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SAKAI'S THEOREM FOR $\mathbb{Q}$-DIVISORS ON SURFACES AND APPLICATIONS*

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Dedicated to Professor Ngaiming Mok on the occasion of his 60th birthday

Abstract. In this paper, we present a characterization of a big $\mathbb{Q}$-divisor D on a smooth projective surface S with $D^2 > 0$ and $H^1(\mathcal{O}_S(-\lceil D \rceil)) \neq 0$, which generalizes a result of Sakai [Sak90] for D integral. As applications of this result for $\mathbb{Q}$-divisors, we prove results on base-point-freeness and very-ampleness of the adjoint linear system $|K_S + \lceil D \rceil|$. These results can be viewed as refinements of previous results on smooth surfaces of Ein-Lazarsfeld [EL93] and Maşek [Maş99].

Key words. $\mathbb{Q}$-divisor, adjoint linear system, vanishing theorem.

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1. Introduction.

1.1. Main result. Being a central object in algebraic geometry, linear systems on projective varieties have been intensively studied over the past decades. One major problem about linear systems, particularly adjoint linear systems, is to determine their base-point-freeness and very-ampleness. Over surfaces, there are three important methods known in the literature. The first one is Reider's method [Rei88] via Bogomolov instability theorem for rank 2 vector bundles on surfaces. The second method is based on a cohomological machinery which uses multiplier ideal sheaves and Kawamata-Viehweg vanishing theorem (see [EL93] for instance). The third one, discovered by Sakai [Sak90], employs a characterization of a big divisor D on a surface S with $D^2 > 0$ and $H^1(S, \mathcal{O}_S(-D)) \neq 0$. That is,

PROPOSITION 1.1 ([Sak90, Proposition 1]). *Let D be a big divisor with $D^2 > 0$ on a smooth projective surface S. If $H^1(S, \mathcal{O}_S(-D)) \neq 0$, then there is a nonzero effective divisor E such that*
(i) $(D - E)E \leq 0$;
(ii) $D - 2E$ *is a big divisor.*

It is now a general philosophy that in birational geometry, we always study $\mathbb{Q}$-divisors rather than merely integral ones. For example, Kawamata-Viehweg vanishing theorem holds for nef and big $\mathbb{Q}$-divisors with simple normal crossing fractional parts, and it has played a crucial role when studying various problems in algebraic geometry. Another example is the development of multiplier ideal sheaves which is mainly aiming at exploring $\mathbb{Q}$-divisors. Having noticed these, we may wonder whether the above result of Sakai is valid also for $\mathbb{Q}$-divisors.

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The main purpose of this paper is to confirm this expectation. We have the following theorem:

THEOREM 1.2. *Let D be a big $\mathbb{Q}$-divisor on a smooth projective surface S. If $H^1(\mathcal{O}_S(-\lceil D\rceil)) \neq 0$, then there is a nonzero effective $\mathbb{Q}$-divisor E such that*

(i) $(D-E)C \leq (\lceil D\rceil - \lceil E\rceil)C \leq 0$ *for any irreducible component C of E;*
(ii) *The intersection matrix of E is negative definite;*
(iii) *For any irreducible component C in E, we have* $\{\mathrm{mult}_C D\} = \{\mathrm{mult}_C E\}$*;*
(iv) $D-2E$ *is a big divisor provided that* $D^2 > 0$.

If D is an integral divisor, then the effective divisor E we construct in Theorem 1.2 is the same as that in Sakai's result. The main differences of Theorem 1.2 from Sakai's result lie in (i) and (iii), where we show that to some extent, the fractional part of E is coherent to that of D. These properties did not show up in Sakai's result, as both D and E therein are just integral without any fractional part. The fact that D and E have the same fractional parts along every component of E does play a crucial role in the following application.

1.2. An application. As mentioned before, Sakai's result applies to the study of adjoint linear systems on surfaces. In this paper, we also apply Theorem 1.2 to the similar problem but for $\mathbb{Q}$-divisors. More precisely, given a $\mathbb{Q}$-divisor D on S, we deduce a base-point-freeness criterion (see Theorem 4.1) and a very-ampleness criterion for the adjoint linear system $|K_S + \lceil D\rceil|$ (see Theorem 5.1 and 6.2).

The (Reider-type) base-point-freeness and very-ampleness results for $\mathbb{Q}$-adjoint linear systems have been investigated for a long time (see [EL93] and [Maş99] for instance). The method used loc. cit. is mainly a combination of multiplier ideal sheaves and the technique of lifting sections from curves. Our method here is completely different. In fact, combining Theorem 1.2 with the Hodge index theorem, we basically transfer the problem into some numerical inequalities. In this way, we are able to recover the previous results on smooth surfaces in [EL93, Maş99] in a much more elementary manner and also provide criteria with weaker intersection conditions in certain cases.

Another feature different from previous results is that, we actually give an explicit characterization of *critical* curves, namely, curves on which we should impose the intersection number conditions (with D). In fact, we find that all *critical* curves that play a role in results of Reider-type are those *smooth* at the point $x \in S$ (resp. the tangent direction $\vec{v} \in T_x(S)$) that we are considering, with the only exception in the separation of tangents case when we need to take into account curves singular at x *of order two* as well.

1.3. A sketchy proof. To illustrate our method more concretely, we sketch in the following the proof of Theorem 5.1 for separating two distinct points. Exactly the same idea applies to separating tangents in Theorem 6.2, and a simpler version is already sufficient for proving the base-point-freeness result in Theorem 4.1.

Let x and y be two distinct points on S, and let $\pi : \tilde{S} \to S$ be the blowing up of S at x and y. Suppose that the linear system $|K_S + \lceil D\rceil|$ fails to separate x and y. We are able to find a big divisor $\tilde{D}$ on $\tilde{S}$ such that $H^1(K_{\tilde{S}} + \lceil \tilde{D}\rceil) \neq 0$. Applying Theorem 1.2 to $\tilde{D}$, we thus construct an effective divisor $\tilde{E}$ on $\tilde{S}$ satisfying all properties therein. Denote by $E = \pi_*\tilde{E}$. An important (and also a bit surprising) observation here is that the Hodge index theorem $D^2E^2 \leq (DE)^2$ for D and E has put lots of constrains on E. For example, we conclude from the above inequality and the property (iii) that there is at most one irreducible component A with its multiplicity $a \leq 1$ in E that passes through x (here we just ignore the interchanging between x and y). Moreover, A is smooth at x if it exists. Combining the assumption on DA and the above Hodge index theorem together, we are able to show that the same situation occurs also at y, i.e., there is at most one irreducible component B (it is possible that B coincides with A) in E that passes through y, and B is smooth at y. Now we combine assumptions on both DA and DB with the Hodge index theorem for one more time, and this time we deduce a contradiction. Hence the proof is finished.

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. In Section 3, we list some preliminary results and notions. Finally, proofs of theorems on base-point-freeness (Theorem 4.1), separating two points (Theorem 5.1) and separating tangents (Theorem 6.2) will be presented in Section 4, 5 and 6 respectively.

Notation and conventions. Throughout this paper, we work over complex numbers $\mathbb{C}$. We always denote by S a smooth projective surface over $\mathbb{C}$ and by D a $\mathbb{Q}$-divisor on S. We will use the following notations:

- $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α.
- $\lfloor \alpha \rfloor$ denotes the largest integer smaller than or equal to α.
- $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ is called the fractional part of α.
- For a $\mathbb{Q}$-divisor $D = \sum \alpha_i D_i$ on S where each D_i a prime divisor and $\alpha_i \in \mathbb{Q}$, we write $\lceil D \rceil = \sum \lceil \alpha_i \rceil D_i$, $\lfloor D \rfloor = \sum \lfloor \alpha_i \rfloor D_i$ and $\{D\} = \sum \{\alpha_i\} D_i$.

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2. Proof of Theorem 1.2. In this section, we present the proof of Theorem 1.2.

The following vanishing result will play a key role here.

LEMMA 2.1 ([Sak84]). *Let S be a smooth projective surface, and let M be a nef and big $\mathbb{Q}$-divisor on S. Then for any $i > 0$, we have*

$$H^i(\mathcal{O}_S(K_S + \lceil M \rceil)) = 0.$$

We now start the proof of Theorem 1.2. First, by Zariski decomposition, we can write $D = P + N$, where P is nef, N is effective with a negative definite intersection matrix, and $PC = 0$ for any irreducible component C of N. Notice that P is also big as D is big. Write $N = \sum_{i=1}^{r} \alpha_i C_i$ with $\alpha_i \in \mathbb{Q}_{>0}$. This gives a decomposition $P = P_1 + P_2$ such that P_1 consists of all irreducible components in P (with their multiplicities) that are supported in N. Therefore, we can write $P_1 = \sum_{i=1}^{r} \beta_i C_i$.[1]

[1]It may happen that $\beta_i = 0$ for some i.

Then we have

$$D = \lceil P_1 \rceil + P_2 + (N + P_1 - \lceil P_1 \rceil) = \lceil P_1 \rceil + P_2 + \sum_{i=1}^{r} \delta_i C_i,$$

where $\delta_i = \alpha_i + \beta_i - \lceil \beta_i \rceil$.

Let $I := \{1, 2, \ldots, r\}$. It is easy to see that $\delta_i > -1$ for any $i \in I$. Let $J := \{i \in I | \delta_i \geq 0\}$. Define $\Delta_+ := \sum\limits_{i \in J} \delta_i C_i$ and $\Delta_- := \sum\limits_{i \in I \setminus J} \delta_i C_i$. Then

$$\lceil D \rceil = \lceil P_1 \rceil + \lceil P_2 \rceil + \lceil \Delta_+ \rceil = \lceil P \rceil + \lceil \Delta_+ \rceil.$$

By Lemma 2.1, we know that

$$H^1(\mathcal{O}_S(-\lceil P \rceil)) = H^1(\mathcal{O}_S(-\lceil D \rceil + \lceil \Delta_+ \rceil)) = 0.$$

This implies that $\Delta_+ > 0$.

Following the idea in [Sak90], we consider any sequence $D_0 = \lceil D \rceil - \lceil \Delta_+ \rceil$, ..., $D_k = D_{k-1} + C_{j_k}$, ..., $D_n = \lceil D \rceil$. There is a short exact sequence for each k:

$$0 \to \mathcal{O}_S(-D_k) \to \mathcal{O}_S(-D_{k-1}) \to \mathcal{O}_{C_{j_k}}(-D_{k-1}) \to 0,$$

which gives

$$H^0(\mathcal{O}_{C_{j_k}}(-D_{k-1})) \to H^1(\mathcal{O}_S(-D_k)) \to H^1(\mathcal{O}_S(-D_{k-1})).$$

If $D_{k-1} C_{j_k} > 0$ for all k, we would inductively get $H^1(\mathcal{O}_S(-\lceil D \rceil)) = 0$. Therefore, there is a sequence D_0, ..., D_k with $k < n$ such that $D_k C_j \leq 0$ for all irreducible components C_j of $\lceil D \rceil - D_k \leq \lceil \Delta_+ \rceil$. Let $K \subseteq J$ be the set of indices so that we can write $\lceil D \rceil - D_k = \sum_{i \in K} m_i C_i$ with each $m_i \in \mathbb{Z}_{>0}$. Then we define

$$E := \sum_{i \in K} b_i C_i, \tag{2.1}$$

where

$$b_i = \begin{cases} m_i - 1 + \{\delta_i\} & \text{if } \{\delta_i\} \neq 0; \\ m_i & \text{if } \{\delta_i\} = 0. \end{cases}$$

Since $k < n$, we deduce that K is non-empty. Hence E is a nonzero effective divisor. The fact that E is supported in $\mathrm{Supp}(N)$ has two consequences. First, the intersection matrix of E is negative definite, which proves (ii). Second, we have $PE = 0$. Moreover, (iii) is also straightforward from the construction of E. In the following, we will prove (i) and (iv).

