action $\mathfrak{T}$ itself is free (i.e., $\mathcal{C}$ is a smooth $W(\mathbb{F})$-scheme). If $p > 2$ and p does not divide $n+1$, then the torsion group of the action $\mathfrak{T}$ has no elements of order p and thus the action $\mathfrak{T}$ is free (see proof of [Va1, Thm. 6.2.2 b)]). In this paper we check that the action $\mathfrak{T}$ is always free i.e., it is free even for the harder cases when either $p = 2$ or p divides $n+1$. The proof relies on the moduli interpretation of $\mathcal{N}_1$ which allows us to make this group action quite explicit (see proof of Theorem 4.3). The cases $p = 2$ and p divides $n+1$ are the hardest due to the following two reasons.

(i) If $p = 2$ and if A is an abelian variety over $\mathbb{F}$ whose 2-rank a is positive, then the group $(\mathbb{Z}/2\mathbb{Z})^{a^2}$ is naturally a subgroup of the group of automorphisms of the formal deformation space $\mathrm{Def}(A)$ of A in such a way that the filtered Dieudonné module of a lift $\star$ of A to $\mathrm{Spf}(W(\mathbb{F}))$ depends only on the orbit under this action of the $\mathrm{Spf}(W(\mathbb{F}))$-valued point of $\mathrm{Def}(A)$ defined by $\star$.

(ii) For a positive integer m divisible by $p-1$ there exist actions of $Z/p\mathbb{Z}$ on $\mathbb{Z}_p[[x_1, \ldots, x_m]]$ such that the induced actions on $\mathbb{Z}_p[[x_1, \ldots, x_m]][\frac{1}{p}]$ are free.

The general case of the Basic Theorem is proved in Corollary 4.4 and Section 5. If $p = 2$ and (G, X) is a Shimura curve, then the Basic Theorem is in essence part of the mathematical folklore (see [Mo2], etc.). We do not know any other previously known cases of the Basic Theorem in which G is a simple, adjoint group of isotypic A_n Dynkin type and p is a divisor of $n+1$. For $p \geq 5$, the Basic Theorem was claimed in [Va1] using a long and technical proof that applied to all Shimura varieties of abelian type and that did not use unitary PEL type embeddings. Unfortunately, loc. cit. had a relevant error in the cases when p divides $n+1$. But the error is now corrected in this paper. In the Appendix we include errata to [Va1].

2. Complements on Shimura varieties. In Subsection 2.1 we gather supplementary notations. In Subsections 2.2 and 2.4 we include a review and complements on Shimura pairs and triples. Lemma 2.3 pertains to reductive groups over $\mathbb{Q}$. Let $p \in \mathbb{N}$ be a prime. Let $n \in \mathbb{N}$.

2.1. Extra notations. If $\mathcal{G}$ is a reductive group scheme over an affine scheme, let $\mathcal{G}^{\mathrm{der}}$, $\mathcal{G}^{\mathrm{ad}}$, $\mathcal{G}^{\mathrm{ab}}$, and $Z(\mathcal{G})$ be the derived group scheme, the adjoint group scheme, the abelianization, and the center (respectively) of $\mathcal{G}$. We have $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}/Z(\mathcal{G})$ and $\mathcal{G}^{\mathrm{ab}} = \mathcal{G}/\mathcal{G}^{\mathrm{der}}$. Let $Z^0(\mathcal{G})$ be the maximal torus of $Z(\mathcal{G})$; the finite, flat group scheme $Z(\mathcal{G})/Z^0(\mathcal{G})$ is of multiplicative type. Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$. We have $\mathbb{S}(\mathbb{R}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$. We identify $\mathbb{S}(\mathbb{C}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) \times \mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$ in such a way that the monomorphism $\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ induces the map $z \to (z, \overline{z})$. For a, $b \in \mathbb{N} \cup \{0\}$, let $\mathbf{SU}(a, b)$ be the simply connected semisimple group over $\mathbb{R}$ whose $\mathbb{R}$–valued points are the $\mathbb{C}$–valued points of $\mathbf{SL}_{a+b,\mathbb{C}}$ that leave invariant the hermitian form $-z_1\overline{z}_1 - \cdots - z_a\overline{z}_a + z_{a+1}\overline{z}_{a+1} + \cdots + z_{a+b}\overline{z}_{a+b}$ on $\mathbb{C}^{a+b}$.

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. Let $W(\mathbb{F})$ be the ring of Witt vectors with coefficients in $\mathbb{F}$. Let $B(\mathbb{F})$ be the field of fractions of $W(\mathbb{F})$.

If M is a free module of finite rank over a commutative ring with unit R, let $\mathbf{GL}_M$ be the reductive group scheme over R of linear automorphisms of M. Let $\mathbf{SL}_M := \mathbf{GL}_M^{\mathrm{der}}$. If λ_M is a perfect alternating form on M, then $\mathbf{GSp}(M, \lambda_M)$ and $\mathbf{Sp}(M, \lambda_M) := \mathbf{GSp}(M, \lambda_M)^{\mathrm{der}}$ are viewed as reductive group schemes over R. If Y (or Y_R or $Y_{*,R}$ with $*$ as an index) is an R-scheme and if $\tilde{R}$ is a commutative R-algebra, let $Y_{\tilde{R}}$ (or $Y_{*,\tilde{R}}$) be the product over R of Y (or Y_R or $Y_{*,R}$) and $\tilde{R}$. Let (W, ψ) be a symplectic space over $\mathbb{Q}$. It is known that there exists a unique $\mathbf{GSp}(W, \psi)(\mathbb{R})$-conjugacy class S of homomorphisms $\mathbb{S} \to \mathbf{GSp}(W, \psi)_{\mathbb{R}}$ that define Hodge $\mathbb{Q}$–structures on W of type $\{(-1, 0), (0, -1)\}$ and that have either $-2\pi i\psi$ or $2\pi i\psi$ as polarizations (see [De1, Example 1.6]). The Shimura variety $\mathrm{Sh}(\mathbf{GSp}(W, \psi), S)$ is called a *Siegel modular variety*.

The *adjoint* and the *toric part* of a Shimura pair (G, X) are denoted as $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$, cf. [Va1, Subsubsect. 2.4.1].

2.2. Complements on Shimura pairs. Let (G, X) be a Shimura pair. Let $x \in X$. Let $\mu_x : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the cocharacter given on complex points by the rule $z \to x_{\mathbb{C}}(z, 1)$. The reflex field $E(G, X)$ of (G, X) is the field of definition of the $G(\mathbb{C})$-conjugacy class of μ_x (see [De1, Subsect. 3.7]). This implies that (cf. also [De1, Prop. 3.8 (i)]):

(i) the field $E(G, X)$ is the composite field of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and $E(G^{\mathrm{ab}}, X^{\mathrm{ab}})$,

(ii) each map $q : (G, X) \to (\tilde{G}, \tilde{X})$ of Shimura pairs induces a natural embedding

$E(\tilde{G},\tilde{X}) \hookrightarrow E(G,X)$, and

(iii) if $q : (G,X) \hookrightarrow (\tilde{G},\tilde{X})$ is an injective map that induces an identity $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}})$ at the level of adjoints of Shimura pairs, then we have $E(G,X) = E(\tilde{G},\tilde{X})$.

We recall that to each map $q : (G,X) \to (\tilde{G},\tilde{X})$ it is associated naturally an $E(\tilde{G},\tilde{X})$-morphism $\mathrm{Sh}(G,X) \to \mathrm{Sh}(\tilde{G},\tilde{X})$, cf. [De1, Cor. 5.4 and Def. 3.13]. From now on until Lemma 2.3 we will assume that G is a simple, adjoint group over $\mathbb{Q}$ such that all simple factors of $G_{\overline{\mathbb{Q}}}$ are of A_n Lie type.

Let F be a totally real number subfield of $\overline{\mathbb{Q}} \subset \mathbb{C}$ such that $G = \mathrm{Res}_{F/\mathbb{Q}} G[F]$, with $G[F]$ as an absolutely simple adjoint group over F (cf. [De2, Subsubsect. 2.3.4 (a)]); the field F is unique up to $\mathrm{Gal}(\mathbb{Q})$-conjugation. Let T be a maximal torus of G. Let $B_{\overline{\mathbb{Q}}}$ be a Borel subgroup of $G_{\overline{\mathbb{Q}}}$ that contains $T_{\overline{\mathbb{Q}}}$. Let $\mathfrak{D}$ be the Dynkin diagram of $\mathrm{Lie}(G_{\overline{\mathbb{Q}}})$ with respect to $T_{\overline{\mathbb{Q}}}$ and $B_{\overline{\mathbb{Q}}}$. It is a disjoint union of connected Dynkin diagrams $\mathfrak{D}_i$ indexed by embeddings $i : F \hookrightarrow \mathbb{R}$; more precisely, $\mathfrak{D}_i$ is the Dynkin diagram of the simple factor $G[F] \times_{F,i} \overline{\mathbb{Q}}$ of $G_{\overline{\mathbb{Q}}}$ with respect to $(G[F] \times_{F,i} \overline{\mathbb{Q}}) \cap T_{\overline{\mathbb{Q}}}$ and $(G[F] \times_{F,i} \overline{\mathbb{Q}}) \cap B_{\overline{\mathbb{Q}}}$. Let $\mathfrak{g}_{\mathfrak{n}}$ be the 1 dimensional Lie subalgebra of $\mathrm{Lie}(B_{\overline{\mathbb{Q}}})$ that corresponds to a node $\mathfrak{n}$ of $\mathfrak{D}$. The Galois group $\mathrm{Gal}(\mathbb{Q})$ acts on $\mathfrak{D}$ as follows. If $\gamma \in \mathrm{Gal}(\mathbb{Q})$, then $\gamma(\mathfrak{n})$ is the node of $\mathfrak{D}$ defined by the equality $\mathfrak{g}_{\gamma(\mathfrak{n})} = i_{g_\gamma}(\gamma(\mathfrak{g}_{\mathfrak{n}}))$, where i_{g_γ} is the inner conjugation of $\mathrm{Lie}(G_{\overline{\mathbb{Q}}})$ by an element $g_\gamma \in G(\overline{\mathbb{Q}})$ which normalizes $T_{\overline{\mathbb{Q}}}$ and for which we have an identity $g_\gamma \gamma(B_{\overline{\mathbb{Q}}}) g_\gamma^{-1} = B_{\overline{\mathbb{Q}}}$.

2.2.1. The field I. Let $J := \mathbf{PGL}_{n+1,\mathbb{Q}}$. Let $\mathrm{Aut}(J)$ be the group over $\mathbb{Q}$ of automorphisms of J. The quotient group $\mathrm{Aut}(J)/J$ is trivial if $n = 1$ and it is $\mathbb{Z}/2\mathbb{Z}$ if $n > 1$. Let I be the smallest field extension of F such that $G[F]_I$ is an inner form of J_I; the degree $[I : F]$ divides the order of $\mathrm{Aut}(J)/J$. If $n = 1$, then $I = F$. Let now $n > 1$. As X is a hermitian symmetric domain, every simple factor G_0 of $G_{\mathbb{R}}$ is an $\mathbf{SU}(a, n+1-a)^{\mathrm{ad}}$ group for some $a \in \{0, \ldots, n+1\}$ (see [He, Ch. X, §6, 2, Table V]). But as $n > 1$, the group $\mathbf{SU}(a, n+1-a)^{\mathrm{ad}}$ is not an inner form of $\mathbf{PGL}_{n+1,\mathbb{R}}$. This implies that for $n > 1$ the field I is a totally imaginary quadratic extension of F.

2.2.2. Definitions. **(a)** We say (G,X) is of *compact type* if the F-rank of $G[F]$ (i.e., the $\mathbb{Q}$–rank of G) is 0.

(b) We say (G,X) is of *strong compact type* if one of the following two disjoint conditions holds:

(b.i) $n = 1$ and there exists a finite prime v of F such that the F_v-rank of $G[F]_{F_v}$ is 0; here F_v is the completion of F with respect to v;

(b.ii) $n > 1$ and the I-rank of $G[F]_I$ is 0.

2.2.3. Lemma. *Let G_0 be a simple factor of $G_{\mathbb{R}}$ that is an $\mathbf{SU}(a, n+1-a)^{\mathrm{ad}}$ group for some $a \in \{1, \ldots, n\}$. Let $x_0 : \mathbb{S} \to G_0$ be the homomorphism defined naturally by an arbitrary element $x \in X$. Let $q \in \mathbb{N}$. Then there exist a reductive group G_{00} over $\mathbb{R}$, a faithful representation $G_{00} \hookrightarrow \mathbf{GL}_{V_{00}}$, and a homomorphism $x_{00} : \mathbb{S} \to G_{00}$ such that the following three properties hold:*

(i) *we have a natural identification $G_{00}^{\mathrm{ad}} = G_0$ under which x_{00} lifts x_0;*

(ii) *the torus G_{00}^{ab} is isomorphic to $\mathbb{S}$;*

(iii) *the Hodge* $\mathbb{R}$*-structure on* V_{00} *defined by* x_{00} *is of type* $\{(-1,0),(0,-1)\}$ *and moreover we have* $\dim_{\mathbb{R}}(V_{00}) = 2q(n+1)$.

Proof. As $a \notin \{0, n+1\}$ and as the Hodge $\mathbb{Q}$–structure on $\mathrm{Lie}(G)$ defined by x has type $\{(-1,1),(0,0),(1,-1)\}$, the image $\mathrm{Im}(x_0)$ is a rank 1 compact torus. Let G_0^{sc} be the simply connected semisimple group cover of G_0. We will identify $G_{0,\mathbb{C}}^{\mathrm{sc}}$ with $\mathbf{SL}_{V_0}$, where $V_0 := \mathbb{C}^{n+1}$. Let $V_{00} := V_0^q$ but viewed as a real vector space. We have a natural faithful representation $G_0^{\mathrm{sc}} \hookrightarrow \mathbf{GL}_{V_{00}}$.

Let G_{00} be the reductive subgroup of $\mathbf{GL}_{V_{00}}$ that is generated by G_0^{sc} and by the center of the double centralizer of G_0^{sc} in $\mathbf{GL}_{V_{00}}$. We have a direct sum decomposition $V_{00} \otimes_{\mathbb{R}} \mathbb{C} = V_{01}^q \oplus V_{02}^q$ into $G_{00,\mathbb{C}}$-modules such that the following two properties hold:

(iv) we can identify $Z(G_{00,\mathbb{C}}) = Z(\mathbf{GL}_{V_{01}^q}) \times_{\mathbb{C}} Z(\mathbf{GL}_{V_{02}^q})$, and

(v) the $G_{00,\mathbb{C}}^{\mathrm{sc}}$-modules V_{01} and V_{02} are irreducible and correspond to the fundamental weights ϖ_1 and ϖ_n (respectively) of the A_n Lie type.

Thus G_{00} is the extension of G_0 by $Z(G_{00})$ and moreover $Z(G_{00})$ is a torus isomorphic to $\mathbb{S}$ (i.e., property (ii) holds). We easily get that there exists a homomorphism $y_{00} : \mathbb{S} \to G_{00}$ that lifts x_0 and such that under it the $\mathbb{G}_{m,\mathbb{R}}$ subtorus of $\mathbb{S}$ gets identified with $Z(\mathbf{GL}_{V_{00}})$. As the Hodge $\mathbb{R}$–structure on $\mathrm{Lie}(G_0)$ defined by x_0 has type $\{(-1,1),(0,0),(1,-1)\}$, we can choose the pair (V_{01}, V_{02}) such that there exists an integer b with the property that the types of V_{01} and V_{02} defined by y_{00} are $\{(b-1,-b),(b,-b-1)\}$ and $\{(-b-1,b),(-b,b-1)\}$ (respectively).

