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A CHARACTERIZATION OF QUADRIC CONSTANT GAUSS-KRONECKER CURVATURE HYPERSURFACES OF SPHERES*

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Abstract. Let $M \subset S^{n+1}$ be a complete orientable hypersurface with constant Gauss-Kronecker curvature G. For any $v \in \mathbf{R}^{n+2}$, let us define the following two real functions $l_v, f_v : M \to \mathbf{R}$ on M by $l_v(x) = \langle x, v\rangle$ and $f_v(x) = \langle \nu(x), v\rangle$ with $\nu : M \to S^{n+1}$ a Gauss map of M. In this paper, we show that if $n = 3$, $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^5$ and some real number λ, then M is either totally umbilical (a Euclidean sphere) or M is a cartesian product of Euclidean spheres. We will also show with an example that the completeness condition is needed in the result we just mentioned. We also show that if $n = 4$, $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^6$ and some real number λ and $(\lambda^2 - 1)^2 + (G - 1)^2 \neq 0$, then M is either totally umbilical (a Euclidean sphere) or M is a cartesian product of Euclidean spheres. Moreover, we will give an example of a complete hypersurface in S^5 with constant Gauss-Kronecker curvature that satisfies the condition $l_v = \lambda f_v$ for some non zero v, which is neither a totally umbilical hypersurface nor a cartesian product of Euclidean spheres.

Key words. Clifford hypersurfaces, Gauss-Kronecker curvature, spheres.

AMS subject classifications. 53C42, 53A10.

1. Introduction. Let $M \subset S^{n+1}$ be an orientable hypersurface. For any $v \in \mathbf{R}^{n+2}$, let us define the following two real functions $l_v, f_v : M \to \mathbf{R}$ on M by

$$l_v(x) = \langle x, v\rangle, \quad f_v(x) = \langle \nu(x), v\rangle$$

where $\nu : M \to S^{n+1}$ is a Gauss map of M. These two family of functions $\{l_v\}_{v\in\mathbf{R}^{n+2}}$ and $\{f_v\}_{v\in\mathbf{R}^{n+2}}$ are very important because they can be defined for *any* oriented immersion on the sphere and they have been widely used to construct test functions to study the spectrum of operators on M such as the Laplace operator and the stability operator when M is minimal, M has constant mean curvature or M has constant scalar curvature. See for example [11], [12], [13], [14], [9], [6] and [4].

In [1], Alías, Brasil and the first author proved that if M is a *complete* hypersurface with constant mean curvature and $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^{n+2}$ and some real number λ, then M is either totally umbilical (a Euclidean sphere) or M is a cartesian product of Euclidean spheres. Thit is, M must be a isoparametric hypersurface of order 1 or 2. We can modify the arguments presented in [1] to show that the completeness was not needed and therefore there are not non-isoparametric locally defined hypersurfaces of S^{n+1} with constant mean curvature that satisfy the condition $l_v = \lambda f_v$. Also in [1] an example of a complete non-isoparametric hypersurface that satisfies $f_v = \lambda l_v$ was presented. After this result, it looked like to fulfill only one of the conditions was easy but only the isoparametric hypersurfaces of order 1 and 2 were the only ones that satisfy an algebraic equation for the principal curvatures and the additional condition $f_v = \lambda l_v$ for some nonzero vector v. In [10], the authors were able to show that, as expected, the conditions constant scalar curvature plus

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$f_v = \lambda l_v$ for some nonzero vector on a *complete* oriented hypersurface of S^4 imply that the hypersurface must be isoparametric of order 1 or 2. The authors proved that the completeness condition was needed this time by showing a non isoparametric hypersurface in S^4 with constant scalar curvature that satisfies the condition $f_v = \lambda l_v$ for some nonzero vector v. Recall that these non complete examples do not exist if we replace the condition, constant scalar curvature by the condition constant mean curvature.

In [3], Cheng, Li and the second author proved that if M has constant Gauss-Kronecker $G = c \neq \pm 1$, $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^{n+2}$ and some real number λ, then M is either totally umbilical (a Euclidean sphere) or M is a cartesian product of Euclidean spheres. A natural question to ask regarding this theorem is if the additional condition $G \neq \pm 1$ is needed? In this paper we show that for hypersurfaces of S^4, the condition $G \neq \pm 1$ is not needed, it is just a technical assumption and, surprisingly, for hypersurfaces in S^5 the condition $G \neq \pm 1$ is needed. More precisely, in this paper we show that the only *complete* hypersurfaces of S^4 with constant Gauss-Kronecker curvature that satisfy $f_v = \lambda l_v$ for some nonzero vector v are the isopametric hypersurfaces with order 1 and 2, and we show an example of a *complete* hypersurface in S^5 with constant Gauss-Kronecker curvature that is not isoparametric and satisfies that $f_v = \lambda l_v$ for some nonzero vector v. We also improve the result in [3] by showing that if a complete non-isoparametric hypersurface in S^5 has constant Gauss-Kronecker G and $f_v = \lambda l_v$ for some nonzero vector v, then G and $|\lambda|$ must be 1.