By the construction of E, we see that $\lceil D \rceil - \lceil E \rceil = D_k$. Also, by the construction of E, we have $\lceil \Delta_+ - E \rceil = \lceil \Delta_+ \rceil - \lceil E \rceil$. Since

$$D - E = \lceil P_1 \rceil + P_2 + \Delta_+ + \Delta_- - E,$$

we obtain that

$$\lceil D - E \rceil = \lceil P_1 \rceil + \lceil P_2 \rceil + \lceil \Delta_+ - E \rceil = \lceil P \rceil + \lceil \Delta_+ \rceil - \lceil E \rceil = \lceil D \rceil - \lceil E \rceil = D_k.$$

It also yields

$$\lceil D \rceil - \lceil E \rceil = D - E + (\lceil P_2 \rceil - P_2) - \Delta_- + \sum_{i \in J \setminus K} (\lceil \delta_i \rceil - \delta_i) C_i.$$

Since E has no component contained in $\lceil D - E \rceil - (D - E)$ by (iii), we conclude that

$$(D - E)C \leq (\lceil D \rceil - \lceil E \rceil)C = D_k C \leq 0 \tag{2.2}$$

for every component C of E. This proves (i).

Finally, if $D^2 > 0$, then (2.2) yields that

$$(D - 2E)^2 = D^2 - 4(D - E)E > 0. \tag{2.3}$$

On the other hand, $PE = 0$ implies that

$$(D - 2E)P = DP = P^2 > 0. \tag{2.4}$$

The above two inequalities guarantee the bigness of $D - 2E$. The proof is completed.

3. Preliminaries. Throughout this section, S is always a smooth projective surface.

3.1. A vanishing result. In the following context, we will frequently use the following result.

Proposition 3.1. *Let L be a (-1)-curve on S. Suppose that D is a divisor on S with $DL = 0$ such that $H^1(\mathcal{O}_S(K_S + D - aL)) = 0$ for an integer $a \geq 0$. Then for any $0 \leq k \leq a$, we also have*

$$H^1(\mathcal{O}_S(K_S + D - kL)) = 0.$$

Proof. Without loss of the generality, we assume that $a \geq 1$. We observe here that

$$H^1(\mathcal{O}_L(K_S + D - kL)) = 0$$

for any $k \geq 0$. For any such k, from the short exact sequence

$$0 \to \mathcal{O}_S(K_S + D - (k+1)L) \to \mathcal{O}_S(K_S + D - kL) \to \mathcal{O}_L(K_S + D - kL) \to 0,$$

we get

$$H^1(\mathcal{O}_S(K_S + D - (k+1)L)) \to H^1(\mathcal{O}_S(K_S + D - kL)) \to H^1(\mathcal{O}_L(K_S + D - kL)).$$

Then the conclusion for $k = a - 1$ follows from the assumption, and the whole proof is completed by iterating the whole process. □

3.2. Further notation. From now on till the end of the paper, we will use the notion of the local ampleness in order to state our results in a more precise manner.

Definition 3.2. *Let x be a closed point on S. We say that a $\mathbb{Q}$-divisor D on S is locally ample at x, if $DC > 0$ for any irreducible curve C on S passing through x.*

Let $x \in S$ be a closed point on S. For any $\mathbb{Q}$-divisor D on S, we define

$$\mu_x := \mathrm{mult}_x(\lceil D \rceil - D).$$

Notice that this notion has been introduced in [EL93] as well as in [Maş99]. Furthermore, if C is a curve on S passing through x, we use $T_x(C)$ to denote the tangent cone of C at x.

4. Base-point-freeness theorem. The main result in this section is the following base-point-free theorem.

THEOREM 4.1. *Let D be a nef and big $\mathbb{Q}$-divisor on a smooth projective surface S. Let $x \in S$ be a closed point. Then $|K_S + \lceil D \rceil|$ is free at x provided that one of the following holds:*

(1) $\mu_x \geq 2$;

(2) $0 \leq \mu_x < 2$, *D is locally ample at x, $D^2 > \beta_2^2$ and $DC \geq \beta_1$ for any irreducible curves C on S smoothly passing through x, where $\beta_2 \geq 2 - \mu_x$ and*

$$\beta_1 = \min\left\{2 - \mu_x, \frac{\beta_2}{\beta_2 - (1 - \mu_x)}\right\}.$$

We remark that in Case (2), we have

$$\beta_1 = \begin{cases} 2 - \mu_x, & \mu_x \geq 1; \\ \dfrac{\beta_2}{\beta_2 - (1 - \mu_x)}, & \mu_x < 1. \end{cases}$$

The first result of such type with $\beta_1 = 2 - \mu_x$ in both cases was proved by Ein-Lazarsfeld [EL93, Theorem 2.3], and the current version was discovered afterwards by Maşek [Maş99, Proposition 3].

Proof. We have the following short exact sequence

$$0 \to \mathcal{O}_S(K_S + \lceil D \rceil) \otimes \mathcal{I}_x \to \mathcal{O}_S(K_S + \lceil D \rceil) \to \mathcal{O}_x(K_S + \lceil D \rceil) \to 0,$$

where $\mathcal{I}_x$ is the ideal sheaf of x. To prove Theorem 4.1, it suffices to show that

$$H^1(\mathcal{O}_S(K_S + \lceil D \rceil) \otimes \mathcal{I}_x) = 0. \tag{4.1}$$

Let $\pi : \tilde{S} \to S$ be the blowing up of S at x and L_x be the exceptional divisor. Then we have

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^* \lceil D \rceil - 2L_x)) \simeq H^1(\mathcal{O}_S(K_S + \lceil D \rceil) \otimes \mathcal{I}_x). \tag{4.2}$$

On the other hand, notice that

$$\pi^* \lceil D \rceil = \lceil \pi^* D \rceil + \lfloor \mu_x \rfloor L_x.$$

By Lemma 2.1, we know that

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^* \lceil D \rceil - \lfloor \mu_x \rfloor L_x)) = H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \lceil \pi^* D \rceil)) = 0.$$

Theorem 4.1 then follows from Proposition 3.1 and (4.2) if $\mu_x \geq 2$.

Now assume that $0 \leq \mu_x < 2$. Let

$$\tilde{D} = \pi^* D - (2 - \mu_x) L_x.$$

By the assumption in Theorem 4.1 (2), $\tilde{D}^2 = D^2 - (2 - \mu_x)^2 > 0$. Since D is big, $\tilde{D}^2 > 0$ implies that $\tilde{D}$ is also big. Moreover,

$$\lceil \tilde{D} \rceil = \lceil \pi^* D + \mu_x L_x \rceil - 2L_x = \pi^* \lceil D \rceil - 2L_x.$$

Suppose that the theorem does not hold true, i.e., $H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \lceil \tilde{D} \rceil)) \neq 0$. Applying Theorem 1.2 to $\tilde{S}$ and $\tilde{D}$, we can find an effective $\mathbb{Q}$-divisor $\tilde{E}$ on $\tilde{S}$ satisfying conditions listed therein. Write

$$\tilde{E} = E + \lambda_x L_x$$

such that L_x is not a component of E. Let $E_1 \leq E$ be the effective $\mathbb{Q}$-divisor such that each irreducible component of E_1 meets L_x properly.

In the following, we present a step-by-step proof. The same strategy also applies to the rest of the paper.

Step 1 We first prove that $E_1 \neq 0$.

Suppose on the contrary that $E_1 = 0$. Then we would have $EL_x \leq 0$ and thus

$$(\tilde{D} - \lambda_x L_x)E = (\pi^* D)E - (2 - \mu_x + \lambda_x)EL_x \geq 0 > E^2.$$

This is impossible by Theorem 1.2 (i).

As a result, $\pi_* E_1$ is a strictly effective $\mathbb{Q}$-divisor passing through x. Hence by the local ampleness assumption on D at x, we have

$$(\pi^* D)E \geq (\pi^* D)E_1 = D(\pi_* E_1) > 0.$$

Step 2 In this step, we deduce several numerical inequalities from Theorem 1.2 and Hodge index theorem.

First, by Theorem 1.2 (i) and (ii), we know that

$$0 > E^2 = \tilde{E}E - \lambda_x EL_x \geq \tilde{D}E - \lambda_x EL_x = (\pi^* D)E - (2 - \mu_x + \lambda_x)EL_x. \tag{4.3}$$

Also from Theorem 1.2 (iii), we obtain that

$$(\lceil E \rceil - E)L_x \leq (\pi^{-1}\lceil D \rceil - \pi^{-1}D)L_x = \mu_x. \tag{4.4}$$

Recall that by Theorem 1.2 (iv), $\tilde{D} - 2\tilde{E}$ is big. Since $\pi^* D$ is nef and big, we conclude that $(\pi^* D)(\tilde{D} - 2\tilde{E}) > 0$, which is equivalent to

$$2(\pi^* D)E < D^2. \tag{4.5}$$

Other inequalities we are going to use in this section are derived from Hodge index theorem. As $\pi^*(\pi_* E) = E + (EL_x)L_x$, we have $(\pi_* E)^2 = E^2 + (EL_x)^2$. By Hodge index theorem, it follows that

$$D^2(E^2 + (EL_x)^2) = D^2(\pi_* E)^2 \leq ((\pi^* D)E)^2.$$

Combine the above inequality with (4.3), and we deduce that

$$D^2 \left((\pi^* D)E - (2 - \mu_x + \lambda_x)EL_x + (EL_x)^2\right) \leq ((\pi^* D)E)^2. \tag{4.6}$$

That is,

$$0 \leq ((\pi^* D)E)^2 - D^2 ((\pi^* D)E) + D^2 \left((2 - \mu_x + \lambda_x)EL_x - (EL_x)^2\right). \tag{4.7}$$

On the other hand, (4.5) and (4.6) also imply that

$$(\pi^* D)E - (2 - \mu_x + \lambda_x)EL_x + (EL_x)^2 \leq \frac{((\pi^* D)E)^2}{D^2} < \frac{(\pi^* D)E}{2},$$

i.e.,

$$(EL_x)^2 - (2 - \mu_x + \lambda_x)EL_x < -\frac{(\pi^* D)E}{2} < 0. \tag{4.8}$$

All inequalities presented above will be used throughout the rest of the proof.

