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LOCAL DONALDSON-THOMAS INVARIANTS OF BLOWUPS OF SURFACES*

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Abstract. Using the degeneration formula for Donaldson-Thomas invariants, we proved a formula for the change of Donaldson-Thomas invariants of local surfaces under blowing up along points.

Key words. Donaldson-Thomas invariant, local surface, degeneration formula, blowup.

AMS subject classifications. 14N35, 14E05.

Introduction. Given a smooth projective Calabi-Yau 3-fold X, the moduli space of stable sheaves on X has virtual dimension zero. Donaldson and Thomas [3] defined the holomorphic Casson invariant of X which essentially counts the number of stable bundles on X. However, the moduli space has positive dimension and is singular in general. Making use of virtual cycle technique (see [1] and [7]), Thomas in [11] showed that one can define a virtual moduli cycle for some X including Calabi-Yau and Fano 3-folds. As a consequence, one can define Donaldson-type invariants of X which are deformation invariant.

Let S be a smooth surface and K_S its canonical bundle. Denote by $Y_S = \mathbb{P}(K_S \oplus \mathcal{O})$ the projective bundle completion of the total space of K_S. The Donaldson-Thomas theory of Y_S is well defined in every rank. Let $\gamma_i \in H^*(Y_S)$, $i = 1, \cdots, r$. Denote by $\tilde{\tau}_{k_i}(\gamma_i)$ the associated descendant fields in Donaldson-Thomas theory defined in [10]. For $\beta \in H_2(Y_S, \mathbb{Z})$ and an integer $n \in \mathbb{Z}$, denote by $\langle \tilde{\tau}_{k_1}(\gamma_1), \cdots, \tilde{\tau}_{k_r}(\gamma_r)\rangle^{Y_S}_{n,\beta}$ descendant Donaldson-Thomas invariant of Y_S. Denote by $Z'_{DT}(S;q)_\beta$ the reduced partition function for the Donaldson-Thomas theory of the local Calabi-Yau geometry of S.

Denote by $p : \tilde{S} \longrightarrow S$ the natural projection of blowup of S at a smooth point $s_0 \in S$. Let $\beta \in H_2(S, \mathbb{Z})$ and $p^!(\beta) = PD_{\tilde{S}} p^* PD_S(\beta) \in H_2(\tilde{S}, \mathbb{Z})$. We view S as the zero locus in K_S , and view K_S as an open subspace of its projective completion Y_S. So we have an embedding $i : S \hookrightarrow Y_S$, and β is often abuse of notation for $i_*\beta \in H_2(Y_S, \mathbb{Z})$. In [4], we use the degeneration formula to study the change of local Gromov-Witten invariants under blowup of a Fano surfaces. Similarly, we observed that Donaldson-Thomas invariants of Y_S of degree β are equal to Donaldson-Thomas invariants of $Y_{\tilde{S}}$ of degree $p^!(\beta)$.

THEOREM 0.1. *Suppose that S is a smooth surface and $\tilde{S}$ is the blown-up surface of S at a smooth point s_0. Let $\beta \in H_2(S, \mathbb{Z})$. Then we have*

$$Z'_{DT}(S;q)_\beta = Z'_{DT}(\tilde{S};q)_{p^!(\beta)}, \tag{1}$$

where $p : \tilde{S} \longrightarrow S$ is the natural projection of blowup.

REMARK 0.2. *Let $\tilde{\mathbb{P}}^2_r$ be blowup of $\mathbb{P}^2$ at r points. Pick one more point p and blow it up, then we obtain $\tilde{\mathbb{P}}^2_{r+1}$ with the map $p : \tilde{\mathbb{P}}^2_{r+1} \longrightarrow \tilde{\mathbb{P}}^2_r$. It is well-known that for $0 \le r \le 3$, $\tilde{\mathbb{P}}^2_r$ is toric, but for $4 \le r \le 8$, $\tilde{\mathbb{P}}^2_r$ is non-toric. In [9], via the localization*

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technique, the authors computed local Donaldson-Thomas invariants of toric surfaces, in particular, their method is valid for del Pezzo surfaces $\tilde{\mathbb{P}}^2_r$ with $0 \leq r \leq 3$. As opposed to toric del Pezzo surfaces, one can not directly use localization with respect to a torus action because there is no torus action on a generic del Pezzo surface $\tilde{\mathbb{P}}^2_r, 4 \leq r \leq 8$. Our Theorem 0.1 implies that for some degrees, we could compute local Donaldson-Thomas invariants of non-toric surfaces $\tilde{\mathbb{P}}^2_r$ with $4 \leq r \leq 8$ from local Donaldson-Thomas invariants of $\tilde{\mathbb{P}}^2_r$ with $0 \leq r \leq 3$.