THEOREM 1.1. *Let $M \subset S^4$ be a complete orientable hypersurface with constant Gauss-Kronecker curvature G. If $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^5$ and some real number λ, then M is either totally umbilical (a Euclidean sphere) or M is a cartesian product of Euclidean spheres.*

THEOREM 1.2. *Let $M \subset S^5$ be a complete orientable hypersurface with constant Gauss-Kronecker curvature G. If $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^6$ and some real number λ, $(\lambda^2 - 1)^2 + (G-1)^2 \neq 0$, then M is either totally umbilical (a Euclidean sphere) or M is a cartesian product of Euclidean spheres. Moreover there exists a complete orientable hypersurface which has constant Gauss-Kronecker curvature , $l_v = f_v$ for some nonzero vector $v \in \mathbf{R}^6$ and which is neither a totally umbilical hypersurface nor a cartesian product of Euclidean spheres. Due to the first part of this theorem this example satisfies $(\lambda^2 - 1)^2 + (G-1)^2 = 0$.*

2. Proofs of Theorems. Without loss of generality, we will assume that M is not totally umbilical. For any fixed vector $v \in \mathbf{R}^{n+2}$, $v^\top : M \to \mathbf{R}^{n+2}$ defined by

$$v^\top(x) = v - l_v(x)x - f_v(x)\nu(x), \quad for\ all\ x \in M$$

is a tangent vector field on M because $\langle v^\top(x), x\rangle = 0$ and $\langle v^\top(x), \nu(x)\rangle = 0$ for every point $x \in M$. By multiplying the equation $l_v = \lambda f_v$ by an appropriated constant, we may assume that $|v| = 1$. In this proof, for any $c \in (-1, 1)$ we will denote by $S^n(c)$ the Euclidean sphere with radius $\sqrt{1-c^2}$ given by $\{x \in S^{n+1} : \langle x, v\rangle = c\}$. We will also assume that l_v is not constant. Otherwise $M \subset S^n(c)$ for some c, and according to the completeness of M, we have $M = S^n(c)$, that is, M is totally umbilical.

Since l_v is not constant, then $\lambda \neq 0$. By changing the Gauss map ν by $-\nu$ if necessary, we will assume that $\lambda > 0$. From [1], we know that

- $N = S^n(0) \cap M$ is a non-empty complete surface of the unit sphere $S^n(0)$. Moreover, viewed as a surface in M, it is totally geodesic.
- For any $x \in N$, $v^\top = v$, therefore v is a unit vector in T_xM.
- For any $x \in N$, if $\beta_x(s)$ is the geodesic in M that satisfies $\beta_x(0) = x$ and $\beta'_x(0) = v$ then, up to a reparametrization, β is an integral curve of $v^\top$.
- For any $x \in N$, we have that if $\lambda_1, \dots, \lambda_{n-1}$ are the principal curvatures of N at x as a hyper surface of $S^n(0)$ and $w = \sqrt{1+\lambda^{-2}}$, then, the principal curvatures of M along $\beta_x(s)$ are

$$\tilde{\lambda}_i(\beta_x(s)) = -\frac{1}{\lambda} + \frac{(1+\lambda^2)(\lambda^{-1}+\lambda_i(x))}{\lambda(\lambda-\lambda_i(x))\cos(ws)+(1+\lambda\lambda_i(x))}, \quad i = 1, 2, \dots, n-1.$$

$$\tilde{\lambda}_n(\beta_x(s)) = -\frac{1}{\lambda}.$$

Let us recall that, since $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, …, $\tilde{\lambda}_n$ are the principal curvatures of M, then we have that the Gauss-Kronecker curvature G satisfies that

$$G = \tilde{\lambda}_1\tilde{\lambda}_2\cdots\tilde{\lambda}_n = \prod_{i=1}^{n}\tilde{\lambda}_i.$$

A direct verification shows that if $\lambda_i(x) = \lambda$ for some $x \in N$, then $\tilde{\lambda}_i(\beta_x(s)) = \lambda$ for all s. Likewise, if $\lambda_i(x) = -\lambda^{-1}$ for some $x \in N$ then, $\tilde{\lambda}_i(\beta_x(s)) = -\lambda^{-1}$ for all s. Therefore, if for all $x \in N$, we have that $\{\lambda_1(x), \dots, \lambda_{n-1}(x)\} \subset \{\lambda, -\lambda^{-1}\}$, we will get that M has either one constant principal curvature equal to $-\frac{1}{\lambda}$ or M has two constant principal curvatures λ and $-\frac{1}{\lambda}$. Then, using a similar argument as in [1] and [3], we have that in the first case, M is a Euclidean sphere and in the second case we get that M is the cartesian product of Euclidean spheres.

Firstly, we will prove Theorem 1.1.

Proof. From the previous paragraph, we have that in order to prove Theorem 1.1, it is enough to show that $\lambda_i(x) \in \{\lambda, -\lambda^{-1}\}$ for all $x \in N$ and $i = 1, 2$.
For any $x \in N$ such that $\lambda_i(x) \notin \{\lambda, -\lambda^{-1}\}$ and let us define

$$a_i(x) = \lambda^{-1} + \lambda_i(x), \quad b_i(x) = \lambda(\lambda - \lambda_i(x)),$$

then,

$$a_i(x) \neq 0, \quad b_i(x) \neq 0.$$

The following sequence of statements will lead us to the conclusion that $\lambda_i(x) \in \{\lambda, -\lambda^{-1}\}$ for all $x \in N$.