Let $C(G^{00})$ be the compact subtorus of $Z(G^{00})$. The homomorphisms $\mathbb{S} \to G_{00}$ that lift x_0 are in natural bijection to $\mathbb{Z} \xrightarrow{\sim} \mathrm{End}(C(G_{00})) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{S}, C(G_{00}))$. We can choose the last isomorphisms such that the homomorphism $y_{c,00} : \mathbb{S} \to G_{00}$ that lifts x_0 and that corresponds to $c \in \mathbb{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbb{S}, C(G_{00}))$, achieves the replacement of b by $b - c$. Therefore $x_{00} := y_{b,00} : \mathbb{S} \to G_{00}$ is the unique homomorphism that lifts x_0 and that defines a Hodge $\mathbb{R}$-structure on V_{00} of type $\{(-1,0),(0,-1)\}$. Thus the properties (i) and (iii) also hold. $\square$

2.2.4. Definition. An injective map $f_1 : (G_1, X_1) \hookrightarrow (\mathbf{GSp}(W,\psi), S)$ of Shimura pairs is called a *unitary PEL type embedding* if the following two axioms hold:

(i) each simple factor of $G_{1,\overline{\mathbb{Q}}}^{\mathrm{ad}}$ is isomorphic to $\mathbf{PGL}_{n,\overline{\mathbb{Q}}}$ for some $n \in \mathbb{N}$, and

(ii) the group G_1 is the subgroup of $\mathbf{GSp}(W,\psi)$ that fixes a semisimple $\mathbb{Q}$–subalgebra of $\mathrm{End}(W)$.

2.3. Lemma. *Let* $\mathcal{G}$ *be a reductive group over* $\mathbb{Q}$*. Let* p *be a prime such that the group* $\mathcal{G}_{\mathbb{Q}_p}$ *is unramified. Let* M *be a free* $\mathbb{Z}_{(p)}$*-module of finite rank. We have:*

(a) *Let* $\mathcal{H}$ *be a hyperspecial subgroup of* $\mathcal{G}_{\mathbb{Q}_p}(\mathbb{Q}_p)$*. Then there exists a unique reductive group scheme* $\mathcal{G}_{\mathbb{Z}_{(p)}}$ *over* $\mathbb{Z}_{(p)}$ *that extends* $\mathcal{G}$ *and such that we have* $\mathcal{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) = \mathcal{H}$.

(b) *Let* $\mathcal{G}$ *be a subgroup of* $\boldsymbol{GL}_{M[\frac{1}{p}]}$*. We assume that the Zariski closures* $\mathcal{G}_{\mathbb{Z}_{(p)}}^{\mathrm{der}}$ *and* $Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}}$ *of* $\mathcal{G}^{\mathrm{der}}$ *and* $Z^0(\mathcal{G})$ *(respectively) in* $\boldsymbol{GL}_M$ *are a semisimple group scheme and a torus (respectively) over* $\mathbb{Z}_{(p)}$*. Then the Zariski closure of* $\mathcal{G}$ *in* $\boldsymbol{GL}_M$ *is a reductive group scheme over* $\mathbb{Z}_{(p)}$*.*

Proof. We prove (a). We know that there exists a unique reductive group scheme $\mathcal{G}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ that extends $\mathcal{G}_{\mathbb{Q}_p}$ and such that we have $\mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p) = \mathcal{H}$, cf. [Ti, Subsects. 3.4.1 and 3.8.1]. But $\mathcal{G}_{\mathbb{Z}_p}$ is the pull back of a reductive group scheme $\mathcal{G}_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, cf. [Va1, Lem. 3.1.3]. Obviously $\mathcal{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) = \mathcal{H}$ and $\mathcal{G}_{\mathbb{Z}_{(p)}}$ is unique. Thus (a) holds.

We prove (b). Let $T_{\mathbb{Z}_{(p)}}$ be a maximal torus of $\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_{(p)}}$. The fibres of the intersections $C := T_{\mathbb{Z}_{(p)}} \cap Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}}$ and $\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \cap Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}}$ coincide. But C is isomorphic to the kernel of the product representation $T_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_M$ and thus it is a flat group scheme of multiplicative type over $\mathbb{Z}_{(p)}$. As $C_{\mathbb{Q}}$ is a finite group, we get that C is a finite, flat group scheme over $\mathbb{Z}_{(p)}$. We consider the closed embedding homomorphism $C \hookrightarrow \mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}}$ which at the level of valued points maps c to $(c, -c)$. As C is a flat, closed subgroup scheme of the center of $\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}}$, the quotient group scheme $\mathcal{G}_{\mathbb{Z}_{(p)}} := (\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} Z^0(\mathcal{G})_{\mathbb{Z}_{(p)}})/C$ exists and is reductive (cf. [DG, Vol. III, Exp. XXII, Prop. 4.3.1]). The fibres of the natural homomorphism $\mathcal{G}_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_M$ are closed embeddings. Thus this homomorphism is a monomorphism (cf. [DG, Vol. I, Exp. VI_B, Cor. 2.11]) and therefore it is also a closed embedding (cf. [DG, Vol. II, Exp. XVI, Cor. 1.5 a)]). But the Zariski closure of $\mathcal{G}$ in $\mathbf{GL}_M$ is $\mathcal{G}_{\mathbb{Z}_{(p)}}$ and thus (b) holds. ∎

2.4. Complements on Shimura triples. Let (G, X, H) and $(\tilde{G}, \tilde{X}, \tilde{H})$ be two Shimura triples with respect to p. Let $\overline{Z(G_{\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}$ be the closure of $Z(G_{\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})$ in $Z(G)(\mathbb{A}_f^{(p)})$. We have (cf. [Mi3, Prop. 4.11])

$$\mathrm{Sh}(G,X)/H(\mathbb{C}) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) \backslash (X \times G(\mathbb{A}_f^{(p)})/\overline{Z(G_{\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}). \tag{2}$$

Let $G_{\mathbb{Z}_{(p)}}$ be the reductive group scheme over $\mathbb{Z}_{(p)}$ that extends G and such that we have $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, cf. Lemma 2.3 (a). The groups $H^{\mathrm{ad}} := G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ and $H^{\mathrm{ab}} := G^{\mathrm{ab}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ are hyperspecial subgroups of $G^{\mathrm{ad}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$ and $G^{\mathrm{ab}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (respectively). The triples $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}})$ and $(G^{\mathrm{ab}}, X^{\mathrm{ab}}, H^{\mathrm{ab}})$ are called the adjoint and toric part (respectively) triples of (G, X, H).

By a map $q : (G, X, H) \to (\tilde{G}, \tilde{X}, \tilde{H})$ of Shimura triples we mean a map $q : (G, X) \to (\tilde{G}, \tilde{X})$ of Shimura pairs such that the homomorphism $q(\mathbb{Q}_p) : G(\mathbb{Q}_p) \to \tilde{G}(\mathbb{Q}_p)$ maps H to $\tilde{H}$. We say $q : (G, X, H) \to (\tilde{G}, \tilde{X}, \tilde{H})$ is a *cover*, if the following two properties hold:

(i) the group G surjects onto $\tilde{G}$, and

(ii) the kernel $\mathrm{Ker}(q)$ is a subtorus of $Z(G)$ with the property that for every field K of characteristic 0 the group $H^1(K, \mathrm{Ker}(q)_K)$ is trivial.

Each cover $q(G, X, H) \to (\tilde{G}, \tilde{X}, \tilde{H})$ induces at the level of adjoint triples an isomorphism $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}) \xrightarrow{\sim} (\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}}, \tilde{H}^{\mathrm{ad}})$.

2.4.1. Lemma. *If $q : (G, X, H) \to (\tilde{G}, \tilde{X}, \tilde{H})$ is a cover, then the natural morphism $\mathrm{Sh}(G,X)/H \to \mathrm{Sh}(\tilde{G}, \tilde{X})_{E(G,X)}/\tilde{H}$ is a pro-étale cover and moreover $\mathrm{Sh}(\tilde{G}, \tilde{X})_{E(G,X)}/\tilde{H}$ is the quotient of $\mathrm{Sh}(G,X)/H$ by $Z(G)(\mathbb{A}_f^{(p)})$.*

Proof. The holomorphic map $X \to \tilde{X}$ is onto (cf. [Mi2, Lem. 4.11]) and locally an isomorphism. The homomorphism $q(\mathbb{A}_f^{(p)}) : G(\mathbb{A}_f^{(p)}) \to \tilde{G}(\mathbb{A}_f^{(p)})$ is onto, cf. [Mi2, Lem. 4.12]. From the last two sentences and (2), we get that we have a natural identification $\mathrm{Sh}(\tilde{G},\tilde{X})_{\mathbb{C}}/\tilde{H} = (\mathrm{Sh}(G,X)_{\mathbb{C}}/H)/Z(G)(\mathbb{A}_f^{(p)})$ and that $\mathrm{Sh}(G,X)_{\mathbb{C}}/H$ is a pro-étale cover of $\mathrm{Sh}(\tilde{G},\tilde{X})_{\mathbb{C}}/\tilde{H}$ (to be compared with [Mi2, Lem. 4.13]). From this the Lemma follows. □

2.4.2. Functoriality of integral canonical models. In this Subsubsection we assume that the integral canonical models $\mathcal{N}$ and $\tilde{\mathcal{N}}$ of (G,X,H) and $(\tilde{G},\tilde{X},\tilde{H})$ (respectively) exist and that we have a map $q:(G,X,H)\to(\tilde{G},\tilde{X},\tilde{H})$ of Shimura triples. As $\mathcal{N}$ is a healthy regular scheme (see Subsection 1.2) and as $\tilde{\mathcal{N}}$ has the extension property, the natural $E(\tilde{G},\tilde{X})$-morphism $\mathrm{Sh}(G,X)/H\to\mathrm{Sh}(\tilde{G},\tilde{X})/\tilde{H}$ defined by q extends uniquely to an $E(\tilde{G},\tilde{X})_{(p)}$-morphism $\mathcal{N}\to\tilde{\mathcal{N}}$.

Suppose q is injective. This implies that $\mathrm{Sh}(G,X)$ is a closed subscheme of $\mathrm{Sh}(\tilde{G},\tilde{X})_{E(G,X)}$, cf. [De1, Prop. 1.15]. Due to the analogy between (1) and (2), the proof of loc. cit. adapts entirely to show that $\mathrm{Sh}(G,X)/H$ is a closed subscheme of $\mathrm{Sh}(\tilde{G},\tilde{X})_{E(G,X)}/\tilde{H}$.

Suppose that q is injective and that q induces an isomorphism $(G^{\mathrm{ad}},X^{\mathrm{ad}}) = (\tilde{G}^{\mathrm{ad}},\tilde{X}^{\mathrm{ad}})$ at the level of adjoint Shimura pairs; thus we can identify $G^{\mathrm{der}}=\tilde{G}^{\mathrm{der}}$. We have $E(G,X)=E(\tilde{G},\tilde{X})$ (cf. property 2.2 (iii)) and $\dim_{\mathbb{C}}(X)=\dim_{\mathbb{C}}(\tilde{X})$. By reasons of dimensions we get that $\mathrm{Sh}(G,X)/H$ is an open closed subscheme of $\mathrm{Sh}(\tilde{G},\tilde{X})/\tilde{H}$. Let $\mathcal{N}'$ be the unique open closed subscheme of $\tilde{\mathcal{N}}$ for which we have an identity $\mathcal{N}'_{E(G,X)}=\mathrm{Sh}(G,X)/H$. Let $\tilde{H}_p$ be a compact, open subgroup of $\tilde{G}(\mathbb{A}_f^{(p)})$ such that $\tilde{\mathcal{N}}$ is a pro-étale cover of $\tilde{\mathcal{N}}/\tilde{H}_p$. Thus if $H_p:=G(\mathbb{A}_f^{(p)})\cap\tilde{H}_p$, then $\mathcal{N}'$ is a pro-étale cover of $\mathcal{N}'/H_p$. Also $\mathcal{N}'$ has the extension property as it is a closed subscheme of $\tilde{\mathcal{N}}$. We get that $\mathcal{N}'$ is the integral canonical model of (G,X,H). Due to the uniqueness of $\mathcal{N}$, we get that $\mathcal{N}=\mathcal{N}'$ and thus that $\mathcal{N}$ is an open closed subscheme of $\tilde{\mathcal{N}}$.

We have the following enlarged version of [Va1, Lem. 6.2.3] that holds for all primes p.

2.4.3. PROPOSITION. *Suppose we have an identification $G^{\mathrm{der}}=\tilde{G}^{\mathrm{der}}$ that induces naturally an identity $(G^{\mathrm{ad}},X^{\mathrm{ad}},H^{\mathrm{ad}})=(\tilde{G}^{\mathrm{ad}},\tilde{X}^{\mathrm{ad}},\tilde{H}^{\mathrm{ad}})$. Let $Spec(\mathbb{Z}_{(p)}^{\mathrm{un}})\to Spec(\mathbb{Z}_{(p)})$ be the maximal connected pro-étale cover of $Spec(\mathbb{Z}_{(p)})$. We have:*

(a) *The integral canonical model $\mathcal{N}$ of (G,X,H) exists if and only if the integral canonical model $\tilde{\mathcal{N}}$ of $(\tilde{G},\tilde{X},\tilde{H})$ exists.*

(b) *If the identification $G^{\mathrm{der}}=\tilde{G}^{\mathrm{der}}$ is defined by a map $q:(G,X,H)\to(\tilde{G},\tilde{X},\tilde{H})$ of Shimura triples and if $\tilde{\mathcal{N}}$ exists, then $\mathcal{N}$ is a pro-étale cover of an open closed subscheme of $\tilde{\mathcal{N}}$ and therefore it is the normalization of $\tilde{\mathcal{N}}$ in the ring of fractions of $\mathrm{Sh}(G,X)/H$.*

(c) *As $E(G,X)_{(p)}$ and $E(\tilde{G},\tilde{X})_{(p)}$ are finite, étale $\mathbb{Z}_{(p)}$-algebras, we view $\mathbb{Z}_{(p)}^{\mathrm{un}}$ as an ind-finite, ind-étale algebra over either $E(G,X)_{(p)}$ or $E(\tilde{G},\tilde{X})_{(p)}$. If $\mathcal{N}$ and $\tilde{\mathcal{N}}$ exist, then the connected components of $\mathcal{N}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ and $\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ are naturally identified.*

Proof. We first show that to prove the Proposition we can assume that we have a map $q:(G,X,H)\to(\tilde{G},\tilde{X},\tilde{H})$ of Shimura triples. We identify X and $\tilde{X}$ with unions of connected components of $X^{\mathrm{ad}}=\tilde{X}^{\mathrm{ad}}$. As $G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ permutes transitively the connected components of X^{ad} (see [Va1, Cor. 3.3.3]), by composing the identification

$G^{\mathrm{der}} = \tilde{G}^{\mathrm{der}}$ with an automorphism of G^{der} defined by an element of $G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$, we can assume that the intersection $X \cap \tilde{X}$ is non-empty. Thus we can speak about a quasi fibre product (see [Va1, Rm. 3.2.7 3)])

$$\begin{array}{ccc} (\tilde{G}_1, \tilde{X}_1, \tilde{H}_1) & \xrightarrow{\tilde{q}_1} & (\tilde{G}, \tilde{X}, \tilde{H}) \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \tilde{q}_{\mathrm{ad}}} \\ (G, X, H) & \xrightarrow{q_{\mathrm{ad}}} & (G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}) = (\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}}, \tilde{H}^{\mathrm{ad}}), \end{array}$$

where q_{ad} and $\tilde{q}_{\mathrm{ad}}$ are the natural morphisms defined by taking adjoints and where $\tilde{X}_1$ contains an a priori chosen connected component of $X \cap \tilde{X}$. The reductive group $\tilde{G}_1$ is a subgroup of $G \times_{\mathbb{Q}} \tilde{G}$ which via the two projections induces isomorphisms $\tilde{G}_1^{\mathrm{ad}} \xrightarrow{\sim} G^{\mathrm{ad}}$ and $\tilde{G}_1^{\mathrm{ad}} \xrightarrow{\sim} \tilde{G}^{\mathrm{ad}}$ at the level of adjoint groups. Thus we can identify naturally $\tilde{G}_1^{\mathrm{der}} = \tilde{G}^{\mathrm{der}} = G^{\mathrm{der}}$. Due to the existence of such a quasi fibre product, to prove the Proposition we can assume that we have a map $q : (G, X, H) \to (\tilde{G}, \tilde{X}, \tilde{H})$ of Shimura triples. Either $\mathcal{N}$ or $\tilde{\mathcal{N}}$ exists and thus we have to consider two cases.