1. Donaldson-Thomas invariant and its degeneration formula. Let X be a smooth projective 3-fold and $\mathcal{I}$ be an ideal sheaf on X. Assume the sub-scheme Y defined by $\mathcal{I}$ has dimension ≤ 1. Here Y is allowed to have embedded points on the curve components. Therefore we have the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

The 1-dimensional components, with multiplicities taken into consideration, determine a homology class

$$[Y] \in H_2(X, \mathbb{Z}).$$

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$\chi(\mathcal{O}_Y) = n, \quad [Y] = \beta \in H_2(X, \mathbb{Z}).$$

$I_n(X, \beta)$ has a virtual fundamental class $[I_n(X, \beta)]^{virt}$ with expected dimension $\int_\beta c_1(T_X)$, see [9, 8] for the details. Donaldson-Thomas invariant $\langle \tilde{\tau}_{k_1}(\gamma_1), \cdots, \tilde{\tau}_{k_r}(\gamma_r)\rangle_{n,\beta}$ is defined as the virtual integration of descendant insertions $\tilde{\tau}_{k_i}(\gamma_i)$. The Donaldson-Thomas partition function with descendant insertions is defined by

$$Z_{DT}(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}))_\beta = \sum_{n \in \mathbb{Z}} < \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}) >_{n,\beta} q^n.$$

The degree 0 moduli space $I_n(X, 0)$ is isomorphic to the Hilbert scheme of n points on X. The degree 0 partition function is $\mathbf{Z}_{DT}(X; q)_0$.

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$Z'_{DT}(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}))_\beta = \frac{Z_{DT}(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_{l_i}))_\beta}{Z_{DT}(X; q)_0}.$$

Let S be a smooth divisor in X. An ideal sheaf $\mathcal{I}$ is said to be relative to S if the morphism

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_S \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_S$$

is injective. The moduli space $I_n(X/S, \beta)$ of ideal sheaves relative to S also has a virtual fundamental class with expected dimension $\int_\beta c_1(X)$. Similar to the absolute case, we can define relative Donaldson-Thomas invariant by integrating absolute descendant insertions $\tilde{\tau}_{k_i}(\gamma_{l_i})$ and relative weighted partition η against the virtual fundamental class. Similarly, we also can define relative reduced partition function as

$$Z'_{DT}(X/S;q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta,\eta} = \frac{Z_{DT}(X/S;q \mid \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta,\eta}}{Z_{DT}(X/S;q)_0}.$$

Let $\pi\colon \mathcal{X} \to C$ be a smooth 4-fold over a smooth irrreducible curve C with a marked point denoted by $\mathbf{0}$ such that $\mathcal{X}_t = \pi^{-1}(t) \cong X$ for $t \neq \mathbf{0}$ and $\mathcal{X}_{\mathbf{0}}$ is a union of two smooth 3-folds X_1 and X_2 intersecting transversely along a smooth surface S. We write $\mathcal{X}_{\mathbf{0}} = X_1 \cup_S X_2$. Assume that C is contractible and S is simply-connected.

Consider the natural maps

$$i_t\colon X = \mathcal{X}_t \to \mathcal{X}, \qquad i_{\mathbf{0}}\colon \mathcal{X}_{\mathbf{0}} \to \mathcal{X},$$

and the gluing map

$$g = (j_1, j_2)\colon X_1 \coprod X_2 \to \mathcal{X}_{\mathbf{0}}.$$

We have

$$H_2(X) \xrightarrow{i_{t*}} H_2(\mathcal{X}) \xleftarrow{i_{0*}} H_2(\mathcal{X}_{\mathbf{0}}) \xleftarrow{g_*} H_2(X_1) \oplus H_2(X_2),$$

where i_{0*} is an isomorphism since there exists a deformation retract from $\mathcal{X}$ to $\mathcal{X}_{\mathbf{0}}$ (see [2]) and g_* is surjective from Mayer-Vietoris sequence. For $\beta \in H_2(X)$, there exist $\beta_1 \in H_2(X_1)$ and $\beta_2 \in H_2(X_2)$ such that

$$i_{t*}(\beta) = i_{0*}(j_{1*}(\beta_1) + j_{2*}(\beta_2)). \tag{2}$$

For simplicity, we write $\beta = \beta_1 + \beta_2$ instead.