Claim 1. *For any $x \in N$, it is impossible to have $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2(x) = -\lambda^{-1}$.*

If this happens, we have that for $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, we infer

$$G(\beta_x(s)) = \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s)) = -\frac{1}{\lambda^3} + \frac{1+\lambda^2}{\lambda^2}\frac{a_1(x)}{b_1(x)\cos(ws)+\lambda a_1(x)}.$$

This means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$\left\{G + \frac{1}{\lambda^3}\right\}(b_1(x)X + \lambda a_1(x)) - \frac{1+\lambda^2}{\lambda^2}a_1(x) = 0.$$

Since the above polynomial equation should only have one root, we derive that the coefficient of X is zero, and, since $b_1(x) \neq 0$, we obtain,

$$G = -\frac{1}{\lambda^3},$$

then $\frac{1+\lambda^2}{\lambda^2} a_1(x) = 0$, it follows that $a_1(x) = 0$. This contradiction implies the claim.

Claim 2. *For any $x \in N$, it is impossible to have $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2(x) = \lambda$.*

Under these assumptions, we have

$$G(\beta_x(s)) = \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s)) = \frac{1}{\lambda} - (1+\lambda^2)\frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}.$$

Using the same arguments as in the proof of Claim 1, we obtain

$$G = \frac{1}{\lambda}, \quad a_1(x) = 0.$$

This contradiction implies the claim.

Claim 3. *If for some $x \in N$, $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2 \notin \{\lambda, -\lambda^{-1}\}$, then $G = -1$, $\lambda = 1$ and $1 - \lambda_1(x)\lambda_2(x) = 0$.*

Under these assumptions, we have

$$\begin{aligned}
G(\beta_x(s)) &= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s)) \\
&= -\frac{1}{\lambda}\left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}\right)\left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_2(x)}{b_2(x)\cos(ws) + \lambda a_2(x)}\right) \\
&= -\frac{1}{\lambda^3} + \frac{(1+\lambda^2)}{\lambda^2}\left(\frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)} + \frac{a_2(x)}{b_2(x)\cos(ws) + \lambda a_2(x)}\right) \\
&\quad - \frac{(1+\lambda^2)^2}{\lambda}\frac{a_1(x)a_2(x)}{(b_1(x)\cos(ws) + \lambda a_1(x))(b_2(x)\cos(ws) + \lambda a_2(x))}.
\end{aligned}$$

This means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$\begin{aligned}
&\left(G + \frac{1}{\lambda^3}\right)(b_1(x)X + \lambda a_1(x))(b_2(x)X + \lambda a_2(x)) \\
&- \frac{(1+\lambda^2)}{\lambda^2}\{a_1(x)(b_2(x)X + \lambda a_2(x)) + a_2(x)(b_1(x)X + \lambda a_1(x))\} \\
&+ \frac{(1+\lambda^2)^2}{\lambda}a_1(x)a_2(x) = 0.
\end{aligned}$$

Since the polynomial equation should only have finite roots, we derive that the coefficients of X^q are zero for any integer $q \in \{0, 1, 2\}$, that is,

$$G = -\frac{1}{\lambda^3}, \quad a_1(x)b_2(x) + a_2(x)b_1(x) = 0, \quad (1-\lambda^2)a_1(x)a_2(x) = 0.$$

Since $a_1(x)a_2(x) \neq 0$, then we get that $\lambda^2 = 1$ and since we are assuming that $\lambda > 0$, we get that $\lambda = 1$. Since $G = -\frac{1}{\lambda^3}$, then we get that $G = -1$. The last equation in Claim 3 follows from these equalities.

$$a_1(x)b_2(x) + a_2(x)b_1(x) = (\lambda^2 - 1)(\lambda_1 + \lambda_2) + 2\lambda(1 - \lambda_1\lambda_2) = 2(1 - \lambda_1\lambda_2) = 0.$$

Claim 4. *For any $x \in N$, it is impossible to have $\lambda_1(x) \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2(x) \notin \{\lambda, -\lambda^{-1}\}$.*

Let us argue by contradiction and let us assume that $\lambda_1(x_0) \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2(x_0) \notin \{\lambda, -\lambda^{-1}\}$. By Claim 3 we must have that the constant value of the scalar curvature must be $G = -1$; moreover, λ must be equal to 1 and, the values $\lambda_1(x_0)$ and $\lambda_2(x_0)$ must satisfy the equation

$$1 - \lambda_1(x)\lambda_2(x) = 0. \tag{2.1}$$

Since $\lambda = 1$, for any $x \in N$ such that $\lambda_i(x) \notin \{\lambda, -\lambda^{-1}\}$, we have that

$$\tilde{\lambda}_i(\beta_x(s)) = -1 + \frac{2(1+\lambda_i(x))}{(1-\lambda_i(x))\cos(ws) + (1+\lambda_i(x))}.$$