Case 1. We first assume that $\tilde{\mathcal{N}}$ exists. It is well known that the integral canonical model $\mathcal{N}^{\mathrm{ab}}$ of $(G^{\mathrm{ab}}, X^{\mathrm{ab}}, H^{\mathrm{ab}})$ exists, cf. either [Mi2, Rm. 2.16] or [Va1, Example 3.2.8]. Let $(\tilde{G}_2, \tilde{X}_2, \tilde{H}_2) := (\tilde{G}, \tilde{X}, \tilde{H}) \times (G^{\mathrm{ab}}, X^{\mathrm{ab}}, H^{\mathrm{ab}})$ and $\tilde{\mathcal{N}}_2 := \tilde{\mathcal{N}}_{E(\tilde{G}_2, \tilde{X}_2)_{(p)}} \times_{E(\tilde{G}_2, \tilde{X}_2)_{(p)}} \mathcal{N}^{\mathrm{ab}}_{E(\tilde{G}_2, \tilde{X}_2)_{(p)}}$. The integral canonical model of $(\tilde{G}_2, \tilde{X}_2, \tilde{H}_2)$ is $\tilde{\mathcal{N}}_2$. Moreover we have a natural injective map $(G, X, H) \hookrightarrow (\tilde{G}_2, \tilde{X}_2, \tilde{H}_2)$. Thus, to prove (a) to (c), we can assume that the homomorphism $G \to \tilde{G}$ is injective. The integral canonical model $\mathcal{N}$ of (G, X, H) is an open closed subscheme of $\tilde{\mathcal{N}}$, cf. the last paragraph of Subsubsection 2.4.2. Obviously this implies that (a) to (c) hold in the Case 1.

Case 2. We now assume that $\mathcal{N}$ exists. Let $(\tilde{G}_2, \tilde{X}_2, \tilde{H}_2)$ be as in Case 1. We have an injective map $(G, X, H) \hookrightarrow (\tilde{G}_2, \tilde{X}_2, \tilde{H}_2)$ of Shimura triples that induces an identity $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (\tilde{G}_2^{\mathrm{ad}}, \tilde{X}_2^{\mathrm{ad}})$. Thus $E(G, X) = E(\tilde{G}_2, \tilde{X}_2)$ and $\mathrm{Sh}(G, X)/H$ is an open closed subscheme of $\mathrm{Sh}(\tilde{G}_2, \tilde{X}_2)/\tilde{H}_2$, cf. Subsubsection 2.4.2. The connected components of $\mathrm{Sh}(\tilde{G}_2, \tilde{X}_2)_{\mathbb{C}}/\tilde{H}_2$ are permuted transitively by $\tilde{G}_2(\mathbb{A}_f^{(p)})$, cf. [Va1, Lem. 3.3.2]. Let $\mathcal{U}$ be a connected component of $\mathcal{N}$. As $\mathcal{N}$ is a healthy regular $E(G, X)_{(p)}$-scheme (cf. Subsection 1.2) that has the extension property, each $E(G, X)$-automorphism of $\mathcal{U}_{E(G,X)}$ defined by a right translation by an element of $\tilde{G}_2(\mathbb{A}_f^{(p)})$, extends uniquely to an $E(G, X)_{(p)}$-automorphism of $\mathcal{U}$ itself. This implies that we can speak about the faithfully flat $E(G, X)_{(p)}$-scheme $\tilde{\mathcal{N}}_2$ whose fibre over $E(G, X)$ is $\mathrm{Sh}(\tilde{G}_2, \tilde{X}_2)/\tilde{H}_2$ and whose connected components are translations by elements of $\tilde{G}_2(\mathbb{A}_f^{(p)})$ of connected components of $\mathcal{N}$. Thus the group $\tilde{G}_2(\mathbb{A}_f^{(p)})$ acts on $\tilde{\mathcal{N}}_2$, $\mathcal{N}$ is an open closed subscheme of $\tilde{\mathcal{N}}_2$, and the $\tilde{G}_2(\mathbb{A}_f^{(p)})$-orbit of $\mathcal{U}$ is $\tilde{\mathcal{N}}_2$.

Let $\mathcal{U}^{\mathrm{ab}}$ be the connected component of $\mathcal{N}^{\mathrm{ab}}$ that is the image of $\mathcal{U}$ in $\mathcal{N}^{\mathrm{ab}}$. Let $\tilde{\mathcal{N}}_2^0$ be the open closed subscheme of $\tilde{\mathcal{N}}_2$ that is the inverse image of $\mathcal{U}^{\mathrm{ab}}$ via the natural morphism $\tilde{\mathcal{N}}_2 \to \mathcal{N}^{\mathrm{ab}}$; the existence of this morphism is guaranteed by the fact that the $E(G^{\mathrm{ab}}, X^{\mathrm{ab}})_{(p)}$-scheme $\mathcal{N}^{\mathrm{ab}}$ has the extension property. Let $\mathcal{A}$ be the group of $E(G, X)_{(p)}$-automorphisms of $\tilde{\mathcal{N}}_2^0$ defined by translations by elements of the subgroup $G^{\mathrm{ab}}(\mathbb{A}_f^{(p)}) = \{1\} \times G^{\mathrm{ab}}(\mathbb{A}_f^{(p)})$ of $\tilde{G}_2(\mathbb{A}_f^{(p)})$. The group $\mathcal{A}$ acts freely on $\tilde{\mathcal{N}}_2^0$ as it does so on $\mathcal{N}^{\mathrm{ab}}$.

Faithfully flat descent. Let $s_1,\ s_2 : \mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}} \times_{E(G,X)_{(p)}} \mathbb{Z}_{(p)}^{\mathrm{un}}) \to \mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}})$ be the two natural projections. We check that the quotient $\tilde{\mathcal{N}}_{E(G,X)_{(p)}}$ of $\tilde{\mathcal{N}}_2^0$ by $\mathcal{A}$ exists, that the morphism $\tilde{\mathcal{N}}_2^0 \to \tilde{\mathcal{N}}_{E(G,X)_{(p)}}$ is a pro-étale cover whose pull back to $\mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}})$ induces isomorphisms at the level of connected components, and that we have $\tilde{\mathcal{N}}_{E(G,X)} = \mathrm{Sh}(\tilde{G},\tilde{X})_{E(G,X)}/\tilde{H}$. As each connected component of $\mathcal{N}_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}$ is geometrically connected over $\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]$, the $\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]$-morphisms $\mathcal{N}_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]} = \mathrm{Sh}(G,X)_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}/H \hookrightarrow \mathrm{Sh}(\tilde{G}_2,\tilde{X}_2)_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}/\tilde{H}_2 \to \mathrm{Sh}(\tilde{G},\tilde{X})_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}/\tilde{H}$ induce isomorphisms at the level of connected components (cf. Subsubsection 2.4.2 for $\mathrm{Sh}(G,X)_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}/H \hookrightarrow \mathrm{Sh}(\tilde{G}_2,\tilde{X}_2)_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}/\tilde{H}_2$). This implies that all the desired properties hold after pull back to $\mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}})$; in other words, the $\mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}})$-scheme $\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ exists and no element of $\mathcal{A}$ produces a non-trivial automorphism of a connected component of $\tilde{\mathcal{N}}^0_{2,\mathbb{Z}_{(p)}^{\mathrm{un}}}$. This last thing implies that $\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ has an open, affine cover that is stable under the isomorphism $s_1^*(\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}) \xrightarrow{\sim} s_2^*(\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}})$ that defines the faithfully flat descent datum on $\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$. Thus this faithfully flat descent datum is effective, cf. [BLR, Ch. 6, 6.1, Thm. 6]. This implies that $\tilde{\mathcal{N}}_{E(G,X)_{(p)}}$ exists and it has all the desired properties.

Galois descent. Let $E_2(\tilde{G},\tilde{X})$ be the Galois closure of $E(G,X)$ over $E(\tilde{G},\tilde{X})$. The finite $\mathbb{Z}_{(p)}$-algebra $E_2(\tilde{G},\tilde{X})_{(p)}$ is étale. The $E(G,X)_{(p)}$-scheme $\tilde{\mathcal{N}}_2^0$ has the extension property and it is a pro-étale cover of $\tilde{\mathcal{N}}_{E(G,X)_{(p)}}$. Thus from [Va1, Rm. 3.2.3.1 6)] we get that the $E(G,X)_{(p)}$-scheme $\tilde{\mathcal{N}}_{E(G,X)_{(p)}}$ has the extension property. Thus the $E_2(\tilde{G},\tilde{X})_{(p)}$-scheme $\tilde{\mathcal{N}}_{E_2(\tilde{G},\tilde{X})_{(p)}}$ has also the extension property; as it is formally smooth over $\mathbb{Z}_{(p)}$, it is also a healthy regular scheme (cf. Subsection 1.2). Thus the natural action of the finite Galois group $\mathrm{Gal}(E_2(\tilde{G},\tilde{X})/E(\tilde{G},\tilde{X}))$ on $\tilde{\mathcal{N}}_{E_2(\tilde{G},\tilde{X})}$ extends naturally to an action of $\mathrm{Gal}(E_2(\tilde{G},\tilde{X})/E(\tilde{G},\tilde{X}))$ on $\tilde{\mathcal{N}}_{E_2(\tilde{G},\tilde{X})_{(p)}}$ that is automatically free. Using Galois descent with respect to the morphism $\mathrm{Spec}(\mathbb{Z}_{(p)}^{\mathrm{un}}) \to \mathrm{Spec}(E(\tilde{G},\tilde{X})_{(p)})$, as in the previous paragraph we argue that the quotient $\tilde{\mathcal{N}}$ of $\tilde{\mathcal{N}}_{E_2(\tilde{G},\tilde{X})_{(p)}}$ by the group $\mathrm{Gal}(E_2(\tilde{G},\tilde{X})/E(\tilde{G},\tilde{X}))$ exists, that we have an identity $\tilde{\mathcal{N}}_{E(\tilde{G},\tilde{X})} = \mathrm{Sh}(\tilde{G},\tilde{X})/\tilde{H}$, and that the natural morphism $\mathcal{N} \to \tilde{\mathcal{N}}$ is a pro-étale cover of its image and moreover its pull back to $\mathbb{Z}_{(p)}^{\mathrm{un}}$ induces an isomorphism at the level of connected components. Thus (b) and (c) hold, provided $\tilde{\mathcal{N}}$ is the integral canonical model of $(\tilde{G},\tilde{X},\tilde{H})$.

Extension property. But it is easy to see that $\tilde{\mathcal{N}}$ is the integral canonical model of $(\tilde{G},\tilde{X},\tilde{H})$. For instance, we will check here that $\tilde{\mathcal{N}}$ has the extension property. Let $\mathcal{Z}$ be a faithfully flat $E(\tilde{G},\tilde{X})_{(p)}$-scheme that is healthy regular. Let $u : \mathcal{Z}_{E(\tilde{G},\tilde{X})} \to \tilde{\mathcal{N}}_{E(\tilde{G},\tilde{X})}$ be a morphism. The scheme $\mathcal{Z}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ is a pro-étale cover of $\mathcal{Z}$ and thus it is a healthy regular scheme, cf. [Va1, Rm. 3.2.2 4), property C)]. But the $\mathbb{Z}_{(p)}^{\mathrm{un}}$-scheme $\tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ is a disjoint union of connected components of $\mathcal{N}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$ (cf. (c)) and thus it also has the extension property. Therefore $u_{\mathbb{Z}_{(p)}^{\mathrm{un}}[\frac{1}{p}]}$ extends uniquely to a morphism $\mathcal{Z}_{\mathbb{Z}_{(p)}^{\mathrm{un}}} \to \tilde{\mathcal{N}}_{\mathbb{Z}_{(p)}^{\mathrm{un}}}$.

This implies that u extends uniquely to a morphism $\mathcal{Z} \to \tilde{\mathcal{N}}$. Therefore $\tilde{\mathcal{N}}$ has the extension property. Thus (a) to (c) also hold in the Case 2. $\square$

3. The existence of unitary PEL type embeddings. Let $p \in \mathbb{N}$ be a prime. Proposition 3.2 presents a $\mathbb{Z}_{(p)}$ version of the embedding results of [Sa1, Subsect. 3.2] and [Sa2, Part III] for unitary Shimura varieties; the approach is close in spirit to [De2, Prop. 2.3.10] and [Va1, Subsects. 6.5 and 6.6]. The setting for Proposition 3.2 is presented in Subsection 3.1. In Subsection 3.3 we include some simple facts.

3.1. The setting. Let (G, X) be a simple, adjoint Shimura pair that is unitary. Thus G is a non-trivial, simple, adjoint group over $\mathbb{Q}$ and there exists $n \in \mathbb{N}$ such that all simple factors of $G_{\overline{\mathbb{Q}}}$ are isomorphic to $\mathbf{PGL}_{n+1,\overline{\mathbb{Q}}}$. Let F and $G[F]$ be as in Subsection 2.2; thus we have $G = \mathrm{Res}_{F/\mathbb{Q}} G[F]$. Let the field I be as in Subsubsection 2.2.1. We assume that the group $G_{\mathbb{Q}_p}$ is unramified. Let H be a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $G_{\mathbb{Z}_{(p)}}$ be the unique adjoint group scheme over $\mathbb{Z}_{(p)}$ that extends G and such that we have $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, cf. Lemma 2.3 (a).

3.2. Proposition. *In the setting of Subsection 3.1, there exists an injective map of Shimura pairs*

$$f_1 \colon (G_1, X_1) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$$

which is a unitary PEL type embedding such that the following two conditions hold:

(i) *the adjoint Shimura pair* $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ *is* (G, X)*;*

(ii) *there exists a* $\mathbb{Z}_{(p)}$*-lattice* $L_{(p)}$ *of* W *for which we get a perfect alternating form* $\psi \colon L_{(p)} \otimes_{\mathbb{Z}_{(p)}} L_{(p)} \to \mathbb{Z}_{(p)}$, *the* $\mathbb{Z}_{(p)}$*-algebra* $\mathcal{O} := End(L_{(p)}) \cap \{e \in End(W) | e$ *is fixed by* $G_1\}$ *is semisimple, and the Zariski closure* $G_{1,\mathbb{Z}_{(p)}}$ *of* G_1 *in* $\boldsymbol{GSp}(L_{(p)}, \psi)$ *is the subgroup scheme of* $\boldsymbol{GSp}(L_{(p)}, \psi)$ *that fixes* $\mathcal{O}$ *and it is a reductive group scheme over* $\mathbb{Z}_{(p)}$ *whose adjoint is* $G_{\mathbb{Z}_{(p)}}$.

Proof. Once $G_{1,\mathbb{Z}_{(p)}}$ is constructed, its derived group scheme will be the simply connected semisimple group scheme cover $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ of $G_{\mathbb{Z}_{(p)}}$. As the proof is quite long, we itemize and boldface its main steps.

Step 1. The construction of the $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$-module $L_{(p)}$. There exists an identity $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} = \mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}} G[F]^{\mathrm{sc}}_{F_{(p)}}$, where $G[F]^{\mathrm{sc}}_{F_{(p)}}$ is a simply connected semisimple group scheme over $F_{(p)}$ that extends the simply connected semisimple group scheme cover $G[F]^{\mathrm{sc}}$ of $G[F]$. Let $T_{\mathbb{Z}_{(p)}}$ be a maximal torus of $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$. Let $T := T_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}$; it is a torus of G whose role is to make all the below data very precisely constructed.

Let F_1 be the smallest Galois extension of $\mathbb{Q}$ with the property that the torus T_{F_1} is split. As T extends to the torus $T_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, F_1 is unramified over p. As F is a subfield of F_1, it is also unramified over p. Let K be a totally imaginary quadratic extension of F unramified over p and disjoint from F_1. If $n = 1$ and (G, X) is of strong compact type, then we choose K such that the group $G[F]_K$ is not split; for instance, if v is a finite prime of F as in the condition (b.i) of Definition 2.2.2 (b), then v is prime to p and thus it suffices to take K such that moreover v splits in it. If $n > 1$ (resp. $n = 1$) let E_1 be F_1 (resp. be $F_1 \otimes_F K$). The $\mathbb{Z}_{(p)}$-algebra $E_{1,(p)}$ is étale.