The degeneration formula of Donaldson-Thomas invariant takes the form

$$\begin{aligned} & Z'_{DT}(\mathcal{X}_t;q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{l_i}))_\beta \\ = & \sum C(\eta) Z'_{DT}(X_1/S;q \mid \prod \tilde{\tau}_0(j_1^*\gamma_{l_i}))_{\beta_1,\eta} \\ & \times Z'_{DT}(X_2/S;q \mid \prod \tilde{\tau}_0(j_2^*\gamma_{l_i}))_{\beta_2,\eta^\vee}, \end{aligned} \tag{3}$$

where the sum is over the splittings $\beta_1 + \beta_2 = \beta$, and cohomology weighted partitions η. There is a compatibility condition

$$|\eta| = \beta_1 \cdot [S] = \beta_2 \cdot [S] \tag{4}$$

and $C(\eta) \in \mathbb{Q}$ is some combinatorial factor depending on the partition underlying η.

For the details, we can see Section 3.4 of [10], Theorem 1.4 of [8] and Section 2 of [5].

2. Main theorems. A technical result in our calculation of this section is the first Chern class of the projective bundle. Let V be a rank r complex vector bundle over a complex manifold M, and $\pi : \mathbb{P}(V) \longrightarrow M$ be the corresponding projective bundle. Let ξ_V be the first Chern class of the tautological bundle in $\mathbb{P}(V)$. A simple calculation shows

$$c_1(\mathbb{P}(V)) = \pi^* c_1(M) + \pi^* c_1(V) - r\xi_V. \tag{5}$$

In this section, we will find a sequence of birational threefolds all of whose invariants are equal. In fact, these birational threefolds are the projective completion Y_S of K_S, blowup $\tilde{Y}_S$ of Y_S along the fiber over s_0, the projective completion $Y_{\tilde{S}}$ of $K_{\tilde{S}}$ and Z, a threefold dominating the last two, obtained by blowing them up along a specific section of the exceptional divisor in $\tilde{S}$. For each pair of spaces, a degeneration is constructed with the goal of comparing absolute invariants of one with relative invariants of the other. Then we prove that the virtual dimension of one of the moduli spaces of relative stable maps appearing in the degeneration formula is negative as soon as there are nontrivial contacts with the relative divisors. Next a second application of the degeneration formula compares such relative invariants with the absolute invariants of the same space. This sequence of comparison results will imply Theorem 0.1. We have

Lemma 2.1. *Suppose that S is a smooth surface. Let $\tilde{Y}_S$ be blowup of Y_S along the fiber over $s_0 \in S$. Then for any $\beta \in H_2(S;\mathbb{Z})$, we have*

$$Z'_{DT}(Y_S;q)_\beta = Z'_{DT}(\tilde{Y}_S/D_1;q)_{p^!(\beta),\emptyset},$$

where $D_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}) \cong \mathbb{P}^1\times\mathbb{P}^1$ is the exceptional divisor in $\tilde{Y}_S$, $p: \tilde{S} \longrightarrow S$ is blowup of S at s_0 and $p^!(\beta) = PD_{\tilde{S}}p^*PD_S(\beta)$.

Proof. Let $\mathcal{X}$ be blowup of $Y_S \times \mathbb{A}^1$ along $F_{s_0}\times\{0\}$, where F_{s_0} is the fiber of Y_S over s_0 and let π be the natural projection from $\mathcal{X}$ to $\mathbb{A}^1$. It is a semistable degeneration of Y_S with the central fiber $\mathcal{X}_0$ being a union of $X_1 = \tilde{Y}_S$ and $X_2 \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}) \cong \mathbb{P}^2\times\mathbb{P}^1$ with the common divisor $D_1 \cong \mathbb{P}^1\times\mathbb{P}^1$.

The degeneration formula (3) takes the form

$$Z'_{DT}(Y_S;q)_\beta = \sum C(\eta) Z'_{DT}(\tilde{Y}_S/D_1;q)_{\beta_1,\eta} Z'_{DT}(X_2/D_1;q)_{\beta_2,\eta^\vee}. \tag{6}$$

Now we need to compute the summands in the right hand side of (6). For this we have the following claim:

Claim. There are only nonzero terms with $\beta_2 = 0$ and $\eta = \emptyset$.