Step 3 We claim that $\lambda_x = 0$, in particular, $\tilde{E} = E$.
Assume on the contrary that $\lambda_x > 0$. Then by Theorem 1.2 (i), we know that

$$(2 - \mu_x + \lambda_x) - EL_x = (\tilde{D} - \tilde{E})L_x \leq 0.$$

However, (4.8) suggests that $EL_x - (2 - \mu_x + \lambda_x) < 0$. This is a contradiction.
Step 4 In this step, we show that $\lceil E \rceil L_x = 1$.
Recall that so far, we have $\tilde{E} = E$ and $\lambda_x = 0$. Thus (4.8) now reads as $EL_x < 2 - \mu_x$. Therefore, by (4.4), we obtain

$$\lceil E \rceil L_x \leq EL_x + \mu_x < 2.$$

This means that $\lceil E \rceil L_x \leq 1$. However, if $\lceil E \rceil L_x = 0$, then $EL_x = 0$ and we would have

$$0 > E^2 \geq (\pi^* D)E > 0$$

by (4.3). This is a contradiction, which forces that $\lceil E \rceil L_x = 1$.
Step 5 In the last step, we prove that $\lceil E \rceil L_x = 1$ yields a contradiction to (4.7).
Notice that $\lceil E \rceil L_x = 1$ simply implies that E_1 has exactly one irreducible component C with its multiplicity $0 < c \leq 1$. Thus $\pi_* C$ is smooth at x. By our assumption, we have

$$(\pi^* D)E \geq (\pi^* D)E_1 \geq c\beta_1.$$

Consider the quadratic polynomial

$$\begin{aligned} F(T) &:= T^2 - (D^2)T + D^2\left((2 - \mu_x)EL_x - (EL_x)^2\right) \\ &= T^2 - (D^2)T + cD^2(2 - \mu_x - c) \end{aligned}$$

in one variable T. Evaluating $F(T)$ at $T = c\beta_1$, we get

$$F(c\beta_1) = c\left(c\beta_1^2 + D^2(2 - \mu_x - \beta_1 - c)\right).$$

If $\mu_x \geq 1$, then the corresponding $\beta_1 = 2 - \mu_x \leq \beta_2$. Notice that $D^2 > \beta_2^2$. Thus we have

$$F(c\beta_1) = c^2(\beta_1^2 - D^2) < 0.$$

If $\mu_x < 1$, then

$$\beta_1 = \frac{\beta_2}{\beta_2 - (1 - \mu_x)} = 1 + \frac{1 - \mu_x}{\beta_2 - (1 - \mu_x)}.$$

Notice that $c = EL_x \geq \lceil E \rceil L_x - \mu_x = 1 - \mu_x > 0$. Thus

$$\begin{aligned} 2 - \mu_x - \beta_1 - c &= (1 - \mu_x) + (1 - \beta_1) - c \\ &= (1 - \mu_x)\left(1 - \frac{\beta_1}{\beta_2}\right) - c \\ &\leq \left(1 - \frac{\beta_1}{\beta_2}\right)c - c \\ &= -\left(\frac{\beta_1}{\beta_2}\right)c. \end{aligned} \tag{4.9}$$

As a result, we have

$$F(c\beta_1) \le c^2 \left(\beta_1^2 - \left(\frac{\beta_1}{\beta_2}\right) D^2\right) < c^2(\beta_1^2 - \beta_1\beta_2) \le 0,$$

i.e., $F(c\beta_1) < 0$ also holds in this case.
Nevertheless, we have got a contradiction. Because $T = \frac{D^2}{2}$ is the axis of symmetry of $F(T)$ and $c\beta_1 \le (\pi^* D)E < \frac{D^2}{2}$ from (4.5), we deduce that

$$F((\pi^* D)E) \le F(c\beta_1) < 0,$$

which contradicts (4.7).

This completes the whole proof. □

5. Separation of two points. The main result in this section is the following:

THEOREM 5.1. *Let D be a nef and big $\mathbb{Q}$-divisor on a smooth projective surface S. Let $x, y \in S$ be two distinct closed points. Then $|K_S + \lceil D \rceil|$ separates x and y provided that one of the following holds:*

(1) $\mu_x, \mu_y \ge 2$

(2) $0 \le \mu_x < 2$, $\mu_y \ge 2$, *D is locally ample at x, $D^2 > \beta_{2,x}^2$ and $DC \ge \beta_{1,x}$ for any irreducible curve C on S smoothly passing through x, where*

$$\beta_{2,x} \ge 2 - \mu_x \quad \textit{and} \quad \beta_{1,x} = \min\left\{2 - \mu_x, \frac{\beta_{2,x}}{\beta_{2,x} - (1 - \mu_x)}\right\}.$$

(3) $\mu_x \ge 2$, $0 \le \mu_y < 2$, *D is locally ample at y, $D^2 > \beta_{2,y}^2$ and $DC \ge \beta_{1,y}$ for any irreducible curve C on S smoothly passing through y, where*

$$\beta_{2,y} \ge 2 - \mu_y \quad \textit{and} \quad \beta_{1,y} = \min\left\{2 - \mu_y, \frac{\beta_{2,y}}{\beta_{2,y} - (1 - \mu_y)}\right\}.$$

(4) $0 \le \mu_x, \mu_y < 2$, *D is locally ample at both x and y, $D^2 > \beta_{2,x}^2 + \beta_{2,y}^2$, $DC \ge \beta_{1,x}$ (resp. $DC \ge \beta_{1,y}$) for any irreducible curve C on S passing through x (resp. y) smoothly, and $DC \ge \beta_{1,x} + \beta_{1,y}$ for any irreducible curve C on S passing through both x and y smoothly. Here $\beta_{1,x}$ and $\beta_{2,x}$ (resp. $\beta_{1,y}$ and $\beta_{2,y}$) are the same as in (2) (resp. in (3)).*

This result refines the previous one proved by Maşek [Maş99, Propositin 4] in the sense that we find that all *critical* curves that we need to consider are only the ones smooth at x or y, or both.

We devote the whole section to the proof of this theorem. At first, we fix some notation that will be used throughout this section. Let $\pi : \tilde{S} \to S$ be the blowing up of S at x and y with exceptional divisors L_x and L_y respectively. Notice that we have

$$\pi^* \lceil D \rceil = \lceil \pi^* D \rceil + \lfloor \mu_x \rfloor L_x + \lfloor \mu_y \rfloor L_y.$$

Similar to the proof of Theorem 4.1, it suffices to prove that

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^* \lceil D \rceil - 2L_x - 2L_y)) = 0. \tag{5.1}$$

5.1. Proof of Case (1). As in the proof of Theorem 4.1, we have

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^* \lceil D \rceil - \lfloor \mu_x \rfloor L_x - \lfloor \mu_y \rfloor L_y)) = 0.$$

Since $\mu_x \ge 2$, by Theorem 3.1,

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^* \lceil D \rceil - 2L_x - \lfloor \mu_y \rfloor L_y)) = 0.$$

Then (5.1) follows simply by applying Theorem 3.1 again to $\lfloor \mu_y \rfloor$.

5.2. Proof of Case (2) and (3). These two cases are quite similar. Here we only prove Case (2), and Case (3) can be proved in the same way.

Recall that $0 \leq \mu_x < 2$. Let $\tilde{D} = \pi^* D - (2-\mu_x)L_x$. Then we have

$$\lceil \tilde{D} \rceil = \pi^* \lceil D \rceil - 2L_x - \lfloor \mu_y \rfloor L_y.$$

Just adopting the same argument as in the proof of Theorem 4.1 (2) verbatim, we conclude that

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^* \lceil D \rceil - 2L_x - \lfloor \mu_y \rfloor L_y)) = 0.$$

We leave the proof here to the interested reader. Finally, since $\mu_y \geq 2$, we can apply Theorem 3.1 again to get (5.1).

5.3. Proof of Case (4). Take $\tilde{D} = \pi^* D - (2-\mu_x)L_x - (2-\mu_y)L_y$. Suppose that the theorem does not hold true. i.e., $H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \lceil \tilde{D} \rceil)) \neq 0$. Then we can find a nonzero effective divisor $\tilde{E}$ as is described in Theorem 1.2. Again, we write

$$\tilde{E} = E + \lambda_x L_x + \lambda_y L_y,$$

and let $E_1 \leq E$ be the effective $\mathbb{Q}$-divisor consisting of all irreducible components which meet either L_x or L_y properly.

The proof here is in the same manner as that of Theorem 4.1.

Step 1 We first prove that $E_1 \neq 0$.

If not, then $EL_x \leq 0$ and $EL_y \leq 0$ and we would have

$$(\tilde{D} - \lambda_x L_x - \lambda_y L_y)E = (\pi^* D)E - (2-\mu_x+\lambda_x)EL_x - (2-\mu_y+\lambda_y)EL_y \geq 0 > E^2,$$

a contradiction to Theorem 1.2 (i).

Similar as before, the local ampleness of D implies that $(\pi^* D)E > 0$.

Step 2 Parallel to the proof of Theorem 4.1 Step 2, here we also have several numerical inequalities.

Similar to (4.3), we have

$$\begin{aligned} 0 > E^2 &= \tilde{E}E - \lambda_x EL_x - \lambda_y EL_y \\ &\geq (\pi^* D)E - (2-\mu_x+\lambda_x)EL_x - (2-\mu_y+\lambda_y)EL_y. \end{aligned} \tag{5.2}$$

Similar to (4.4), here we have two inequalities as follows:

$$(\lceil E \rceil - E)L_x \leq \mu_x, \tag{5.3}$$
$$(\lceil E \rceil - E)L_y \leq \mu_y. \tag{5.4}$$

It is easy to see that (4.5) also holds here, i.e.,

$$2(\pi^* D)E < D^2. \tag{5.5}$$

Notice that in this case $\pi^*(\pi_* E) = E + (EL_x)L_x + (EL_y)L_y$. Then $(\pi_* E)^2 = E^2 + (EL_x)^2 + (EL_y)^2$. Using the same technique as in the proof of Theorem 4.1 Step 2, we deduce that

$$\begin{aligned} 0 \leq &\left((\pi^* D)E\right)^2 - D^2\left((\pi^* D)E\right) + \\ &D^2\left((2-\mu_x+\lambda_x)EL_x + (2-\mu_y+\lambda_y)EL_y - (EL_x)^2 - (EL_y)^2\right), \end{aligned} \tag{5.6}$$

as well as

$$(EL_x)^2 + (EL_y)^2 - (2 - \mu_x + \lambda_x)EL_x - (2 - \mu_y + \lambda_y)EL_y < 0. \tag{5.7}$$

As a consequence of (5.7), one of the following inequalities must be true:

$$EL_x < 2 - \mu_x + \lambda_x, \tag{5.8}$$
$$EL_y < 2 - \mu_y + \lambda_y. \tag{5.9}$$

Without loss of the generality, from now on, we assume that (5.8) holds.
Step 3 In this step, we show that $\lambda_x = 0$ and thus $\lceil E \rceil L_x \leq 1$.
Otherwise, L_x is contained in $\tilde{E}$. By Theorem 1.2 (i),

$$0 \geq (\tilde{D} - \tilde{E})L_x = 2 - \mu_x + \lambda_x - EL_x.$$

However, this contradicts (5.8).
As a result, (5.8) now reads as $EL_x < 2 - \mu_x$. Together with (5.3), we deduce that

$$\lceil E \rceil L_x \leq EL_x + \mu_x < 2,$$

i.e., $\lceil E \rceil L_x \leq 1$.
Step 4 In this step, we prove the theorem when $\lceil E \rceil L_x = 0$.
In this case, we have $EL_x = 0$. Then (5.7) implies that (5.9) holds here. Using a similar argument to that in Step 3, we conclude that $\lambda_y = 0$ and $\lceil E \rceil L_y \leq 1$. Notice that (5.7) also guarantees that $EL_y > 0$. Therefore, we conclude that there is only one irreducible component, say A, in E_1. Moreover, $AL_x = 0$ and $AL_y = 1$. This implies that $\pi_* A$ is smooth at y but not passing through x.
Now we apply the same argument as in the proof of Theorem 4.1 Step 5 to get a contradiction. We leave the proof to the interested reader as it is just identical to the proof of Theorem 4.1.
Step 5 In this step, we prove the theorem when $\lceil E \rceil L_x = 1$.
Recall that $\lceil E \rceil L_x = 1$ means that there is only one irreducible component A (with its multiplicity $0 < a \leq 1$) in E_1 that meets L_x with $AL_x = 1$.
Consider the following quadratic polynomial

$$\begin{aligned} G(T) &:= T^2 - (D^2)T + D^2\left((2 - \mu_x - \lambda_x)EL_x + (2 - \mu_y + \lambda_y)EL_y - (EL_x)^2 - (EL_y)^2\right) \\ &= T^2 - (D^2)T + D^2\left((2 - \mu_x)a + (2 - \mu_y + \lambda_y)EL_y - a^2 - (EL_y)^2\right) \end{aligned}$$

in one variable T. Notice that the axis of symmetry of $G(T)$ is $T = \frac{D^2}{2}$. We evaluate $G(T)$ at $T = a\beta_{1,x}$, and it follows that

$$\begin{aligned} G\left(a\beta_{1,x}\right) &= a^2\beta_{1,x}^2 - a\beta_{1,x}D^2 + D^2\left((2 - \mu_x)a + (2 - \mu_y + \lambda_y)EL_y - a^2 - (EL_y)^2\right) \\ &= a\left(a\beta_{1,x}^2 + (2 - \mu_x - \beta_{1,x} - a)D^2\right) + D^2\left((2 - \mu_y + \lambda_y)EL_y - (EL_y)^2\right). \end{aligned}$$