Using the fact that $\frac{1+\lambda_i(x)}{1-\lambda_i(x)}$ is between -1 and 1 when $\lambda_i(x) \leq 0$, we conclude that $\lambda_i(x) > 0$; otherwise, for some value of s the expression $\tilde{\lambda}_i(\beta_x(s))$ would blow up. Let us show for all $x \in N$, $\lambda_1(x)$ and $\lambda_2(x)$ are positive and $\lambda_1(x)\lambda_2(x) = 1$. Let $x \in N$ be any point in N. Since N is complete there is a geodesic γ in N such that $\gamma(0) = x_0$ and $\gamma(t_0) = x$. By claim 2 we have that, if for some t, $\lambda_1(\gamma(t)) = 1$ then $\lambda_2(\gamma(t)) = 1$. By continuity it follows that $\lambda_1(x)$ and $\lambda_2(x)$ are positive and $\lambda_1(x)\lambda_2(x) = 1$. From the previous arguments, we have that the principal curvatures of N as a surface of the unit three dimensional sphere are greater than 0 and they satisfy equation 2.1, hence the sectional curvature K of N is $K = 1 + \lambda_1\lambda_2 = 2$, then N is compact from Myers's theorem. From the result of Cheng-Yau (see Theorem 2 of [5], Page 200), we obtain that N is a totally umbilical sphere. This is a contradiction with the fact that $\lambda_1(x_0) \notin \{\lambda, -\lambda^{-1}\} = \{1, -1\}$, because the only positive value $t = \lambda_1 = \lambda_2$ that satisfies equation 2.1 is $t = 1$.

From Claims 1, 2 and 4 we conclude that the principal curvatures on N are constant and contained in the set $\{\lambda, -\lambda^{-1}\}$. As pointed out before, we can infer from this fact that the principal curvature on M are constant and contained in the set $\{\lambda, -\lambda^{-1}\}$ and therefore Theorem 1.1 follows. □

Secondly, we will prove Theorem 1.2.

Proof. For any $x \in N$ such that $\lambda_i(x) \notin \{\lambda, -\lambda^{-1}\}$ and let us define

$$a_i(x) = \lambda^{-1} + \lambda_i(x), \quad b_i(x) = \lambda(\lambda - \lambda_i(x)),$$

then,

$$a_i(x) \neq 0, \quad b_i(x) \neq 0.$$

Claim 1. *For any $x \in N$, it is impossible to have $\lambda_1(x) \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2(x) = \lambda_3(x) = -\lambda^{-1}$.*

If this happens, we have that for $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, we infer

$$\begin{aligned} G(\beta_x(s)) &= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s))\tilde{\lambda}_4(\beta_x(s)) \\ &= \frac{1}{\lambda^4} - \frac{1+\lambda^2}{\lambda^3}\frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}. \end{aligned}$$

This means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$\left\{G - \frac{1}{\lambda^4}\right\}(b_1(x)X + \lambda a_1(x)) + \frac{1+\lambda^2}{\lambda^3} a_1(x) = 0.$$

Since the above polynomial equation should only have one root, we derive that the coefficient of X is zero, and, since $b_1(x) \neq 0$, we obtain,

$$G = \frac{1}{\lambda^4},$$

then $\frac{1+\lambda^2}{\lambda^3} a_1(x) = 0$, it follows that $a_1(x) = 0$. This contradiction implies the claim.

Claim 2. *For any $x \in N$, it is impossible to have $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$, $\lambda_2(x) = \lambda$ and $\lambda_3(x) = -\lambda^{-1}$.*

Under these assumptions, we have

$$\begin{aligned} G(\beta_x(s)) &= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s))\tilde{\lambda}_4(\beta_x(s)) \\ &= -\frac{1}{\lambda^2} + \frac{1+\lambda^2}{\lambda} \frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}. \end{aligned}$$

Using the same arguments as in the proof of Claim 1, we obtain

$$G = -\frac{1}{\lambda^2}, \quad a_1(x) = 0.$$

This contradiction implies the claim.

Claim 3. *For any $x \in N$, it is impossible to have $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_2(x) = \lambda_3(x) = \lambda$.*

Under these assumptions, we have

$$\begin{aligned} G(\beta_x(s)) &= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s))\tilde{\lambda}_4(\beta_x(s)) \\ &= 1 - \lambda(1+\lambda^2)\frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}. \end{aligned}$$

Using the same arguments as in the proof of Claim 1, we obtain

$$G = 1, \quad a_1(x) = 0.$$

This contradiction implies the claim.

Claim 4. *If for some $x \in N$, $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$, $\lambda_2 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_3 = -\lambda^{-1}$, then $G = 1$, $\lambda = 1$ and $1 - \lambda_1(x)\lambda_2(x) = 0$.*

Under these assumptions, we have

$$\begin{aligned} G(\beta_x(s)) &= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s))\tilde{\lambda}_4(\beta_x(s)) \\ &= \frac{1}{\lambda^2}\left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}\right)\left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_2(x)}{b_2(x)\cos(ws) + \lambda a_2(x)}\right) \\ &= \frac{1}{\lambda^4} - \frac{(1+\lambda^2)}{\lambda^3}\left(\frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)} + \frac{a_2(x)}{b_2(x)\cos(ws) + \lambda a_2(x)}\right) \\ &\quad + \frac{(1+\lambda^2)^2}{\lambda^2}\frac{a_1(x)a_2(x)}{(b_1(x)\cos(ws) + \lambda a_1(x))(b_2(x)\cos(ws) + \lambda a_2(x))}. \end{aligned}$$