In fact, if $\eta \neq \emptyset$, then $\beta_2 \neq 0$ because $\beta_2 \cdot D_1 = |\eta|$. Any summand of (6) which is not zero must satisfy

$$\mathrm{vdim} I_n(X_1/D_1;\beta_1) = \deg\eta,$$

where $\deg\eta \in \mathbb{N}$ is the total degree of the cohomology weighted partition η. From Section 2, we know

$$\mathrm{vdim} I_n(X_1/D_1;\beta) = c_1(X)\cdot\beta, \tag{7}$$

where $c_1(X)$ is the first Chern class of X.

Applying (5) to X_2, we obtain

$$c_1(X_2) = \pi^*\mathcal{O}_{\mathbb{P}^1}(2) - 3\xi,$$

where ξ is the first Chern class of tautological bundle in X_2. Since the homology class β_2 may be decomposed into the sum of base class $\beta_2^{\mathbb{P}^1}$ and fiber class β_2^f, so we have

$$\begin{aligned} c_1(X_2)\cdot\beta_2 &= \mathrm{vdim} I_n(X_2/D_1,\beta_2) \\ &= \pi^*\mathcal{O}_{\mathbb{P}^1}(2)\cdot\beta_2^{\mathbb{P}^1} - 3\xi\cdot\beta_2^f \geq 3|\eta|. \end{aligned} \tag{8}$$

In the last inequality, we use the fact that $-\xi$ is infinite section, so $-\xi \cdot \beta_2^f = |\eta|$ and $\pi^*\mathcal{O}_{\mathbb{P}^1}(2) \cdot \beta_2^{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2) \cdot \pi_* \beta_2^{\mathbb{P}^1} \geq 0$. Since $\beta \in H_2(S, \mathbb{Z})$, from (5), we have

$$c_1(Y_S) \cdot \beta = \mathrm{vdim} I_n(Y_S, \beta) = 0.$$

It is a general feature of the degeneration formula, see Lemma 2.2 of [5], that

$$\mathrm{vdim} I(\mathcal{X}_t, \beta) = \mathrm{vdim} I(X_1/D_1, \beta_1) + \mathrm{vdim} I(X_2/D_1, \beta_2) - 2\beta_2 \cdot D_1. \tag{9}$$

From (9), we have

$$c_1(Y_S) \cdot \beta = c_1(X_1) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|. \tag{10}$$

Therefore, from (8) we obtian

$$\deg \eta + |\eta| \leq 0.$$

This is a contradiction unless $\eta = \emptyset$. Therefore $|\eta| = 0$, i. e. $\eta = \emptyset$. This implies $\beta_2^f = 0$. So $\beta_2 = \beta_2^{\mathbb{P}^1}$.

On the other hand, if $\eta = \emptyset$, it is easy to know $c_1(X_1) \cdot \beta_1 = 0$. If $\beta_2 = \beta_2^{\mathbb{P}^1} \neq 0$, we have

$$\pi^*\mathcal{O}_{\mathbb{P}^1}(2) \cdot \beta_2^{\mathbb{P}^1} > 0.$$

So the claim is proved.

This claim implies that the nonzero summands must have $\beta_2 = 0$ and $\beta_1 = p^!(\beta)$.

By the degeneration formula, we have

$$Z'_{DT}(Y_S; q)_\beta = Z'_{DT}(\tilde{Y}_S/D_1; q)_{p^!(\beta), \emptyset}.$$

This proves the lemma. □

Lemma 2.2. *Under the assumption of Lemma 2.1, Then for $\beta \in H_2(S; \mathbb{Z})$, we have*

$$Z'_{DT}(\tilde{Y}_S; q)_{p!(\beta)} = Z'_{DT}(\tilde{Y}_S/D_1; q)_{p!(\beta), \emptyset}$$

Proof. Let $\mathcal{X}$ be the blow up of $\tilde{Y}_S \times \mathbb{A}^1$ along $D_1 \times \{0\}$. Let $\pi : \mathcal{X} \longrightarrow \mathbb{C}$ be the natural projection. Thus we get a semi-stable degeneration of $\tilde{Y}_S$ whose central fiber is a union of $X_1 \cong \tilde{Y}_S$ and $X_2 = \mathbb{P}_{\mathbb{P}^1}(N_{D_1} \oplus \mathcal{O})$, where the normal bundle of the divisor D_1 is $N_{D_1} = \mathcal{O}(-1, -1)$.