We start the whole proof in this case from the following two claims.
Claim I. We always have

$$a\left(a\beta_{1,x}^2 + (2 - \mu_x - \beta_{1,x} - a)D^2\right) < 0.$$

In fact, when $\mu_x \geq 1$, then $\beta_{1,x} = 2 - \mu_x$. Hence by the assumption on D^2, it follows that

$$a\left(a\beta_{1,x}^2 + (2 - \mu_x - \beta_{1,x} - a)D^2\right) = a^2\left(\beta_{1,x}^2 - D^2\right) < 0.$$

If $\mu_x < 1$, then we know that $a \geq \lceil E \rceil L_x - \mu_x = 1 - \mu_x$. Similar to (4.9), we deduce that

$$2 - \mu_x - \beta_{1,x} - a \leq -\left(\frac{\beta_{1,x}}{\beta_{2,x}}\right) a. \tag{5.10}$$

As a result, we deduce that

$$\begin{aligned} & a\left(a\beta_{1,x}^2 + (2-\mu_x-\beta_{1,x}-a)D^2\right) \\ \leq\ & a^2\left(\beta_{1,x}^2 - \left(\frac{\beta_{1,x}}{\beta_{2,x}}\right) D^2\right) < a^2\left(\beta_{1,x}^2 - \beta_{1,x}\beta_{2,x}\right) \leq 0. \end{aligned}$$

This completes the proof of Claim I.
Claim II. We have

$$G\left(a\beta_{1,x}\right) \geq 0.$$

Actually, by (5.5), $(\pi^* D)E$ lies on the left of the axis of symmetry of $G(T)$, and by (5.6), $G(T) \geq 0$ when $T = (\pi^* D)E$. Since $(\pi^* D)E \geq (\pi^* D)E_1 \geq a\beta_{1,x}$, we conclude that $G\left(a\beta_{1,x}\right) \geq 0$, which is just the desired result.
It is fairly obvious that the above two claims imply that

$$D^2\left((2-\mu_y+\lambda_y)EL_y - (EL_y)^2\right) > 0,$$

i.e.

$$(2-\mu_y+\lambda_y)EL_y - (EL_y)^2 > 0.$$

This is equivalent to $0 < EL_y < 2 - \mu_y + \lambda_y$. In particular, (5.9) holds. Similar to Step 3, we obtain that $\lambda_y = 0$ and $\lceil E \rceil L_y \leq 1$.
Step 5.1. Here we give the proof when

$$\lceil E \rceil L_y = 1.$$

In this case, there is exactly one irreducible component B (with its multiplicity $0 < b \leq 1$) in E_1 that meets L_y with $BL_y = 1$. We have the following two possibilities: $A \neq B$ or $A = B$.
We first consider the case when $A \neq B$. The proof here will apply to all the other cases, even when $\lceil E \rceil L_y = 0$.
In this case, $E_1 = aA + bB$, and

$$(\pi^* D)E \geq (\pi^* D)E_1 \geq a\beta_{1,x} + b\beta_{1,y}.$$

Our approach is to evaluate $G(T)$ at $T = a\beta_{1,x} + b\beta_{1,y}$. Similar to Claim II in the above, $(\pi^* D)E \geq a\beta_{1,x} + b\beta_{1,y}$ implies that $G(a\beta_{1,x} + b\beta_{1,y}) \geq 0$. In the following, we will finish the proof by showing that $G(a\beta_{1,x} + b\beta_{1,y}) < 0$ in any case.
In fact, we have

$$\begin{aligned} & G(a\beta_{1,x} + b\beta_{1,y}) \\ =\ & (a\beta_{1,x} + b\beta_{1,y})^2 - (a\beta_{1,x} + b\beta_{1,y})D^2 + D^2\left((2-\mu_x)a + (2-\mu_y)b - a^2 - b^2\right) \\ =\ & (a\beta_{1,x} + b\beta_{1,y})^2 + (2-\mu_x-\beta_{1,x}-a)aD^2 + (2-\mu_y-\beta_{1,y}-b)bD^2. \end{aligned}$$

Suppose first that $\mu_x \geq 1$. Then the corresponding $\beta_{1,x} = 2 - \mu_x \leq \beta_{2,x}$. If $\mu_y \geq 1$, then $\beta_{1,y} = 2 - \mu_y \leq \beta_{2,y}$. Thus we have

$$\begin{aligned} G(a\beta_{1,x} + b\beta_{1,y}) &= (a\beta_{1,x} + b\beta_{1,y})^2 - (a^2 + b^2)D^2 \\ &< (a\beta_{1,x} + b\beta_{1,y})^2 - (a^2 + b^2)(\beta_{2,x}^2 + \beta_{2,y}^2) \\ &\leq 0. \end{aligned}$$

If $\mu_y < 1$, then by (5.4), we know that $b = EL_y \geq \lceil E \rceil L_y - \mu_y = 1 - \mu_y$. Similar to (5.10), we deduce that

$$2 - \mu_y - \beta_{1,y} - b \leq -\left(\frac{\beta_{1,y}}{\beta_{2,y}}\right) b. \tag{5.11}$$

As a result, we have

$$\begin{aligned} G(a\beta_{1,x} + b\beta_{1,y}) &\leq (a\beta_{1,x} + b\beta_{1,y})^2 - \left(a^2 + \left(\frac{\beta_{1,y}}{\beta_{2,y}}\right) b^2\right) D^2 \\ &< (a\beta_{1,x} + b\beta_{1,y})^2 - \left(a^2 + \left(\frac{\beta_{1,y}}{\beta_{2,y}}\right) b^2\right) (\beta_{2,x}^2 + \beta_{2,y}^2) \\ &\leq (a\beta_{1,x} + b\beta_{1,y})^2 - \left(a\beta_{2,x} + b\beta_{2,y}\sqrt{\frac{\beta_{1,y}}{\beta_{2,y}}}\right)^2 \\ &\leq (a\beta_{1,x} + b\beta_{1,y})^2 - (a\beta_{2,x} + b\beta_{1,y})^2 \\ &\leq 0. \end{aligned}$$

Therefore, $G(a\beta_{1,x} + b\beta_{1,y}) < 0$ when $\mu_x \geq 1$.

Now assume that $\mu_x < 1$. If $\mu_y \geq 1$, then the above proof for $\mu_x \geq 1$ and $\mu_y < 1$ is also applicable here just by interchanging x and y. If $\mu_y < 1$, then both (5.10) and (5.11) hold in this case. Therefore, we have

$$\begin{aligned} G(a\beta_{1,x} + b\beta_{1,y}) &\leq (a\beta_{1,x} + b\beta_{1,y})^2 - \left(\left(\frac{\beta_{1,x}}{\beta_{2,x}}\right) a^2 + \left(\frac{\beta_{1,y}}{\beta_{2,y}}\right) b^2\right) D^2 \\ &< (a\beta_{1,x} + b\beta_{1,y})^2 - \left(\left(\frac{\beta_{1,x}}{\beta_{2,x}}\right) a^2 + \left(\frac{\beta_{1,y}}{\beta_{2,y}}\right) b^2\right) (\beta_{2,x}^2 + \beta_{2,y}^2) \\ &\leq (a\beta_{1,x} + b\beta_{1,y})^2 - \left(a\beta_{2,x}\sqrt{\frac{\beta_{1,x}}{\beta_{2,x}}} + b\beta_{2,y}\sqrt{\frac{\beta_{1,y}}{\beta_{2,y}}}\right)^2 \\ &\leq (a\beta_{1,x} + b\beta_{1,y})^2 - (a\beta_{1,x} + b\beta_{1,y})^2 \\ &= 0. \end{aligned}$$

Thus we see that $G(a\beta_{1,x} + b\beta_{1,y}) < 0$ when $\mu_x < 1$. This completes the proof when $A \neq B$.

Now the proof for $A = B$ becomes very easy. Notice that $A = B$ implies that $E_1 = aA$, where $0 < a \leq 1$ and $\pi_* A$ is smooth at both x and y. Then the proof for this case is identical to the previous one by simply letting $b = a$.

Step 5.2. Finally, we consider the case when $\lceil E \rceil L_y = 0$. Then there is no component in E that meets L_y. The proof here is already very straightforward, and we just need to repeat the proof in Step 5.1 by setting $b = 0$.

6. Separation of tangent directions. In this section, we consider the problem about separating tangent directions by the adjoint linear system $|K_S + \lceil D \rceil|$. We fix some notation first. Let x be a closed point on S and $\vec{v}$ a tangent direction at x. Let $f : S' \to S$ be the blowing up of S at x with the exceptional curve L'_x. Denote by $p \in L'_x$ the closed point on L'_x corresponding to the tangent direction $\vec{v}$. Let $g : \tilde{S} \to S'$ be the blowing up of S' at p with the exceptional curve L_p. Write $L_x = g^{-1}L'_x$. Then $L_x^2 = -2$ and $L_x L_p = 1$.

For any $\mathbb{Q}$-divisor D on S, we define

$$\hat{\mu}_{\vec{v}} := \mathrm{mult}_p(\lceil f^{-1}D \rceil - f^{-1}D).$$

Denote by $\pi = g \circ f : \tilde{S} \to S$ the composition of f and g. We thus have

$$\begin{aligned} \mu_x &= (\lceil f^{-1}D \rceil - f^{-1}D)L'_x = (\lceil \pi^{-1}D \rceil - \pi^{-1}D)(L_x + L_p), \\ \hat{\mu}_{\vec{v}} &= (g^{-1}(\lceil f^{-1}D \rceil - f^{-1}D))L_p = (\lceil \pi^{-1}D \rceil - \pi^{-1}D)L_p. \end{aligned}$$

Write $\mu_{\vec{v}} := \mu_x + \hat{\mu}_{\vec{v}}$. We observe that $\mu_x - \hat{\mu}_{\vec{v}} = (\lceil \pi^{-1}D \rceil - \pi^{-1}D)L_x \geq 0$. This implies that

$$\hat{\mu}_{\vec{v}} \leq \mu_x \quad \text{and} \quad 2\hat{\mu}_{\vec{v}} \leq \mu_{\vec{v}} \leq 2\mu_x. \tag{6.1}$$

All above notation are also used in [Maş99].