This means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$\left(G - \frac{1}{\lambda^4}\right)(b_1(x)X + \lambda a_1(x))(b_2(x)X + \lambda a_2(x))$$
$$+ \frac{(1+\lambda^2)}{\lambda^3}\{a_1(x)(b_2(x)X + \lambda a_2(x)) + a_2(x)(b_1(x)X + \lambda a_1(x))\}$$
$$- \frac{(1+\lambda^2)^2}{\lambda^2}a_1(x)a_2(x) = 0.$$

Since the polynomial equation should only have finite roots, we derive that the coefficients of X^q are zero for any integer $q \in \{0, 1, 2\}$, that is,

$$G = \frac{1}{\lambda^4}, \quad a_1(x)b_2(x) + a_2(x)b_1(x) = 0, \quad (\lambda^2 - 1)a_1(x)a_2(x) = 0.$$

Since $a_1(x)a_2(x) \neq 0$, then we get that $\lambda^2 = 1$ and since we are assuming that $\lambda > 0$, we get that $\lambda = 1$. Since $G = \frac{1}{\lambda^4}$, then we get that $G = 1$. The last equation in Claim 4 follows from these equalities.

$$a_1(x)b_2(x) + a_2(x)b_1(x) = (\lambda^2 - 1)(\lambda_1 + \lambda_2) + 2\lambda(1 - \lambda_1\lambda_2) = 2(1 - \lambda_1\lambda_2) = 0.$$

Claim 5. *If for some $x \in N$, $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$, $\lambda_2 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_3 = \lambda$, then $G = -1$, $\lambda = 1$ and $1 - \lambda_1(x)\lambda_2(x) = 0$.*

Under these assumptions, we have

$$\begin{aligned} G(\beta_x(s)) &= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s))\tilde{\lambda}_4(\beta_x(s)) \\ &= -\left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)}\right)\left(-\frac{1}{\lambda} + \frac{(1+\lambda^2)a_2(x)}{b_2(x)\cos(ws) + \lambda a_2(x)}\right) \\ &= -\frac{1}{\lambda^2} + \frac{(1+\lambda^2)}{\lambda}\left(\frac{a_1(x)}{b_1(x)\cos(ws) + \lambda a_1(x)} + \frac{a_2(x)}{b_2(x)\cos(ws) + \lambda a_2(x)}\right) \\ &\quad -(1+\lambda^2)^2\frac{a_1(x)a_2(x)}{(b_1(x)\cos(ws) + \lambda a_1(x))(b_2(x)\cos(ws) + \lambda a_2(x))}. \end{aligned}$$

This means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$\left(G + \frac{1}{\lambda^2}\right)(b_1(x)X + \lambda a_1(x))(b_2(x)X + \lambda a_2(x))$$
$$- \frac{(1+\lambda^2)}{\lambda}\{a_1(x)(b_2(x)X + \lambda a_2(x)) + a_2(x)(b_1(x)X + \lambda a_1(x))\}$$
$$+ (1+\lambda^2)^2a_1(x)a_2(x) = 0.$$

Since the polynomial equation should only have finite roots, we derive that the coefficients of X^q are zero for any integer $q \in \{0, 1, 2\}$, that is,

$$G = -\frac{1}{\lambda^2}, \quad a_1(x)b_2(x) + a_2(x)b_1(x) = 0, \quad (1 - \lambda^2)a_1(x)a_2(x) = 0.$$

Since $a_1(x)a_2(x) \neq 0$, then we get that $\lambda^2 = 1$ and since we are assuming that $\lambda > 0$, we get that $\lambda = 1$. Since $G = -\frac{1}{\lambda^2}$, then we get that $G = -1$. The last equation in Claim 5 follows from these equalities.

$$a_1(x)b_2(x) + a_2(x)b_1(x) = (\lambda^2 - 1)(\lambda_1 + \lambda_2) + 2\lambda(1 - \lambda_1\lambda_2) = 2(1 - \lambda_1\lambda_2) = 0.$$

Claim 6. *For any $x \in N$, it is impossible to have $\lambda_1 \notin \{\lambda, -\lambda^{-1}\}$, $\lambda_2 \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_3 \notin \{\lambda, -\lambda^{-1}\}$.*

Under these assumptions, we have

$$
\begin{aligned}
&G(\beta_x(s))\\
&= \tilde{\lambda}_1(\beta_x(s))\tilde{\lambda}_2(\beta_x(s))\tilde{\lambda}_3(\beta_x(s))\tilde{\lambda}_4(\beta_x(s))\\
&= -\frac{1}{\lambda}\left(-\frac{1}{\lambda}+\frac{(1+\lambda^2)a_1(x)}{b_1(x)\cos(ws)+\lambda a_1(x)}\right)\left(-\frac{1}{\lambda}+\frac{(1+\lambda^2)a_2(x)}{b_2(x)\cos(ws)+\lambda a_2(x)}\right)\\
&\quad\times\left(-\frac{1}{\lambda}+\frac{(1+\lambda^2)a_3(x)}{b_3(x)\cos(ws)+\lambda a_3(x)}\right)\\
&= \frac{1}{\lambda^4}-\frac{(1+\lambda^2)}{\lambda^3}\left(\frac{a_1(x)}{b_1(x)\cos(ws)+\lambda a_1(x)}+\frac{a_2(x)}{b_2(x)\cos(ws)+\lambda a_2(x)}\right.\\
&\quad\left.+\frac{a_3(x)}{b_3(x)\cos(ws)+\lambda a_3(x)}\right)\\
&\quad+\frac{(1+\lambda^2)^2}{\lambda^2}\left\{\frac{a_1(x)a_2(x)}{(b_1(x)\cos(ws)+\lambda a_1(x))(b_2(x)\cos(ws)+\lambda a_2(x))}\right.\\
&\quad+\frac{a_1(x)a_3(x)}{(b_1(x)\cos(ws)+\lambda a_1(x))(b_3(x)\cos(ws)+\lambda a_3(x))}\\
&\quad\left.+\frac{a_2(x)a_3(x)}{(b_2(x)\cos(ws)+\lambda a_2(x))(b_3(x)\cos(ws)+\lambda a_3(x))}\right\}\\
&\quad-\frac{(1+\lambda^2)^3}{\lambda}\frac{a_1(x)a_2(x)a_3(x)}{(b_1(x)\cos(ws)+\lambda a_1(x))(b_2(x)\cos(ws)+\lambda a_2(x))(b_3(x)\cos(ws)+\lambda a_3(x))}.
\end{aligned}
$$