The degeneration formula (3) takes the form

$$Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)} = \sum C(\eta) Z'_{DT}(\tilde{Y}_S/D_1; q)_{\beta_1, \eta} Z'_{DT}(X_2/D_1; q)_{\beta_2, \eta^\vee}. \tag{11}$$

Similar to the proof of Lemma 2.1, we need to prove that there are only terms with $\beta_2 = 0$ in the right hand side of (11).

In fact, Note that $X_2 = \mathbb{P}_{D_1}(N_{D_1} \oplus \mathcal{O})$ and $D_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$. Denote by $F_{s_0} \cong \mathbb{P}^1$ the fiber of Y_S at the point s_0. Applying (5) to X_2 and D_1, we obtain

$$\begin{aligned} c_1(X_2) &= \pi^* c_1(D_1) + \pi^* c_1(N_{D_1}) - 2\xi \\ &= \pi^* c_1(F_{s_0}) + \pi^* c_1(N_{F_{s_0}|Y_S}) - 2\xi_1 + \pi^* c_1(N_{D_1}) - 2\xi, \end{aligned}$$

where ξ_1 and ξ are the first Chern classes of tautological bundles in $\mathbb{P}(N_{F_{s_0}|Y_S})$ and $\mathbb{P}(N_{D_1} \oplus \mathcal{O})$ respectively. Here we denote the Chern class and its pullback by the same symbol. It is well-known that the normal bundle to D_1 in $\tilde{Y}_S$ is just tautological line bundle on $D_1 \cong \mathbb{P}(N_{F_{s_0}|Y_S})$. Therefore $c_1(N_{D_1}) = \xi_1$. So we have

$$c_1(X_2) = \pi^* c_1(F_{p_0}) - \xi_1 - 2\xi.$$

where $-\xi$ is infinite section which has positive intersections with the effective curve classes.

Note that X_2 is a projective bundle over D_1 with fiber $\mathbb{P}^1$. Let L be the class of a line in the fiber $\mathbb{P}^1$ and e be the class of a line in the fiber $\mathbb{P}^1$ in $D_1 = \mathbb{P}(N_{F_{s_0}|Y_S})$. Denote by $\beta_2^{F_{s_0}}$ the homology class of the projection in F_{s_0} of the curve component. Denote by β_2^f the difference of β_2 and $\beta_2^{F_{s_0}}$, i. e. $\beta_2^f = \beta_2 - \beta^{F_{s_0}}$. Then it is easy to know $\beta_2^f = aL + be$. Since $(-\xi)\cdot\beta_2 = |\eta|$, $(-\xi)\cdot\beta_2^f = a = |\eta|$. On the other hand, since all curves of class β_2 come from the curve of class $p^!(\beta)$ by the degeneration and the degeneration only happens away from the divisor D_1, from $p^!(\beta)\cdot D_1 = 0$, we have $D_1 \cdot \beta_2 = 0$. Thus we have $D_1 \cdot \beta_2^f = a - b = 0$. Therefore, we have $a = b = |\eta|$. So we have $\beta_2^f = |\eta|(L + e)$. Since $c_1(F_{s_0}) + c_1(N_{F_{s_0}|Y_S}) = c_1(F_{s_0}) \geq 0$, we have

$$c_1(X_2)\cdot\beta_2 \geq 4|\eta|.$$

From (7)and (9), we have

$$c_1(\tilde{Y}_S)\cdot p^!(\beta) = c_1(X_1)\cdot\beta_1 + c_1(X_2)\cdot\beta_2 - 2|\eta|. \tag{12}$$

If $\eta \neq \emptyset$, then $|\eta| \neq 0$. Therefore,

$$\deg\eta + 2|\eta| > 0.$$

This contradicts to (12). Thus $\eta = \emptyset$.

Therefore, from the discussion above, we have $\beta_2 = \beta^{F_{s_0}}$. So $c_1(X_2)\cdot\beta_2 = c_1(F_{s_0})(\beta^{F_{s_0}})$. Thus $c_1(X_2)\cdot\beta_2 > 0$ if $\beta^{F_{s_0}} \neq 0$. Furthermore, if $\beta_2 = \beta^{F_{s_0}} \neq 0$, by definition, we have

$$Z'_{DT}(X_2/D_1; q)_{\beta_2,\emptyset} = 0.$$

Therefore, we have proved that there are only terms with $\beta_2 = 0$ in the right hand side of (11).