Let $W_{1,(p)}$ be a free $F_{1,(p)}$-module of rank $n+1$. Let $W_{2,(p)} := W_{1,(p)} \otimes_{F_{1,(p)}} E_{1,(p)}$. When we view $W_{2,(p)}$ as a free $\mathbb{Z}_{(p)}$-module we denote it by $W_{3,(p)}$. We identify $G[F]^{\mathrm{sc}}_{F_{1,(p)}} = \mathbf{SL}_{W_{1,(p)}}$ and $G[F]^{\mathrm{sc}}_{E_{1,(p)}} = \mathbf{SL}_{W_{2,(p)}}$. Let $m : G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathbf{SL}_{W_{3,(p)}}$ be the composite of the natural monomorphisms $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{Res}_{E_{1,(p)}/\mathbb{Z}_{(p)}} G[F]^{\mathrm{sc}}_{E_{1,(p)}} \hookrightarrow \mathbf{SL}_{W_{3,(p)}}$, the second one being defined via the mentioned identifications. Let $W^*_{3,(p)} := \mathrm{Hom}_{\mathbb{Z}_{(p)}}(W_{3,(p)}, \mathbb{Z}_{(p)})$ and

$$L_{(p)} := W_{3,(p)} \oplus W^*_{3,(p)} \text{ and } W := L_{(p)}[\frac{1}{p}].$$

Let $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathbf{Sp}(L_{(p)}, \tilde{\psi})$ be the composite of m with standard monomorphisms $\mathbf{SL}_{W_{3,(p)}} \hookrightarrow \mathbf{GL}_{W_{3,(p)}} \hookrightarrow \mathbf{Sp}(L_{(p)}, \tilde{\psi})$, where $\tilde{\psi}$ is a perfect alternating form on $L_{(p)}$ such that we have $\tilde{\psi}(W_{3,(p)} \otimes W_{3,(p)}) = \tilde{\psi}(W^*_{3,(p)} \otimes W^*_{3,(p)}) = 0$.

Step 2. The construction of $G_{1,\mathbb{Z}_{(p)}}$. Let $\mathcal{S}$ be the set of extremal nodes of the Dynkin diagram $\mathfrak{D}$ of $\mathrm{Lie}(G_{\overline{\mathbb{Q}}})$ with respect to $\mathrm{Lie}(T_{\overline{\mathbb{Q}}})$ and some fixed Borel Lie subalgebra of $\mathrm{Lie}(G_{\overline{\mathbb{Q}}})$ that contains $\mathrm{Lie}(T_{\overline{\mathbb{Q}}})$. The Galois group $\mathrm{Gal}(\mathbb{Q})$ acts on $\mathcal{S}$ (see Subsection 2.2). Thus if $n > 1$ we can identify $\mathcal{S}$ with the $\mathrm{Gal}(\mathbb{Q})$-set $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{K}, \overline{\mathbb{Q}})$, where $\mathcal{K} := I$ is a totally imaginary quadratic extension of F. If $n = 1$ let $\mathcal{K} := K$. We have $[\mathcal{K} : \mathbb{Q}] = 2[F : \mathbb{Q}]$. Always $\mathcal{K}$ is a subfield of E_1 and thus $L_{(p)}$ has a natural structure of a $\mathcal{K}_{(p)}$-module. Thus the torus $\mathcal{T} := \mathrm{Res}_{\mathcal{K}_{(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{m,\mathcal{K}_{(p)}}$ acts on $L_{(p)}$. If $n > 1$ this action over the Witt ring $W(\mathbb{F})$ introduced in Subsection 2.1, can be described as follows:

(*) if $\mathcal{L}$ is a direct summand of $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ which is a simple $G^{\mathrm{sc}}_{W(\mathbb{F})}$-module, then the highest weight of $\mathcal{L}$ is a fundamental weight associated to an extremal node $\mathfrak{n} \in \mathcal{S}$ and moreover $\mathcal{T}_{W(\mathbb{F})}$ acts on $\mathcal{L}$ via the character of $\mathcal{T}_{W(\mathbb{F})}$ that corresponds naturally to $\mathfrak{n}$.

As $\mathcal{K}$ is a totally imaginary quadratic extension of F, for $n \geq 1$ the extension to $\mathbb{R}$ of the quotient torus $\mathcal{T}/\mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{m,F_{(p)}}$ is compact. This implies that the maximal subtorus $\mathcal{T}_c$ of $\mathcal{T}$ that over $\mathbb{R}$ is compact, is isomorphic to $\mathcal{T}/\mathrm{Res}_{F_{(p)}/\mathbb{Z}_{(p)}} \mathbb{G}_{m,F_{(p)}}$.

We view $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ as a closed subgroup scheme of $\mathbf{Sp}(L_{(p)}, \tilde{\psi})$. Let $C_{\mathbb{Z}_{(p)}}$ be the centralizer of $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ in $\mathbf{GL}_{L_{(p)}}$, cf. [DG, Vol. II, Exp. XI, Cor. 6.11]. If $n > 1$, then $C_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$ and the torus $Z(C_{\mathbb{Z}_{(p)}}) = \mathcal{T}$ has rank $2[F : \mathbb{Q}]$ (cf. (*) and the definition of the representation $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_{L_{(p)}}$). If $n = 1$, then $C_{\mathbb{Z}_{(p)}}$ is a reductive group scheme and $Z(C_{\mathbb{Z}_{(p)}})$ is a torus of rank $[F : \mathbb{Q}]$ (cf. the definition of the representation $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_{L_{(p)}}$); we view $\mathcal{T}_c$ as a torus of $C_{\mathbb{Z}_{(p)}}$.

Let $\mathcal{T}_{c,+}$ be the subtorus of $C_{\mathbb{Z}_{(p)}}$ generated by $\mathcal{T}_c$ and $Z(\mathbf{GL}_{L_{(p)}})$; its rank is $[F : \mathbb{Q}] + 1$ and it commutes with $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$. Let G_1 be the subgroup of $\mathbf{GL}_W$ generated by $\mathcal{T}_{c,+,\mathbb{Q}}$ and $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}$. Let $G_{1,\mathbb{Z}_{(p)}}$ be the Zariski closure of G_1 in $\mathbf{GL}_{L_{(p)}}$. As $\mathcal{T}_{c,+}$ and $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ are a torus and a semisimple group scheme (respectively), from Lemma 2.3 (b) we get that $G_{1,\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$.

The representation of $G_{1,W(\mathbb{F})}$ on $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ is a direct sum of rank $n+1$ irreducible representations. This implies that the centralizer $C_{1,\mathbb{Z}_{(p)}}$ of $G_{1,\mathbb{Z}_{(p)}}$ in $\mathbf{GL}_{L_{(p)}}$ is also a reductive group scheme; its center is $\mathcal{T}$. Let $\mathcal{O}$ be the semisimple $\mathbb{Z}_{(p)}$-subalgebra of $\mathrm{End}(L_{(p)})$ defined by the elements of $\mathrm{Lie}(C_{1,\mathbb{Z}_{(p)}})$.

Step 3. Real and complex representations. Let $\mathcal{R} := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$. For $i \in \mathcal{R}$ let V_i be the real vector subspace of $W \otimes_{\mathbb{Q}} \mathbb{R}$ generated by its simple $G[F]^{\mathrm{sc}} \times_{F,i} \mathbb{R}$-

submodules. We have a product decomposition $\mathcal{T}_{\mathbb{R}} = \prod_{i\in\mathcal{R}} T_i$ into 2 dimensional tori such that $T_i := \mathrm{Res}_{\mathcal{K}\otimes_{F,i}\mathbb{R}/\mathbb{R}}\mathbb{G}_{m,\mathcal{K}\otimes_{F,i}\mathbb{R}}$ acts trivially on $V_{i'}$ if and only if $i' \in \mathcal{R}\setminus\{i\}$. Each torus T_i is isomorphic to $\mathbb{S}$ and acts faithfully on V_i. We have $\mathcal{T}_{c,\mathbb{R}} = \prod_{i\in\mathcal{R}} T_{c,i}$, where $T_{c,i}$ is the rank 1 compact subtorus of T_i. We have a direct sum decomposition

$$W \otimes_{\mathbb{Q}} \mathbb{R} = \oplus_{i\in\mathcal{R}} V_i$$

into $G_{1,\mathbb{R}}$-modules. We also have a unique direct sum decomposition

$$V_i \otimes_{\mathbb{R}} \mathbb{C} = V_i^+ \oplus V_i^-$$

into $G_{1,\mathbb{C}}$-modules such that $\mathcal{T}_{\mathbb{C}}$ acts on each V_i^u via a unique character; here $u \in \{-,+\}$. Each $G_{1,\mathbb{C}}^{\mathrm{der}}$-module V_i^u is isotypic i.e., it is a direct sum of isomorphic simple $G_{1,\mathbb{C}}^{\mathrm{der}}$-modules. More precisely, the highest weight of the representation of the factor $G[F]^{\mathrm{sc}} \times_{F,i} \mathbb{C}$ of $G_{1,\mathbb{C}}^{\mathrm{der}}$ on V_i^u is $\varpi_{s_i(u)}$, where $s_i : \{+,-\} \to \{1,n\}$ is a surjective function. Thus the torus $T_{ii} := \mathrm{Im}(T_i \to \mathbf{GL}_{V_i})$ is the center of the centralizer of $C_{1,\mathbb{R}}$ in $\mathbf{GL}_{V_i}$. Moreover, the $G_{1,\mathbb{C}}^{\mathrm{der}}$-modules V_i^+ and V_i^- are dual to each other.

Step 4. The construction of the Shimura pair (G_1, X_1). Let $x \in X$. We will construct a monomorphism $x_1 : \mathbb{S} \hookrightarrow G_{1,\mathbb{R}}$ such that the Hodge $\mathbb{Q}$–structure on W defined by it has type $\{(-1,0),(0,-1)\}$ and the resulting homomorphism $\mathbb{S} \to G_{1,\mathbb{R}}^{\mathrm{ad}} = G_{\mathbb{R}}$ is x. We will take x_1 such that its restriction to the split subtorus $\mathbb{G}_{m,\mathbb{R}}$ of $\mathbb{S}$ induces an isomorphism $\mathbb{G}_{m,\mathbb{R}} \xrightarrow{\sim} Z(\mathbf{GL}_{W\otimes_{\mathbb{Q}}\mathbb{R}})$ and the following two properties hold:

(iii.a) if $i \in \mathcal{R}$ is such that the group $G[F] \times_{F,i} \mathbb{R}$ is non-compact, then the homomorphism $\mathbb{S} \to \mathbf{GL}_{V_i}$ defined by x_1 is constructed as in the proof of [De2, Prop. 2.3.10] and it is unique; more precisely, the faithful representation $\mathrm{Im}(G_{1,\mathbb{R}} \to \mathbf{GL}_{V_i}) \hookrightarrow \mathbf{GL}_{V_i}$ is isomorphic to the faithful representation $G_{00} \hookrightarrow \mathbf{GL}_{V_{00}}$ of Lemma 2.2.3 for $q = [E_1 : F]$ and thus the homomorphism $\mathbb{S} \to \mathrm{Im}(G_{1,\mathbb{R}} \to \mathbf{GL}_{V_i})$ defined by x_1 is obtained in the same way we constructed x_{00} in Lemma 2.2.3;

(iii.b) if $i \in \mathcal{R}$ is such that $G[F] \times_{F,i} \mathbb{R}$ is compact, then the homomorphism $\mathbb{S} \to \mathbf{GL}_{V_i}$ defined by x_1 is a monomorphism whose image is naturally identified with T_{ii}.

For the rest of the proof it is irrelevant which one of the two possible natural identifications of (iii.b) we choose; one such choice is obtained naturally from the other choice via the standard non-trivial automorphism of the compact subtorus of $\mathbb{S}$.

Let X_1 be the $G_1(\mathbb{R})$-conjugacy class of x_1. As x_1 lifts $x \in X$, the pair (G_1, X_1) is a Shimura pair whose adjoint is (G, X). Thus the property (i) holds.

Step 5. Centralizing properties. For every $i \in \mathcal{R}$, the two characters of $\mathcal{T}_{c,\mathbb{C}}$ (equivalently of $T_{c,i,\mathbb{C}}$) that define the actions of $\mathcal{T}_{c,\mathbb{C}}$ on V_i^+ and V_i^- are non-trivial and their product is the trivial character. Moreover, the representations of $G_{1,\mathbb{C}}^{\mathrm{der}}$ on V_i^+ and V_i^- are dual to each other. The last two sentences imply that $\mathcal{T}_{c,i}$ is a subgroup of the subgroup $\mathbf{GL}_{W_{3,(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{R}}$ of $\mathbf{Sp}(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R}, \tilde{\psi})$. Therefore $\mathcal{T}_c$ is a torus of $\mathbf{Sp}(L_{(p)}, \tilde{\psi})$. Thus $G_{1,\mathbb{Z}_{(p)}}$ is a closed subgroup scheme of $\mathbf{GSp}(L_{(p)}, \tilde{\psi})$. The representation of $G_{1,W(\mathbb{F})}$ on $L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ is a direct sum of rank $n+1$ irreducible representations that are isotropic with respect to $\tilde{\psi}$. The number of pairwise non-isomorphic such irreducible representations is $[\mathcal{K} : \mathbb{Q}] = 2[F : \mathbb{Q}]$ (for $n > 1$ cf. (*)). Thus the subgroup scheme $G'_{1,W(\mathbb{F})}$ of $\mathbf{GSp}(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F}), \tilde{\psi})$ that centralizes $\mathcal{O} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ (equivalently $C_{1,W(\mathbb{F})}$) is a reductive group scheme that has the following three properties: $G_{1,W(\mathbb{F})}$ is a subgroup scheme of $G'_{1,W(\mathbb{F})}$, $G'^{,\mathrm{der}}_{1,W(\mathbb{F})}$ is isomorphic to $\mathbf{SL}^{[F:\mathbb{Q}]}_{n+1,W(\mathbb{F})}$, and $Z^0(G'_{1,W(\mathbb{F})})$ is isomorphic to $\mathbb{G}^{[F:\mathbb{Q}]+1}_{m,W(\mathbb{F})}$. Thus by reasons of dimensions we get that $G_{1,W(\mathbb{F})} = G'_{1,W(\mathbb{F})}$. Therefore the subgroup scheme of $\mathbf{GSp}(L_{(p)}, \tilde{\psi})$ that centralizes $\mathcal{O}$ (equivalently $C_{1,\mathbb{Z}_{(p)}}$) is $G_{1,\mathbb{Z}_{(p)}}$.