This means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the following polynomial equation on X:

$$
\begin{aligned}
&\left(G-\frac{1}{\lambda^4}\right)(b_1(x)X+\lambda a_1(x))(b_2(x)X+\lambda a_2(x))(b_3(x)X+\lambda a_3(x))\\
&\quad+\frac{(1+\lambda^2)}{\lambda^3}\Big\{a_1(x)(b_2(x)X+\lambda a_2(x))(b_3(x)X+\lambda a_3(x))+a_2(x)(b_1(x)X\\
&\quad+\lambda a_1(x))(b_3(x)X+\lambda a_3(x))+a_3(x)(b_1(x)X+\lambda a_1(x))(b_2(x)X+\lambda a_2(x))\Big\}\\
&\quad-\frac{(1+\lambda^2)^2}{\lambda^2}\Big\{a_1(x)a_2(x)(b_3(x)X+\lambda a_3(x))+a_1(x)a_3(x)(b_2(x)X+\lambda a_2(x))\\
&\quad+a_2(x)a_3(x)(b_1(x)X+\lambda a_1(x))\Big\}\\
&\quad+\frac{(1+\lambda^2)^3}{\lambda}a_1(x)a_2(x)a_3(x)=0.
\end{aligned}
$$

Since the polynomial equation should only have finite roots, we derive that the coefficients of X^q are zero for any integer $q \in \{0,1,2,3\}$, that is,

$$G=\frac{1}{\lambda^4},\quad a_1(x)b_2(x)b_3(x)+a_2(x)b_1(x)b_3(x)+a_3(x)b_1(x)b_2(x)=0,$$

$$(\lambda^2-1)(a_1(x)a_2(x)b_3(x)+a_1(x)b_2(x)a_3(x)+b_1(x)a_2(x)a_3(x))=0,$$

$$(-\lambda^4+\lambda^2-1)a_1(x)a_2(x)a_3(x)=0.$$

Since $-\lambda^4+\lambda^2-1<0$, then we get that $a_1(x)a_2(x)a_3(x)=0$. This contradiction implies the claim.

REMARK 2.1. From claims 1, 2, 3 and 6, it follows that at every point in N, either exactly two principal curvatures of N are not in the set $\{\lambda, -\frac{1}{\lambda}\}$ or all three principal curvatures are in the set $\{\lambda, -\frac{1}{\lambda}\}$.

CLAIM 7. *For any $x \in N$, it is impossible to have $\lambda_1(x) \notin \{\lambda, -\lambda^{-1}\}$, $\lambda_2(x) \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_3(x)=\lambda$.*

Let us argue by contradiction and let us assume that $\lambda_1(x_0) \notin \{\lambda, -\lambda^{-1}\}$, $\lambda_2(x_0) \notin \{\lambda, -\lambda^{-1}\}$ and $\lambda_3(x_0)=\lambda$. By Claim 5 we must have that the constant value of the Gauss-Kronecker curvature must be $G=-1$, and moreover, λ must be equal to 1 and, the values $\lambda_1(x_0)$ and $\lambda_2(x_0)$ must satisfy the equation

$$1-\lambda_1(x)\lambda_2(x)=0. \tag{2.2}$$

Using a similar argument as the one used in the proof of Claim 4 of Theorem 1.1, we obtain that the principal curvatures of N as a hypersurface of the unit four dimensional sphere are greater than 0 and they satisfy equation 2.2, hence the sectional curvature of N along each plane spanned by two principal directions is $R_{ijij}=1+\lambda_i\lambda_j>1$, we can prove that the sectional curvature is positive for all the planes. Therefore N is compact from Myers's theorem. A direct computation shows that equation 2.2 can be written as

$$r_N = H_N + 1,$$

where r_N is the normalized scalar curvature of N and H_N is the mean curvature of N. From the result of Li-Suh-Wei (see Theorem 1.3 of [8], Page 322), we obtain that N is a totally umbilical sphere. This is a contradiction with the fact that $\lambda_1(x_0) \notin \{\lambda, -\lambda^{-1}\} = \{1, -1\}$, because the only positive value $t=\lambda_1=\lambda_2$ that satisfies equation 2.2 is $t=1$.