From (11), we have

$$Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)} = Z'_{DT}(\tilde{Y}_S/D_1; q)_{p^!(\beta),\emptyset}.$$

This proves the lemma. □

Summarizing Lemma 2.1 and 2.2, we have

Theorem 2.3.

$$Z'_{DT}(Y_S; q)_\beta = Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)}.$$

Denote by E the exceptional divisor in $\tilde{S}$. In $\tilde{Y}_S$, take a section, σ , corresponding to $\mathcal{O} \longrightarrow \mathcal{O}\oplus K_S$, of the exceptional divisor D_1 over E and blow it up. Denote by Z the blown-up manifold, then Z has a natural projection π to $\tilde{S}$ given by the composition of blowup projection $Z \longrightarrow \tilde{Y}_S$ and the bundle projection $\tilde{Y}_S \longrightarrow \tilde{S}$. It is easy to see that the fiber $\pi^{-1}(E)$ has two normal crossing components: $D_1 \cong \mathbb{F}_0$ and $D_2 \cong \mathbb{F}_1$ intersecting along a section σ with the normal bundle $N_{\sigma|\mathbb{F}_0} \cong \mathcal{O}$ and $N_{\sigma|\mathbb{F}_1} \cong \mathcal{O}(-1)$ respectively.

Next, we want to compare Donaldson-Thomas invariants $Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)}$ of $\tilde{Y}_S$ to Donaldson-Thomas invariants of Z. In fact, we have

Theorem 2.4.

$$Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)} = Z'_{DT}(Z; q)_{p^!(\beta)}.$$

Proof. In $\tilde{Y}_S$, take the section σ of the exceptional divisor D_1 over the old exceptional divisor E, then $\sigma \cong \mathbb{P}^1$ and the normal bundle to σ in $\tilde{Y}_S$ is $N_{\sigma|\tilde{Y}_S} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}$. Let $\mathcal{X}$ be blowup of $\tilde{Y}_S \times \mathbb{A}^1$ along $\sigma \times \{0\}$. Let $\pi : \mathcal{X} \longrightarrow \mathbb{C}$ be the natural projection. Thus we get a semi-stable degeneration of $\tilde{Y}_S$ whose central fiber is a union of $X_1 \cong Z$ and $X_2 = \mathbb{P}_\sigma(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O} \oplus \mathcal{O})$ with the Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}_\sigma(\mathcal{O}(-1) \oplus \mathcal{O})$ as the common divisor.

By the degeneration formula (3), we may express absolute Donaldson-Thomas invariants of $\tilde{Y}_S$ in terms of relative Donaldson-Thomas invariants of $(X_1, \mathbb{F}_1)$ and $(X_2, \mathbb{F}_1)$ as follows

$$Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)} = \sum C(\eta) Z'_{DT}(Z/\mathbb{F}_1; q)_{\beta_1,\eta} Z'_{DT}(X_2/\mathbb{F}_1; q)_{\beta_2,\eta^\vee}. \tag{13}$$

Similar to the proof of Lemma 2.1, we need to prove that there are only terms with $\beta_2 = 0$ in the right hand side of (13).

Note that $X_2 = \mathbb{P}_\sigma(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O} \oplus \mathcal{O})$ and $\mathbb{F}_1 = \mathbb{P}_\sigma(\mathcal{O}(-1) \oplus \mathcal{O})$. Therefore, from (5), we have

$$c_1(X_2) = \pi^* c_1(\mathcal{O}_\sigma(1)) - 3\xi$$

where ξ is the first Chern class of tautological line bundle over X_2. It is easy to see that

$$c_1(X_2) \cdot \beta_2 \geq 3|\eta|.$$

From (7)and (9), we have

$$c_1(\tilde{Y}_S) \cdot p^!(\beta) = c_1(Z) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|.$$

Therefore,

$$c_1(\tilde{Y}_S) \cdot p^!(\beta) \geq \deg \eta + |\eta| > 0.$$

Since $c_1(\tilde{Y}_S) \cdot p^!(\beta) = 0$, so this is a contradiction. Thus $|\eta| = 0$.