Definition 6.1. *Let C be an irreducible curve passing through x. We say that C passes through $\vec{v}$ smoothly, if p is a smooth point on $f^{-1}C$.*

Note that $g^*(f^{-1}C) = \pi^{-1}C + ((\pi^{-1}C)L_p)L_p$. Hence C passes through $\vec{v}$ smoothly if and only if $(\pi^{-1}C)L_p = \mathrm{mult}_p(f^{-1}C) = 1$.

Here is the main result in this section.

Theorem 6.2. *Let D be a nef and big $\mathbb{Q}$-divisor on a smooth projective surface S. Let $x \in S$ be a closed point and let $0 \neq \vec{v} \in T_x(S)$ be a tangent direction at x. Then $|K_S + \lceil D \rceil|$ separates $\vec{v}$ at x, provided that one of the following holds:*

(1) $\mu_x \geq 3$.
(2) $\mu_{\vec{v}} \geq 4$.
(3) $2 \leq \mu_x < 3$, $2 \leq \mu_{\vec{v}} < 4$, *D is locally ample at x, $D^2 > \beta_2^2$ and $DC \geq 4 - \mu_{\vec{v}}$ for every irreducible curve C passing through $\vec{v}$ smoothly, where $\beta_2 \geq 4 - \mu_{\vec{v}}$. When $\hat{\mu}_{\vec{v}} < 1$, we further assume that $DC \geq 2\beta_1$ for every irreducible curve C singular at x of order two and $\vec{v} \notin T_x(C)$, and $DC \geq \beta_1$ for every irreducible curve C passing through x smoothly and $\vec{v} \notin T_x(C)$, where*

$$\beta_1 = \min\left\{3 - \mu_x, \frac{\beta_2(1 - \hat{\mu}_{\vec{v}})}{\beta_2 - (2 + \hat{\mu}_{\vec{v}} - \mu_x)}\right\}.$$

(4) $0 \leq \mu_x < 2$, *D is locally ample at x, $D^2 > \beta_{2,x}^2 + \beta_{2,p}^2$, $DC \geq \beta_1$ for every irreducible curve C passing through x smoothly and $\vec{v} \notin T_x(C)$, and $DC \geq 2\beta_1$ for every irreducible curve C passing through $\vec{v}$ smoothly. When $\hat{\mu}_{\vec{v}} < 1$, we further assume that $DC \geq 2\beta_1$ for every irreducible curve C singular at x of order two and $\vec{v} \notin T_x(C)$. Here $\beta_{2,x}$, $\beta_{2,p}$ and β_1 are real numbers such that $\beta_{2,x} \geq 2 - \mu_x$, $\beta_{2,p} \geq 2 - \hat{\mu}_{\vec{v}}$, and*

$$\beta_1 = \min\left\{\frac{1}{2}(4 - \mu_{\vec{v}}), \frac{\beta_{2,x} + \beta_{2,p}}{\beta_{2,x} + \beta_{2,p} - (2 - \mu_{\vec{v}})}\right\}.$$

Before stating the proof, we would like to remark that in Case (3), we have

$$\beta_1 = \begin{cases} 3-\mu_x, & \mu_x - \hat{\mu}_{\vec{v}} \geq 2; \\ \dfrac{\beta_2(1-\hat{\mu}_{\vec{v}})}{\beta_2 - (2+\hat{\mu}_{\vec{v}} - \mu_x)}, & \mu_x - \hat{\mu}_{\vec{v}} < 2. \end{cases}$$

In particular, we always have $\beta_1 \leq \frac{1}{2}(4-\mu_{\vec{v}})$, and the number $\frac{1}{2}(4-\mu_{\vec{v}})$ was the β_1 deduced in [Maş99, Proposition 5]. Moreover, the above inequality is strict once $\mu_x - \hat{\mu}_{\vec{v}} \neq 2$.

In Case (4), we have

$$\beta_1 = \begin{cases} \dfrac{1}{2}(4-\mu_{\vec{v}}), & \mu_{\vec{v}} \geq 2; \\ \dfrac{\beta_{2,x}+\beta_{2,p}}{\beta_{2,x}+\beta_{2,p} - (2-\mu_{\vec{v}})}, & 0 \leq \mu_{\vec{v}} < 2. \end{cases}$$

To prove Theorem 6.2, it is equivalent to prove that $|K_{S'} + f^*\lceil D\rceil - 2L'_x|$ is base point free at p. Using the same observation as in Theorem 4.1, we only need to prove that

$$H^1\left(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^*\lceil D\rceil - 2L_x - 4L_p)\right) = 0. \tag{6.2}$$

Write $\nu_x = \mathrm{mult}_x D$ and $\hat{\nu}_{\vec{v}} = \mathrm{mult}_p(f^{-1}D)$. Notice that $\mu_{\vec{v}} = \mu_x + \hat{\mu}_{\vec{v}}$. Then we have

$$f^*\lceil D\rceil = \lceil f^*D\rceil + \lfloor \mu_x\rfloor L'_x$$

and

$$\begin{aligned} \pi^*\lceil D\rceil &= g^*\left(f^{-1}\lceil D\rceil + (\nu_x+\mu_x)L'_x\right) \\ &= \pi^{-1}\lceil D\rceil + (\hat{\nu}_{\vec{v}} + \hat{\mu}_{\vec{v}})L_p + (\nu_x+\mu_x)(L_x+L_p) \\ &= \lceil g^*(f^{-1}D) + \hat{\mu}_{\vec{v}}L_p + (\nu_x+\mu_x)(L_x+L_p)\rceil \\ &= \lceil \pi^*D + \mu_x(L_x+L_p) + \hat{\mu}_{\vec{v}}L_p\rceil \\ &= \lceil \pi^*D\rceil + \lfloor\mu_x\rfloor L_x + \lfloor \mu_{\vec{v}}\rfloor L_p. \end{aligned}$$

6.1. Proof of Case (1). By Lemma 2.1, we have

$$H^1(\mathcal{O}_{S'}(K_{S'} + f^*\lceil D\rceil - \lfloor\mu_x\rfloor L'_x)) = H^1(\mathcal{O}_{S'}(K_{S'} + \lceil f^*D\rceil)) = 0.$$

Since $\mu_x \geq 3$, by Proposition 3.1, we obtain

$$H^1(\mathcal{O}_{S'}(K_{S'} + f^*\lceil D\rceil - 3L'_x)) = 0.$$

Therefore, the following map

$$H^0(\mathcal{O}_{S'}(K_{S'} + f^*\lceil D\rceil - 2L'_x)) \to H^0(\mathcal{O}_{L'_x}(K_{S'} + f^*\lceil D\rceil - 2L'_x))$$

is surjective. Notice that $\mathcal{O}_{L'_x}(K_{S'} + f^*\lceil D\rceil - 2L'_x) = \mathcal{O}_{L'_x}(-L'_x)$ is globally generated. Thus the above surjectivity implies that $|K_{S'} + f^*\lceil D\rceil - 2L'_x|$ is base point free at any closed point on L'_x, in particular, at p.

6.2. Proof of Case (2). Since $\mu_{\vec{v}} \geq 4$, by (6.1), we have $\mu_x \geq 2$. We may assume that $\mu_x < 3$, otherwise we may go back to Case (1). Thus $\lfloor \mu_x \rfloor = 2$. Similar to Case (1), we have

$$H^1(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \pi^*\lceil D \rceil - 2L_x - \lfloor \mu_{\vec{v}} \rfloor L_p)) = 0.$$

This implies (6.2) by using Proposition 3.1 again.

6.3. Proof of Case (3). In this case, we take

$$\tilde{D} = \pi^* D - (4 - \mu_{\vec{v}})L_p.$$

By the assumption, $\tilde{D}^2 > 0$ and $\tilde{D}$ is big. Moreover, since $\lfloor \mu_x \rfloor = 2$, we have

$$\lceil \tilde{D} \rceil = \lceil \pi^* D \rceil + \lfloor \mu_{\vec{v}} \rfloor L_p - 4L_p = \pi^* \lceil D \rceil - 2L_x - 4L_p.$$

Assume that (6.2) does not hold. Then $H^1(\mathcal{O}_{\tilde{S}}(-\lceil \tilde{D} \rceil)) \neq 0$. Again, by Theorem 1.2, we can find a divisor $\tilde{E}$ as before. We write

$$\tilde{E} = E + \lambda_x L_x + \lambda_p L_p,$$

and let $E_1 \leq E$ be the effective $\mathbb{Q}$-divisor consisting of all irreducible components which meet either L_x or L_p properly. Notice that by Theorem 1.2 (iii) and the construction of $\tilde{D}$, we know that λ_p and $\lambda_x + \mu_x$ are both integers.

Step 1 In this step, we prove that $E_1 \neq 0$.

Suppose on the contrary that $E_1 = 0$. Then we have $\lambda_x > \lambda_p$. Otherwise, since

$$\tilde{E}L_p = (\lambda_x L_x + \lambda_p L_p)L_p = \lambda_x - \lambda_p \leq 0,$$

we would get a contradiction as follows:

$$0 > \tilde{E}^2 \geq (\pi^* D)\tilde{E} - (4 - \mu_{\vec{v}})\tilde{E}L_p \geq (4 - \mu_{\vec{v}})(\lambda_p - \lambda_x) \geq 0.$$

In particular, $\lambda_x > 0$. It implies that L_x is contained in $\tilde{E}$. Thus it follows that

$$(\lceil \lambda_x \rceil L_x + \lambda_p L_p)L_x = \lceil \tilde{E} \rceil L_x \geq \lceil \tilde{D} \rceil L_x = 0,$$

i.e., $\lambda_p - 2\lceil \lambda_x \rceil \geq 0$. However, this contradicts with $\lambda_x > \lambda_p$.

In the following steps, we always assume that $E_1 \neq 0$. Then $\pi_* E_1$ is a strictly effective divisor passing through x. Hence by the local ampleness of D at x, we have $(\pi^* D)E > 0$ as before.