From Claims, 1, 2, 3, 4, 6 and 7, we conclude that the principal curvatures on N are constant and contained in the set $\{\lambda, -\lambda^{-1}\}$ if $(\lambda^2-1)^2+(G-1)^2 \neq 0$. As pointed out before, we can infer from this fact that the principal curvature on M are constant and contained in the set $\{\lambda, -\lambda^{-1}\}$ and therefore Theorem 1.2 follows. □

3. Some hypersurfaces in $S^{n+1}(1)$ with constant Gauss-Kronecker curvature and with $l_v = \lambda f_v$ for some nonzero vector $v \in \mathbf{R}^{n+2}$ and some real number $\lambda \in \mathbf{R}^1$.

EXAMPLE 1. *A non complete hypersurface in $S^4(1)$ with constant Gauss-Kronecker curvature $G=-1$ that satisfies $l_v = f_v$ for some nonzero vector $v \in \mathbf{R}^5$.*

Let us assume that $S \subset S^3(1) \subset \mathbf{R}^5$ is an open piece of a minimal surface with principal curvatures a and $-a$. Let us assume that for all $x \in S$, we have that $0<a(x)<1$. Assume that $\phi : U \subset \mathbf{R}^2 \to S$ is a conformal parametrization and $N : U \to S^3(1)$ is Gauss map of the immersion. Let us also assume that

$$\frac{\partial N}{\partial t_1} = -a(\phi(t_1,t_2))\frac{\partial \phi}{\partial t_1}, \quad \frac{\partial N}{\partial t_2} = a(\phi(t_1,t_2))\frac{\partial \phi}{\partial t_2}.$$

This is, we will assume that $\frac{\partial \phi}{\partial t_1}$ and $\frac{\partial \phi}{\partial t_2}$ are principal directions.

Let us define $\xi : U \to S^3(1)$ by

$$\xi(t_1,t_2) = \frac{1}{\sqrt{2}}(\phi(t_1,t_2) + N(t_1,t_2)).$$

A direct computation shows that

$$\nu(t_1,t_2) = \frac{1}{\sqrt{2}}(\phi(t_1,t_2) - N(t_1,t_2))$$

is a Gauss map of ξ and moreover, since

$$\frac{\partial \xi}{\partial t_1} = \frac{1-a}{\sqrt{2}}\frac{\partial \phi}{\partial t_1}, \quad \frac{\partial \xi}{\partial t_2} = \frac{1+a}{\sqrt{2}}\frac{\partial \phi}{\partial t_2}$$

and

$$\frac{\partial \nu}{\partial t_1} = \frac{1+a}{\sqrt{2}}\frac{\partial \phi}{\partial t_1}, \quad \frac{\partial \nu}{\partial t_2} = \frac{1-a}{\sqrt{2}}\frac{\partial \phi}{\partial t_2}.$$

We conclude that the differential of ν satisfies

$$d\nu\left(\frac{1-a}{\sqrt{2}}\frac{\partial \phi}{\partial t_1}\right) = \frac{1+a}{\sqrt{2}}\frac{\partial \phi}{\partial t_1}, \quad d\nu\left(\frac{1+a}{\sqrt{2}}\frac{\partial \phi}{\partial t_2}\right) = \frac{1-a}{\sqrt{2}}\frac{\partial \phi}{\partial t_2}.$$

Therefore the principal curvatures of ξ are $\frac{1+a}{a-1}$ and $\frac{a-1}{a+1}$.

Let us define $\varphi : U \times I \to S^4$ by

$$\varphi(t_1,t_2,t_3) = \left(\frac{1}{2}\big(\cos(\sqrt{2}t_3) - 1\big)(\xi+\nu) + \xi, \frac{1}{\sqrt{2}}\sin(\sqrt{2}t_3)\right).$$

We have that

$$\frac{\partial \varphi}{\partial t_1} = \left(\frac{1}{\sqrt{2}}\big(\cos(\sqrt{2}t_3) - a\big)\frac{\partial \phi}{\partial t_1}, 0\right),$$

$$\frac{\partial \varphi}{\partial t_2} = \left(\frac{1}{\sqrt{2}}\big(\cos(\sqrt{2}t_3) + a\big)\frac{\partial \phi}{\partial t_2}, 0\right),$$

$$\frac{\partial \varphi}{\partial t_3} = \left(-\frac{1}{\sqrt{2}}\sin(\sqrt{2}t_3)(\xi+\nu), \cos(\sqrt{2}t_3)\right).$$

From the computations above we get that $\frac{\partial \varphi}{\partial t_3}$ is a unit vector. Let us define,

$$\zeta(t_1,t_2,t_3) = \left(\frac{1}{2}\big(\cos(\sqrt{2}t_3) + 1\big)(\xi+\nu) - \xi, \frac{1}{\sqrt{2}}\sin(\sqrt{2}t_3)\right),$$

A direct computation shows that ζ is a Gauss map for the immersion φ, therefore we can easily see that $f_v = l_v$ for $v = (0,0,0,0,1)$. Let us compute the principal curvatures of the immersion φ. We have that

$$\frac{\partial \zeta}{\partial t_1} = \left(\frac{1}{\sqrt{2}}\big(\cos(\sqrt{2}t_3) + a\big)\frac{\partial \phi}{\partial t_1}, 0\right),$$

$$\frac{\partial \zeta}{\partial t_2} = \left(\frac{1}{\sqrt{2}}\big(\cos(\sqrt{2}t_3) - a\big)\frac{\partial \phi}{\partial t_2}, 0\right),$$

$$\frac{\partial \zeta}{\partial t_3} = \left(-\frac{1}{\sqrt{2}}\sin(\sqrt{2}t_3)(\xi + \nu), \cos(\sqrt{2}t_3)\right) = \frac{\partial \varphi}{\partial t_3}.$$