The same argument as in the proof of Lemma 2.2 shows that $\beta_2 = 0$. Therefore, from (13), we have

$$Z'_{DT}(\tilde{Y}_S; q)_{p^!(\beta)} = Z'_{DT}(Z/\mathbb{F}_1; q)_{p^!(\beta),\emptyset}. \tag{14}$$

Now it remains to prove

$$Z'_{DT}(Z; q)_{p^!(\beta)} = Z'_{DT}(Z/\mathbb{F}_1; q)_{p^!(\beta),\emptyset}.$$

To prove this, we degenerate Z along the exceptional divisor $\mathbb{F}_1$. Then we obtain two smooth 3-folds

$$X_1 = Z, \qquad X_2 = \mathbb{P}_{\mathbb{F}_1}(N_{\mathbb{F}_1} \oplus \mathcal{O}),$$

intersecting along the exceptional divisor $\mathbb{F}_1$ in Z and the infinite section of the $\mathbb{P}^1$-bundle X_2.

Applying the degeneration formula to $Z'_{DT}(Z;q)_{p^!(\beta)}$, we have

$$Z'_{DT}(Z;q)_{p^!(\beta)} = \sum C(\eta) Z'_{DT}(Z/\mathbb{F}_1;q)_{\beta_1,\eta} Z'_{DT}(X_2/\mathbb{F}_1;q)_{\beta_2,\eta^\vee}. \tag{15}$$

Note that $X_2 = \mathbb{P}_{\mathbb{F}_1}(N_{\mathbb{F}_1} \oplus \mathcal{O})$ and $\mathbb{F}_1 = \mathbb{P}_\sigma(\mathcal{O}(-1) \oplus \mathcal{O})$. Applying (5) to X_2 and $\mathbb{F}_1$, we obtain

$$\begin{aligned} c_1(X_2) &= \pi^* c_1(\mathbb{F}_1) + \pi^* c_1(N_{\mathbb{F}_1}) - 2\xi \\ &= \pi^* c_1(\mathcal{O}_\sigma(1)) - \xi_1 - 2\xi, \end{aligned}$$

where ξ_1 and ξ are the first Chern classes of tautological bundles in $\mathbb{P}_\sigma(\mathcal{O}(-1) \oplus \mathcal{O})$ and $\mathbb{P}(N_{\mathbb{F}_1} \oplus \mathcal{O})$ respectively. Here we denote the Chern class and its pullback by the same symbol. The same calculation as in the proof of Lemma 2.2 shows that

$$c_1(X_2) \cdot \beta_2 \geq 4|\eta|.$$

From (7) and (9), we have

$$c_1(Z) \cdot p^!(\beta) = c_1(X_1) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|.$$

Therefore, if $|\eta| \neq 0$, we have

$$\deg \eta + 2|\eta| > 0.$$

This is a contradiction because $c_1(Z) \cdot p^!(\beta) = 0$. Thus $|\eta| = 0$.

The same argument shows that $\beta_2 = \beta_\sigma$, i. e. the class of a curve in σ. So $c_1(X_2) \cdot \beta_2 = c_1(\mathcal{O}_\sigma(1))(\beta_\sigma)$. Thus $c_1(X_2) \cdot \beta_2 > 0$ if $\beta_\sigma \neq 0$. Furthermore, if $\beta_2 = \beta_\sigma \neq 0$, then, by definition, we have

$$Z'_{DT}(X_2/\mathbb{F}_1;q)_{\beta_2,\emptyset} = 0.$$

Therefore, we have proved that there are only terms with $\beta_2 = 0$ in the right hand side of (15).

This implies that

$$Z'_{DT}(Z;q)_{p^!(\beta)} = Z'_{DT}(Z/\mathbb{F}_1;q)_{p^!(\beta),\emptyset}.$$

This proves the theorem. □

Next, we consider the projective completion $Y_{\tilde{S}}$. Since the restriction $K_{\tilde{S}}|_E$ of the canonical bundle $K_{\tilde{S}}$ to the exceptional divisor E in $\tilde{S}$ is isomorphic to $\mathcal{O}(-1)$, so we can pick up a section, σ_1, of the restriction of $Y_{\tilde{S}}$ to E satisfying $\sigma_1^2 = -1$. Then we blow this section σ_1 up, and it is easy to know that the blown-up manifold is Z.

Finally, we want to prove the following theorem

THEOREM 2.5.

$$Z'_{DT}(Y_{\tilde{S}};q)_{p!(\beta)} = Z'_{DT}(Z;q)_{p!(\beta)}.$$

Proof. Take a section $\sigma_1 \cong \mathbb{P}^1$ of $Y_{\tilde{S}} \mid_E = \mathbb{F}_1$ such that $\sigma_1^2 = -1$. Then we degenerate $Y_{\tilde{S}}$ along the section σ_1 and obtain two 3-folds,

$$X_1 = Z, \qquad X_2 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}),$$

with the common divisor $\mathbb{F}_0 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$.