Let $\mathfrak{A}$ be the free $\mathbb{Z}_{(p)}$-module of alternating forms on $L_{(p)}$ that are fixed by $G_{1,\mathbb{Z}_{(p)}} \cap \mathbf{Sp}(L_{(p)}, \tilde{\psi})$. There exist elements of $\mathfrak{A} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R}$ that define polarizations of the Hodge $\mathbb{Q}$–structure on W defined by $x_1 \in X_1$, cf. [De2, Cor. 2.3.3]. Thus the real vector space $\mathfrak{A} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R}$ has a non-empty, open subset of such polarizations (cf. [De2, Subsubsect. 1.1.18 (a)]). A standard application to $\mathfrak{A}$ of the approximation theory for independent valuations, implies the existence of an alternating form $\psi \in \mathfrak{A}$ that is congruent modulo p to $\tilde{\psi}$ and that defines a polarization of the Hodge $\mathbb{Q}$–structure on W defined by $x_1 \in X_1$. As ψ is congruent modulo p to $\tilde{\psi}$, it is a perfect, alternating form on $L_{(p)}$. Moreover, the subgroup scheme $\tilde{G}_{1,\mathbb{Z}_{(p)}}$ of $\mathbf{GSp}(L_{(p)}, \psi)$ that centralizes $\mathcal{O}$ contains $G_{1,\mathbb{Z}_{(p)}}$ and its special fibre is $G_{1,\mathbb{F}_p}$. This implies that $\tilde{G}_{1,\mathbb{Z}_{(p)}} = G_{1,\mathbb{Z}_{(p)}}$. Thus the condition (ii) also holds. □

3.3. Simple facts. The semisimple $\mathbb{Q}$–algebra $\mathcal{B} := \mathcal{O}[\frac{1}{p}]$ has $\mathcal{K}$ as its center and thus it is simple. The double centralizer DC_1 of G_1 in $\mathbf{GL}_W$ is such that the group $DC_{1,\overline{\mathbb{Q}}}$ is isomorphic to $\mathbf{GL}^{2[F:\mathbb{Q}]}_{n+1,\mathbb{Q}}$ and DC_1^{ad} is $\mathrm{Res}_{\mathcal{K}/\mathbb{Q}} G[F]_{\mathcal{K}}$. Thus if (G, X) is of strong compact type, then the $\mathbb{Q}$–rank of DC_1^{ad} is 0 (for $n = 1$, cf. the choice of $K = \mathcal{K}$).

4. The proof of the Basic Theorem, part I. Let $p \in \mathbb{N}$ be a prime. All continuous actions of this Section are in the sense of [De2, Subsubsect. 2.7.1] and are right actions. Thus if a locally compact totally discontinuous group Γ acts continuously on a scheme Y, then for each compact, open subgroup $\dagger$ of Γ the geometric quotient scheme $Y/\dagger$ exists and the epimorphism $Y \twoheadrightarrow Y/\dagger$ is pro-finite; moreover, we have $Y = \mathrm{proj.lim.}_{\dagger} Y/\dagger$. In this Section we apply Proposition 3.2 to prove the Basic Theorem 1.3 in the case when (G, X) is a simple, adjoint, unitary Shimura pair.

Until Section 5 we work under the setting of Subsection 3.1 and we also use the notations of Proposition 3.2 and its proof. Thus (G, X) is a simple, adjoint, unitary Shimura pair, we write $G = \mathrm{Res}_{F/\mathbb{Q}} G[F]$ where $G[F]$ is an absolutely simple, adjoint group over a totally real number field F, $G_{\mathbb{Z}_{(p)}}$ is an adjoint group scheme over $\mathbb{Z}_{(p)}$ that extends G, $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, we have an injective map $f_1 : (G_1, X_1) \hookrightarrow (\mathbf{GSp}(W, \psi), S)$ of Shimura pairs such that $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) = (G, X)$, the reductive group scheme $C_{1,\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ is the centralizer of $G_{1,\mathbb{Z}_{(p)}}$ in $\mathbf{GL}_{L_{(p)}}$, $G_{1,\mathbb{Z}_{(p)}}$ is the closed subgroup scheme of $\mathbf{GSp}(L_{(p)}, \psi)$ that fixes a semisimple $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{O}$ of $\mathrm{End}(L_{(p)})$ and it is a reductive group scheme over $\mathbb{Z}_{(p)}$, as Lie algebras we can identify $\mathcal{O} = \mathrm{Lie}(C_{1,\mathbb{Z}_{(p)}})$, etc. Let $g := \frac{1}{2}\dim_{\mathbb{Q}}(W) \in \mathbb{N}$. Let $\mathcal{B} := \mathcal{O}[\frac{1}{p}]$. Let $U := \mathbf{GSp}(L_{(p)}, \psi)(\mathbb{Z}_p)$; it is a hyperspecial subgroup of $\mathbf{GSp}(W \otimes_{\mathbb{Q}} \mathbb{Q}_p, \psi)(\mathbb{Q}_p)$. Let $H_1 := U \cap G_1(\mathbb{Q}_p) = G_{1,\mathbb{Z}_p}(\mathbb{Z}_p)$; it is a hyperspecial subgroup of $G_{1,\mathbb{Q}_p}(\mathbb{Q}_p)$. Let L be a $\mathbb{Z}$-lattice of W such that we have $L_{(p)} = L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and ψ induces a perfect alternating form on L.

In Subsection 4.1 we introduce the integral canonical model $\mathcal{N}_1$ of the Shimura triple (G_1, X_1, H_1). In Subsection 4.2 we introduce another integral canonical model $\mathcal{N}'_1$ that has $\mathcal{N}_1$ as an open closed subscheme. Theorem 4.3 constructs the $E(G, X)_{(p)}$-scheme $\mathcal{N}$ which will turn out to be the integral canonical model of (G, X, H). Corollary 4.4 proves the existence of the integral canonical models of those Shimura triples whose adjoints are isomorphic to (G, X, H).

4.1. The scheme $\mathcal{N}_1$. Let $N \in \mathbb{N}\backslash(\{1,2\}\cup p\mathbb{N})$. Let $\psi_N : L/NL\otimes_{\mathbb{Z}/N\mathbb{Z}} L/NL \to \mathbb{Z}/N\mathbb{Z}$ be the reduction modulo N of ψ. If (C, λ_C) is a principally polarized abelian scheme of relative dimension g over a $\mathbb{Z}[\frac{1}{N}]$-scheme Y and if $\lambda_{C[N]} : C[N] \times_Y C[N] \to \mu_{N,Y}$ is the Weil pairing induced by λ_C, then by a *level-N symplectic similitude structure* of (C, λ_C) we mean an isomorphism $\kappa_N : (L/NL)_Y \xrightarrow{\sim} C[N]$ of finite, étale group schemes over Y such that there exists an element $\nu \in \mu_{N,Y}(Y)$ with the property that for all points $a, b \in (L/NL)_Y(Y)$ we have an identity $\nu^{\psi_N(a\otimes b)} = \lambda_{C[N]}(\kappa_N(a), \kappa_N(b))$ between elements of $\mu_{N,Y}(Y)$.

Let $\mathcal{A}_{g,1,N}$ be Mumford's moduli $\mathbb{Z}_{(p)}$-scheme mentioned before Subsection 1.4. Let

$$\mathcal{M} := \text{proj.lim.}_{N\in\mathbb{N}\backslash(\{1,2\}\cup p\mathbb{N})}\mathcal{A}_{g,1,N}.$$

Thus $\mathcal{M}$ is the moduli $\mathbb{Z}_{(p)}$-scheme that parametrizes isomorphism classes of principally polarized abelian schemes which are of relative dimension g and are endowed with compatible level-N symplectic similitude structures for all numbers $N \in \mathbb{N} \setminus p\mathbb{N}$.

We can identify $\mathcal{M}_{\mathbb{Q}} = \text{Sh}(\mathbf{GSp}(W, \psi), S)/U$, cf. [De1, Thm. 4.21]. Thus $\mathcal{M}$ together with the continuous action of $\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})$ on it defined naturally by the choice of the $\mathbb{Z}$-lattice L of W, is an integral canonical model of $(\mathbf{GSp}(W, \psi), S, U)$, cf. either [Mi2, Thm. 2.10] or [Va1, Example 3.2.9 and Subsect. 4.1].

We recall that $\text{Sh}(G_1, X_1)/H_1$ is a closed subscheme of $\text{Sh}(\mathbf{GSp}(W, \psi), S)_{E(G_1,X_1)}/U = \mathcal{M}_{E(G_1,X_1)}$, cf. Subsubsection 2.4.2. We have the following Corollary to Proposition 3.2.

4.1.1. Corollary. *Let $\mathcal{N}_1$ be the normalization of the Zariski closure of $\text{Sh}(G_1, X_1)/H_1$ in $\mathcal{M}_{E(G_1,X_1)_{(p)}}$; the group $G_1(\mathbb{A}_f^{(p)})$ acts continuously on the $E(G_1, X_1)_{(p)}$-scheme $\mathcal{N}_1$.*

(a) *Then $\mathcal{N}_1$ is the integral canonical model of (G_1, X_1, H_1). Moreover $\mathcal{N}_1$ is quasi-projective and a closed subscheme of $\mathcal{M}_{E(G_1,X_1)_{(p)}}$.*

(b) *If moreover (G, X) is of strong compact type, then $\mathcal{N}_1$ is in fact projective.*

Proof. From [Va1, Prop. 3.4.1] and its proof we get that the following three properties hold:

(i) the $E(G_1, X_1)_{(p)}$-scheme $\mathcal{N}_1$ has the extension property of Definition 1.1 (a);

(ii) axiom (i) of Definition 1.1 (b) holds for $\mathcal{N}_1$;

(iii) there exists a compact, open subgroup U_p of $\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})$ such that if $H_{1,p} := U_p \cap G_1(\mathbb{A}_f^{(p)})$, then $\mathcal{N}_1$ is a pro-étale cover of $\mathcal{N}_1/H_{1,p}$ and $\mathcal{N}_1/H_{1,p}$ is a finite $\mathcal{M}_{E(G_1,X_1)_{(p)}}/U_p$-scheme.

It is well known that the following two properties also hold:

(iv) $\mathcal{N}_1$ is in fact a closed subscheme of $\mathcal{M}_{E(G_1,X_1)_{(p)}}$;

(v) if U_p is small enough, then the $E(G_1, X_1)_{(p)}$-scheme $\mathcal{N}_1/H_{1,p}$ is smooth and quasi-projective (see [Zi, Subsect. 3.5]; see also [LR] and [Ko, Sect. 5]).

This implies that the axiom (ii) of Definition 1.1 (b) also holds. Thus $\mathcal{N}_1$ is the integral canonical model of (G_1, X_1, H_1) and it is quasi-projective. Thus (a) holds.

We now assume (G, X) is of strong compact type. The $\mathbb{Q}$–algebra $\mathcal{B}$ is simple and the $\mathbb{Q}$–rank of the adjoint group of the centralizer DC_1 of $\mathcal{B}$ in $\mathbf{GL}_W$ is 0, cf. Subsection 3.3. Therefore the $\mathbb{Q}$–algebra $\mathrm{End}_{\mathcal{B}}(W)$ is a division $\mathbb{Q}$–algebra. Thus by taking U_p to be small enough, we can assume that moreover $\mathcal{N}_1/H_{1,p}$ is a projective $E(G_1, X_1)_{(p)}$-scheme (cf. [Mo1, Thm. 2]; see also [Ko, end of Sect. 5]). Thus (b) holds. □

4.2. The scheme $\mathcal{N}'_1$. Let G'_1 be the subgroup of $\mathbf{GL}_W$ generated by G_1^{der} and $Z^0(C_1) = Z(C_1)$. From Lemma 2.3 (b) we get that the Zariski closure $G'_{1,\mathbb{Z}_{(p)}}$ of G'_1 in $\mathbf{GL}_{L_{(p)}}$ is a reductive group scheme. Thus the group scheme $Z(G'_{1,\mathbb{Z}_{(p)}}) = Z^0(C_{1,\mathbb{Z}_{(p)}})$ is the torus $\mathcal{T} = \mathrm{Res}_{\mathcal{K}_{(p)}/\mathbb{Z}_{(p)}}\mathbb{G}_{m,\mathcal{K}_{(p)}}$ of the proof of Proposition 3.2, $G_{1,\mathbb{Z}_{(p)}}$ is a closed, normal subgroup scheme of $G'_{1,\mathbb{Z}_{(p)}}$, and we have $G^{\mathrm{der}}_{1,\mathbb{Z}_{(p)}} = G'^{,\mathrm{der}}_{1,\mathbb{Z}_{(p)}}$. Let X'_1 be such that we get an injective map $(G_1, X_1) \hookrightarrow (G'_1, X'_1)$ of Shimura pairs. Let $q'_1 : (G'_1, X'_1) \to (G, X) = (G'^{,\mathrm{ad}}_1, X'^{,\mathrm{ad}}_1)$ be the resulting map of Shimura pairs. The subgroup $H'_1 := G'_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ of $G'_{1,\mathbb{Q}_p}(\mathbb{Q}_p)$ is hyperspecial. Let $\mathcal{N}'_1$ be the integral canonical model of (G'_1, X'_1, H'_1), cf. Proposition 2.4.3 (a) and the first part of Corollary 4.1.1 (a). Thus $\mathcal{N}_1$ is an open closed subscheme of $\mathcal{N}'_1$, cf. Subsubsection 2.4.2. The connected components of $\mathcal{N}'_1$ are permuted transitively by $G'_1(\mathbb{A}_f^{(p)})$, cf. [Va1, Lem. 3.3.2]. From the last two sentences and the second part of Corollary 4.1.1 (a), we get that $\mathcal{N}'_1$ is quasi-projective. If $\mathcal{N}_1$ is projective, then $\mathcal{N}'_1$ is also projective.

4.3. Theorem. *There exists a unique pro-étale cover $\mathcal{N}'_1 \to \mathcal{N}$ of $E(G, X)_{(p)}$-schemes that extends the natural pro-étale cover $\mathrm{Sh}(G'_1, X'_1)/H'_1 \to \mathrm{Sh}(G, X)/H$ of $E(G, X)$-schemes. Moreover, the $E(G, X)_{(p)}$-scheme $\mathcal{N}$ has the extension property.*

Proof. The uniqueness part is obvious. As the argument for the existence of $\mathcal{N}$ is quite long, in this paragraph we outline its main parts. The scheme $\mathcal{N}_{E(G_1,X_1)_{(p)}}$ will be the quotient of $\mathcal{N}'_1$ by $Z(G'_1)(\mathbb{A}_f^{(p)})$. The difficult part will be to check that $Z(G'_1)(\mathbb{A}_f^{(p)})$ acts freely on $\mathcal{N}'_1$ (equivalently, that this quotient is a smooth $E(G_1, X_1)_{(p)}$-scheme). In order to achieve this, we will rely heavily on the moduli interpretation of $\mathcal{N}'_1$ and on the explicit description of the right action of $Z(G'_1)(\mathbb{A}_f^{(p)})$ on $\mathcal{N}'_1$ (see Step 1 below). The scheme $\mathcal{N}$ will be obtained from $\mathcal{N}_{E(G_1,X_1)_{(p)}}$ via standard descent whose very essence will be the fact that the $E(G_1, X_1)_{(p)}$-scheme $\mathcal{N}_{E(G_1,X_1)_{(p)}}$ has the extension property (see Step 2 below).

As $Z(G'_1) = \mathcal{T}_{\mathbb{Q}} = \mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}}$, for every field K of characteristic 0 the group $H^1(K, Z(G'_1)_K)$ is trivial. Thus $q'_1 : (G'_1, X'_1, H'_1) \to (G, X, H)$ is a cover in the sense of Subsection 2.4. We consider an arbitrary $\mathbb{Z}_{(p)}$-monomorphism $E(G_1, X_1)_{(p)} \hookrightarrow W(\mathbb{F})$ and we use it to view also $W(\mathbb{F})$ as an $E(G, X)_{(p)}$-algebra. We have the following two main steps; as they are quite long, its main parts are itemized and baldfaced.

Step 1. We first consider the case when the field $E(G_1, X_1) = E(G'_1, X'_1)$ is $E(G, X)$ (see Subsection 2.2 for the last identity). As $E(G'_1, X'_1) = E(G, X)$ and as q'_1 is a cover, we have $\mathrm{Sh}(G, X)/H = \mathrm{Sh}(G'_1, X'_1)/(H'_1 \times Z(G'_1)(\mathbb{A}_f^{(p)}))$ (cf. Lemma 2.4.1).

If $H'_{1,p}$ is a compact, open subgroup of $G'_1(\mathbb{A}_f^{(p)})$, then the following quotient group $Q'_{1,p} := H'_{1,p}Z(G'_1)(\mathbb{A}_f^{(p)})/H'_{1,p}\overline{Z(G'_{1,\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}$ is finite. Thus as $\mathcal{N}'_1/H'_{1,p}$ is a quasi-projective $E(G, X)_{(p)}$-scheme, its quotient $(\mathcal{N}'_1/H'_{1,p})^{Q'_{1,p}}$ by $Q'_{1,p}$ exists and is a normal, quasi-projective $E(G,X)_{(p)}$-scheme (cf. [DG, Vol. I, Exp. V, Thm. 4.1]). Let $\mathcal{N}$ be the projective limit of the $E(G,X)_{(p)}$-schemes $(\mathcal{N}_1'/H_{1,p}')^{Q_{1,p}'}$ indexed by the groups $H_{1,p}'$.