Step 2 Similar to (5.2), we have

$$\begin{aligned} 0 > E^2 &= \tilde{E}E - \lambda_x EL_x - \lambda_p EL_p \\ &\geq (\pi^* D)E - \lambda_x EL_x - (4 - \mu_{\vec{v}} + \lambda_p)EL_p. \end{aligned} \tag{6.3}$$

Corresponding to (5.3) and (5.4) for separating two points, we still have the following two inequalities based on the definitions of μ_x and $\hat{\mu}_{\vec{v}}$ at the beginning:

$$(\lceil E \rceil - E)L_p \leq \left(\pi^{-1}(\lceil D \rceil - D)\right) L_p = \hat{\mu}_{\vec{v}}, \tag{6.4}$$

$$(\lceil E \rceil - E)L_x \leq \left(\pi^{-1}(\lceil D \rceil - D)\right) L_x = \mu_x - \hat{\mu}_{\vec{v}}. \tag{6.5}$$

Also, the inequality

$$2(\pi^* D)E < D^2 \tag{6.6}$$

holds here.
Notice that in this case, we have $\pi^*(\pi_* E) = E + (EL_x + EL_p)(L_x + L_p) + (EL_p)L_p$. As $(L_p + L_x)L_p = 0$ and $(L_p + L_x)^2 = L_p^2 = -1$, we obtain

$$(\pi_* E)^2 = E^2 + (EL_p)^2 + (EL_x + EL_p)^2.$$

Applying the same technique for obtaining (4.7) and (5.6), we deduce that

$$0 \leq ((\pi^* D)E)^2 - D^2((\pi^* D)E) + D^2 \left(\lambda_x EL_x + (4 - \mu_{\vec{v}} + \lambda_p)EL_p - (EL_x + EL_p)^2 - (EL_p)^2\right). \tag{6.7}$$

The analogue of (4.8) and (5.7) in this case becomes

$$\begin{aligned} 0 &> (EL_x + EL_p)^2 + (EL_p)^2 - \lambda_x EL_x - (4 - \mu_{\vec{v}} + \lambda_p)EL_p \\ &= (EL_x + EL_p)^2 + (EL_p)^2 - \lambda_x(EL_x + EL_p) - (4 - \mu_{\vec{v}} + \lambda_p - \lambda_x)EL_p. \end{aligned} \tag{6.8}$$

Up to now, the proof goes in a similar way as that of Theorem 5.1. However, the two points x and y in Theorem 5.1 are interchangeable in some sense, while here no "symmetry" lies in between x and $\vec{v}$. This can be seen, for example, by comparing (5.7) with (6.8). Therefore, in order to proceed the proof, we have to *take a detour*, and our argument will be subject to λ_x. In particular, we have to take some effort to deal with the case when $\lambda_x = 3 - \mu_x$ which is quite different from the proofs before. Also in the proof of Case (4), there is the same issue.
However, there are also similarities. For example, we still rely on the analysis using a quadratic polynomial. The polynomial we are going to employ in this case is

$$H(T) := T^2 - (D^2)T + D^2 \left(\lambda_x EL_x + (4 - \mu_{\vec{v}} + \lambda_p)EL_p - (EL_x + EL_p)^2 - (EL_p)^2\right). \tag{6.9}$$

Here T is a real variable.
Step 3 Throughout this step, we prove the theorem when $\lambda_x = 0$.
Substituting $\lambda_x = 0$ into (6.8), we obtain that

$$(EL_p)^2 - (4 - \mu_{\vec{v}} + \lambda_p)EL_p < 0,$$

i.e., $0 < EL_p < 4 - \mu_{\vec{v}} + \lambda_p$. This inequality implies that $\lambda_p = 0$. Otherwise, L_p is contained in E and by Theorem 1.2 (i), we would have

$$EL_p = \tilde{E}L_p + \lambda_p \geq \tilde{D}L_p + \lambda_p = 4 - \mu_{\vec{v}} + \lambda_p.$$

This is a contradiction.
Now $\lambda_p = 0$ too. By (6.4), we deduce that

$$\lceil E \rceil L_p \leq EL_p + \hat{\mu}_{\vec{v}} < 4 - \mu_{\vec{v}} + \hat{\mu}_{\vec{v}} = 4 - \mu_x \leq 2,$$

i.e., $\lceil E \rceil L_p \leq 1$. Notice that $\lceil E \rceil L_p \geq EL_p > 0$. It forces that $\lceil E \rceil L_p = 1$. This implies that there is exactly one irreducible component, say A, in E with its multiplicity $0 < a \leq 1$ that meets L_p. Moreover, $AL_p = 1$. Hence by our assumption, $(\pi^* D)E \geq a(\pi^* D)A \geq a(4 - \mu_{\vec{v}})$.
The polynomial (6.9) under the current setting becomes

$$H(T) = T^2 - (D^2)T + D^2 \left((4 - \mu_{\vec{v}})a - (EL_x + a)^2 - a^2\right).$$

Evaluate $H(T)$ at $T = a(4-\mu_{\vec{v}})$, and it follows that

$$\begin{aligned} H\left(a(4-\mu_{\vec{v}})\right) &= a^2(4-\mu_{\vec{v}})^2 - D^2\left((EL_x+a)^2+a^2\right) \\ &< a^2(4-\mu_{\vec{v}})^2 - (4-\mu_{\vec{v}})^2\left((EL_x+a)^2+a^2\right) \\ &= -(4-\mu_{\vec{v}})^2(EL_x+a)^2 \\ &\le 0. \end{aligned}$$

On the other hand, similar to the observation before, we have $a(4-\mu_{\vec{v}}) \le (\pi^*D)E < \frac{D^2}{2}$ from (6.6) and $H\left((\pi^*D)E\right) \ge 0$ from (6.7). These imply that $H\left(a(4-\mu_{\vec{v}})\right) \ge 0$. This is a contradiction. Therefore, the proof is completed in this case.

Step 4 In this step as well as the next one, we assume that $\lambda_x > 0$. Our goal in this step is to prove that

$$\lambda_p = 0, \quad \lambda_x = 3-\mu_x, \quad EL_p = 0, \quad EL_x < 3-\mu_x, \quad \lceil E\rceil L_x = 2, \quad \hat{\mu}_{\vec{v}} < 1. \quad (6.10)$$

We first prove that $\lambda_p = 0$. Otherwise, $\lambda_p > 0$ and by Theorem 1.2 (i), we have

$$EL_p = \tilde{E}L_p + \lambda_p - \lambda_x \ge \tilde{D}L_p + \lambda_p - \lambda_x = 4 - \mu_{\vec{v}} + \lambda_p - \lambda_x.$$

This, together with (6.8), implies that

$$(EL_x + EL_p)^2 - \lambda_x(EL_x + EL_p) < 0,$$

i.e., $0 < EL_x + EL_p < \lambda_x$. On the other hand, by Theorem 1.2 (i) again, both $\lambda_x, \lambda_p > 0$ implies that

$$0 = \tilde{D}(L_x + L_p) \le \tilde{E}(L_x + L_p) = EL_x + EL_p - \lambda_x.$$

Therefore, we get a contradiction. As a result, $\tilde{E} = E + \lambda_x L_x$ now.
Second, we prove that $\lambda_x = 3 - \mu_x$ and $EL_p = 0$. Since $\lambda_x > 0$, by Theorem 1.2 (i) once more, we know that

$$EL_x \ge \tilde{D}L_x - \lambda_x L_x^2 \ge 2\lambda_x - (4 - \mu_{\vec{v}}).$$

We claim that $EL_p < 4 - \mu_{\vec{v}} - \lambda_x$. This is again from (6.8) and similar to the above proof for $\lambda_p = 0$. Suppose on the contrary that $EL_p \ge 4 - \mu_{\vec{v}} - \lambda_x$. Then by (6.8), $EL_x + EL_p < \lambda_x$. Combine this with the above lower bound of EL_x, and we have

$$EL_p < \lambda_x - EL_x \le 4 - \mu_{\vec{v}} - \lambda_x.$$

This is a contradiction. Hence the claim holds. A simple consequence of the above claim is that $\lambda_x + EL_p < 4 - \mu_{\vec{v}} < 4 - \mu_x$. Recall that $\lambda_x + \mu_x$ must be an integer and $\mu_x \ge 2$. This forces that $\lambda_x + \mu_x = 3$, i.e.,

$$\lambda_x = 3 - \mu_x.$$

Substitute this equality into the above inequality, we deduce that

$$EL_p < 4 - \mu_{\vec{v}} - (3 - \mu_x) = 1 - \hat{\mu}_{\vec{v}}.$$

Hence by (6.4), it follows that $\lceil E\rceil L_p < EL_p + \hat{\mu}_{\vec{v}} < 1$. This gives $\lceil E\rceil L_p = 0$ and $EL_p = 0$.

Now we are ready to prove the rest of (6.10). First, notice that $\lceil \tilde{E} \rceil = \lceil E \rceil + L_x$. By Theorem 1.2 (i), we have

$$\lceil E \rceil L_x = (\lceil \tilde{E} \rceil - L_x)L_x \geq \lceil \tilde{D} \rceil L_x + 2 = 2.$$

On the other hand, (6.8) simply reads as $(EL_x)^2 - \lambda_x EL_x < 0$ in this case. It yields $0 < EL_x < \lambda_x = 3 - \mu_x$. Combine this with (6.5) and the above lower bound of $\lceil E \rceil L_x$, we deduce that

$$2 \leq \lceil E \rceil L_x \leq EL_x + \mu_x - \hat{\mu}_{\vec{v}} < 3 - \hat{\mu}_{\vec{v}} \leq 3.$$

This forces that $\lceil E \rceil L_x = 2$ and $\hat{\mu}_{\vec{v}} < 1$. The proof of (6.10) is completed now.

Step 5 In this step, we complete the whole proof based on (6.10).

In fact, (6.10) has put lots of constraints on E and E_1. It is straightforward to see there are only two possibilities for E_1:

(i) $E_1 = b_1B_1 + b_2B_2$, where $b_1, b_2 > 0$, $b_1 + b_2 < 3 - \mu_x$, $B_1L_x = B_2L_x = 1$ and $B_1L_p = B_2L_p = 0$;

(ii) $E_1 = bB$, where $0 < 2b < 3 - \mu_x$, $BL_x = 2$ and $BL_p = 0$.

To unify the notation here, we simply denote $\frac{b_1+b_2}{2}$ by b if we are in Case (i). Thus $EL_x = 2b$ for both cases. Recall that $\hat{\mu}_{\vec{v}} < 1$ now. By the assumption, we have

$$(\pi^* D)E \geq 2b\beta_1$$

in any case. Moreover, the polynomial (6.9) under (6.10) becomes

$$\begin{aligned} H(T) &= T^2 - (D^2)T + D^2((3 - \mu_x)EL_x - (EL_x)^2) \\ &= T^2 - (D^2)T + 2bD^2(3 - \mu_x - 2b). \end{aligned}$$

By (6.7), $H((\pi^* D)E) \geq 0$. Similar argument as before gives $H(2b\beta_1) \geq 0$.

To get a contradiction, in the following, we show that $H(2b\beta_1) < 0$. Evaluate $H(T)$ at $T = 2b\beta_1$. It follows that

$$\begin{aligned} H(2b\beta_1) &= 4b^2\beta_1^2 - 2b\beta_1 D^2 + bD^2(6 - 2\mu_x - 4b) \\ &= 4b^2\beta_1^2 + 2bD^2(3 - \mu_x - \beta_1 - 2b). \end{aligned}$$

When $\mu_x - \hat{\mu}_{\vec{v}} \geq 2$, we have $\beta_1 = 3 - \mu_x$. Notice that $\hat{\mu}_{\vec{v}} < 1$. Thus $3 - \mu_x < 4 - \mu_{\vec{v}} \leq \beta_2$. Therefore, it follows that

$$H(2b\beta_1) = 4b^2(\beta_1^2 - D^2) < 4b^2(\beta_1^2 - \beta_2^2) < 0.$$

When $\mu_x - \hat{\mu}_{\vec{v}} < 2$, we deduce that

$$\beta_1 = \frac{\beta_2(1 - \hat{\mu}_{\vec{v}})}{\beta_2 - (2 + \hat{\mu}_{\vec{v}} - \mu_x)} = (1 - \hat{\mu}_{\vec{v}}) + (2 + \hat{\mu}_{\vec{v}} - \mu_x)\frac{\beta_1}{\beta_2}.$$

In the meantime, we have

$$\frac{\beta_1}{\beta_2} = \frac{1 - \hat{\mu}_{\vec{v}}}{\beta_2 - (2 + \hat{\mu}_{\vec{v}} - \mu_x)} \leq \frac{1 - \hat{\mu}_{\vec{v}}}{(4 - \mu_{\vec{v}}) - (2 + \hat{\mu}_{\vec{v}} - \mu_x)} = \frac{1 - \hat{\mu}_{\vec{v}}}{2 - 2\hat{\mu}_{\vec{v}}} = \frac{1}{2} < 1.$$

Notice that by (6.5), $2b = EL_x \geq \lceil E \rceil L_x - \mu_x + \hat{\mu}_{\vec{v}} = 2 - \mu_x + \hat{\mu}_{\vec{v}}$. Combine these inequalities together, and it follows that

$$\begin{aligned} 3-\mu_x-\beta_1-2b &= (2+\hat{\mu}_{\vec{v}}-\mu_x)+(1-\hat{\mu}_{\vec{v}}-\beta_1)-2b \\ &= (2+\hat{\mu}_{\vec{v}}-\mu_x)\left(1-\frac{\beta_1}{\beta_2}\right)-2b \\ &\le 2b\left(1-\frac{\beta_1}{\beta_2}\right)-2b \\ &= -2b\left(\frac{\beta_1}{\beta_2}\right). \end{aligned}$$

As a result, we deduce that

$$H(2b\beta_1) \le 4b^2\left(\beta_1^2 - D^2\left(\frac{\beta_1}{\beta_2}\right)\right) < 4b^2\left(\beta_1^2-\beta_2^2\left(\frac{\beta_1}{\beta_2}\right)\right) = 4b^2(\beta_1^2-\beta_1\beta_2) < 0.$$

Therefore, we get $H(2b\beta_1) < 0$ in any case. Thus the proof is completed.