Applying the degeneration formula (3) to $Z'_{DT}(Y_{\tilde{S}}; q)_{p!(\beta)}$, we have

$$Z'_{DT}(Y_{\tilde{S}}; q)_{p^!(\beta)} = \sum C(\eta) Z'_{DT}(Z/\mathbb{F}_0; q)_{\beta_1,\eta} Z'_{DT}(X_2/\mathbb{F}_0; q)_{\beta_2,\eta^\vee}. \tag{16}$$

Similar to the proof of Lemma 2.1, we need to prove that the summand with nonzero contribution in the right hand side of (16) must have the trivial partition $\eta = \emptyset$.

Note that $X_2 = \mathbb{P}_{\sigma_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O})$. From (5), it is easy to know

$$c_1(X_2) = \pi^* c_1(\sigma_1) + \pi^* c_1(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) - 3\xi = -3\xi,$$

where ξ is the first Chern class of tautological line bundle over X_2. Therefore, we have

$$c_1(X_2) \cdot \beta_2 = 3|\eta|.$$

From (7)and (9), we have

$$c_1(Y_{\tilde{S}}) \cdot p^!(\beta) = c_1(X_1) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|.$$

Therefore, if $|\eta| \neq 0$, we have

$$c_1(Y_{\tilde{S}}) \cdot p!(\beta) = \deg \eta + |\eta| > 0.$$

This is a contradiction because $c_1(Y_{\tilde{S}}) \cdot p^!(\beta) = 0$. This means that the summand with nonzero contribution in the right hand side of (16) must have $\eta = \emptyset$.

Since the section σ_1 and the old exceptional divisor E have the same homology class in $Y_{\tilde{S}}$, so from $\beta_1 + \beta_2 = p^!(\beta)$ and $\eta = \emptyset$, we have $\beta_2 = 0$.

Therefore, from (16), we have

$$Z'_{DT}(Y_{\tilde{S}}; q)_{p^!(\beta)} = Z'_{DT}(Z/\mathbb{F}_0; q)_{p^!(\beta),\emptyset}.$$

Now it remains to prove

$$Z'_{DT}(Z; q)_{p^!(\beta)} = Z'_{DT}(Z/\mathbb{F}_0; q)_{p^!(\beta),\emptyset}. \tag{17}$$

To prove this, we degenerate Z along the exceptional divisor $\mathbb{F}_0$. Then we obtain two 3-folds

$$X_1 = Z, \qquad X_2 = \mathbb{P}_{\mathbb{F}_0}(N_{\mathbb{F}_0} \oplus \mathcal{O}).$$

Note that $X_2 = \mathbb{P}_{\mathbb{F}_0}(N_{\mathbb{F}_0} \oplus \mathcal{O})$. Applying (5) to X_2 and $\mathbb{F}_0$, we have

$$\begin{aligned} c_1(X_2) &= \pi^* c_1(\mathbb{F}_0) + \pi^* c_1(N_{\mathbb{F}_0}) - 2\xi \\ &= -\xi_1 - 2\xi, \end{aligned}$$

where ξ_1 and ξ are the first Chern classes of tautological bundles in $\mathbb{P}_{\sigma_1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$ and $\mathbb{P}(N_{\mathbb{F}_0}\oplus\mathcal{O})$ respectively. The same calculation as in the proof of Lemma 2.2 shows that

$$c_1(X_2)\cdot\beta_2 = 4|\eta|.$$

From (7)and (9), we have

$$\begin{aligned} c_1(Z)\cdot p^!(\beta) &= c_1(X_1)\cdot\beta_1 + c_1(X_2)\cdot\beta_2 - 2|\eta| \\ &= \deg\eta + 2|\eta| > 0. \end{aligned}$$

This is a contradiction because $c_1(Z)\cdot p^!(\beta)=0$. Thus $|\eta|=0$.

Since $\mathbb{F}_0\cdot p^!(\beta)=0$, the same argument as above shows that $\beta_2=0$. As before, this implies (17). This completes the proof of the theorem. □

Remark 2.6. *From Theorem 2.4 and 2.5, it is easy to know that Theorem 0.1 holds.*

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