Step 1.1. Connected components. The $E(G,X)_{(p)}$-scheme $\mathcal{N}$ is the quotient of $\mathcal{N}_1'$ by $Z(G_1')(\mathbb{A}_f^{(p)})$ and it is a faithfully, flat $E(G,X)_{(p)}$-scheme whose generic fibre is $\mathrm{Sh}(G,X)/H$. Let $\mathcal{C}$ be a connected component of $\mathcal{N}_{W(\mathbb{F})}$ that is dominated by a connected component $\mathcal{C}_1$ of $\mathcal{N}_{1,W(\mathbb{F})}$. The connected components of $\mathcal{N}_{B(\mathbb{F})}$ (resp. of $\mathcal{N}_{1,B(\mathbb{F})}'$) and thus also of $\mathcal{N}_{W(\mathbb{F})}$ (resp. of $\mathcal{N}_{1,W(\mathbb{F})}'$) are permuted transitively by $G(\mathbb{A}_f^{(p)})$ (resp. by $G_1'(\mathbb{A}_f^{(p)})$), cf. [Va1, Lem. 3.3.2]. Thus as the homomorphism $q_1'(\mathbb{A}_f^{(p)}) : G_1'(\mathbb{A}_f^{(p)}) \to G(\mathbb{A}_f^{(p)})$ is onto, to show that $\mathcal{N}_1'$ is a pro-étale cover of $\mathcal{N}$ it is enough to show that $\mathcal{C}_1$ is a pro-finite Galois cover of $\mathcal{C}$.

The scheme $\mathcal{C}$ is the quotient of $\mathcal{C}_1$ by a group of $\mathcal{C}$-automorphisms $\mathfrak{Q}$ of $\mathcal{C}_1$ defined by right translations by elements of a subgroup of $Z(G_1')(\mathbb{A}_f^{(p)})$. It is known that there exists $N_0 \in \mathbb{N}$ such that $\mathfrak{Q}$ is an N_0-torsion group, cf. [Va1, p. 493, Fact]. Let $t \in \mathfrak{Q}$ be an element that fixes a point $y \in \mathcal{C}_1(\mathbb{F})$. We denote also by t an arbitrary element of $Z(G_1')(\mathbb{A}_f^{(p)})$ that defines it. The point y gives birth to a quadruple

$$Q_y = (A_y, \lambda_{A_y}, \mathcal{O}, (\kappa_N)_{N \in \mathbb{N} \setminus p\mathbb{N}}),$$

where (A_y, λ_{A_y}) is a principally polarized abelian variety over $\mathbb{F}$ of dimension g, endowed with a $\mathbb{Z}_{(p)}$-algebra of endomorphisms denoted also by $\mathcal{O}$, and having in a compatible way a level-N symplectic similitude structure κ_N for all $N \in \mathbb{N} \setminus p\mathbb{N}$.

Step 1.2. Axioms. The quadruple Q_y is subject to some axioms, cf. the standard interpretation of $\mathcal{N}_1$ as a moduli scheme (see [Zi, Subsect. 3.5]; see also [LR] and [Ko, Sect. 5]). Briefly, the axioms say (for instance cf. [Ko, Sects. 5 and 8]):

(i) if $\{\alpha_1, \ldots, \alpha_m\}$ is a $\mathbb{Z}_{(p)}$-basis of $\mathcal{O}$ and if $X_1, \ldots, X_m$ are independent variables, then the determinant of the linear endomorphism $\sum_{j=1}^m X_j \alpha_j$ of $\mathrm{Lie}(A_y)$ is the extension to $\mathbb{F}$ of a universal determinant over $E(G_1, X_1)_{(p)}$ that is of a similar nature and it is associated naturally to the faithful representation $\mathcal{O} \hookrightarrow \mathrm{End}(L_{(p)})$;

(ii) for each prime $l \in \mathbb{N} \setminus \{p\}$, the following symplectic similitude isomorphism $\kappa_{l^\infty} : (W \otimes_{\mathbb{Q}} \mathbb{Q}_l, \psi) \xrightarrow{\sim} (T_l(A_y) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \lambda_{A_y})$ induced naturally by κ_{l^m}'s ($m \in \mathbb{N}$), is also a $\mathcal{B}$-linear isomorphism; here we denote also by λ_{A_y} the perfect alternating form on the l-adic Tate-module $T_l(A_y)$ of A_y induced by λ_{A_y};

(iii) under an $E(G,X)_{(p)}$-monomorphism $W(\mathbb{F}) \hookrightarrow \mathbb{C}$, the principally polarized abelian schemes over $W(\mathbb{F})$ that are endowed with a $\mathbb{Z}_{(p)}$-algebra of endomorphisms and that lift the triple $(A_y, \lambda_{A_y}, \mathcal{O})$ give birth to principally polarized abelian varieties over $\mathbb{C}$ that are endowed with a $\mathbb{Z}_{(p)}$-algebra of endomorphisms and that are naturally associated through Riemann's theorem to triples of the following form

$$(L_1 \backslash W \otimes_{\mathbb{Q}} \mathbb{C}/F_{x_1}^{0,-1}, \varepsilon_1 \psi, \mathcal{O}).$$

Here $W \otimes_{\mathbb{Q}} \mathbb{C} = F_{x_1}^{-1,0} \oplus F_{x_1}^{0,-1}$ is the Hodge decomposition defined by an element $x_1 \in X_1$, L_1 is a $\mathbb{Z}$-lattice of W such that we have $h_1(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) = L_1 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ for some element $h_1 \in G_1(\mathbb{A}_f^{(p)})$, and ε_1 is the unique non-zero rational number such that $\varepsilon_1 \psi : L_1 \otimes_{\mathbb{Z}} L_1 \to \mathbb{Z}$ is a principal polarization of the Hodge $\mathbb{Z}$-structure on L_1 defined by x_1.

Step 1.3. Moduli interpretation of $\mathcal{N}_1'$. Let $G'_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})^{X_1}$ be the maximal subgroup of $G'_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ that normalizes X_1. The group $G_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ permutes transitively the connected components of X_1, cf. [Va1, Cor. 3.3.3]. As the group of connected components of X_1' (or of X_1) is abelian, every element of $G_1'(\mathbb{R})$ that takes a connected component of X_1 into another connected component of X_1, will in fact take X_1 onto X_1. From (2) and the last two sentences we get a natural identification

$$\mathrm{Sh}(G_1', X_1')/H_1'(\mathbb{C}) = G'_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})^{X_1}\backslash(X_1 \times G_1'(\mathbb{A}_f^{(p)})/\overline{Z(G'_{1,\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})}). \tag{3}$$

Formula (3) implies that we also have a standard moduli interpretation of $\mathcal{N}_1'$, provided we work in an $F_{(p)}$-polarized context (see [De1, Variant 4.14] for the case of $\mathbb{C}$-valued points, stated in terms of isogeny classes). In such a context we speak about an abelian scheme B which is endowed with an $F_{(p)}$-principal polarization λ_B and with a $\mathbb{Z}_{(p)}$-algebra of endomorphisms denoted also by $\mathcal{O}$ and which has in an F-compatible way level-N symplectic similitude structures $\kappa_{B,N}$ for all $N \in \mathbb{N} \setminus p\mathbb{N}$. If B is over an algebraically closed field or over a complete discrete valuation ring that has an algebraically closed residue field, then the F-compatibility refers here to the fact that for every prime $l \in \mathbb{N}\setminus\{p\}$, there exists a $\mathcal{B}$-isomorphism $\kappa_{B,l^\infty} : W \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} T_l(B) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ induced naturally by κ_{B,l^m}'s ($m \in \mathbb{N}$) and such that for all $a, b \in W \otimes_{\mathbb{Q}} \mathbb{Q}_l$ we have $\psi(a \otimes b) = \chi_l \lambda_B(\kappa_{B,l^\infty}(a), \kappa_{B,l^\infty}(b))$, where $\chi_l \in \mathbb{G}_{m,F}(F \otimes_{\mathbb{Q}} \mathbb{Q}_l)$ and where we denote also by λ_B the non-degenerate alternating form on $T_l(B) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ induced by λ_B. As $\mathcal{B}$ is an F-algebra, it makes sense to speak about the action of F on $T_l(B) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Also by an $F_{(p)}$-principal polarization λ_B of B, we mean the set of $\mathbb{G}_{m,\mathbb{Z}_{(p)}}(F_{(p)})$-multiples of a polarization of B that induces for every $m \in \mathbb{N}$ an isomorphism between $B[p^m]$ and its Cartier dual. The composite embedding $F \hookrightarrow \mathcal{B} \hookrightarrow \mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ defines naturally an action of F on $\mathrm{Hom}(B, B^t) \otimes_{\mathbb{Z}} \mathbb{Q}$, where B^t is the abelian scheme that is the dual of B. Thus the $\mathbb{G}_{m,\mathbb{Z}_{(p)}}(F_{(p)})$-multiples of λ_B are well defined as elements of $\mathrm{Hom}(B, B^t) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Step 1.4. Crystalline setting. Due to this moduli interpretation of $\mathcal{N}_1'$, the right translation of y by t gives birth to a quadruple

$$Q_y' = (A_y', \lambda_{A_y'}, \mathcal{O}, (\kappa_N')_{N \in \mathbb{N}\setminus p\mathbb{N}}),$$

where the pair $(A_y', \lambda_{A_y'})$ is an abelian variety over $\mathbb{F}$ that is endowed with an $F_{(p)}$-principal polarization and it is naturally $\mathbb{Z}_{(p)}$-isomorphic to (A_y, λ_{A_y}). Thus we can identify

$$M := H^1_{\mathrm{crys}}(A_y/W(\mathbb{F})) = H^1_{\mathrm{crys}}(A_y'/W(\mathbb{F})).$$

The fact that t fixes y means that the quadruples Q_y and Q_y' are isomorphic under an isomorphism $a : A_y \xrightarrow{\sim} A_y'$. Let ϕ be the Frobenius endomorphism of M and let λ_M be the perfect alternating form on M defined by λ_{A_y}. Let $a_M : M \xrightarrow{\sim} M$ be the crystalline realization of a. We check that the following property holds:

(iv) the element $a_M \in \mathrm{End}(M[\frac{1}{p}])$ belongs to the $\mathbb{Q}$–vector space generated by crystalline realizations of $\mathbb{Q}$–endomorphisms of A_y defined naturally by elements of $\mathrm{Lie}(Z(G_1'))$.

To check (iv), it suffices to show that for a prime $l \neq p$, the $\mathbb{Q}_l$-étale realization of a belongs to the $\mathbb{Q}$–vector space generated by $\mathbb{Q}_l$-étale realizations of $\mathbb{Q}$–endomorphisms of A_y defined naturally by elements of $\mathrm{Lie}(Z(G_1'))$. But this is a direct consequence of the facts that $t \in Z(G_1')(\mathbb{A}_f^{(p)})$ and that the isomorphism $a : A_y \xrightarrow{\sim} A_y'$ is compatible with the level structures κ_{l^m} and κ_{l^m}' for all $m \in \mathbb{N}$.

Due to (iv), the automorphism a_M normalizes each direct summand F^1 of M which is the Hodge filtration of the abelian scheme over $W(\mathbb{F})$ that correspond to a $W(\mathbb{F})$-valued point z of $\mathcal{C}_1$ that lifts y. If $p > 2$ such a lift z is uniquely determined by F^1, cf. the deformation theory of polarized abelian varieties endowed with endomorphisms (see the Serre–Tate and Grothendieck–Messing deformation theories of [Me, Chs. 4 and 5]). Thus based on the moduli interpretation of $\mathcal{N}_1$, for $p > 2$ we get that t fixes all these lifts. Therefore for $p > 2$ we have $t = 1_{\mathcal{C}_1}$.

Step 1.5. The case $p = 2$**.** In the remaining part of the Step 1 we show that we have $t = 1_{\mathcal{C}_1}$ even for $p = 2$. In order to achieve this we will first show that there exists a lift $z_0 \in \mathcal{N}_1(W(\mathbb{F}))$ of the point $y \in \mathcal{N}_1(\mathbb{F})$ which is uniquely determined in some sense by the Hodge filtration F_0^1 of M it defines. We begin by constructing first the direct summand F_0^1 of M.

Let $M = M_0 \oplus M_1 \oplus M_2$ be the direct sum decomposition left invariant by ϕ and such that we have $\phi(M_0) = M_0$, $\phi(M_1) = pM_1$, and all slopes of (M_2, ϕ) belong to the interval $(0, 1)$. This decomposition is left invariant by the crystalline realization of each $\mathbb{Z}_{(p)}$-endomorphism of A_y and thus by the crystalline realizations of elements of $\mathcal{O}$. Thus the opposite $\mathbb{Z}_{(2)}$-algebra $\mathcal{O}^{\text{opp}}$ of $\mathcal{O}$ acts on M_0, M_1, and M_2. The subgroup scheme $\tilde{G}_{1,W(\mathbb{F})}$ of $\mathbf{GSp}(M, \lambda_M)$ that centralizes the crystalline realizations of $\mathcal{O}^{\text{opp}} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$ is a reductive group scheme isomorphic to $G_{1,W(\mathbb{F})}$. Let $\mu : \mathbb{G}_{m,W(\mathbb{F})} \to \tilde{G}_{1,W(\mathbb{F})}$ be a cocharacter such that we have a direct sum decomposition $M = F_0^1 \oplus F_0^0$ for which the following two properties hold:

(v) the cocharacter μ fixes F_0^0 and acts on F_0^1 via the inverse of the identical character of $\mathbb{G}_{m,W(\mathbb{F})}$;

(vi) the kernel of ϕ modulo p is F_0^1/pF_0^1.

The functorial aspects of [Wi, p. 513] imply that the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [Wi, p. 512] normalizes the $W(\mathbb{F})$-span of λ_M and it commutes with the crystalline realizations of $\mathcal{O}^{\text{opp}}$; thus as μ we can take the factorization through $\tilde{G}_{1,W(\mathbb{F})}$ of the inverse of the canonical split cocharacter of (M, F^1, ϕ).

Let $\nu : \mathbb{G}_{m,W(\mathbb{F})} \to \mathbf{GL}_M$ be the cocharacter that fixes M_0, that acts on M_1 as the second power of the identity character of $\mathbb{G}_{m,W(\mathbb{F})}$, and that acts on M_2 as the identity character of $\mathbb{G}_{m,W(\mathbb{F})}$. The cocharacter ν factors through $\tilde{G}_{1,W(\mathbb{F})}$.