6.4. Proof of Case (4). We start with defining

$$\tilde{D} = \pi^*D - (2-\mu_x)(L_x+L_p) - (2-\hat{\mu}_{\vec{v}})L_p = \pi^*D - (2-\mu_x)L_x - (4-\mu_{\vec{v}})L_p.$$

Direct calculations show that $\tilde{D}^2 > 0$ and

$$\lceil \tilde{D} \rceil = \pi^*\lceil D \rceil - 2L_x - 4L_p.$$

Similar to the proof of Case (3), we assume that (6.2) does not hold. Then $H^1(\mathcal{O}_{\tilde{S}}(-\lceil \tilde{D} \rceil)) \neq 0$. Again, by Theorem 1.2, we can find a divisor $\tilde{E}$ as before. We write

$$\tilde{E} = E + \lambda_x L_x + \lambda_p L_p,$$

and let $E_1 \le E$ be the effective $\mathbb{Q}$-divisor consisting of all irreducible components which meet either L_x or L_p properly. Similar but slightly different from Case (3), here both λ_p and λ_x are integers.

In fact, the proof here is similar to that of Case (3), and we are going to deduce the same contradiction. However, for the convenience of the reader, we still present our proof in details and follow the same line as that of Case (3).

Step 1 We start the proof again by showing that $E_1 \neq 0$.

If not, then $E_1 = 0$. It follows that

$$\begin{aligned} \tilde{E}^2 &\ge (\pi^*D)\tilde{E} - (2-\mu_x)\tilde{E}(L_x+L_p) - (2-\hat{\mu}_{\vec{v}})\tilde{E}L_p \\ &\ge (2-\mu_x)\lambda_x - (2-\hat{\mu}_{\vec{v}})(\lambda_x-\lambda_p) \\ &= (2-\hat{\mu}_{\vec{v}})\lambda_p - (\mu_x-\hat{\mu}_{\vec{v}})\lambda_x \\ &> (2-\hat{\mu}_{\vec{v}})\lambda_p - (2-\hat{\mu}_{\vec{v}})\lambda_x. \end{aligned}$$

This implies that $\lambda_x > \lambda_p$. Using the same argument as in the proof of Case (3) Step 1, we will get a contradiction. Hence $E_1 \neq 0$ and we still have $(\pi^*D)E > 0$.

Step 2 We still have several inequalities here as analogues to (6.7) and (6.8) in Case (3) but with slight changes. We just list them in the following and leave their proofs to the interested reader.

For simplicity, we denote

$$\alpha := (EL_x + EL_p)^2 + (EL_p)^2 - (2 - \mu_x + \lambda_x)(EL_x + EL_p) - (2 - \hat{\mu}_{\vec{v}} + \lambda_p - \lambda_x)EL_p.$$

The first inequality corresponding to (6.7) is

$$0 \le ((\pi^* D)E)^2 - D^2((\pi^* D)E) - \alpha D^2, \tag{6.11}$$

and the second one corresponding to (6.8) is simply

$$\alpha < 0. \tag{6.12}$$

Moreover, (6.6), (6.4) and (6.5) also hold true here.
Similar to Case (3), we will frequently use the following quadratic polynomial in T:

$$K(T) := T^2 - (D^2)T - \alpha D^2. \tag{6.13}$$

Step 3 In this step, we prove the theorem when $\lambda_x = 0$.
Notice that now

$$\alpha = (EL_x + EL_p)^2 + (EL_p)^2 - (2 - \mu_x)(EL_x + EL_p) - (2 - \hat{\mu}_{\vec{v}} + \lambda_p)EL_p.$$

Following the manner of the proof for Case (3), we first claim that

$$EL_p < 2 - \hat{\mu}_{\vec{v}} + \lambda_p. \tag{6.14}$$

Otherwise, $EL_p \ge 2-\hat{\mu}_{\vec{v}}+\lambda_p$, and the inequality (6.12) now implies that $EL_x+EL_p < 2 - \mu_x$. This is a contradiction, because

$$2 - \mu_x \le 2 - \hat{\mu}_{\vec{v}} + \lambda_p \le EL_p \le EL_x + EL_p < 2 - \mu_x.$$

As a result of the above claim, we have $\lambda_p = 0$. In fact, if $\lambda_p > 0$, then by Theorem 1.2 (i), we deduce that

$$EL_p = \tilde{E}L_p + \lambda_p \ge \tilde{D}L_p + \lambda_p = 2 - \hat{\mu}_{\vec{v}} + \lambda_p.$$

This is impossible. Hence $\lambda_p = 0$ and (6.14) now becomes $EL_p < 2 - \hat{\mu}_{\vec{v}}$. Applying (6.4), we have

$$\lceil E \rceil L_p \le EL_p + \hat{\mu}_{\vec{v}} < 2,$$

i.e., $\lceil E \rceil L_p \le 1$.
Step 3.1. We first study the case when $\lceil E \rceil L_p = 0$. The proof here is similar to that of Theorem 4.1. Therefore, we just sketch it here and mention only the key ingredients, because all the reasoning here follows in the same way.
Now $EL_p = 0$. Then (6.12) becomes

$$(EL_x)^2 - (2 - \mu_x)EL_x < 0,$$

i.e., $0 < EL_x < 2-\mu_x$. By (6.5), we obtain that $\lceil E \rceil L_x = 1$, and thus $E_1 = bB$, where $BL_x = 1$, $BL_p = 0$ and $0 < b \le 1$. Since $\pi_* B$ is smooth at x, we have $(\pi^* D)E \ge b\beta_1$. Now (6.13) simply reads as

$$K(T) = T^2 - (D^2)T - D^2\left(b^2 - (2 - \mu_x)b\right) = T^2 - (D^2)T + bD^2(2 - \mu_x - b).$$

Applying the previous reasoning that we always use, in order to finish the proof here, we only need to prove that

$$b^2\beta_1^2 + bD^2(2-\mu_x-\beta_1-b) = K(b\beta_1) < 0.$$

When $\mu_{\vec{v}} \geq 2$, we have $\beta_1 = \frac{1}{2}(4-\mu_{\vec{v}})$. Thus

$$2-\mu_x-\beta_1-b = -\frac{\mu_x-\hat{\mu}_{\vec{v}}}{2} - b \leq -b.$$

Then it is easy to see that

$$K(b\beta_1) \leq b^2\beta_1^2 - b^2D^2 < \frac{1}{2}b^2\left((4-\mu_{\vec{v}})^2 - 2(2-\mu_x)^2 - 2(2-\hat{\mu}_{\vec{v}})^2\right) \leq 0.$$

When $\mu_{\vec{v}} < 2$, we have

$$\beta_1 = \frac{\beta_{2,x}+\beta_{2,p}}{\beta_{2,x}+\beta_{2,p}-(2-\mu_{\vec{v}})} = 1 + (2-\mu_{\vec{v}})\frac{\beta_1}{\beta_{2,x}+\beta_{2,p}}. \tag{6.15}$$

Notice that by (6.5),

$$b = EL_x \geq \lceil E\rceil L_x - \mu_x + \hat{\mu}_{\vec{v}} = 1 - \mu_x + \hat{\mu}_{\vec{v}}.$$

Also, we have

$$\frac{\beta_1}{\beta_{2,x}+\beta_{2,p}} = \frac{1}{\beta_{2,x}+\beta_{2,p}-(2-\mu_{\vec{v}})} \leq \frac{1}{(4-\mu_{\vec{v}})-(2-\mu_{\vec{v}})} = \frac{1}{2}. \tag{6.16}$$

Therefore, as an analogue of (4.9) in the proof of Theorem 4.1, here we deduce that

$$\begin{aligned}
2-\mu_x-\beta_1-b &= (1-\mu_x+\hat{\mu}_{\vec{v}}) + (1-\hat{\mu}_{\vec{v}}-\beta_1) - b \\
&= (1-\mu_x+\hat{\mu}_{\vec{v}}) - \left((2-\mu_{\vec{v}})\frac{\beta_1}{\beta_{2,x}+\beta_{2,p}} + \hat{\mu}_{\vec{v}}\right) - b \\
&\leq (1-\mu_x+\hat{\mu}_{\vec{v}}) - \frac{\beta_1}{\beta_{2,x}+\beta_{2,p}}(2-\mu_{\vec{v}}+2\hat{\mu}_{\vec{v}}) - b \\
&\leq (1-\mu_x+\hat{\mu}_{\vec{v}}) - \frac{\beta_1}{\beta_{2,x}+\beta_{2,p}}(2-2\mu_x+2\hat{\mu}_{\vec{v}}) - b \\
&= (1-\mu_x+\hat{\mu}_{\vec{v}})\left(1-\frac{2\beta_1}{\beta_{2,x}+\beta_{2,p}}\right) - b \\
&\leq b\left(1-\frac{2\beta_1}{\beta_{2,x}+\beta_{2,p}}\right) - b \\
&= -\left(\frac{2\beta_1}{\beta_{2,x}+\beta_{2,p}}\right)b.
\end{aligned}$$

Moreover, by our assumption, $D^2 > \beta_{2,x}^2 + \beta_{2,p}^2 \geq \frac{(\beta_{2,x}+\beta_{2,p})^2}{2}$. Thus it follows that

$$K(b\beta_1) \leq b^2\beta_1^2 - b^2D^2\left(\frac{2\beta_1}{\beta_{2,x}+\beta_{2,p}}\right) < b^2\left(\beta_1^2 - \beta_1(\beta_{2,x}+\beta_{2,p})\right) < 0.$$

This completes the proof for this case.