It follows that the principal curvatures of ζ are

$$k_1 = \frac{\cos(\sqrt{2}t_3) + a}{a - \cos(\sqrt{2}t_3)}, \quad k_2 = \frac{a - \cos(\sqrt{2}t_3)}{\cos(\sqrt{2}t_3) + a}, \quad k_3 = -1.$$

EXAMPLE 2. *A complete hypersurface in $S^5(1)$ with constant Gauss-Kronecker curvature $G = 1$ and with $l_v = f_v$ for some nonzero vector $v \in \mathbf{R}^6$.*

Let $f : \mathbf{R}^5 \to \mathbf{R}$ be the function given by

$$f(x_1, \dots, x_5) = x_1^3 - 3x_1x_2^2 + \frac{3}{2}x_1(x_3^2 + x_4^2 - 2x_5^2) + \frac{3\sqrt{3}}{2}x_2(x_3^2 - x_4^2) + 3\sqrt{3}x_3x_4x_5,$$

one has that

$$M = \{x \in S^4 : f(x) = 0\}$$

is a minimal isoparametric hypersurface with principal curvatures $\sqrt{3}$, 0 and $-\sqrt{3}$.

Let

$$\Sigma = \left\{\big(\frac{1}{\sqrt{2}}x\cos s + \frac{1}{\sqrt{2}}\nu(x), \frac{1}{\sqrt{2}}\sin s\big) \colon x \in M, s \in \mathbf{R}\right\},$$

where $x \in M \subset S^4(1) \subset \mathbf{R}^5$, $\nu : M \to S^4(1) \subset \mathbf{R}^5$ denotes the Gauss map of M.

Let $\phi : M \times \mathbf{R} \to \Sigma$ is given by

$$\phi(x, s) = \big(\frac{1}{\sqrt{2}}x\cos s + \frac{1}{\sqrt{2}}\nu(x), \frac{1}{\sqrt{2}}\sin s\big).$$

By a direct calculation, one can show that Σ is compact and embedded because Σ is one connected component of $h^{-1}(0)$, where $h(x_1, \dots, x_6) = \sqrt{2}f(x_1, \dots, x_5) + 3x_6^2 - 1$ and we can check that 0 is a regular value of the function h restricted to $S^5(1) \subset \mathbf{R}^6$.

Let us consider a vector $v_1 \in T_xM$ such that $A(v_1) = -d\nu(v_1) = \sqrt{3}v_1$, one has

$$w_1 = d\phi_{(x,s)}(v_1) = \frac{1}{\sqrt{2}}((-\sqrt{3} + \cos s)v_1, 0) \in T_{\phi(x,s)}\Sigma.$$

The Gauss map at the point $\phi(x, s)$ is given by

$$N(\phi(x, s)) = \left(\frac{1}{\sqrt{2}}x\cos s - \frac{1}{\sqrt{2}}\nu(x), \frac{1}{\sqrt{2}}\sin s\right).$$

A direct computation shows that

$$dN_{\phi(x,s)}(w_1) = \frac{1}{\sqrt{2}}((\cos s + \sqrt{3})v_1, 0) = \frac{\cos s + \sqrt{3}}{\cos s - \sqrt{3}}w_1,$$

then $\dfrac{\cos s + \sqrt{3}}{\sqrt{3} - \cos s}$ is a principal curvature of Σ at $\phi(x, s)$.

If we consider a vector v_2 such that $d\nu(v_2) = \sqrt{3}v_2$, then one gets that $\frac{\sqrt{3}-\cos s}{\sqrt{3}+\cos s}$ is a principal curvature of Σ by using the same computation as that of v_1. If we consider a vector v_3 such that $d\nu(v_3) = 0 \times v_3$, then one gets that -1 is a principal curvature of Σ.

Since

$$d\phi_{(x,s)}(\frac{\partial}{\partial s}) = (-\frac{1}{\sqrt{2}}x\sin s, \frac{1}{\sqrt{2}}\cos s)$$

and

$$dN_{\phi(x,s)}(w_4) = (-\frac{1}{\sqrt{2}}x\sin s, \frac{1}{\sqrt{2}}\cos s) = w_4,$$

we have -1 is another principal curvature of Σ.

Therefore Σ has principal curvatures

$$\frac{\cos s+\sqrt{3}}{\sqrt{3}-\cos s},\ \frac{\sqrt{3}-\cos s}{\sqrt{3}+\cos s},\ -1, \text{and } -1,$$

and Gauss-Kronecker curvature is

$$G = \frac{\cos s+\sqrt{3}}{\sqrt{3}-\cos s} \times \frac{\sqrt{3}-\cos s}{\sqrt{3}+\cos s} \times (-1) \times (-1) = 1.$$

From the expression of $\phi(x,s)$ and $N(\phi(x,s))$, we obtain

$$l_v = f_v, \qquad \text{for } v = (0,0,0,0,0,1) \in \mathbf{R}^6.$$

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