The intersection $\tilde{J}_{1,W(\mathbb{F})} := \tilde{G}_{1,W(\mathbb{F})} \cap (\mathbf{GL}_{M_0} \times_{W(\mathbb{F})} \mathbf{GL}_{M_1} \times_{W(\mathbb{F})} \mathbf{GL}_{M_2})$ is the centralizer in $\tilde{G}_{1,W(\mathbb{F})}$ of $\mathrm{Im}(\nu)$ and thus it is a reductive, closed subgroup scheme of $\tilde{G}_{1,W(\mathbb{F})}$ (cf. [DG, Vol. III, Exp. XIX, Subsect. 2.8 and Prop. 6.3]). The special fibre $\nu_{\mathbb{F}}$ of ν factors through the parabolic subgroup of $\tilde{J}_{1,\mathbb{F}}$ that normalizes F_0^1/pF_0^1. This implies that up to a replacement of μ by its conjugate under an element of $\tilde{G}_{1,W(\mathbb{F})}(W(\mathbb{F}))$ that normalizes F_0^1/pF_0^1, we can assume that the special fibre $\mu_{\mathbb{F}}$ of μ factors through $\tilde{J}_{1,\mathbb{F}}$. Based on [DG, Vol. II, Exp. IX, Thms. 3.6 and 7.1], by performing a similar replacement of μ we can assume that μ itself factors through $\tilde{J}_{1,W(\mathbb{F})}$. Thus we have a direct sum decomposition $F_0^1 = \oplus_{i=0}^2 M_i \cap F_0^1$. For $i \in \{0, 1, 2\}$, let D_i be the unique p-divisible group over $W(\mathbb{F})$ whose filtered Dieudonné module is $(M_i, F_0^1 \cap M_i, \phi)$, cf. [Fo, Ch. IV, §1, Prop. 1.6]; we emphasize that, strictly speaking, loc. cit. is stated in terms of Honda triples $(M_i, \phi(\frac{1}{p}F_0^1 \cap M_i), \phi)$. Let $D := \prod_{i=0}^2 D_i$. Loc. cit. also implies that there exists a unique principal quasi-polarization λ_D of D whose crystalline realization is λ_M; it is a direct sum of principal quasi-polarizations of $D_0 \oplus D_1$ and D_2. From loc. cit. we also get that the crystalline realizations of elements of $\mathcal{O}^{\text{opp}}$ are crystalline realizations of endomorphisms of D. Thus from Serre–Tate deformation theory and the standard moduli interpretation of $\mathcal{N}_1$, we get that there exists a $W(\mathbb{F})$-valued point z_0 of $\mathcal{C}_1$ such that the principally quasi-polarized p-divisible group of the principally polarized abelian scheme over $\text{Spec}(W(\mathbb{F}))$ that corresponds to z_0 and that lifts (A_y, λ_{A_y}), is (D, λ_D). As the pair $(D = \prod_{i=0}^2 D_i, \lambda_D)$ is uniquely determined by the Hodge filtration $F_0^1 = a_M(F_0^1)$ of M and as the right translation of z by $t \in Z(G_1')(\mathbb{A}_f^{(p)})$ gives birth to an analogous pair, we conclude that t fixes z_0. As $\mathcal{C}_{1,B(\mathbb{F})}$ is a pro-finite Galois cover of $\mathcal{C}_{B(\mathbb{F})}$, each $\mathcal{C}$-automorphism of $\mathcal{C}_1$ either acts freely on $\mathcal{C}_{1,B(\mathbb{F})}$ or is $1_{\mathcal{C}_1}$. Therefore we have $t = 1_{\mathcal{C}_1}$ even for $p = 2$.

Step 1.6. Conclusion. Thus regardless of what p is, we have $t = 1_{\mathcal{C}_1}$ and therefore $\mathfrak{Q}$ acts freely on $\mathcal{C}_1$. Thus $\mathcal{C}_1$ is a pro-finite Galois cover of $\mathcal{C}$ and therefore the desired pro-étale cover $\mathcal{N}_1' \to \mathcal{N}$ exists.

Step 2. We now consider the general case; thus $E(G_1', X_1')$ does not necessarily coincide with $E(G, X)$. As in Step 1 we argue that there exists a pro-étale cover $\mathcal{N}_1' \to \mathcal{N}_{E(G_1',X_1')_{(p)}}$ of $E(G_1', X_1')_{(p)}$-schemes that extends the pro-étale cover $\text{Sh}(G_1', X_1')/H_1' \to \text{Sh}(G, X)_{E(G_1',X_1')}/H$ of $E(G_1', X_1')$-schemes. Let $\mathcal{Q}$ be the Galois group of the Galois extension $E_1'(G, X)$ of $E(G, X)$ generated by $E(G_1', X_1')$. The finite $E(G, X)_{(p)}$-algebra $E_1'(G, X)_{(p)}$ is étale over $\mathbb{Z}_{(p)}$.

Step 2.1. Extension property. In this paragraph we recall the argument (see [Va1, pp. 493–494]) that the $E_1'(G, X)_{(p)}$-scheme $\mathcal{N}_{E_1'(G,X)_{(p)}}$ has the extension property. Let $\mathcal{Z}$ be a faithfully flat $E_1'(G, X)_{(p)}$-scheme that is healthy regular. Let $u : \mathcal{Z}_{E_1'(G,X)} \to \mathcal{N}_{E_1'(G,X)}$ be an $E_1'(G, X)$-morphism. Let $\mathcal{D}$ be a local ring of $\mathcal{Z}$ that is a discrete valuation ring. Let $\mathcal{W}$ and $\mathcal{E}$ be the normalizations of $\mathcal{Z}$ and $\mathcal{D}$ (respectively) in $\mathcal{Z}_{E_1'(G,X)} \times_{\mathcal{N}_{E_1'(G,X)}} \mathcal{N}'_{1,E_1'(G,X)}$. If $\mathcal{W}$ is a pro-étale cover of $\mathcal{Z}$, then $\mathcal{W}$ is a healthy regular scheme (cf. [Va1, Rm. 3.2.2 4), property C)]) and thus from the fact that $\mathcal{N}_1'$ has the extension property we get that the morphism $\mathcal{Z}_{E_1'(G,X)} \times_{\mathcal{N}_{E_1'(G,X)_{(p)}}} \mathcal{N}'_{1,E_1'(G,X)} \to \mathcal{N}'_{1,E_1'(G,X)}$ extends uniquely to a morphism $\mathcal{W} \to \mathcal{N}'_{1,E_1'(G,X)_{(p)}}$. This last thing implies that u extends uniquely to an $E_1'(G, X)_{(p)}$-morphism $\mathcal{Z} \to \mathcal{N}_{E_1'(G,X)_{(p)}}$. Thus to end the argument that $\mathcal{N}_{E_1'(G,X)_{(p)}}$ has the extension property, it suffices to show that $\mathcal{W}$ is a pro-étale cover of $\mathcal{Z}$. Based on the classical purity theorem of Zariski and Nagata (see [Gr, Exp. X, Thm. 3.4 (i)]), it suffices to show that $\text{Spec}(\mathcal{E})$ is a pro-étale cover of $\text{Spec}(\mathcal{D})$. To check this, we can assume that $\mathcal{D}$ is a complete, local ring that has mixed characteristic $(0, p)$ and an algebraically closed residue field. Thus we can assume that the morphism $\text{Spec}(\mathcal{D}[\frac{1}{p}]) \to \mathcal{N}_{E_1'(G,X)}$ factors through $\mathcal{N}_{B(\mathbb{F})}$. If $\mathcal{F}$ is the field of fractions of a connected component of $\text{Spec}(\mathcal{E})$, let $(\mathcal{V}_{\mathcal{F}}, \lambda_{\mathcal{F}})$ be the principally polarized abelian variety over $\mathcal{F}$ which is associated naturally to the morphism $\text{Spec}(\mathcal{F}) \to \mathcal{N}_1'$; it has a level-$N$ structure for all $N \in \mathbb{N} \setminus p\mathbb{N}$. From [Va1, p. 493, fact] we get that there exists $N_0 \in \mathbb{N}$ such that the Galois group $\text{Gal}(\mathcal{F}/\mathcal{D}[\frac{1}{p}])$ is an N_0-torsion group (to be compared with the fourth paragraph of Step 1). Let $l \in \mathbb{N}$ be a prime that does not divide pN_0. Each N_0-torsion subgroup of an l-adic Lie group is finite. Thus the image of the l-adic representation of an open subgroup of $\mathrm{Gal}(\mathcal{F}/\mathcal{D}[\frac{1}{p}])$ associated naturally to a model of $\mathcal{V}_{\mathcal{F}}$ over a finite field extension of $\mathcal{D}[\frac{1}{p}]$, is an l-adic Lie group (cf. [Se, Thms. 1 and 2]) which is an N_0-torsion group. Thus this image is finite. From this and the Néron–Ogg–Shafarevich criterion of good reduction of abelian varieties (see [BLR, Ch. 7, 7.4, Thm. 5]), we get that $\mathcal{V}_{\mathcal{F}}$ has an abelian scheme model $\mathcal{V}_1$ over the ring of integers $\mathcal{O}_1$ of a subfield $\mathcal{F}_1$ of $\mathcal{F}$ which is a finite extension of $\mathcal{D}[\frac{1}{p}]$. But each level-N structure, polarization, or endomorphism of $\mathcal{V}_{1,\mathcal{F}_1}$ extends uniquely to a level-N structure, polarization, or endomorphism (respectively) of $\mathcal{V}_1$ (cf. [FC, Ch. I, 2, Prop. 2.7] for endomorphisms). From the last two sentences and the moduli interpretation of $\mathcal{N}'_1$, we easily get that the morphism $\mathrm{Spec}(\mathcal{E}[\frac{1}{p}]) \to \mathcal{N}'_{1,E'_1(G,X)}$ defined by u extends to a morphism $\mathrm{Spec}(\mathcal{E}) \to \mathcal{N}'_{1,E'_1(G,X)_{(p)}}$. This implies that u extends to a morphism $\mathrm{Spec}(\mathcal{D}) \to \mathcal{N}_{E'_1(G,X)_{(p)}}$. Thus $\mathrm{Spec}(\mathcal{E}) = \mathcal{D} \times_{\mathcal{N}_{E'_1(G,X)_{(p)}}} \mathcal{N}'_{1,E'_1(G,X)_{(p)}}$ is a pro-étale cover of $\mathrm{Spec}(\mathcal{D})$. This ends the argument that the $E'_1(G,X)_{(p)}$-scheme $\mathcal{N}_{E'_1(G,X)_{(p)}}$ has the extension property.

Step 2.2. Galois descent. As $\mathcal{N}_{E'_1(G,X)_{(p)}}$ is a healthy regular scheme (cf. Subsection 1.2) that has the extension property, the canonical action of $\mathcal{Q}$ on $\mathcal{N}_{E'_1(G,X)}$ extends uniquely to a free action of $\mathcal{Q}$ on $\mathcal{N}_{E'_1(G,X)_{(p)}}$. Thus as $\mathcal{N}_{E'_1(G,X)_{(p)}}$ is a pro-étale cover of a quasi-projective $E'_1(G,X)_{(p)}$-scheme, the quotient scheme $\mathcal{N}$ of $\mathcal{N}_{E'_1(G,X)_{(p)}}$ by $\mathcal{Q}$ exists and the quotient morphism $\mathcal{N}_{E'_1(G,X)_{(p)}} \to \mathcal{N}$ is an étale cover (cf. [DG, Vol. I, Exp. V, Thm. 4.1]). This implies that the $E(G,X)_{(p)}$-morphism $\mathcal{N}'_1 \to \mathcal{N}$ is a pro-étale cover that extends the $E(G,X)$-morphism $\mathrm{Sh}(G'_1,X'_1)/H'_1 \to \mathrm{Sh}(G,X)/H$.

Step 2.3. Conclusion. Due to the fact that the $E'_1(G,X)_{(p)}$-scheme $\mathcal{N}_{E'_1(G,X)_{(p)}}$ has the extension property, as in the last paragraph of the proof of Proposition 2.4.3 we argue that $\mathcal{N}$ itself has the extension property. □

4.3.1. Remark. For $p > 2$, the above usage of $\mathcal{N}'_1$ and of the moduli interpretations of $\mathcal{N}_1$ and $\mathcal{N}'_1$, can be entirely avoided as follows. Let $G_2 := G \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$. Thus $H_2 := H \times \mathbb{G}_{m,\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ is a hyperspecial subgroup of $G_{2,\mathbb{Q}_p}(\mathbb{Q}_p)$. Let G_1^0 be the subgroup of G_1 that fixes ψ. We consider the homomorphism $q_1 : G_1 \to G_2$ that lifts the two natural epimorphisms $G_1 \twoheadrightarrow G_1^{\mathrm{ad}} = G$ and $G_1 \twoheadrightarrow G_1/G_1^0 = \mathbb{G}_{m,\mathbb{Q}}$. Let X_2 be such that q_1 defines a map $q_1 : (G_1, X_1) \to (G_2, X_2)$ of Shimura pairs. The torus $\mathrm{Ker}(q_1)$ is the torus $\mathcal{T}_{c,\mathbb{Q}}$ of the proof of Proposition 3.2 and thus it is the $\mathrm{Res}_{F/\mathbb{Q}}$ of a rank 1 torus which over $\mathbb{R}$ is compact. Thus for each field K of characteristic 0, the group $H^1(K, \mathcal{T}_{c,K})$ is a 2-torsion group which in general (like for $K = \mathbb{R}$) is non-trivial. From this and (2), we easily get that the image of $\mathcal{C}_{1,\mathbb{C}}$ in the quotient of $\mathrm{Sh}(G_1, X_1)_{\mathbb{C}}/H_1$ by $\mathcal{T}_{c,\mathbb{Q}}(\mathbb{A}_f^{(p)})$ is a (potentially infinite) Galois cover of $\mathrm{Im}(\mathcal{C}_{1,\mathbb{C}} \to \mathrm{Sh}(G_2, X_2)_{\mathbb{C}}/H_2)$, whose Galois group is a 2-torsion group; here $\mathcal{C}_{1,\mathbb{C}}$ is obtained via extension of scalars through an $E(G_1, X_1)_{(p)}$-monomorphism $W(\mathbb{F}) \hookrightarrow \mathbb{C}$. Thus it suffices to show that $t = 1_{\mathcal{C}_1}$ under the extra assumption that there exists $s \in \mathbb{N}$ such that t^{2^s} is the automorphism of $\mathcal{C}_1$ defined by an element of $\mathcal{T}_{c,\mathbb{Q}}(\mathbb{A}_f^{(p)})$ and thus also of $Z(G_1)(\mathbb{A}_f^{(p)})$. But if $t^{2^s} = 1_{\mathcal{C}_1}$, then, as we assumed $p > 2$, we get that $t = 1_{\mathcal{C}_1}$ (cf. [Va1, Prop. 3.4.5.1]). Thus the part of the proof of Theorem 4.3 that involves t can be worked out only in terms of t^{2^s} and thus of the map $q_1 : (G_1, X_1) \to (G_2, X_2)$ and not of the map $q'_1 : (G'_1, X'_1) \to (G, X)$.

4.4. Corollary. *Let $q_3 : (G_3, X_3, H_3) \to (G, X, H)$ be a map of Shimura triples* *that induces an isomorphism* $(G_3^{\mathrm{ad}}, X_3^{\mathrm{ad}}, H_3^{\mathrm{ad}}) \xrightarrow{\sim} (G, X, H)$*. Then the normalization* $\mathcal{N}_3$ *of* $\mathcal{N}$ *in the ring of fractions of* $\mathrm{Sh}(G_3, X_3)/H_3$ *is a pro-étale cover of an open closed subscheme of* $\mathcal{N}$*. Moreover,* $\mathcal{N}_3$ *together with the natural continuous action of* $G_3(\mathbb{A}_f^{(p)})$ *on it, is the integral canonical model of* (G_3, X_3, H_3) *and it is quasi-projective. If the integral canonical model* $\mathcal{N}_1$ *of Subsection 4.1 is projective, then* $\mathcal{N}_3$ *is projective too.*

Proof. We consider the fibre product (cf. [Va1, Subsect. 2.4 and Rm. 3.2.7 3)])

$$\begin{array}{ccc} (G'_3, X'_3, H'_3) & \xrightarrow{s_1} & (G'_1, X'_1, H'_1) \\ {\scriptstyle s_3}\downarrow & & \downarrow{\scriptstyle q'_1} \\ (G_3, X_3, H_3) & \xrightarrow{q_3} & (G, X, H). \end{array}$$

As G_1^{der} is simply connected (cf. proof of Proposition 3.2), we have $G_3'^{,\mathrm{der}} = G_1'^{,\mathrm{der}} = G_1^{\mathrm{der}}$. Thus by applying Proposition 2.4.3 (a) and (b) to (G'_3, X'_3, H'_3) and (G'_1, X'_1, H'_1), we get that the normalization $\mathcal{N}'_3$ of $\mathcal{N}'_1$ in the ring of fractions of $\mathrm{Sh}(G'_3, X'_3)/H'_3$ together with the natural continuous action of $G'_3(\mathbb{A}_f^{(p)})$ on it, is the integral canonical model of (G'_3, X'_3, H'_3). We consider an arbitrary $\mathbb{Z}_{(p)}$-embedding $E(G_3, X_3)_{(p)} \hookrightarrow W(\mathbb{F})$.