Step 3.2. We then study the case when $\lceil E\rceil L_p = 1$. This means that there is exactly one irreducible component A (with the multiplicity $0 < a \le 1$) in E_1 that meets L_p with $AL_p = 1$. Moreover, g_*A is smooth at p. Hence $(\pi^*D)E \ge 2a\beta_1$.
Rather than analogue to the proof of Theorem 4.1, the proof here is more similar to that of Theorem 5.1 Step 5. Again, we just sketch the proof here and leave some of the details to the interested reader.
Recall that now (6.13) becomes

$$\begin{aligned} K(T) &= T^2 - (D^2)T + D^2\left((2-\hat{\mu}_{\vec{v}})a - a^2 + (2-\mu_x)(EL_x + a) - (EL_x + a)^2\right) \\ &= T^2 - (D^2)T + D^2\left((4-\mu_{\vec{v}})a - 2a^2 + (2-\mu_x - 2a)EL_x - (EL_x)^2\right). \end{aligned}$$

The fact that $(\pi^*D)E \ge 2a\beta_1$ guarantees that $K(2a\beta_1) \ge 0$. That is,

$$\begin{aligned} 0 &\le 4a^2\beta_1^2 - 2a\beta_1 D^2 + D^2\left((4-\mu_{\vec{v}})a - 2a^2 + (2-\mu_x-2a)EL_x - (EL_x)^2\right) \\ &= a\left(4a\beta_1^2 + (4-\mu_{\vec{v}} - 2\beta_1 - 2a)D^2\right) + D^2\left((2-\mu_x-2a)EL_x - (EL_x)^2\right). \end{aligned}$$

Here we claim that

$$4a\beta_1^2 + (4-\mu_{\vec{v}} - 2\beta_1 - 2a)D^2 < 0.$$

Again, our discussion is based on $\mu_{\vec{v}}$. If $\mu_{\vec{v}} \ge 2$, then $\beta_1 = \frac{1}{2}(4-\mu_{\vec{v}})$. We simply have

$$4 - \mu_{\vec{v}} - 2\beta_1 - 2a = -2a$$

and thus

$$4a\beta_1^2 + (4-\mu_{\vec{v}} - 2\beta_1 - 2a)D^2 = a((4-\mu_{\vec{v}})^2 - 2D^2) < 0.$$

Now we consider the case when $\mu_{\vec{v}} < 2$. By (6.4), $a = EL_p \ge \lceil E\rceil L_p - \hat{\mu}_{\vec{v}} = 1 - \hat{\mu}_{\vec{v}}$. Together with (6.15) and (6.16) again, we have

$$\begin{aligned} 4 - \mu_{\vec{v}} - 2\beta_1 - 2a &= (2-\mu_{\vec{v}}) + 2(1-\beta_1) - 2a \\ &= (2-\mu_{\vec{v}})\left(1 - \frac{2\beta_1}{\beta_{2,x}+\beta_{2,p}}\right) - 2a \\ &\le 2a\left(1 - \frac{2\beta_1}{\beta_{2,x}+\beta_{2,p}}\right) - 2a \\ &= -4a\left(\frac{\beta_1}{\beta_{2,x}+\beta_{2,p}}\right). \end{aligned}$$

Therefore, using the fact that $D^2 > \frac{(\beta_{2,x}+\beta_{2,p})^2}{2}$ and (6.16) again, we have

$$4a\beta_1^2 + (4-\mu_{\vec{v}} - 2\beta_1 - 2a)D^2 < 4a\left(\beta_1^2 - \frac{\beta_1(\beta_{2,x}+\beta_{2,p})}{2}\right) \le 0.$$

The proof of the claim is completed.
A consequence of the above claim is that

$$(2-\mu_x-2a)EL_x - (EL_x)^2 > 0,$$

i.e., $0 < EL_x < 2 - \mu_x - 2a$. Together with (6.5) and the fact that $a \ge 1 - \hat{\mu}_{\vec{v}}$, we see that

$$1 \le \lceil E\rceil L_x < 2 - \mu_x - 2a + \mu_x - \hat{\mu}_{\vec{v}} = 2 - (a + \hat{\mu}_{\vec{v}}) - a \le 1 - a < 1.$$

This is a contradiction. Hence the proof of this case is completed.

Step 4 From now on till the end of this section, we assume that $\lambda_x > 0$. Our goal in this step is to prove that

$$\lambda_p = 0, \quad \lambda_x = 1, \quad EL_p = 0, \quad EL_x < 3 - \mu_x, \quad \lceil E \rceil L_x = 2, \quad \hat{\mu}_{\vec{v}} < 1. \tag{6.17}$$

The proof is in the same flavor as that of Case (3). Since there are some differences, we give an explicit proof here.

We first prove that $\lambda_p = 0$. If not, then $\lambda_p > 0$ and by Theorem 1.2 (i), we have

$$EL_p = \tilde{E}L_p + \lambda_p - \lambda_x \geq \tilde{D}L_p + \lambda_p - \lambda_x = 2 - \hat{\mu}_{\vec{v}} + \lambda_p - \lambda_x.$$

By (6.12), the above inequality forces that

$$(EL_x + EL_p)^2 - (2 - \mu_x + \lambda_x)(EL_x + EL_p) < 0,$$

i.e., $0 < EL_x + EL_p < 2 - \mu_x + \lambda_x$. On the other hand, both $\lambda_x, \lambda_p > 0$ now. Thus Theorem 1.2 (i) also implies that

$$2 - \mu_x = \tilde{D}(L_x + L_p) \leq \tilde{E}(L_x + L_p) = EL_x + EL_p - \lambda_x.$$

It is a contradiction.

We then prove that $\lambda_x = 1$ and $EL_p = 0$. In fact, by Theorem 1.2 (i), $\lambda_x > 0$ implies

$$EL_x = \tilde{E}L_x - \lambda_x L_x^2 \geq \tilde{D}L_x + 2\lambda_x = (2 - \mu_x) - (2 - \hat{\mu}_{\vec{v}}) + 2\lambda_x = 2\lambda_x - \mu_x + \hat{\mu}_{\vec{v}}.$$

We claim that $EL_p < 2 - \hat{\mu}_{\vec{v}} - \lambda_x$. Suppose on the contrary that $EL_p \geq 2 - \hat{\mu}_{\vec{v}} - \lambda_x$. Then by (6.12), $EL_x + EL_p < 2 - \mu_x + \lambda_x$. Thus we have

$$EL_p < 2 - \mu_x + \lambda_x - EL_x \leq 2 - \mu_x + \lambda_x - (2\lambda_x - \mu_x + \hat{\mu}_{\vec{v}}) = 2 - \hat{\mu}_{\vec{v}} - \lambda_x,$$

which is again a contradiction. Hence the claim holds. Since λ_x is an integer, this claim simply implies that $\lambda_x = 1$ and $EL_p < 1 - \hat{\mu}_{\vec{v}}$. Moreover, by (6.4), we deduce that $\lceil E \rceil L_p \leq EL_p + \hat{\mu}_{\vec{v}} < 1$, i.e., $\lceil E \rceil L_p = 0$ and $EL_p = 0$.

For the rest of (6.17), we notice that $\lceil \tilde{E} \rceil = \lceil E \rceil + L_x$. Thus Theorem 1.2 (i) yields

$$\lceil E \rceil L_x = (\lceil \tilde{E} \rceil - L_x)L_x \geq \lceil \tilde{D} \rceil L_x + 2 = 2.$$

On the other hand, now (6.12) becomes $(EL_x)^2 - (3 - \mu_x)EL_x < 0$ in this case, which is equivalent to $0 < EL_x < 3 - \mu_x$. Combine this with (6.5), we deduce that

$$\lceil E \rceil L_x \leq EL_x + \mu_x - \hat{\mu}_{\vec{v}} < 3 - \hat{\mu}_{\vec{v}}.$$

All the above inequalities force that $\lceil E \rceil L_x = 2$ and thus $\hat{\mu}_{\vec{v}} < 3 - \lceil E \rceil L_x = 1$. This completes the proof of this step.

Step 5 In this step, we complete the whole proof when $\lambda_x > 0$. The proof can be reduced to that for Case (3), so we just sketch it here.

In fact, (6.17) shows that there are only two possibilities for E_1:

(i) $E_1 = b_1 B_1 + b_2 B_2$, where $b_1, b_2 > 0$, $b_1 + b_2 < 3 - \mu_x$, $B_1 L_x = B_2 L_x = 1$ and $B_1 L_p = B_2 L_p = 0$;

(ii) $E_1 = bB$, where $0 < 2b < 3 - \mu_x$, $BL_x = 2$ and $BL_p = 0$.

We still denote $b := \frac{b_1 + b_2}{2}$ if we are in Case (i). Then $EL_x = 2b$ and (6.13) can be simplified as

$$K(T) = T^2 - (D^2)T - D^2\left(4b^2 - 2(3 - \mu_x)b\right).$$

Notice that this is exactly the same as $H(T)$ in Case (3) when $\lambda_x > 0$.
The rest of the proof is routine, and we only need to show that

$$4b^2\beta_1^2 + 2bD^2(3 - \mu_x - \beta_1 - 2b) = K(2b\beta_1) < 0.$$

A key fact here is that by (6.5), we have

$$2b = EL_x \geq \lceil E \rceil L_x - \mu_x + \hat{\mu}_{\vec{v}} = 2 - \mu_x + \hat{\mu}_{\vec{v}}.$$

When $\mu_{\vec{v}} \geq 2$, $\beta_1 = \frac{1}{2}(4 - \mu_{\vec{v}})$. Thus the above inequality yields

$$3 - \mu_x - \beta_1 - 2b = 1 - \frac{\mu_x - \hat{\mu}_{\vec{v}}}{2} - 2b \leq b - 2b = -b.$$

Therefore, we have

$$K(2b\beta_1) \leq 4b^2\beta_1^2 - 2b^2D^2 < b^2\left((4 - \mu_{\vec{v}})^2 - 2(2 - \mu_x)^2 - 2(2 - \hat{\mu}_{\vec{v}})^2\right) \leq 0.$$

When $\mu_{\vec{v}} < 2$, we deduce that

$$\beta_1 = \frac{\beta_{2,x} + \beta_{2,p}}{\beta_{2,x} + \beta_{2,p} - (2 - \mu_{\vec{v}})} = 1 + (2 - \mu_{\vec{v}})\frac{\beta_1}{\beta_{2,x} + \beta_{2,p}}.$$

Notice that we have

$$\frac{\beta_1}{\beta_{2,x} + \beta_{2,p}} = \frac{1}{\beta_{2,x} + \beta_{2,p} - (2 - \mu_{\vec{v}})} \leq \frac{1}{(4 - \mu_{\vec{v}}) - (2 - \mu_{\vec{v}})} = \frac{1}{2}.$$

Therefore, we can just adopt the proof in Case (3) almost identically to conclude that

$$\begin{aligned} 3 - \mu_x - \beta_1 - 2b &= (2 + \hat{\mu}_{\vec{v}} - \mu_x) + (1 - \hat{\mu}_{\vec{v}} - \beta_1) - 2b \\ &= (2 + \hat{\mu}_{\vec{v}} - \mu_x) - \left((2 - \mu_{\vec{v}})\frac{\beta_1}{\beta_{2,x} + \beta_{2,p}} + \hat{\mu}_{\vec{v}}\right) - 2b \\ &\leq (2 + \hat{\mu}_{\vec{v}} - \mu_x) - \frac{\beta_1}{\beta_{2,x} + \beta_{2,p}}(2 + \hat{\mu}_{\vec{v}} - \mu_x) - 2b \\ &\leq 2b\left(1 - \frac{\beta_1}{\beta_{2,x} + \beta_{2,p}}\right) - 2b \\ &= -2b\left(\frac{\beta_1}{\beta_{2,x} + \beta_{2,p}}\right). \end{aligned}$$

Thus it follows that

$$K(2b\beta_1) < 4b^2\left(\beta_1^2 - \frac{\beta_1(\beta_{2,x} + \beta_{2,p})}{2}\right) < 0.$$

This completes the whole proof.

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