We can identify each connected component $\mathcal{C}'_3$ of $\mathcal{N}'_{3,W(\mathbb{F})}$ with a connected component $\mathcal{C}'_1$ of $\mathcal{N}'_{1,W(\mathbb{F})}$, cf. Proposition 2.4.3 (c). Let $\mathcal{C}_3$ and $\mathcal{C}$ be the connected components of $\mathcal{N}_{3,W(\mathbb{F})}$ and $\mathcal{N}_{W(\mathbb{F})}$ (respectively) dominated by $\mathcal{C}'_3$. The composite morphism $\mathcal{C}'_3 = \mathcal{C}'_1 \to \mathcal{C}_3 \to \mathcal{C}$ of pro-finite covers, is a pro-étale cover (cf. Theorem 4.3). Thus $\mathcal{C}_3$ is a pro-étale cover of $\mathcal{C}$. As q'_1 is a cover, s_3 is also a cover. Therefore the homomorphism $s_3(\mathbb{A}_f^{(p)}) : G'_3(\mathbb{A}_f^{(p)}) \to G_3(\mathbb{A}_f^{(p)})$ is onto. As the connected components of $\mathcal{N}_{3,W(\mathbb{F})}$ are permuted transitively by $G_3(\mathbb{A}_f^{(p)})$ (cf. [Va1, Lem. 3.3.2]), by using $G'_3(\mathbb{A}_f^{(p)})$-translates of $\mathcal{C}'_3$ (and thus $G'_3(\mathbb{A}_f^{(p)})$-translates of $\mathcal{C}_3$) we get that $\mathcal{N}_{3,W(\mathbb{F})}$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}_{W(\mathbb{F})}$. Thus $\mathcal{N}_3$ is a pro-étale cover of an open closed subscheme of $\mathcal{N}$. As $\mathcal{N}$ has the extension property (cf. Theorem 4.3), each closed subscheme of it which is flat over $E(G, X)_{(p)}$ has also the extension property. From the last two sentences we get that the $E(G_3, X_3)_{(p)}$-scheme $\mathcal{N}_3$ has the extension property, cf. [Va1, Rm. 3.2.3.1 6)].

It is easy to see that there exists a compact, open subgroup $H_{3,p}$ of $G_3(\mathbb{A}_f^{(p)})$ such that the morphism $\mathcal{N}_3 \to \mathcal{N}_3/H_{3,p}$ is a pro-étale cover. As $\mathcal{N}'_1$ is quasi-projective, we easily get that $\mathcal{N}_3/H_{3,p}$ is a smooth, quasi-projective $E(G_3, X_3)_{(p)}$-scheme. Thus $\mathcal{N}_3$ is the integral canonical model of (G_3, X_3, H_3) and it is quasi-projective.

If $\mathcal{N}_1$ is projective, then $\mathcal{N}'_1$ is projective (cf. Subsection 4.2) and thus $\mathcal{N}$ is projective (cf. Theorem 4.3); this implies that $\mathcal{N}_3$ is projective. □

5. The proof of the Basic Theorem, part II. We have the following stronger form of the Basic Theorem 1.3:

5.1. Basic Theorem. *Let* (G, X, H) *be a Shimura triple with respect to* p *such that the Shimura pair* (G, X) *is unitary. Then the following two properties hold:*

(a) *The integral canonical model* $\mathcal{N}$ *(resp.* $\mathcal{N}^{\mathrm{ad}}$*) of the Shimura triple* (G, X, H)

(resp. $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}})$*) over* $E(G,X)_{(p)}$ *(resp. over* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{(p)}$*) exists and it is quasi-projective.*

(b) *The* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$*-morphism* $\mathrm{Sh}(G,X)/H \to \mathrm{Sh}(G^{\mathrm{ad}}, X^{\mathrm{ad}})/H^{\mathrm{ad}}$ *extends uniquely to an* $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{(p)}$*-morphism* $m : \mathcal{N} \to \mathcal{N}^{\mathrm{ad}}$ *that is a pro-étale cover of its image.*

(c) *If each simple factor of* $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ *is of strong compact type, then* $\mathcal{N}$ *is projective.*

Proof. Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}) = \prod_{i\in\mathcal{J}}(G^i, X^i, H^i)$ be the product decomposition into simple factors. Let $\mathcal{N}^i$ be the integral canonical model of (G^i, X^i, H^i) over $E(G^i, X^i)_{(p)}$; it is quasi-projective (cf. Corollary 4.4). We consider the product $\mathcal{N}^{\mathrm{ad}} := \prod_{i\in\mathcal{J}} \mathcal{N}^i_{E(G^{\mathrm{ad}},X^{\mathrm{ad}})_{(p)}}$ of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{(p)}$-schemes. Let $\mathcal{N}$ be the normalization of $\mathcal{N}^{\mathrm{ad}}$ in the ring of fractions of $\mathrm{Sh}(G,X)/H$.

We check that the natural $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{(p)}$-morphism $m : \mathcal{N} \to \mathcal{N}^{\mathrm{ad}}$ is a pro-étale cover of its image. Let $q_4 : (G_4, X_4, H_4) \to (G, X, H)$ be a cover such that at the level of reflex fields we have $E(G_4, X_4) = E(G,X)$ and the semisimple group G_4^{der} is simply connected, cf. [Va1, Rm. 3.2.7 10)]. Similarly we consider a cover $q_4^i : (G_4^i, X_4^i, H_4^i) \to (G^i, X^i, H^i)$ such that at the level of reflex fields we have $E(G_4^i, X_4^i) = E(G^i, X^i)$ and the semisimple group $G_4^{i,\mathrm{der}}$ is simply connected. The morphisms $\mathrm{Sh}(G_4, X_4)/H_4 \to \mathrm{Sh}(G,X)/H$ and $\mathrm{Sh}(G_4^i, X_4^i)/H_4^i \to \mathrm{Sh}(G^i, X^i)/H^i$ are pro-étale covers, cf. Lemma 2.4.1. In particular, we get that to check that m is a pro-étale cover of its image, we can assume G^{der} is simply connected. Let $(G_5, X_5, H_5) := \prod_{i\in\mathcal{J}}(G_4^i, X_4^i, H_4^i)$. We have $(G_5^{\mathrm{ad}}, X_5^{\mathrm{ad}}, H_5^{\mathrm{ad}}) = (G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}})$ and $G_5^{\mathrm{der}} = G^{\mathrm{der}}$. Based on Proposition 2.4.3 (a) and (c), we can also assume that we have $(G_5, X_5, H_5) = (G, X, H)$. Thus to check that m is a pro-étale cover, we can assume that $\mathcal{J}$ has one element (i.e., that G^{ad} is a simple, adjoint group over $\mathbb{Q}$) and this case follows from Corollary 4.4.

As in the end of the proof of Corollary 4.4 we argue that $\mathcal{N}$ is the integral canonical model of (G,X,H) and it is quasi-projective. See Subsubsection 2.4.2 for the uniqueness of m. Thus (a) and (b) hold. Based on (b), to check (c) we can assume G is a simple, adjoint group. But this case follows from Corollary 4.1.1 (b) and Theorem 4.3. □

Appendix: Errata to [Va1]. We now include errata to [Va1].

E.1. On [Va1, Prop. 3.1.2.1 c)]. Gopal Prasad pointed out to us that the result [Va1, Prop. 3.1.2.1 c)] and its proof are partially wrong for the prime $p = 2$; see [Va3, Thm. 1.1] for a correction of this. It is easy to see that [Va3, Rm. 3.4 (b)] implies that [Va1, Lem. 3.1.6] remains true even if $p = 2$.

E.2. On [Va1, Subsubsect. 3.2.17]. Faltings argument reproduced in the last paragraph of [Va1, Subsubsect. 3.2.17, Step B] is incorrect. A correct argument that implicitly validates all of [Va1, Subsubsect. 3.2.17] is presented in [Va2, Prop. 4.1 or Rm. 4.2]. Loc. cit. proves a more general result: every p-healthy regular scheme is also healthy regular.

E.3. On [Va1, Thm. 6.2.2]. Paragraphs [Va1, proof of Thm. 6.2.2, F) to H)] are wrong. But the only cases of [Va1, Thm. 6.2.2 b)] that can not be easily reduced based on [Va1, Lem. 6.2.3 and Rm. 3.2.7 11)] to [Va1, Thm. 6.2.2 a)], are the ones that involve simple, adjoint, unitary Shimura pairs of A_n type and an odd prime p

which divides $n+1$. Thus the case $p>2$ of Corollary 4.4 indirectly completes the proof of [Va1, Thm. 6.2.2].

E.4. On [Va1, Subsubsubsect. 6.6.5.1]. The third paragraph of [Va1, p. 512] referring to PEL type embeddings is incorrect. It is corrected by Proposition 3.2.

E.5. On [Va1, Subsubsect. 6.4.11]. Remark [Va1, Rm. 6.4.1.1 2)] is incorrect. This invalidates [Va1, Subsubsect. 6.4.11 and Cor. 6.8.3]. However, [Va1, Subsubsect. 6.4.11 and Cor. 6.8.3] hold for Shimura pairs (G,X) with the property that each simple factor (G_0,X_0) of $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is either unitary of strong compact type (cf. Theorem 5.1 (c)) or such that $G_{0,\mathbb{R}}$ has simple, compact factors (cf. [Va4, Cor. 4.3 and Rm. 4.6 (b)]).

E.6. On [Va1, Example 4.3.11]. In [Va1, Example 4.3.11], as we worked with the natural trace form on the Lie algebra $\mathfrak{gsp}$ of a **GSp** group scheme, the condition p does not divide the rank r_L of L must be added in order to have this form perfect (i.e., in order that the fourth paragraph of [Va1, p. 469] applies). If p is odd and divides r_L, then, provided we work with **Sp** group schemes instead of **GSp** group schemes, [Va1, Example 4.3.11] applies entirely (cf. [Va1, Lem. 3.1.6]). Thus the application [Va1, Example 5.6.3] of [Va1, Example 4.3.11] does not require modifications, as it pertains to primes $p\geq 3$.

E.7. On [Va1, Rm. 4.3.6 3)]. The condition of [Va1, Rm. 4.3.6 3)] on the existence of Lie does not suffice; the reason is: it misses data required to relate ${H'_1}_{R/I_1+I_2}$ with ${H'_2}_{R/I_1+I_2}$. Loc. cit. was thought to take aside part of the argument of [Va1, Rm. 4.3.7 4)]; thus in loc. cit. we had in mind the context of [Va1, Rm. 4.3.7 4)]. It turns out that [Va1, Rms. 4.3.7 4) and 5)] require also extra assumptions. The impact of this to [Va1, Prop. 4.3.10] is: the sentence between parentheses which contains "it is instructive not to do so" and which was used before [Va1, proof of Prop. 4.3.10, Case a)], has to be deleted.

E.8. On [Va1, p. 496, Lem.]. A great part of the proof of [Va1, p. 496, Lem.] is wrong and in fact there exist counterexamples to [Va1, p. 496, Lem.] (for instance, with H_O of A_{p-1} Lie type). The error in [Va1, p. 496, Lem.] implies that corrections are required for [Va1, Prop. 6.2.2.1, Cor. 6.2.4.1, and Lem. 6.4.5] as well. The simplest way to correct these subsubsections is to eliminate all primes $p\in\mathbb{N}$ that are not greater than $1+\max\{\dim(G_0)|(G_0,X_0)$ is a simple factor of $(G^{\mathrm{ad}},X^{\mathrm{ad}})\}$ (as [Va1, p. 496, Lem.] has no content if $H_O(O)$ has no element of order p and thus if $p>\dim(H_O)$).

For minute and very general corrections to E.7 and E.8 we refer to math.NT/0307 098 (to be published later). The corrections E.2 to E.8 are incorporated in math.NT/03 07098.

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REFERENCES

[BB] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2), 84 (1966), pp. 442–528.

[BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 21, Springer-Verlag, Berlin, 1990.

[De1] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, pp. 123–165, Lecture Notes in Math., Vol. 244, Springer, Berlin, 1971.

[De2] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, OR, 1977), Part 2, pp. 247–289, Proc. Sympos. Pure Math., 33, Amer. Math. Soc., Providence, RI, 1979.

[DG] M. Demazure, A. Grothendieck, et al., *Schémas en groupes*, Vol. I–III, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Lecture Notes in Math., Vol. 151–153, Springer-Verlag, Berlin-New York, 1970.

[Fo] J.-M. Fontaine, *Groupes p-divisibles sur les corps locaux*, J. Astérisque, Soc. Math. de France, Paris (1977), pp. 47–48.

[FC] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Math. und ihrer Grenzgebiete (3), 22, Springer-Verlag, Heidelberg, 1990.

[Gr] A. Grothendieck et al., *Cohomologie locale des faisceau cohérents et théorèmes de Lefschetz locaux et globaux*, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2. North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968.

[He] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, Vol. 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[Ko] R. E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. of Amer. Math. Soc., 5 (1992), pp. 373–444.

[LR] R. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. reine angew. Math., 378 (1987), pp. 113–220.

[Me] W. Messing, *The crystals associated to Barsotti-Tate groups with applications to abelian schemes*, Lecture Notes in Math., Vol. 264, Springer-Verlag, Berlin-New York, 1972.

[Mi1] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic Forms, Shimura varieties and L-functions, Vol. I (Ann Arbor, MI, 1988), pp. 283–414, Perspectives in Math., Vol. 10, Academic Press, Inc., Boston, MA, 1990.

[Mi2] J. S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The Zeta functions of Picard modular surfaces, pp. 153–255, Univ. Montréal, Montreal, Quebec, 1992.

[Mi3] J. S. Milne, *Shimura varieties and motives*, Motives (Seattle, WA, 1991), Part 2, pp. 447–523, Proc. Sympos. Pure Math., Vol. 55, Amer. Math. Soc., Providence, RI, 1994.

[Mi4] J. S. Milne, *Descent for Shimura varieties*, Mich. Math. J., 46 (1999), pp. 203–208.

[Mo1] Y. Morita, *On potential good reduction of abelian varieties*, J. Fac. Sci. Univ. Tokyo Sect. I A Math., 22 (1975), pp. 437–447.

[Mo2] Y. Morita, *Reduction mod $\mathfrak{B}$ of Shimura curves*, Hokkaido Math. J., 10 (1981), pp. 209–238.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory. Third enlarged edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34, Springer-Verlag, Berlin, 1994.

[Sa1] I. Satake, *Holomorphic imbeddings of symmetric domains into a Siegel space*, Amer. J. Math., 87 (1965), pp. 425–461.

[Sa2] I. Satake, *Symplectic representations of algebraic groups satisfying a certain analyticity condition*, Acta Math., 117 (1967), pp. 215–279.

[Se] S. Sen, *Lie algebras of Galois groups arising from Hodge-Tate modules*, Ann. of Math. (2), 97 (1973), pp. 160–170.

[Ti] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, OR, 1977), Part 1, pp. 29–69, Proc. Sympos. Pure Math., Vol. 33, Amer. Math. Soc., Providence, RI, 1979.

[Va1] A. Vasiu, *Integral canonical models for Shimura varieties of preabelian type*, Asian J. Math., 3 (1999), pp. 401–518.

[Va2] A. Vasiu, *A purity theorem for abelian schemes*, Mich. Math. J., 54 (2004), pp. 71–81.

[Va3] A. Vasiu, *On two theorems for flat, affine group schemes over a discrete valuation ring*, Centr. Eur. J. Math., 3 (2005), pp. 14–25.

[Va4] A. Vasiu, *Projective integral models of Shimura varieties of Hodge type with compact factors*, J. reine angew. Math., 518 (2008), 25 pages.

[Zi] T. Zink, *Isogenieklassen von Punkten von Shimuramannigfaltigkeiten mit Werten in einem endlichen Körper*, Math. Nachr., 112 (1983), pp. 103–124.

[Wi] J.-P. Wintenberger, *Un scindage de la filtration de Hodge pour certaines varietés algébriques sur les corps locaux*, Ann. of Math. (2), 119 (1984), pp. 511–548.